

An inequality for longitudinal and transverse wave attenuation coefficients

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Total absorption, defined as the net flux of energy out of a bounded region averaged over one cycle for time harmonic motion, must be non-negative when there are no sources of energy within the region. This passivity condition places constraints on the non-dimensional absorption coefficients of longitudinal and transverse waves, γ_L and γ_T , in isotropic linearly viscoelastic materials. Typically, γ_L, γ_T are small, in which case the constraints imply that coefficients of attenuation per unit length, α_L, α_T , must satisfy the inequality $\alpha_L/\alpha_T \geq 4c_T^3/3c_L^3$ where c_L, c_T are the wave speeds. This inequality, which as far as the author is aware, has not been presented before, provides a relative bound on wave speed in terms of attenuation, or vice versa. It also serves as a check on the consistency of ultrasonic measurements from the literature, with most but not all of the data considered passing the positive absorption test. © 2017 Acoustical Society of America.

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I. INTRODUCTION

When an acoustic wave is incident on a passive obstacle, i.e., one with no active sources of energy present, the scattering process should not result in more energy than that of the incident wave itself. If the object is viscoelastic then the total energy should decrease by virtue of the passive absorbing properties of viscoelastic solids. The absorption of wave energy in solids has been considered from many points of view, ranging from its thermodynamic and molecular origins (Ref. 1, Chaps. 11–13), to sub-wavelength scattering effects at the crystalline and granular scale (Ref. 2, Chap. 9), to physically consistent mathematical models (Ref. 3, Chaps. 3.E, 5.C–E and Ref. 4, Chap. 2). The interest here is in linearly viscoelastic materials, and the implications of positive absorption, also known as passivity.⁵ In particular, we derive a new and useful relation between the attenuation coefficients for longitudinal and transverse waves isotropic solids.

We begin in Sec. II with a background review of absorption and viscoelasticity theory. The main results are presented in Sec. III, where they are discussed in the context of published data on ultrasonic attenuation coefficients.

II. BACKGROUND REVIEW

A. Absorption

The energy lost in a passive target subject to an incident time harmonic acoustic wave is defined by the time-averaged outward net flux over the bounding surface of the object,

$$P_{\text{abs}} = - \int_S \langle p \mathbf{v} \rangle \cdot d\mathbf{s} \geq 0. \quad (1)$$

Here, p denotes acoustic pressure, $\mathbf{v}$ the particle velocity, S is the enclosing surface, $d\mathbf{s} = \mathbf{n}dS$ is the surface element with the unit normal $\mathbf{n}$ outwards, and $\langle \cdot \rangle$ indicates the average over a period. To be specific consider a single solid object of volume V . Traction continuity across S implies $-p\mathbf{n} = \boldsymbol{\sigma}\mathbf{n}$ where $\boldsymbol{\sigma}$ is the symmetric stress in the solid. Normal velocity is also continuous, and the divergence theorem therefore implies the equivalent definition:

$$P_{\text{abs}} = \int_V \text{div} \langle \boldsymbol{\sigma} \mathbf{v} \rangle dV. \quad (2)$$

B. Viscoelasticity

In order to accommodate a viscoelastic constitutive relation it is necessary to work with complex-valued quantities. The real and imaginary parts of material properties, such as density ρ and elastic stiffness $\mathbf{C}$, are denoted in standard fashion using single and double primes:

$$\rho = \rho' + i\rho'', \quad \mathbf{C} = \mathbf{C}' + i\mathbf{C}''. \quad (3)$$

The density is also considered complex-valued as this represents a better alternative to using viscoelastic moduli in certain materials encountered especially in geophysical acoustics, such as poroelastic continua where the complex ρ includes Darcy-like flow effects on the overall inertia.^{6,7}

Notwithstanding the danger of confusion, we now let the field variables $\boldsymbol{\sigma}$ and $\mathbf{v}$ (until now assumed to be real quantities) denote complex-valued amplitudes with the time dependence $e^{-i\omega t}$, $\omega > 0$, understood. The real physical quantities are $\text{Re} \boldsymbol{\sigma}(\mathbf{x})e^{-i\omega t}$ and $\text{Re} \mathbf{v}(\mathbf{x})e^{-i\omega t}$. With $*$ denoting the complex-conjugate, Eq. (2) becomes

$$P_{\text{abs}} = \frac{1}{2} \text{Re} \int_V \text{div} \boldsymbol{\sigma} \mathbf{v}^* dV. \quad (4)$$

