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Elastodynamics of radially inhomogeneous spherically anisotropic elastic materials in the Stroh formalism

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A method for solving elastodynamic problems in radially inhomogeneous elastic materials with spherical anisotropy is presented, i.e. materials having $c_{ijkl} = c_{ijkl}(r)$ in a spherical coordinate system $\{r, \theta, \phi\}$. The time-harmonic displacement field $\mathbf{u}(r, \theta, \phi)$ is expanded in a separation of variables form with dependence on θ, ϕ described by vector spherical harmonics with r-dependent amplitudes. It is proved that such separation of variables solution is generally possible only if the spherical anisotropy is restricted to transverse isotropy (TI) with the principal axis in the radial direction, in which case the amplitudes are determined by a first-order ordinary differential system. Restricted forms of the displacement field, such as $\mathbf{u}(r, \theta)$, admit this type of separation of variables solution for certain lower material symmetries. These results extend the Stroh formalism of elastodynamics in rectangular and cylindrical systems to spherical coordinates.

Keywords: elastodynamics; spherical anisotropy; vector spherical harmonics; Stroh

1. Introduction

The Stroh formalism (Wu *et al.* 1991), which recasts equations of time-harmonic elastodynamics in the form of a first-order ordinary differential system (ODS), is a powerful technique in dealing with elastic materials inhomogeneous in one coordinate. The method was originally established for rectangular coordinates (e.g. Ting 1996; Wu 1998; Shuvalov 2000; Shuvalov *et al.* 2004) and has been developed for applications in cylindrical coordinate systems (Shuvalov 2003*a*; Norris & Shuvalov 2010). One complicating factor for cylindrical, when compared with rectangular, anisotropy is that the radial and azimuthal basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend on the angular coordinate θ. This, however, does not hamper separation of variables and allows for a Stroh-like ODS provided that the material coefficients depend either on the radial or axial coordinate r or z. The situation is quite different for spherical anisotropy. The lowest anisotropy that supports general displacement fields which can be described by a separation of variables appears to be transverse isotropy (TI) with the axis of symmetry in the radial direction, as was assumed in the derivations by Hu (1954) for statics and Shul'ga

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et al. (1988) for dynamics (the term 'spherically isotropic' used by Hu (1954) and others is equivalent to TI). A state space system was developed for radially inhomogeneous TI by Shul'ga *et al.* (1988), who also identified two distinct types of wave motion solutions: an uncoupled pure shear motion and a coupled radial–angular solution pair. The state vector approach has been applied to vibrations of thick-walled TI shells (Chen & Ding 2001; Hasheminejad & Maleki 2009) and further developed for piezoelectric shells (Scandrett 2002). Lower symmetry can support specific types of kinematically restricted deformation. The most general form of spherical anisotropy which admits static solutions of the form $\mathbf{u} = u(r)\mathbf{e}_r$ in spherical coordinates (r, θ, ϕ) is described by Ting (1998).

The purpose of this paper is to present a method for solving elastodynamic problems in radially inhomogeneous elastic materials with spherical anisotropy, i.e. materials having $c_{ijkl} = c_{ijkl}(r)$ in a spherical coordinate system (Lekhnitskii 1963). No *a priori* restrictions are made on the form of the displacement field. The main departure from previous studies is the use of vector spherical harmonic functions as the set of vector basis functions. We show that the most general type of spherical symmetry, for which the basis of vector spherical harmonics always yields a separable solution, is transverse isotropy about $\mathbf{e}_r$ (certain lower symmetries admit such solutions as well but only under appropriate kinematic restrictions). Such anisotropy restriction may actually be not so severe, since any spherical anisotropic material with properties independent of the polar-axis orientation must be TI anyway (see below). The spherical TI problem reduces to an ODS in the radial coordinate r with a system matrix possessing hermiticity properties that guarantee physical attributes such as energy conservation. The key feature of this analysis is the set of basis functions, vector spherical harmonics, which allow for the first time application of the full Stroh formalism to spherical elasticity for arbitrary displacement fields.

The paper is organized as follows. The concept of spherical anisotropy is revisited in §2. Vector spherical harmonic basis functions are introduced in §3. The main result, which is the Stroh-like ODS for the state vector comprising the r-dependent components of displacement and radial traction in the basis of vector spherical harmonics, is described in §4 and proved in detail in §5. Explicit solutions and their properties are discussed in §6. Conclusions and further prospects are presented in §7.

2. Elastic anisotropy in cylindrical and spherical coordinates

The concept of cylindrical and spherical elastic anisotropy was introduced by Saint-Venant and subsequently developed by Lekhnitskii (1963). The concept is motivated by the existence of special materials possessing either a physically distinguished direction aligned with the Z-axis of the cylindrical coordinate system or a point that can be identified as the origin O of the spherical system. It is physically relevant to consider such materials in terms of tensor fields in an orthogonal curvilinear coordinate system. A tensor $\boldsymbol{\Phi}$ of order $p \in \mathbb{N}$ is defined at every point $\mathbf{r}$ of a material body $\mathcal{P}$ by the array $\{\Phi_{i_1 \ldots i_p}\}$ $(i_j = 1, 2, 3;\ j = 1, \ldots, p)$ of its components in some frame of orthogonal basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ associated with $\mathbf{r} \in \mathbb{R}^3$. The components change in the usual manner under a change of basis (see Nair 2009, §2.6). Recall that the frame $\{\mathbf{e}\} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of a rectangular

(orthogonal rectilinear) coordinate system in $\mathbb{R}^3$ is independent of $\mathbf{r}$ and of the origin point O, whereas an orthogonal curvilinear coordinate system implies a varying frame, i.e. the frame $\{\mathbf{e}(\theta)\} = (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ with a fixed longitudinal axis $Z \parallel \mathbf{e}_z$ for the cylindrical system, and the frame $\{\mathbf{e}(\theta, \phi)\} = (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ with a fixed origin O and polar axis Z for the spherical system. The essential difference between the rectilinear and curvilinear arrays of components is related to the meaning of a uniform tensor. If the array of components $\{\Phi_{i_1 \ldots i_p}\}$ in some rectangular frame is uniform, i.e. $\Phi_{i_1 \ldots i_p} = \text{const.}$ for all i_j, j and $\mathbf{r} \in \mathcal{P}$, then the tensor $\boldsymbol{\Phi}$ is also uniform, i.e. it is the same for all $\mathbf{r}$, and vice versa. By contrast, a uniform array of components $\{\Phi_{i_1 \ldots i_p}\}$ referred to a curvilinear frame $\{\mathbf{e}\}$ describes a tensor $\boldsymbol{\Phi}$ which generally depends on $\mathbf{r}$ since $\{\mathbf{e}\}$ does. For example, $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\theta)$ for a uniform cylindrical array of components and $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\theta, \phi)$ for uniform spherical components, unless these tensors are isotropic (components invariant to change of basis under SO(3)) or if the cylindrical one is transversely isotropic (invariant under SO(2) about $\mathbf{e}_z$). As a matter of definition, a tensor $\boldsymbol{\Phi}$ associated with cylindrical or spherical components which is not isotropic in the above sense is called *cylindrically or spherically anisotropic*, respectively.

The components of a tensor may certainly be non-uniform so that $\{\Phi_{i_1 \ldots i_p}\} = \{\Phi_{i_1 \ldots i_p}(\mathbf{r})\}$ (where $\mathbf{r}$ does not have to be related to the same coordinate system as the frame $\{\mathbf{e}\}$). Many applications deal with a specific type of inhomogeneous body $\mathcal{P}$ in which the curvilinear components $\Phi_{i_1 \ldots i_p}(\mathbf{r})$ of a non-uniform tensor vary with position but maintain the same anisotropy (symmetry class) at every $\mathbf{r} \in \mathcal{P}$. This is the general framework considered in this paper. A widely encountered example is the case of a cylindrical or spherical, radially inhomogeneous elastic tensor with $c_{ijkl} = c_{ijkl}(r)$, where r may be measured from any point of the fixed axis Z of the cylindrical system (since c_{ijkl} do not depend on z) or from the fixed origin O of the spherical system.

Another particular aspect of cylindrical and spherical tensor components $\{\Phi_{i_1 \ldots i_p}\}$ stems from the fact that the *orientation* of frame vectors is undefined at certain points, namely (i) of the pair $(\mathbf{e}_r, \mathbf{e}_\theta)$ of the cylindrical frame at the points $r = 0$ lying on the longitudinal axis Z, (ii) of the pair $(\mathbf{e}_\theta, \mathbf{e}_\phi)$ of the spherical frame at the points $\theta = 0, \pi$ of the polar axis Z and (iii) of the whole spherical frame $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ at the point $r = 0$ of the origin O. Now suppose that a given body $\mathcal{P}$ contains either the axis Z or the point O and has a tensor $\boldsymbol{\Phi}$ described by the array of components $\{\Phi_{i_1 \ldots i_p}(\mathbf{r})\}$ which is single-valued at every $\mathbf{r} \in \mathcal{P}$ including the above-mentioned special points. Then in case (i), the array $\{\Phi_{i_1 \ldots i_p}\}$ that is independent of r must at every $\mathbf{r}$ be invariant to the orientation of the pair $(\mathbf{e}_r, \mathbf{e}_\theta)$, i.e. $\boldsymbol{\Phi}$ must be transversely isotropic about $Z \parallel \mathbf{e}_z$; in case (ii), $\{\Phi_{i_1 \ldots i_p}\}$ that is independent of θ must at every $\mathbf{r}$ be invariant to the orientation of the pair $(\mathbf{e}_\theta, \mathbf{e}_\phi)$, i.e. $\boldsymbol{\Phi}$ must be transversely isotropic about $\mathbf{e}_r$; in case (iii), $\{\Phi_{i_1 \ldots i_p}\}$ that is independent of r must at every $\mathbf{r}$ be invariant to the orientation of the spherical frame $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$, i.e. $\boldsymbol{\Phi}$ must be isotropic. These restrictions on cylindrical or spherical anisotropy, which result from eliminating a singularity that exists only at isolated (axial or origin) points, are neither immanent nor physically binding and can certainly be circumvented formally, say, by assuming a small cavity surrounding each singular point. However, in the spherical case, it is physically reasonable to single out the class of spherically anisotropic materials invariant with respect to any orientation of the polar axis which may thus be

called materials with *complete* spherical anisotropy. The remedy for case (ii) described above must then be provided at every $\mathbf{r}$. Hence, material tensors in a body with complete spherical anisotropy can only be either uniform or radially inhomogeneous and, unless isotropic, they must be transversely isotropic about $\mathbf{e}_r$. They are well defined everywhere except a single origin point O where (iii) needs to be addressed.

3. Governing equations

(a) *Elastodynamic equations*

The dynamic equilibrium vector equation for a linearly elastic material when expressed in spherical coordinates is

$$\left.\begin{aligned} &r^{-2}(r^2\mathbf{t}_r)_{,r} + (r\sin\theta)^{-1}[(\sin\theta\mathbf{t}_\theta)_{,\theta} + \mathbf{t}_{\phi,\phi} + \sin\theta\mathbf{K}\mathbf{t}_\theta + \mathbf{H}\mathbf{t}_\phi] = \rho\ddot{\mathbf{u}} \\ &\text{with} \\ &\mathbf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (= -\mathbf{K}^{\mathrm{T}}), \quad \mathbf{H} = \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & -\cos\theta \\ \sin\theta & \cos\theta & 0 \end{pmatrix} (= -\mathbf{H}^{\mathrm{T}}). \end{aligned}\right\} \tag{3.1}$$

Here $\rho = \rho(\mathbf{x})$ is the mass density, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ the displacement and $\mathbf{t}_i = \mathbf{t}_i(\mathbf{x}, t) = \mathbf{e}_i^{\mathrm{T}}\boldsymbol{\sigma}$ $(i = r, \theta, \phi)$ the traction vectors defined by the elements of stress $\boldsymbol{\sigma}(\mathbf{x}, t)$ in the orthonormal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ of the spherical coordinates (r, θ, ϕ). The left-hand side term in (3.1), $\operatorname{div}\boldsymbol{\sigma} = \sum_i \boldsymbol{\nabla}\cdot(\mathbf{e}_i \otimes \mathbf{t}_i)$, follows using $(3.3)_1$ plus the identities $\partial\mathbf{a}/\partial\theta = \mathbf{a}_{,\theta} + \mathbf{K}\mathbf{a}$ and $\partial\mathbf{a}/\partial\phi = \mathbf{a}_{,\phi} + \mathbf{H}\mathbf{a}$, where the comma suffix notation indicates partial differentiation of components only: $\mathbf{a}_{,\varphi} \equiv \sum_i a_{i,\varphi}\mathbf{e}_i$ for $\varphi = \theta, \phi$. The stress elements in the basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ are $\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$ where $c_{ijkl} = c_{ijkl}(\mathbf{x})$ are the components of the spherically anisotropic elastic tensor and ε_{kl} are the components of the strain $\boldsymbol{\varepsilon}(\mathbf{x}, t) = \frac{1}{2}[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}]$, with summation on repeated indices assumed and T for transpose. The traction vectors can therefore be written as

$$\begin{pmatrix} \mathbf{t}_r \\ \mathbf{t}_\theta \\ \mathbf{t}_\phi \end{pmatrix} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} & \mathbf{P} \\ \mathbf{R}^{\mathrm{T}} & \mathbf{T} & \mathbf{S} \\ \mathbf{P}^{\mathrm{T}} & \mathbf{S}^{\mathrm{T}} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{,r} \\ r^{-1}(\mathbf{u}_{,\theta} + \mathbf{K}\mathbf{u}) \\ (r\sin\theta)^{-1}(\mathbf{u}_{,\phi} + \mathbf{H}\mathbf{u}) \end{pmatrix} \tag{3.2a}$$

with

$$\begin{pmatrix} \mathbf{Q} & \mathbf{R} & \mathbf{P} \\ \mathbf{R}^{\mathrm{T}} & \mathbf{T} & \mathbf{S} \\ \mathbf{P}^{\mathrm{T}} & \mathbf{S}^{\mathrm{T}} & \mathbf{M} \end{pmatrix} = \begin{pmatrix} (e_r e_r) & (e_r e_\theta) & (e_r e_\phi) \\ (e_\theta e_r) & (e_\theta e_\theta) & (e_\theta e_\phi) \\ (e_\phi e_r) & (e_\phi e_\theta) & (e_\phi e_\phi) \end{pmatrix}, \tag{3.2b}$$

where, in the notation of Lothe & Barnett (1976), the matrix (ab) has components $(ab)_{jk} = a_i c_{ijkl} b_l$ for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$.

(b) *Vector spherical harmonics*

Our objective here is to develop separation of variables vector solutions in the form $\mathbf{v}(r, \theta, \phi) = \sum_{\mathbf{A}} V_{\mathbf{A}}(r)\mathbf{A}(\theta, \phi)$ where the three vectors $\mathbf{A}(\mathbf{e}_r)$ are independent of r and provide a complete basis for representing vectorial functions of the spherical angles. Vector spherical harmonics are one such set of functions.

It is useful to first introduce the angular parts $\mathbf{D}$ and $\mathbf{D}\cdot\mathbf{D}\equiv D^2$ of the vector differential operators $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\equiv\nabla^2(\equiv\Delta)$ in spherical coordinates:

$$\left.\begin{aligned}\boldsymbol{\nabla}&=\mathbf{e}_r\frac{\partial}{\partial r}+\frac{1}{r}\mathbf{D}\left(\Rightarrow\mathbf{D}=\mathbf{e}_\theta\frac{\partial}{\partial\theta}+\frac{\mathbf{e}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right),\\ \Delta f(\mathbf{r})&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}D^2f\\ \text{and}\qquad\Delta\mathbf{f}(\mathbf{r})&=\sum_i\left[\mathbf{e}_i\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f_i}{\partial r}\right)\right]+\frac{1}{r^2}D^2\mathbf{f},\end{aligned}\right\}\tag{3.3}$$

so that $\Delta f=r^{-2}D^2f$ and $\Delta\mathbf{f}=r^{-2}D^2\mathbf{f}$ for pure angular functions $f(\mathbf{e}_r)$ and $\mathbf{f}(\mathbf{e}_r)$, respectively. Another ingredient is the set of (scalar) spherical harmonics Y_n^m of polar order n and azimuthal order m, for which there are several slightly different notations in use. Following Martin (2006, p. 64), let

$$\left.\begin{aligned}Y_n^m(\mathbf{e}_r)\equiv Y_n^{me}+iY_n^{mo}&=A_n^mP_n^m(\cos\theta)\mathrm{e}^{im\phi}\\ \text{with}\qquad A_n^m&=(-1)^m\sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}},\end{aligned}\right\}\tag{3.4}$$

where $P_n^m(\cos\theta)$ are the associated Legendre functions of the first kind. The functions $Y_n^m(\mathbf{e}_r)$ satisfy the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n^m}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y_n^m}{\partial\phi^2}(\equiv D^2Y_n^m)=-\lambda^2Y_n^m\quad\text{with }\lambda=[n(n+1)]^{1/2}.\tag{3.5}$$