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Using the equation of motion,

$$\operatorname{div} \boldsymbol{\sigma} + i\omega \rho \mathbf{v} = 0, \quad (5)$$

the absorption can be expressed:

$$P_{\text{abs}} = \frac{\omega}{2} \int_V \rho'' |\mathbf{v}|^2 dV + \frac{1}{2} \operatorname{Re} \int_V \operatorname{tr}(\boldsymbol{\sigma} \operatorname{grad} \mathbf{v}^*) dV. \quad (6)$$

Strain is the symmetric part of the displacement gradient $\boldsymbol{\varepsilon} = \operatorname{Sym} \mathbf{U}$, $\mathbf{U} = \operatorname{grad} \mathbf{u}$, where $\mathbf{u} = (-i\omega)^{-1} \mathbf{v}$ is the displacement.

The viscoelastic linear constitutive relation between stress and strain in its most general form (Ref. 4, Chap. 4) posits stress as a convolution of strain with a time-dependent stiffness. The relation is then linear in the frequency domain,

$$\boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon} \iff \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (7)$$

with complex-valued stiffness $\mathbf{C}$ defined by the Fourier transform of the time-dependent moduli. The latter are assumed to have the symmetries associated with a symmetric strain and a symmetric stress, implying $C_{ijkl} = C_{ijlk}$, $C_{ijkl} = C_{jikl}$. Hence, $\boldsymbol{\sigma} = \mathbf{C} \mathbf{U}$, and $\operatorname{Re} \operatorname{tr}(\boldsymbol{\sigma} \operatorname{grad} \mathbf{v}^*) = -\omega \operatorname{Im} \operatorname{tr} \mathbf{U}^* \mathbf{C} \mathbf{U}$. In purely elastic solids the stiffness $\mathbf{C}$ is real-valued and satisfies the usual symmetry in terms of the interchange of the “major indices” associated with a reversible strain energy function. This property does not extend to viscoelasticity. However, it is expected in the quasistatic limit, and it is therefore reasonable to assume that it holds for the real part of $\mathbf{C}$, at least in some range of frequencies, but is not valid for the imaginary part of $\mathbf{C}$. We therefore split the imaginary part into symmetric and anti-symmetric parts,

$$\begin{aligned} C'_{ijkl} &= C'_{klij}, \quad \mathbf{C}'' = \mathbf{C}''_S + \mathbf{C}''_A, \\ C''_{Sijkl} &= \mathbf{C}''_{Sklij}, \quad \mathbf{C}''_{Aijkl} = -\mathbf{C}''_{Aklij}. \end{aligned} \quad (8)$$

The absorption can then be written

$$P_{\text{abs}} = \frac{\omega}{2} \int_V \left(\rho'' |\mathbf{v}|^2 - \operatorname{tr} \mathbf{U}^* \mathbf{C}''_S \mathbf{U} \right) dV. \quad (9)$$

This is always non-negative if and only if ρ'' is non-negative and $\mathbf{C}''_S$ is negative semi-definite in the sense that $\operatorname{tr} \mathbf{A}^* \mathbf{C}''_S \mathbf{A} \leq 0$ for all $\mathbf{A} = \mathbf{A}^T \neq 0$. In summary,

$$P_{\text{abs}} \geq 0 \iff \rho'' \geq 0 \text{ and } \mathbf{C}''_S \text{ is negative semi-definite.} \quad (10)$$

The absorption is identically zero if the density is purely real and $\mathbf{C}''_S$ vanishes. The latter condition is equivalent to the requirement that $\mathbf{C}$ is Hermitian, i.e., $C_{ijkl} = C_{klij}^*$. Hence,

$$P_{\text{abs}} = 0 \iff \rho \text{ is real and } \mathbf{C} \text{ is Hermitian.} \quad (11)$$

Note that the present results do not rely upon the necessary consequences of causality on the analytic properties of the complex-valued moduli, a topic that has been addressed well elsewhere, e.g., Ref. 8.

1. Isotropic viscoelasticity

The moduli have standard form with two complex-valued Lamé moduli, λ and μ ,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (12)$$

This stiffness satisfies Eq. (8) with $\mathbf{C}''_A = 0$, i.e., $\mathbf{C}$ is not Hermitian, and it is therefore expected that P_{abs} will be non-zero. The absorption becomes

$$P_{\text{abs}} = \frac{\omega}{2} \int_V \left[\rho'' |\mathbf{v}|^2 - \kappa'' |\operatorname{tr} \boldsymbol{\varepsilon}|^2 - \mu'' \operatorname{tr}(\boldsymbol{\varepsilon}_d \boldsymbol{\varepsilon}_d^*) \right] dV, \quad (13)$$

where $\kappa = \lambda + (2/3)\mu$ is the bulk modulus and $\boldsymbol{\varepsilon}_d = \boldsymbol{\varepsilon} - \mathbf{I}(1/3) \operatorname{tr} \boldsymbol{\varepsilon}$ is the deviatoric strain. Hence,

$$P_{\text{abs}} \geq 0 \iff \rho'' \geq 0, \kappa'' \leq 0 \text{ and } \mu'' \leq 0. \quad (14)$$

This places constraints on the imaginary parts of the elastic moduli.

For instance, the Kelvin–Voigt model assumes that the stress is of the form

$$\boldsymbol{\sigma} = (\lambda_e + \lambda_v \partial_t) (\operatorname{tr} \boldsymbol{\varepsilon}) \mathbf{I} + 2(\mu_e + \mu_v \partial_t) \boldsymbol{\varepsilon}, \quad (15)$$

where λ_e, μ_e are the elastic moduli, λ_v, μ_v are generalized viscosities, all real quantities. Hence, $\lambda'' = -i\omega \lambda_v$, $\mu'' = -i\omega \mu_v$, and the constraints [Eq. (14)] are satisfied if $\lambda_v + (2/3)\mu_v > 0$ and $\mu_v > 0$.

Positive absorption has implications for other elastic moduli. For any complex-valued elastic modulus $M = M' + iM''$, the loss factor (Ref. 9, p. 7) is defined as $d_M = -M''/M'$, so that $M = (1 - id_M)M'$. The constraints [Eq. (14)] imply that $d_\mu \geq 0$, $d_\kappa \geq 0$. The longitudinal modulus, Poisson’s ratio, Young’s modulus, and area modulus¹⁰ are $L = \lambda + 2\mu$, $\nu = \lambda/[2(\lambda + \mu)]$, $E = 2(1 + \nu)\mu$, and $A = (1 - \nu)^{-1}(1 + \nu)\mu$, respectively. Their loss factors are, to leading order in d_μ and d_κ ,

$$d_A = c_1 d_\mu + (1 - c_1) d_\kappa, \quad (16a)$$

$$d_L = (1 - c_1) d_\mu + c_1 d_\kappa, \quad (16b)$$

$$d_E = c_2 d_\mu + (1 - c_2) d_\kappa, \quad (16c)$$

$$d_\nu = (3\nu')^{-1} (1 + \nu') (1 - 2\nu') (d_\mu - d_\kappa), \quad (16d)$$

where $c_1 = (1/3)(1 + \nu')/(1 - \nu')$, $c_2 = (2/3)(1 + \nu')$. Positive definiteness of the elastic strain energy requires that $-1 < \nu' < 1/2$, and hence $0 < c_1 < c_2 < 1$ and the loss factors d_L , d_E , and d_A are always non-negative with values between d_μ and d_κ . The Poisson’s ratio loss factor may in principle be of either sign, although reported values, e.g., for rubber,⁸ are positive indicating $d_\mu > d_\kappa$.

III. ELASTIC WAVE DAMPING

A. Constraints on absorption coefficients

The complex-valued longitudinal and transverse wave-numbers, $\tilde{k}_L$ and $\tilde{k}_T$, are

$$\tilde{k}_L = \frac{\omega}{\tilde{c}_L}, \quad \tilde{k}_T = \frac{\omega}{\tilde{c}_T}, \quad \tilde{c}_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \tilde{c}_T = \sqrt{\frac{\mu}{\rho}}. \quad (17)$$

A common method for characterizing viscoelasticity is via ultrasonic measurement of the complex-valued wavenumbers. Specifically, we assume that the viscoelastic moduli are defined in terms of two real-valued wave speeds c_L , c_T , and two non-dimensional absorption coefficients γ_L , γ_T :

$$\tilde{k}_M = k_M(1 + i\gamma_M) \text{ with } k_M = \frac{\omega}{c_M}, \quad M = L, T. \quad (18)$$