In these terms, the vector spherical harmonics are defined as

$$\left.\begin{aligned}\mathbf{P}_{mn}(\mathbf{e}_r)&=\mathbf{e}_rY_n^m(\mathbf{e}_r),\\ \mathbf{B}_{mn}(\mathbf{e}_r)&=\lambda^{-1}\boldsymbol{D}Y_n^m(\mathbf{e}_r)\\ \text{and}\qquad\mathbf{C}_{mn}(\mathbf{e}_r)&=\mathbf{B}_{mn}(\mathbf{e}_r)\times\mathbf{e}_r\end{aligned}\right\}\tag{3.6}$$

(see Martin 2006, §3.17 for a literature review). The vector harmonics are pointwise orthogonal

$$\mathbf{P}_{mn}\cdot\mathbf{B}_{mn}=\mathbf{B}_{mn}\cdot\mathbf{C}_{mn}=\mathbf{C}_{mn}\cdot\mathbf{P}_{mn}=0\tag{3.7}$$

and orthonormal when integrated by $\mathrm{d}\Omega=\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$:

$$\left.\begin{aligned}\int_\Omega\mathrm{d}\Omega\mathbf{P}_{mn}\cdot\mathbf{B}^*_{\mu\nu}=\int_\Omega\mathrm{d}\Omega\mathbf{B}_{mn}\cdot\mathbf{C}^*_{\mu\nu}&=\int_\Omega\mathrm{d}\Omega\mathbf{C}_{mn}\cdot\mathbf{P}^*_{\mu\nu}=0\\ \text{and}\qquad\int_\Omega\mathrm{d}\Omega\mathbf{P}_{mn}\cdot\mathbf{P}^*_{\mu\nu}=\int_\Omega\mathrm{d}\Omega\mathbf{B}_{mn}\cdot\mathbf{B}^*_{\mu\nu}&=\int_\Omega\mathrm{d}\Omega\mathbf{C}_{mn}\cdot\mathbf{C}^*_{\mu\nu}=\delta_{m\mu}\delta_{n\nu},\end{aligned}\right\}\tag{3.8}$$

where the latter two identities hold at $n, \nu \neq 0$, see §6*d*. Using Morse & Feshbach (1953, p. 1901), the vector spherical harmonics can be shown to satisfy the following identities where $f = f(r)$ and indices m and n are suppressed:

$$\left.\begin{aligned} &\boldsymbol{\nabla}\cdot f\mathbf{P} = r^{-2}(r^2 f)' Y, \quad \boldsymbol{\nabla}\cdot f\mathbf{B} = -r^{-1} f\lambda Y, \quad \boldsymbol{\nabla}\cdot f\mathbf{C} = 0;\\ &\Delta(f\mathbf{A}) = (\Delta f)\mathbf{A} + r^{-2} f D^2\mathbf{A} \quad \text{where} \quad \mathbf{A} \equiv \mathbf{P}, \mathbf{B}, \mathbf{C}\\ \text{and} \quad &D^2\mathbf{P} = 2\lambda\mathbf{B} - (\lambda^2+2)\mathbf{P}, \quad D^2\mathbf{B} = 2\lambda\mathbf{P} - \lambda^2\mathbf{B}, \quad D^2\mathbf{C} = -\lambda^2\mathbf{C}. \end{aligned}\right\} \tag{3.9}$$

4. The Stroh formalism in spherical coordinates

We consider time-harmonic motion with the time dependence $\mathrm{e}^{-i\omega t}$ omitted but understood. It is useful to re-write the equilibrium equation (3.1) in the form distinguishing terms with radial and angular derivatives, namely

$$(r^2\mathbf{t}_r)_{,r} + r\,\boldsymbol{\tau} = -r^2\rho\omega^2\mathbf{u} \tag{4.1a}$$

with

$$\boldsymbol{\tau} \equiv (\sin\theta)^{-1}[(\sin\theta\mathbf{t}_\theta)_{,\theta} + \mathbf{t}_{\phi,\phi} + \sin\theta\mathbf{K}\mathbf{t}_\theta + \mathbf{H}\mathbf{t}_\phi], \tag{4.1b}$$

where $\mathbf{t}_i$ are defined by (3.2*a*). In the following, the density and elastic coefficients are assumed to be radially inhomogeneous. Suppressing the indices m and n, denote $\mathbf{A}(\mathbf{e}_r) \equiv \mathbf{P}, \mathbf{B}, \mathbf{C}$ and let

$$\mathbf{u} = \sum_{\mathbf{A}} U_{\mathbf{A}}\mathbf{A}, \quad \mathbf{t}_r = \sum_{\mathbf{A}} T_{\mathbf{A}}\mathbf{A}, \quad \boldsymbol{\tau} = \sum_{\mathbf{A}} \mathit{\Gamma}_{\mathbf{A}}\mathbf{A} \tag{4.2a}$$

and

$$\mathbf{U} = (U_{\mathbf{P}}, U_{\mathbf{B}}, U_{\mathbf{C}})^{\mathrm{T}}, \quad \mathbf{T} = (T_{\mathbf{P}}, T_{\mathbf{B}}, T_{\mathbf{C}})^{\mathrm{T}}, \quad \boldsymbol{\mathit{\Gamma}} = (\mathit{\Gamma}_{\mathbf{P}}, \mathit{\Gamma}_{\mathbf{B}}, \mathit{\Gamma}_{\mathbf{C}})^{\mathrm{T}}. \tag{4.2b}$$

Our objective in this paper is twofold: first to find the most general symmetry which admits the separation of variables solution $\mathbf{u} = \sum_{\mathbf{A}} U_{\mathbf{A}}(r)\mathbf{A}$; and, second, to obtain these separation of variables solutions when they are possible.

In view of equations (4.1)–(4.2), the first problem implies answering the following question: given the ansatz $\mathbf{U} = \mathbf{U}(r)$, what symmetry yields simultaneous conditions $\mathbf{T} = \mathbf{T}(r)$ and $\boldsymbol{\mathit{\Gamma}} = \boldsymbol{\mathit{\Gamma}}(r)$? Direct calculation of $T_{\mathbf{A}} = \mathbf{t}_r \cdot \mathbf{A}$ from (3.2*a*) with $\mathbf{u} = \sum_{\mathbf{A}} U_{\mathbf{A}}(r)\mathbf{A}$ shows that $\mathbf{T} = \mathbf{T}(r)$ holds for tetragonal, cubic and transversely isotropic symmetries if their principal axes are parallel to $\mathbf{e}_r$, but is invalid for any other cases including the above symmetries with non-radial principal axes, the trigonal symmetry and certainly any lower symmetries. Note that the other traction vectors $\mathbf{t}_\theta$ and $\mathbf{t}_\phi$ do not admit a similar expansion in $\mathbf{A}$ with scalar coefficients depending on r even if the material is isotropic and uniform; however, this is not so for their combination $\boldsymbol{\tau}$. Calculation of $\mathit{\Gamma}_{\mathbf{A}} = \boldsymbol{\tau} \cdot \mathbf{A}$ shows that $\boldsymbol{\mathit{\Gamma}} = \boldsymbol{\mathit{\Gamma}}(r)$ is possible, but it holds only for TI with the principal axis along $\mathbf{e}_r$. Thus, altogether the separation of variables solution $\mathbf{u} = \sum_{\mathbf{A}} U_{\mathbf{A}}(r)\mathbf{A}$ is ensured only in the presence of TI about $\mathbf{e}_r$; otherwise, the

coefficients $U_{\mathbf{A}}$ must depend on the spherical angles (apart from the theoretical possibility of coincidental equalities between material constants that is of no practical interest).

Regarding the second problem, it is solved by deriving the first-order ODS in r (solutions of such ODS are standard; see §6*b*). This is done in detail in the next section. For now, we simply formulate the overall result.