According to this definition, $c_M = (\text{Re } \tilde{c}_M^{-1})^{-1}$, $M = L, T$. It follows from the imaginary parts of the identities $\tilde{c}_M^2 = c_M^2 / (1 + i\gamma_M)^2$, $M = L, T$ that

$$\text{Im} \frac{\mu}{\rho} = -\frac{2c_T^2\gamma_T}{(1 + \gamma_T^2)^2}, \quad (19a)$$

$$\text{Im} \frac{\kappa}{\rho} = -\frac{2c_L^2\gamma_L}{(1 + \gamma_L^2)^2} + \frac{8c_T^2\gamma_T}{3(1 + \gamma_T^2)^2}. \quad (19b)$$

The three constraints of Eq. (14) imply that the left members in Eq. (19) are non-positive, and hence, we obtain the main result of the paper:

$$\gamma_T \geq 0, \quad \gamma \geq 0; \quad \gamma \equiv \frac{\gamma_L}{(1 + \gamma_L^2)^2} - \frac{4c_T^2}{3c_L^2} \frac{\gamma_T}{(1 + \gamma_T^2)^2}. \quad (20)$$

The first is usually satisfied because both γ_T and γ_L are specifically taken as non-negative. The condition for γ places a constraint on γ_T and γ_L that depends upon the ratio of the undamped wave speeds. The value of γ for measurements on Polymethylmethacrylate (PMMA) and other polymers are given in Table I, all satisfying the condition $\gamma > 0$.

In practice the values of γ_L and γ_T are small, so that Eq. (20)₂ ($\gamma \geq 0$) can be safely replaced by

$$\frac{\gamma_L}{\gamma_T} \geq \frac{4c_T^2}{3c_L^2} \quad \text{for } \gamma_L, \gamma_T \ll 1. \quad (21)$$

Thus, the ratio of the wave absorption factors must satisfy a strict but simple inequality when the attenuation is small. Note that the parameter depends upon the real part of the Poisson's ratio,

TABLE I. An example of some absorption coefficients and the associated value of γ from Eq. (20). The numerical values from Ref. 11 are for measurements at room temperature of $\alpha\lambda = 2\pi\gamma$, the attenuation per wavelength in dB, and uses the relation (Ref. 13) $\alpha\lambda$ (dB) = $40\pi\gamma/\ln 10$. Those of Ref. 12 are based on measurements of the attenuation coefficients α_L and α_T (dB/cm) at 25 °C and frequency 1.8 MHz. Speeds are in m/s.

Material	Source	c_L	c_T	γ_L	γ_T	γ
PMMA	Ref. 11	2690	1340	0.0035	0.0053	0.0017
Polyethylene	Ref. 11	2430	950	0.0073	0.0220	0.0028
Phenolic polymer	Ref. 12	2840	1320	0.0119	0.0255	0.0045

$$\frac{4c_T^2}{3c_L^2} = \frac{2(1 - 2\nu')}{3(1 - \nu')}. \quad (22)$$

B. Constraints on attenuation coefficients

The amplitude of either wave type decays as $e^{-k_M\gamma_M x}$, $M = L$ or T . Attenuation as measured in dB/cm, for instance, defines the logarithm of the amplitude, and is therefore equivalent to measurement of $\alpha_M \equiv k_M\gamma_M$, $M = L$ or T , since $\alpha = k\gamma 20/\ln 10$, where k is wavenumber in cm^{-1} . The multiplicative factor is irrelevant if we are only concerned with the ratio of the two attenuations. In the small attenuation regime Eq. (21) then implies

$$\frac{\alpha_L}{\alpha_T} \geq \frac{4c_T^3}{3c_L^3}. \quad (23)$$

We consider the passivity constraint Eq. (23) in light of some reported ultrasonic data^{14,15} in Table II. The values of absorption indicates loss moduli of 1% or less than the real parts, i.e., small attenuation for which the criterion Eq. (23) applies, and is met for the data in Table II. Velocity and attenuation data for styrene-butadiene rubber¹⁶ at 1 MHz over a temperature range from 0° to 20° is consistent with Eq. (23).