Theorem 4.1. *Time-harmonic solutions of the equations of linear elasticity in a spherically anisotropic radially inhomogeneous body with $\rho(r)$ and $c_{ijkl}(r)$ admit the separation of variables using the vector spherical harmonic functions only if the material is transversely isotropic about $\mathbf{e}_r$. In this case, the separable solution for the displacement and radial-traction vectors is in the form $\mathbf{u}=\sum_{\mathbf{A}} U_{\mathbf{A}}(r)\mathbf{A}$, $\mathbf{t}_r=\sum_{\mathbf{A}} T_{\mathbf{A}}(r)\mathbf{A}$ (with $\mathbf{A}\equiv\mathbf{P},\mathbf{B},\mathbf{C}$ and the indices m and n being omitted). The amplitudes $\mathbf{U}(r)=(U_{\mathbf{P}}, U_{\mathbf{B}}, U_{\mathbf{C}})^{\mathrm{T}}$ and $\mathbf{T}(r)=(T_{\mathbf{P}}, T_{\mathbf{B}}, T_{\mathbf{C}})^{\mathrm{T}}$ are defined by the Stroh-like ODS*

$$\boldsymbol{\eta}'=\frac{i\mathbf{G}}{r^2}\boldsymbol{\eta}\quad\text{with }\boldsymbol{\eta}(r)=\begin{pmatrix}\mathbf{U}\\ ir^2\mathbf{T}\end{pmatrix},$$

$$\mathbf{G}(r)=\begin{pmatrix} ir\mathcal{T}^{-1}\mathcal{R}^{\mathrm{T}} & -\mathcal{T}^{-1}\\ r^2(\mathcal{Q}-\mathcal{R}\mathcal{T}^{-1}\mathcal{R}^{\mathrm{T}})-r^4\rho\omega^2\mathbf{I} & -ir\mathcal{R}\mathcal{T}^{-1}\end{pmatrix}, \tag{4.3a}$$

where $'=\mathrm{d}/\mathrm{d}r$,

$$\mathcal{T}=\mathrm{diag}[c_{11},\ c_{66},\ c_{66}],\quad \mathcal{R}=\begin{pmatrix}2c_{12} & \lambda c_{66} & 0\\ -\lambda c_{12} & -c_{66} & 0\\ 0 & 0 & -c_{66}\end{pmatrix} \tag{4.3b}$$

and

$$\mathcal{Q}=\begin{pmatrix}\lambda^2c_{66}+4(c_{22}-c_{44}) & \lambda(2c_{44}-2c_{22}-c_{66}) & 0\\ \lambda(2c_{44}-2c_{22}-c_{66}) & \lambda^2c_{22}+c_{66}-2c_{44} & 0\\ 0 & 0 & \lambda^2c_{44}+c_{66}-2c_{44}\end{pmatrix}=\mathcal{Q}^{\mathrm{T}}, \tag{4.3c}$$

and $\lambda=[n(n+1)]^{1/2}$. For real material parameters (and real ω^2), $\mathbf{G}=\mathbb{T}\mathbf{G}^{+}\mathbb{T}$ where $^{+}$ means Hermitian conjugation and $\mathbb{T}$ is a matrix with zero diagonal and identity off-diagonal 3×3 blocks.

5. Derivation of the Stroh-like ODS for spherical transverse isotropy

(*a*) *Elastic coefficients and stress*

Having established that the separation of variables solution via the use of harmonics works only for TI, our purpose here is to obtain the explicit form of the ODS which is stated by Theorem 4.1. The derivation proceeds by splitting the problem into the isotropic and anisotropic parts based on the standard form

of the transversely isotropic elastic coefficients $c_{ijkl}(r)$ (Fedorov 1968):

$$\left.\begin{aligned} c_{ijkl} &= c^{(\mathrm{iso})}_{ijkl} + c^{(\mathrm{anis})}_{ijkl}: \ c^{(\mathrm{iso})}_{ijkl} = c_{23}\delta_{ij}\delta_{kl} + c_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),\\ c^{(\mathrm{anis})}_{ijkl} &= a_1(\delta_{ik}\delta_{j1}\delta_{l1} + \delta_{il}\delta_{j1}\delta_{k1} + \delta_{jk}\delta_{i1}\delta_{l1} + \delta_{jl}\delta_{i1}\delta_{k1})\\ &\quad + a_2\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + a_3(\delta_{ij}\delta_{k1}\delta_{l1} + \delta_{kl}\delta_{i1}\delta_{j1}) \end{aligned}\right\} \tag{5.1}$$

with

$$\left.\begin{aligned} a_1 &= c_{66} - c_{44}, \ \ a_2 = c_{11} + c_{22} - 2c_{12} - 4c_{66},\\ a_3 &= c_{12} - c_{23}, \ \ c_{22} = c_{23} + 2c_{44}. \end{aligned}\right\}$$

These partitions stress as

$$\left.\begin{aligned} &\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(\mathrm{iso})} + \boldsymbol{\sigma}^{(\mathrm{anis})}, \quad \boldsymbol{\sigma}^{(\mathrm{iso})} = c_{23}(\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{I} + 2c_{44}\boldsymbol{\varepsilon}\\ &\text{and}\\ &\boldsymbol{\sigma}^{(\mathrm{anis})} = 2a_1(\mathbf{e}_r\otimes\boldsymbol{\gamma}_r + \boldsymbol{\gamma}_r\otimes\mathbf{e}_r) + a_2u_{r,r}\mathbf{e}_r\otimes\mathbf{e}_r + a_3[u_{r,r}\mathbf{I} + (\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{e}_r\otimes\mathbf{e}_r] \end{aligned}\right\} \tag{5.2}$$

where

$$\boldsymbol{\gamma}_r = \mathbf{e}_r\boldsymbol{\varepsilon} = \tfrac{1}{2}(\mathbf{u}_{,r} + \boldsymbol{\nabla}u_r - r^{-1}\mathbf{u}_\perp) \quad \text{with } \mathbf{u}_\perp \equiv u_\theta\mathbf{e}_\theta + u_\phi\mathbf{e}_\phi.$$

Another useful split occurs due to uncoupling of the shear wave motion with $\mathbf{u}$ polarized along $\mathbf{C}$. We begin with this observation and subsequently examine solutions with $\mathbf{u}$ spanned by $\mathbf{P}$ and $\mathbf{B}$.

(*b*) *Uncoupled shear-horizontal solution*

Keeping the conventional term 'shear-horizontal' (SH) for the waves with $\mathbf{u}$ parallel to $\mathbf{C}$, assume that

$$\mathbf{u}_{\mathrm{SH}} = U_C(r)\mathbf{C}. \tag{5.3}$$

Inserting the ansatz (5.3) into (5.2) readily determines the SH radial traction as

$$\mathbf{t}_r = T_C\mathbf{C} \quad \text{with } T_C = c_{66}(U_C' - r^{-1}U_C) \tag{5.4}$$

(note that the SH tractions $\mathbf{t}_\theta$ and $\mathbf{t}_\phi$ are not aligned with $\mathbf{C}$). Applying the divergence identities (3.9) to the isotropic and anisotropic parts of the stress gives the equalities

$$\left.\begin{aligned} &\operatorname{div}\boldsymbol{\sigma}^{(\mathrm{iso})}_{\mathrm{SH}} = [c_{44}'(U_C' - r^{-1}U_C) + c_{44}(\Delta U_C - \lambda^2 r^{-2}U_C)]\mathbf{C}\\ &\text{and}\\ &\operatorname{div}\boldsymbol{\sigma}^{(\mathrm{anis})}_{\mathrm{SH}} = [a_1'(U_C' - r^{-1}U_C)\mathbf{C} + a_1(\Delta U_C - 2r^{-2}U_C)]\mathbf{C}. \end{aligned}\right\} \tag{5.5}$$

Adding them leads to the equation of SH motion $\operatorname{div}\boldsymbol{\sigma}_{\mathrm{SH}} = -\rho\omega^2\mathbf{u}_{\mathrm{SH}}$ in the form

$$(r^2c_{66}U_C')' + (r^2\rho\omega^2 - \lambda^2c_{44} + 2(c_{44} - c_{66}) - rc_{66}')U_C = 0. \tag{5.6}$$

The latter can be recast, using (5.4), as

$$(r^2c_{66}U_C')' - (2c_{66} + rc_{66}')U_C = (r^2T_C)' + rT_c. \tag{5.7}$$

Thus, (5.4) is the first and (5.7) the second equations of the following ODS for SH waves:

$$\begin{pmatrix} U_C \\ ir^2 T_C \end{pmatrix}' = i \begin{pmatrix} -ir^{-1} & -(r^2 c_{66})^{-1} \\ (\lambda^2 - 2)c_{44} - r^2\rho\omega^2 & ir^{-1} \end{pmatrix} \begin{pmatrix} U_C \\ ir^2 T_C \end{pmatrix}. \tag{5.8}$$

This is clearly the same as the pair of equations in the third and sixth rows of system (4.3*a*).

(*c*) *In-plane problem*

We now consider $\mathbf{U} = \sum_{\mathbf{A}} U_{\mathbf{A}}\mathbf{A}$ and $\mathbf{T} = \sum_{\mathbf{A}} T_{\mathbf{A}}\mathbf{A}$ where $\mathbf{A} = \mathbf{P}, \mathbf{B}$ only, in which sense this case may be referred to as the 'in-plane' problem. It implies restricting attention to the upper 2×2 block of the matrices in (4.3).

(i) *Isotropic part*

Assume

$$\mathbf{u}_{\mathrm{in-plane}} = \sum_{\mathbf{A}} U_{\mathbf{A}}(r)\mathbf{A} \quad \text{with } \mathbf{A} = \mathbf{P}, \mathbf{B}. \tag{5.9}$$

The equation for $\mathbf{t}_r = \mathbf{e}_r\boldsymbol{\sigma}$ readily follows from $\boldsymbol{\sigma}^{(\mathrm{iso})}$ in $(5.2)_1$ to yield the isotropic part of the first equation of the in-plane ODS as

$$r\mathbf{T}_{\mathrm{iso}} = r\mathcal{T}_{\mathrm{iso}}\mathbf{U}' + \mathcal{R}_{\mathrm{iso}}^{\mathrm{T}}\mathbf{U} \quad \text{with } \mathcal{T}_{\mathrm{iso}} = \begin{pmatrix} c_{23} & 0 \\ 0 & c_{44} \end{pmatrix}, \quad \mathcal{R}_{\mathrm{iso}}^{\mathrm{T}} = \begin{pmatrix} 2c_{23} & -\lambda c_{23} \\ \lambda c_{44} & -c_{44} \end{pmatrix}. \tag{5.10}$$

In order to formulate the second equation, which is based on the equilibrium condition $\operatorname{div}\boldsymbol{\sigma} = -\rho\omega^2\mathbf{u}$, it is convenient to use the following identity for the radially inhomogeneous medium

$$(r^2\mathbf{t}_r)_{,r} + r^2\rho\omega^2\mathbf{u} = [(r^2\mathbf{t}_r)_{,r} - r^2\operatorname{div}\boldsymbol{\sigma}]_{\mathrm{homo}}, \tag{5.11}$$

where the 'homogeneous' term on the right-hand side is understood to be evaluated as if the elastic moduli are independent of r, i.e. derivatives of $c_{ijkl}(r)$ are ignored. This reduces the task to concentrating on the right-hand side term in (5.11), namely, to expanding $\operatorname{div}\boldsymbol{\sigma}_{\mathrm{homo}}$ in vector spherical harmonics given that $\mathbf{u} = \sum_{\mathbf{A}} U_{\mathbf{A}}\mathbf{A}$:

$$\operatorname{div}\boldsymbol{\sigma}_{\mathrm{homo}} = \sum_{\mathbf{A}} F_{\mathbf{A}}\mathbf{A} \quad \text{with } \mathbf{A} = \mathbf{P}, \mathbf{B}. \tag{5.12}$$

Using $\operatorname{div}\boldsymbol{\sigma}_{\mathrm{homo}}^{(\mathrm{iso})} = (c_{23} + c_{44})\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + c_{44}\boldsymbol{\nabla}^2\mathbf{u}$ and the identities (3.9) yields the isotropic part $\mathbf{F}_{\mathrm{iso}}$ of the vector $\mathbf{F} = (F_{\mathbf{P}}, F_{\mathbf{B}})^{\mathrm{T}}$ as

$$\mathbf{F}_{\mathrm{iso}} = \begin{pmatrix} c_{22}\Delta - \dfrac{1}{r^2}(2c_{22} + \lambda^2 c_{44}) & \dfrac{\lambda}{r}\left[(c_{44} - c_{22})\dfrac{\mathrm{d}}{\mathrm{d}r} + \dfrac{1}{r}(c_{22} + c_{44})\right] \\ \dfrac{\lambda}{r}\left[(c_{22} - c_{44})\dfrac{\mathrm{d}}{\mathrm{d}r} + \dfrac{1}{r}c_{22}\right] & c_{44} - \dfrac{\lambda^2}{r^2}c_{22} \end{pmatrix}\mathbf{U}. \tag{5.13}$$

Combining (5.13) with (5.10) and substituting into (5.11) gives

$$\left.\begin{aligned} &(r^2\mathbf{T}_{\rm iso})' = r\mathcal{R}_{\rm iso}\mathbf{U}' + (\mathcal{Q}_{\rm iso} - r^2\rho\omega^2\mathbf{I})\mathbf{U} \\ &\text{with} \\ &\mathcal{Q}_{\rm iso} = \begin{pmatrix} 4(c_{22}-c_{44}) + \lambda^2 c_{44} & \lambda(c_{44}-2c_{22}) \\ \lambda(c_{44}-2c_{22}) & -c_{44} + \lambda^2 c_{22} \end{pmatrix}. \end{aligned}\right\} \tag{5.14}$$

(ii) *Anisotropic part*

From $(5.2)_2$ and the definition of a_j in (5.1), the anisotropic part to be added to the first equation is

$$\left.\begin{aligned} &r\mathbf{T}_{\rm anis} = r\mathcal{T}_{\rm anis}\mathbf{U}' + \mathcal{R}_{\rm anis}^{\rm T}\mathbf{U} \\ &\text{with} \\ &\mathcal{T}_{\rm anis} = \begin{pmatrix} c_{11}-c_{23} & 0 \\ 0 & c_{66}-c_{44} \end{pmatrix}, \quad \mathcal{R}_{\rm anis}^{\rm T} = \begin{pmatrix} 2(c_{12}-c_{23}) & \lambda(c_{23}-c_{12}) \\ \lambda(c_{66}-c_{44}) & c_{44}-c_{66} \end{pmatrix}. \end{aligned}\right\} \tag{5.15}$$

Again using $(5.2)_2$, now with the identities

$$\left.\begin{aligned} \boldsymbol{\nabla}[(\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{e}_r\otimes\mathbf{e}_r] &= \left(\Delta + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right) u_{,r}\mathbf{e}_r, \\ \boldsymbol{\nabla}[f(\mathbf{r})\mathbf{e}_r\otimes\mathbf{e}_r] &= \left(f_{,r} + \frac{2}{r}f\right)\mathbf{e}_r \\ \text{and} \qquad \boldsymbol{\nabla}[f(r)(\mathbf{e}_r\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{e}_r)] &= f'\mathbf{B} + f\boldsymbol{\nabla}(\mathbf{e}_r\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{e}_r) \\ &= f'\mathbf{B} + r^{-1}f(3\mathbf{B} - \lambda\mathbf{P}), \end{aligned}\right\} \tag{5.16}$$

yields the anisotropic part $\mathbf{F}_{\rm anis}$ of the vector $\mathbf{F} = (F_{\mathbf{P}}, F_{\mathbf{B}})^{\rm T}$ defined in (5.12):

$$\mathbf{F}_{\rm anis} = \begin{pmatrix} (a_2+2a_3)\Delta + \dfrac{1}{r^2}(2a_3 - 4a_1 - \lambda^2 a_1) & \dfrac{\lambda}{r}\left[(a_1-a_3)\dfrac{\mathrm{d}}{\mathrm{d}r} + \dfrac{1}{r}(3a_1-a_3)\right] \\ \dfrac{\lambda}{r}\left((a_3-a_1)\dfrac{\mathrm{d}}{\mathrm{d}r} + \dfrac{2}{r}a_1\right) & a_1\left(\Delta - \dfrac{2}{r^2}\right) \end{pmatrix}\mathbf{U}. \tag{5.17}$$

Combining (5.17) with (5.15) and substituting the result into (5.11) gives

$$(r^2\mathbf{T}_{\rm anis})' = r\mathcal{R}_{\rm anis}\mathbf{U}' + (\mathcal{Q}_{\rm anis} - r^2\rho\omega^2\mathbf{I})\mathbf{U} \quad \text{with } \mathcal{Q}_{\rm anis} = (c_{66}-c_{44})\begin{pmatrix} \lambda^2 & -\lambda \\ -\lambda & 1 \end{pmatrix}. \tag{5.18}$$

(iii) *Result*

The isotropic and anisotropic parts of the in-plane solution can now be superimposed. Adding (5.15) to (5.10) and (5.18) to (5.14) gives the in-plane part of (4.3*a*). This, together with (5.8), completes the proof of Theorem 4.1.

6. Discussion

(*a*) *Radially uniform materials*

Consider first the SH motion. Assuming radially uniform material coefficients and denoting $U_C(r) \to u(R)$ with $R = r\omega(\rho/c_{66})^{1/2}$, the uncoupled equation (5.6) becomes

$$(R^2u')' + [R^2 - \mu(\mu+1)]\,u = 0, \quad \mu(\mu+1) = [n(n+1)-2]\frac{c_{44}}{c_{66}} + 2, \tag{6.1}$$

the solutions of which are spherical Bessel functions $j_\mu(R)$ and $y_\mu(R)$ (alternatively, Hankel functions $h_\mu^{(1)}(R)$ and $h_\mu^{(2)}(R)$). The identity $(6.1)_2$ was obtained by Hu (1954) for static equilibrium, and the vector function defined by (5.3) with Bessel or Hankel function solutions for $U_C(r)$ is equivalent to the vector function $\boldsymbol{M}(r,\theta,\phi)$ of Morse & Feshbach (1953) (see also Dassios & Rigou 1995).