Layman *et al.*¹⁷ provide curve-fitted equations, Eqs. (15)–(18) in Ref. 17, for all four of the parameters in Eq. (23) based on ultrasonic measurements on a particulate composite sample over a broad frequency range (2–10 MHz). It may be easily verified that these wave speeds and attenuations satisfy the condition (23) over the entire range of frequencies considered. Measurements of high frequency (25–65 MHz) velocities and attenuation in passive materials for ultrasonic transducers at room temperature are given in Ref. 18. The materials include alumina/EPO-TEK 301 composites and tungsten/EPOTEK 301 composites. We have checked that all of the parameters reported satisfy Eq. (23). Pinton *et al.*¹⁹ measured attenuation and absorption of ultrasound in skull bone. They reported longitudinal absorption of 2.7 dB/cm and shear absorption of 5.4 dB/cm at the assumed longitudinal and shear wave speeds of 3000 m/s and 1500 m/s, respectively, which clearly satisfies the positive absorption condition (23).

TABLE II. $f = \omega/2\pi$. Velocity and attenuation data for PMMA at 22.2 °C and atmospheric pressure (Ref. 14). The data for the other polymers are from Ref. 15, where polymer #1 is poly(4-methyl pentene-1) and polymer #2 is poly(acrylonitrile-butadiene-styrene).

Material	f MHz	c_L m/s	c_T m/s	α_L dB/cm	α_T dB/cm	Eq. (23)
PMMA	6	2756.4	1401.5	4.97	13.64	✓
PMMA	10	2760.5	1404.8	7.69	23.99	✓
PMMA	18	2764.2	1405.1	12.68	37.21	✓
PMMA	20	2765.1	1405.7	12.64	44.28	✓
PMMA	30	2765.5	1406.1	19.64	63.94	✓
polymer #1	1.8	2180	1080	1.4	6.7	✓
polymer #2	1.8	2040	830	1.8	15	✓

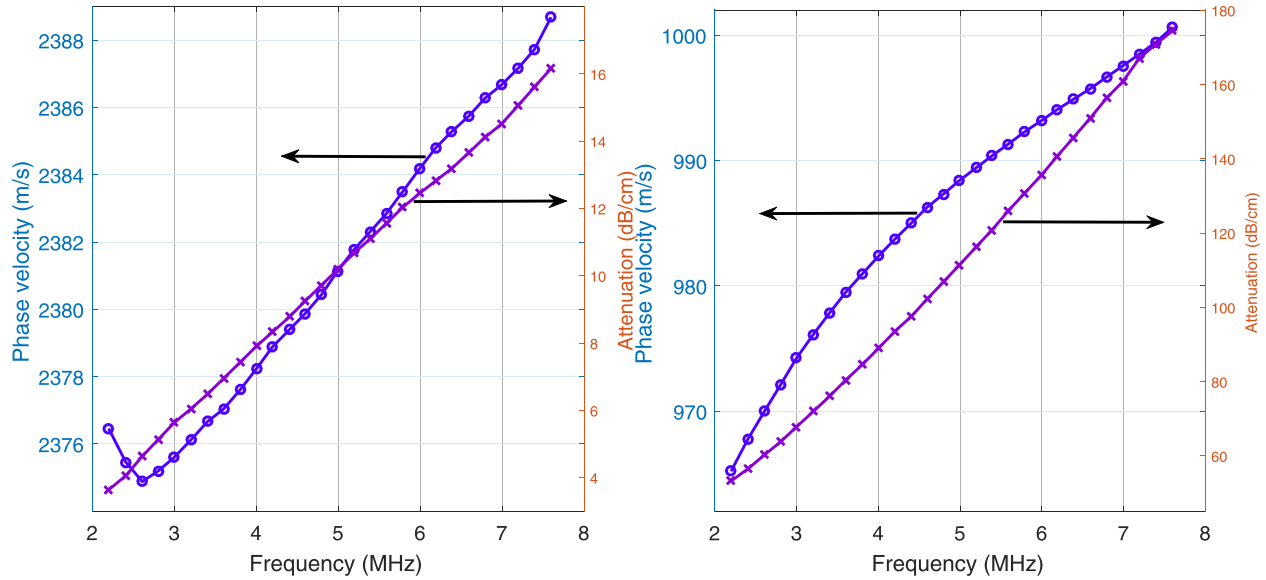


FIG. 1. (Color online) Phase velocity and attenuation as functions of frequency for a high-density polyethylene sample: longitudinal (left) and transverse (right). Data from Ref. 20, Fig. 4.