The in-plane differential equations for displacement and traction amplitudes may be recast as second-order differential equations for U_P and U_B. In the case of radially uniform materials, these equations reduce to

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix} &= \begin{pmatrix}c_{11} & 0\\ 0 & c_{66}\end{pmatrix}\begin{pmatrix}r^2U_P'\\ r^2U_B'\end{pmatrix}' + r\lambda(c_{12}+c_{66})\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}U_P'\\ U_B'\end{pmatrix}\\ &\quad + \begin{pmatrix}\rho\omega^2r^2 - 2(c_{22}+c_{23}-c_{12}) - \lambda^2c_{66} & \lambda(c_{22}+c_{23}-c_{12}+c_{66})\\ \lambda(c_{22}+c_{23}+2c_{66}) & \rho\omega^2r^2 + 2(c_{44}-c_{66}) - \lambda^2c_{22}\end{pmatrix}\\ &\quad \times \begin{pmatrix}U_P\\ U_B\end{pmatrix}.\end{aligned} \tag{6.2}$$

Even though the material coefficients are constant, this coupled pair of equations does not seem to admit an evident explicit solution in terms of special functions and should be solved by the means discussed in §6*b*. For isotropic materials with $c_L^2 = c_{11}/\rho$ and $c_T^2 = c_{66}/\rho$, equation (6.2) further simplify to the form

$$(r^2U_P')' + \left(\frac{\omega^2}{c_L^2}r^2 - \lambda^2 - 2\right)U_P + 2\lambda U_B - \lambda c_L^{-2}(c_L^2 - c_T^2)[(rU_B)' - \lambda U_P] = 0, \tag{6.3a}$$

and

$$(r^2U_B')' + \left(\frac{\omega^2}{c_T^2}r^2 - \lambda^2\right)U_B + 2\lambda U_P + \lambda c_T^{-2}(c_L^2 - c_T^2)\left[\frac{1}{r}(r^2U_P)' - \lambda U_B\right] = 0, \tag{6.3b}$$

which leads to the known isotropic solutions. They may be elicited as follows. Set the final term in (6.3*a*) to zero by assuming for some as yet unknown function v that

$$U_P = v' \quad \text{and} \quad U_B = \lambda\frac{v}{r}. \tag{6.4}$$

Substituting this into equation (6.3) recasts them, respectively, as

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{2}{r}\right)M_L v=0,\quad \frac{\lambda}{r}\frac{c_L^2}{c_T^2}M_L v=0\quad\text{with } M_L v\equiv (r^2v')'+\left(\frac{\omega^2}{c_L^2}r^2-\lambda^2\right)v. \tag{6.5}$$

Both equalities are satisfied if v is a solution of the same spherical Bessel equation, $M_L v=0$, which has solutions $j_n(r\omega/c_L)$ and $y_n(r\omega/c_L)$. Similarly, setting to zero the final term in (6.3b) by taking

$$U_P=\lambda\frac{w}{r},\quad U_B=\frac{1}{r}(rw)', \tag{6.6}$$

for some function w, we find that (6.3a) and (6.3b) become, respectively,

$$\frac{\lambda}{r}\frac{c_T^2}{c_L^2}M_T w=0,\quad \left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{r}\right)M_T w=0,\quad\text{with } M_T w\equiv (r^2w')'+\left(\frac{\omega^2}{c_T^2}r^2-\lambda^2\right)w. \tag{6.7}$$

These are satisfied if w is a solution of the spherical Bessel equation of order n, $M_T w=0$.

The uncoupled longitudinal and transverse wave solutions (6.4) and (6.6) for the uniform and isotropic case are consistent with the potential representation using the Helmholtz decomposition. The vector functions of r,θ and ϕ formed from (5.9) with (6.4) and (6.6) can be identified as the vector functions $\boldsymbol{L}$ and $\boldsymbol{N}$, respectively, of Morse & Feshbach (1953) (see also Dassios & Rigou 1995).

(*b*) *Radially inhomogeneous materials*

The first-order ODS (4.3a) with material coefficients depending on r admits a general solution in the standard form $\boldsymbol{\eta}(r)=\mathbf{M}(r,r_0)\boldsymbol{\eta}(r_0)$, where $\boldsymbol{\eta}(r_0)$ is the initial data and the matricant $\mathbf{M}(r,r_0)$ may be evaluated by the Peano series (Pease 1965)

$$\mathbf{M}(r,r_0)=\mathbf{I}+\int_{r_0}^{r}\mathrm{d}x\frac{i\mathbf{G}(x)}{x^2}+\int_{r_0}^{r}\mathrm{d}x\frac{i\mathbf{G}(x)}{x^2}\int_{r_0}^{x}\mathrm{d}x_1\frac{i\mathbf{G}(x_1)}{x_1^2}+\cdots. \tag{6.8}$$

The matricant solution applies for $r,r_0\neq 0$. The case where the solution needs to be extended to the origin point $r=0$ requires a special treatment based on the theory of ODS with an irregular singular point (Wasow 1965). A similar state of affairs arises for the Stroh-like ODS in cylindrical coordinates at the axial points $r=0$ (except that they are regular singular points; Shuvalov 2003*b*, Norris & Shuvalov 2010). In the following, we assume $r,r_0\neq 0$.

By analogy with the cylindrical case, the algebraic symmetry $\mathbf{G}=\mathbb{T}\mathbf{G}^{+}\mathbb{T}$ (see Theorem 4.1) yields $\mathbf{M}^{+}\mathbb{T}\mathbf{M}=\mathbb{T}$ and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathcal{N}^{+}\mathbb{T}\mathcal{N}\right)=\mathbf{0}\quad\text{with } \mathcal{N}(r)\equiv\{\boldsymbol{\eta}^{(\alpha)}\}, \tag{6.9}$$

where the 6×6 matrix $\mathcal{N}(r)$ is composed of any six linearly independent solutions $\boldsymbol{\eta}^{(\alpha)}(r)=(\mathbf{U}^{(\alpha)},\ ir^2\mathbf{T}^{(\alpha)})^{\mathrm{T}}$, $\alpha=1\ldots 6$, of (4.3*a*). Thus, $\mathcal{N}^{+}(r)\mathbb{T}\mathcal{N}(r)$ is a constant matrix which can be chosen to provide the partial solutions $\boldsymbol{\eta}^{(\alpha)}(r)$ with appropriate pointwise orthogonality in the sense of a product (6.9) (Shuvalov 2003*a*).

Recalling the angular indices of the vector spherical harmonics $\mathbf{A}_{mn}(\mathbf{e}_r)\equiv\mathbf{P}_{mn}$, $\mathbf{B}_{mn}$ and $\mathbf{C}_{mn}$ defines the displacement-traction modes in full as

$$\left.\begin{aligned}\boldsymbol{\eta}^{(\alpha)}(\mathbf{r})&=\begin{pmatrix}\mathbf{u}^{(\alpha)}\\ ir^2\mathbf{t}_r^{(\alpha)}\end{pmatrix}=\sum_{n=0}^{\infty}\boldsymbol{\eta}_n^{(\alpha)}\quad(\alpha=1\ldots 6)\\ \text{with}\qquad\\ \boldsymbol{\eta}_n^{(\alpha)}(\mathbf{r})&=\begin{pmatrix}\mathbf{u}_n^{(\alpha)}\\ ir^2\mathbf{t}_{nr}^{(\alpha)}\end{pmatrix}=\sum_{\mathbf{A}}\begin{pmatrix}U^{(\alpha)}_{\mathbf{A}_{mn},n}(r)\sum_{|m|<n}\mathbf{A}_{mn}\\ ir^2\,T^{(\alpha)}_{\mathbf{A}_{mn},n}(r)\sum_{|m|<n}\mathbf{A}_{mn}\end{pmatrix},\end{aligned}\right\}\tag{6.10}$$

where it is taken into account that $U^{(\alpha)}_{\mathbf{A}_{mn},n}$ and $T^{(\alpha)}_{\mathbf{A}_{mn},n}(r)$ are independent of m (see §6*c*). There is no pointwise orthogonality of the 'full' modes in general owing to the complex conjugation in (6.9) and its absence in (3.7). At the same time, one can make use of (6.9) along with the integral orthogonality of vector harmonics (3.8). This enables evaluation of the angular (and time-period) average of the radial component of energy flux associated with the mode $\boldsymbol{\eta}^{(\alpha)}(\mathbf{r})$ as

$$P_r^{(\alpha)}\equiv-\int_{\Omega}\langle\mathbf{t}_r^{(\alpha)}\cdot\dot{\mathbf{u}}^{(\alpha)}\rangle_t\mathrm{d}\Omega=-\frac{\omega}{4r^2}\int_{\Omega}\boldsymbol{\eta}^{(\alpha)+}(\mathbf{r})\mathbb{T}\boldsymbol{\eta}^{(\alpha)}(\mathbf{r})\mathrm{d}\Omega.\tag{6.11}$$

This may be further reduced by using (3.8), (6.9) and (6.11),

$$\begin{aligned}P_r^{(\alpha)}&=\sum_{n=0}^{\infty}P_{nr}^{(\alpha)}\quad\text{where } P_{nr}^{(\alpha)}\equiv-\int_{\Omega}\langle\mathbf{t}_{rn}^{(\alpha)}\cdot\dot{\mathbf{u}}_n^{(\alpha)}\rangle_t\mathrm{d}\Omega\\ &=-\frac{\omega}{4r^2}(2n+1)\boldsymbol{\eta}_n^{(\alpha)+}(r)\mathbb{T}\boldsymbol{\eta}_n^{(\alpha)}(r).\end{aligned}\tag{6.12}$$

Thus, the fluxes $P_{nr}^{(\alpha)}$ carried by the modes $\boldsymbol{\eta}_n^{(\alpha)}(\mathbf{r})$ add up to give $P_r^{(\alpha)}$ of (6.11). According to (6.9), these fluxes do not depend on r which is consistent with energy conservation for the assumed case of real material parameters.

(*c*) *The case* $\mathbf{u}(r,\theta)$ $(m=0)$

The separation of variables using vector spherical harmonics involves a single auxiliary equation (3.5) which does not depend on m, and hence Theorem 4.1 holds for any m (Shul'ga *et al.* 1988). This does not, however, preclude the possibility that some lower symmetry permits such separation for the specific case of $m=0$. The derivation along the lines of §4 shows that 'radially tetragonal' symmetry, with $\mathbf{e}_r$ parallel to the fourfold axis and with $c_{24},\ c_{34}=0$, admits separation of variables for $m=0$. The result for this tetragonal symmetry at $m=0$ amounts to replacing $2c_{44}$ by $c_{22}-c_{23}$ in all entries of $\mathcal{Q}$ in (4.3*c*), except the term

$\lambda^2 c_{44}$ in the last diagonal component. Any other tetragonal symmetry (including that with $\mathbf{e}_r \parallel$ fourfold axis but with $c_{24} = -c_{34} \neq 0$), any trigonal symmetry and indeed all lower symmetries prevent separation of variables even for $m = 0$.

The same question may be raised concerning the SH uncoupling for $m = 0$. The above consideration implies that (5.8) with $m = 0$ can be extended to 'radially tetragonal' symmetry. Moreover, it can be shown that orthorhombic symmetry aligned with the coordinate planes also leads to SH uncoupling for $m = 0$, with the equation of motion obtained from (5.8) by replacing c_{66} with c_{55}. In this case, the SH modes admit the separation of variables form $\mathbf{u}_{\mathrm{SH}}(r,\theta) = U_{\mathbf{C}}(r)\lambda^{-1}P_{n,\theta}(\theta)\mathbf{e}_\phi$ while the modes $\mathbf{u}(r,\theta)$ polarized in the plane $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ do not. SH uncoupling is generally precluded for tetragonal symmetry with $c_{24} = -c_{34} \neq 0$ and for trigonal symmetry even if $m = 0$.

(*d*) *Solutions for* $n = 0$

Consider the term $\boldsymbol{\eta}_0^{(\alpha)}(\mathbf{r})$ with $n = 0$ (and hence $m = 0$) of the series (6.10). The corresponding spherical harmonics are

$$\mathbf{P}_{00} = \frac{1}{2\sqrt{\pi}}\,\mathbf{e}_r, \quad \mathbf{B}_{00} = \mathbf{0} \text{ and } \quad \mathbf{C}_{00} = \mathbf{0}. \tag{6.13}$$

Vanishing of $\mathbf{B}_{00}$ and $\mathbf{C}_{00}$ may be formally inferred from the definition (3.6) by taking the limit of $[\nu(\nu+1)]^{-1/2}(\partial/\partial\theta)P_\nu(\cos\theta)$ as $\nu \to 0$. It is also consistent with another framework that defines $\mathbf{B}_{mn}$ and $\mathbf{C}_{mn}$ without a normalization factor λ^{-1}. Note however that the symmetry $\mathbf{G} = \mathbb{T}\mathbf{G}^+\mathbb{T}$ of the system matrix in (4.3) is not preserved under any change in the definition (3.6) unless it implies multiplying all three harmonics $\mathbf{A}_{mn}$ by the same factor.

By (6.13), the sextet $\boldsymbol{\eta}_0^{(\alpha)}$ $(\alpha = 1 \ldots 6)$ for $n = 0$ contains only two non-zero modes which are the modes $\alpha = l1, l2$ of the longitudinal (radially polarized) wave

$$\mathbf{u} = U(r)\mathbf{e}_r \text{ and } \quad \mathbf{t}_r = T(r)\mathbf{e}_r, \tag{6.14}$$

where we drop the indices used in (6.10). Equation (4.3*a*) with $n = 0$ provides an uncoupled system of two equations for the amplitudes U and T:

$$\begin{pmatrix} U \\ ir^2 T \end{pmatrix}' = i \begin{pmatrix} 2ir^{-1}c_{12}c_{11}^{-1} & -r^{-2}c_{11}^{-1} \\ 4\left(c_{22} - c_{44} - \dfrac{c_{12}^2}{c_{11}}\right) - r^2\rho\omega^2 & -2ir^{-1}c_{12}c_{11}^{-1} \end{pmatrix} \begin{pmatrix} U \\ ir^2 T \end{pmatrix}, \tag{6.15}$$

which reduces to a single second-order equation for $U(r)$,

$$(r^2 c_{11} U')' + (r^2\rho\omega^2 + 4(c_{44} - c_{22}) + 2(rc_{12})')\,U = 0. \tag{6.16}$$

Note that the wave (6.14) exerts the tractions $\mathbf{t}_i = f(r)\mathbf{e}_i$ for $i = \theta, \phi$, where $f = c_{12}U' + r^{-1}(c_{22} - c_{44})U$. Equations (6.15) and (6.16) are similar to the SH-wave equations (5.8) and (5.7) formally taken with $n = 0$. Since the wave (6.14) also implies $m = 0$, it follows from §6*c* that equations (6.15) and (6.16) obtained from the transversely isotropic system (4.3*b*) and (4.3*c*) can be extended to the tetragonal case, by replacing $2c_{44}$ with $c_{22} - c_{23}$. The same results can be obtained by inserting $\mathbf{u} = U(r)\mathbf{e}_r$ in the initial elastodynamic equations (3.1)–(3.2) with

tetragonal symmetry. Note, in this regard, that the elastostatic solution of the form (6.14) was analysed for various symmetries in Ting (1998), Antman & Negron-Marrero (1987) and Antman & Ting (2001).

Interestingly, replacing λ^{-1} in the harmonics definition (3.6) with some other power of λ may lead to peculiar solutions at $n=0$ different from (6.14). This is a problem of interest in its own right. Consider the set of spherical harmonics defined as

$$\tilde{\mathbf{P}}_{mn}=\mathbf{P}_{mn},\quad \tilde{\mathbf{B}}_{mn}=\lambda^{-1}\mathbf{B}_{mn}\quad\text{and}\quad \tilde{\mathbf{C}}_{mn}=\lambda^{-1}\mathbf{C}_{mn}. \tag{6.17}$$

The corresponding spherical harmonic functions for $n=0$ are (dropping the $m=0$ subscript)

$$\tilde{\mathbf{P}}_0=A_0\mathbf{e}_r,\quad \tilde{\mathbf{B}}_0=-A_0\tan\frac{\theta}{2}\mathbf{e}_\theta,\quad \tilde{\mathbf{C}}_0=A_0\tan\frac{\theta}{2}\mathbf{e}_\phi\quad\text{with } A_0=\frac{1}{2\sqrt{\pi}}, \tag{6.18}$$

which follow from (6.17) using first an exchange of limits,

$$\lim_{\nu\to 0}\frac{1}{\nu(\nu+1)}\frac{\mathrm{d}}{\mathrm{d}\theta}P_\nu(\cos\theta)=\frac{\mathrm{d}}{\mathrm{d}\theta}\lim_{\nu\to 0}\frac{P_\nu(\cos\theta)-P_0(\cos\theta)}{\nu(\nu+1)}, \tag{6.19}$$

and then Jolliffe's formula (Jolliffe 1919) to evaluate the derivative with respect to ν,

$$\left.\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}\nu}\right|_{\nu=n}\equiv F_n(z)=-P_n(z)\ln\frac{z+1}{2}+\frac{2}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left((z^2-1)^n\ln\frac{z+1}{2}\right). \tag{6.20}$$

By (6.18), $\tilde{\mathbf{P}}_0$ corresponds to purely radial motion, while $\tilde{\mathbf{B}}_0$ and $\tilde{\mathbf{C}}_0$ (non-zero in contrast to (6.13)) represent shearing and twisting about the polar axis, with zero at one pole and singularity at the other. The latter can be avoided by introducing a conical cut centred at $r=0$ of arbitrarily small angular extent at $\theta=\pi$. This obviates the singularity allowing the otherwise unnormalized spherical harmonics $\tilde{\mathbf{B}}_0$ and $\tilde{\mathbf{C}}_0$ to satisfy the orthonormality conditions (3.8). Note that the spherical harmonics $\tilde{\mathbf{B}}_0$ and $\tilde{\mathbf{C}}_0$ are derived from and related with the Legendre polynomials of the first kind which define the spherical harmonic $\tilde{\mathbf{P}}_0$, although the Legendre polynomials of the second kind may be represented as $Q_n(z)=\frac{1}{2}[F_n(z)-(-1)^nF_n(-z)]$ (see equation (6.20); Jolliffe 1919).