As a final example we consider the velocity and attenuation data shown in Fig. 1, which reproduces as accurately as possible the measurements reported in Ref. 20 for a high-density polyethylene sample. The calculated values of the two terms in Eq. (23) are plotted in Fig. 2. The smooth curve for α_L/α_T uses a fitted power law model for attenuation proposed by Szabo and Wu²¹ and applied to the data of Wu.²⁰ The model assumes the attenuation has frequency dependence of the form

$$\alpha_M = \alpha_{0M} + \alpha_{1M} |f|^{y_M}, \quad M = L, T, \quad (24)$$

where the coefficients for the high-density polyethylene sample are in Table III. Note that the formula corresponding to Eq. (24) in Szabo and Wu²¹ has $|\omega|$ instead of $|f|$, but we find that the numbers reported there are for Eq. (24).

It is evident from the relative positions of the curves in Fig. 2 that the passivity inequality (23) is not satisfied at any of the frequencies considered. We note that Eq. (23) is an

approximation valid for small values of attenuation. The precise condition $\gamma \geq 0$ may be written in similar form as

$$\frac{\alpha_L}{\alpha_T} \geq \frac{4c_T^3}{3c_L^3} I \quad \text{where } I = \left(\frac{1 + \gamma_L^2}{1 + \gamma_T^2} \right)^2. \quad (25)$$

Generally, the factor I is close to but slightly less than unity, with $0.9964 \leq I \leq 0.9970$ for the data of Fig. 1. The effect of including this term in Fig. 2 is almost imperceptible, i.e., the data is in violation of the passivity condition for the bulk modulus. We can only conclude that the attenuation data for high-density polyethylene is not consistent with a passive linear viscoelastic model with frequency dependent complex-valued density and elastic moduli.

We note, however, that for the other data sets reported in Ref. 20, for samples of low-density polyethylene and Lexan Plexiglas, we find the passivity condition (23) is satisfied.

IV. CONCLUSION

The main finding is the constraint on the non-dimensional absorption parameter γ in Eq. (20). For given values of wave speeds and shear absorption γ_T this sets a lower bound on the longitudinal absorption γ_L . The inequality $\gamma \geq 0$ has direct interpretation when absorption is small ($\gamma_L, \gamma_T \ll 1$), implying that the ratio of the attenuations per unit length, α_L/α_T , has a lower bound that depends on the

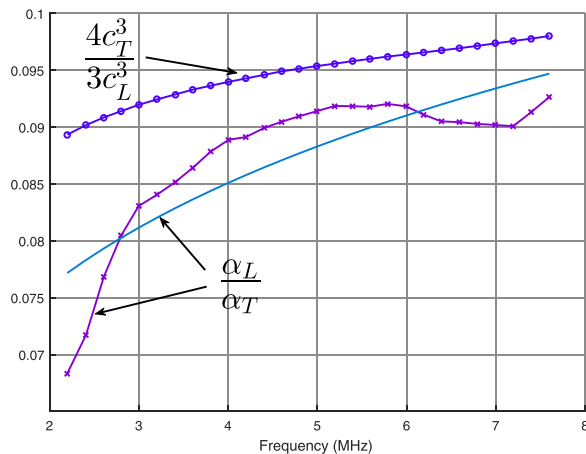


FIG. 2. (Color online) The two terms in the passivity inequality Eq. (23) calculated from the data of Fig. 1. The continuous curve for α_L/α_T is based on Eq. (24) and Table III.

TABLE III. Absorption coefficients α_{0M} (dB/cm), α_{1M} [dB/(MHz) ^{y_M} - cm], and exponential powers y_M for high density polyethylene. Speeds, in m/s, are for the reference frequency 4.8 MHz. Data from Ref. 21 except for α_{0T} which is not given there but is found here by least square fitting, and y_T which is found to provide better accuracy for α_T than the value 0.95 in Ref. 21.

c_L	c_T	α_{0L}	α_{1L}	y_L	α_{0T}	α_{1T}	y_T
2380	987	0	1.522	1.171	-0.517	22.80	1.00

ratio of the wave speeds, Eq. (23). The lower bound tends to zero as the Poisson's ratio of the material tends to $1/2$ ($\Leftrightarrow c_T/c_L \rightarrow 0$). For instance, measurements of ultrasonic properties of soft tissues and tissue-like materials²² shows shear wave attenuation coefficients on the order of 10^4 times the longitudinal wave attenuation coefficients. Equation (23) then implies, under the small absorption assumption, that the transverse wave speed must be less than 4.2% the value of the longitudinal speed.

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