Based on the above, it is of interest to consider specific $n=0$ solutions in the form

$$\mathbf{u}=\sum_{\tilde{\mathbf{A}}_0}\tilde{U}_{\tilde{\mathbf{A}}_0}(r)\tilde{\mathbf{A}}_0,\quad \mathbf{t}_r=\sum_{\tilde{\mathbf{A}}_0}\tilde{T}_{\tilde{\mathbf{A}}_0}(r)\tilde{\mathbf{A}}_0\ \text{and}\quad \tilde{\mathbf{A}}_0=\tilde{\mathbf{P}}_0,\tilde{\mathbf{B}}_0,\tilde{\mathbf{C}}_0. \tag{6.21}$$

The system (4.3) modified with respect to the spherical harmonics $\tilde{\mathbf{A}}$ of (6.17) and taken for $n=0$ defines the displacement amplitudes $\tilde{U}_{\tilde{\mathbf{A}}_0}(r)$ by the following equations:

$$\begin{aligned}&(r^2c_{11}\tilde{U}'_{\tilde{\mathbf{P}}_0})'+(r^2\rho\omega^2+4(c_{44}-c_{22})+2(rc_{12})')\tilde{U}_{\tilde{\mathbf{P}}_0}\\&\quad=\left[2\left(c_{44}-c_{22}+\frac{c_{12}^2}{c_{11}}\right)-rc'_{66}\right]\tilde{U}_{\tilde{\mathbf{B}}_0}+r^2[r^{-1}(c_{66}+c_{12})\tilde{U}_{\tilde{\mathbf{B}}_0}]'\end{aligned} \tag{6.22a}$$

and

$$(r^2 c_{66}\tilde{U}'_{\tilde{\mathbf{A}}_0})' + (r^2\rho\omega^2 + 2(c_{44} - c_{66}) - rc'_{66})\,\tilde{U}_{\tilde{\mathbf{A}}_0} = 0, \quad \tilde{\mathbf{A}}_0 = \tilde{\mathbf{B}}_0, \tilde{\mathbf{C}}_0. \tag{6.22b}$$

It is seen that equation (6.22*b*) for the amplitudes $\tilde{U}_{\tilde{\mathbf{B}}_0}$ and $\tilde{U}_{\tilde{\mathbf{C}}_0}$ are the same as the SH wave equation (5.6) with $n = 0$, and that $\tilde{U}_{\tilde{\mathbf{B}}_0}$ provides a forcing term in equation (6.22*a*) for $\tilde{U}_{\tilde{\mathbf{P}}_0}$ (note also that (6.22*a*) reduces to (6.16) if $\tilde{U}_{\tilde{\mathbf{B}}_0} = 0$). Thus the SH solution $\mathbf{u} = \tilde{U}_{\tilde{\mathbf{C}}_0}\tilde{\mathbf{C}}_0(\parallel \mathbf{e}_\phi)$ is completely uncoupled. However, the SH motion polarized in the $\tilde{\mathbf{B}}_0 \parallel \mathbf{e}_\theta$ direction drives the radial motion and this results in the coupled wave $\mathbf{u} = \tilde{U}_{\tilde{\mathbf{P}}_0}\tilde{\mathbf{P}}_0 + \tilde{U}_{\tilde{\mathbf{B}}_0}\tilde{\mathbf{B}}_0$. At the same time, non-zero angular motion requires that the issue of singularities at $\theta = \pi$ be resolved. Under normal circumstances, e.g. a solid sphere or a complete shell, this is not the case, and any angular motion is precluded, i.e. $\tilde{U}_{\tilde{\mathbf{B}}_0} = \tilde{U}_{\tilde{\mathbf{C}}_0} = 0$ and only radial motion $\mathbf{u} = U(r)\mathbf{e}_r$ occurs (where U is defined by (6.16)). If the small conical cut device is introduced, then the surface of the cone must apply a force and a moment sufficient to maintain the dynamic tractions required of the solutions. The magnitudes of the tractions for different types of motion depend critically on the cone angle $\varepsilon \ll 1$. For purely longitudinal motion the traction is independent of ε, i.e. O(1). The $(\mathbf{e}_r, \mathbf{e}_\theta)$-coupled motion requires normal traction $\mathbf{t}_\theta$ of $\mathrm{O}(\tan^2(\theta/2)|_{\theta=\pi-\varepsilon}) = \mathrm{O}(\varepsilon^{-2})$ and is therefore ruled out as a viable $n = 0$ dynamic solution. In the case of the pure SH motion $\mathbf{u} = \tilde{U}_{\tilde{\mathbf{C}}_0}\tilde{\mathbf{C}}_0$, the normal traction $\mathbf{t}_\theta = r^{-1}c_{44}\tan(\theta/2)\,\mathbf{u}$ is $\mathrm{O}(\varepsilon^{-2})$, a pure twist in the $\mathbf{e}_\phi$ direction corresponding to a net torque about the polar axis of order unity ($r\varepsilon \times 2\pi r\varepsilon \times \varepsilon^{-2}$). This suggests that pure twisting motion may be induced in a solid sphere with a fixed polar axis by application of torque to a small conical insert.

7. Conclusion

The central result of the paper, Theorem 4.1, shows that spherically anisotropic radially inhomogeneous materials admit elastodynamic solutions $\mathbf{u}(r, \theta, \phi)$ in a separation of variables form. The angular dependence (on θ and ϕ) is described by the vector spherical harmonics while the radial dependence (on r) is separated and determined by the Stroh-like first-order ODS that is solvable by standard means. It is proved that such separation of variables solution is generally possible only if the spherical anisotropy is restricted to TI with the principal axis in the radial direction $\mathbf{e}_r$. TI about $\mathbf{e}_r$ distinguishes a class of materials with *complete* spherical anisotropy, which is a physically natural model of spherical anisotropy since it ensures invariance of material properties with respect to any orientation of the polar axis.

The separable of variables solution $\mathbf{u}(r, \theta, \phi)$ for the transverse isotropic case does not explicitly depend on the azimuthal order m. At the same time, dependence on m reveals itself in that the solution of the form $\mathbf{u}(r, \theta)$ (i.e. with $m = 0$) admits separation of variables via spherical harmonics not only for TI but also for lower symmetry—but only 'up to' tetragonal with $\mathbf{e}_r$ along the fourfold axis and with c_{24}, $c_{34} = 0$. Note that the solutions $\mathbf{u}(r, \theta, \phi)$ for TI and $\mathbf{u}(r, \theta)$ for the above tetragonal symmetry uncouple the shear modes parallel to the vector

harmonic $\mathbf{C}$ ($\|\,\mathbf{e}_\phi$ at $m=0$) from the in-plane modes orthogonal to $\mathbf{C}$. Moreover, shear modes of the form $\mathbf{u}(r,\theta)$ are uncoupled also for orthorhombic symmetry but not for trigonal symmetry nor for tetragonal if $c_{24},\ c_{34}\neq 0$.

The establishment of the Stroh format for the elastodynamic equations in spherical coordinates opens the door for applications to various boundary value and scattering problems. For instance, solutions for acoustic and elastic wave scattering from solid spheres and shells, which have been limited to isotropic materials (see Martin 2006, §4.10 for a review) or transversely isotropic shells with $m=0$ (Hasheminejad & Maleki 2009), can be generated for arbitrarily layered shells and solids using standard solution techniques outlined in §6*b*. Other possible approaches that can be explored with the Stroh formalism include impedance matrices for spherical shells and solids, the use of which simplifies the formulation of boundary value problems, such as determining modal frequencies, solving radiation and scattering problems. By analogy with the cylindrical situation (Norris & Shuvalov 2010) it should be possible to formulate a matrix Riccati ordinary differential equation for the impedance matrix as a function of the spherical radius r, with a unique solution at the origin that depends only on the elastic constants at $r=0$. The Stroh formalism is particularly suited to solution of elastodynamic problems with forcing, for example, from thermal expansion via laser excitation with application to non-destructive testing. More exotic issues could be addressed with the Stroh system, such as modelling and simulation of fully elastic three-dimensional 'radial wave crystals' (Torrent & Sánchez-Dehesa 2009), i.e. shells of radially periodic materials that exhibit Bloch wave effects normally associated with rectangular periodic crystals.

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