Comment on "Design of acoustic devices with isotropic material via conformal transformation" [Appl. Phys. Lett. 97, 044101 (2010)]

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The wave equation for an acoustic medium with uniform density ρ_0 and bulk modulus λ_0 is, assuming time dependence $e^{-i\omega t}$,

$$\nabla \cdot \nabla p + \omega^2(\rho_0/\lambda_0)p = 0. \tag{1}$$

Ren *et al.*[1] demonstrated that under the conformal transformation $x^i \to x^{i'}$, $z(=x^1 + ix^2) \to \zeta(z)$, Eq. (1) becomes, for $p'(x^{i'}) = p(x^i)$,

$$\nabla' \cdot \nabla' p' + \omega^2(\rho'/\lambda')p' = 0, \tag{2}$$

where

$$\lambda'/\rho' = |\zeta'(z)|^2 \lambda_0/\rho_0. \tag{3}$$

Ren *et al.*[1] then imposed the constraint that the impedance in the transformed domain remains unchanged, i.e.,

$$\lambda'\rho' = \lambda_0\rho_0, \tag{4}$$

which together with Eq. (3) implies the identities (Eq. (6) of Ref. 1)

$$\rho' = \rho_0/|\zeta'(z)|, \quad \lambda' = \lambda_0|\zeta'(z)|. \tag{5}$$

The expressions (5) are not correct as we explain next.

The equations of motion for an acoustic fluid of density ρ' and bulk modulus λ' are

$$-\omega^2\rho'\mathbf{u}' = -\nabla'\mathbf{p}', \quad p' = -\lambda'\nabla' \cdot \mathbf{u}'. \tag{6}$$

Eliminating the displacement $\mathbf{u}'$ yields

$$\rho'\nabla' \cdot (1/\rho')\nabla' p' + \omega^2(\rho'/\lambda')p' = 0. \tag{7}$$

Comparison of Eqs. (2) and (7) implies that the former is the reduced equation for the pressure in an acoustic fluid only if ρ' is constant. That is, Eqs. (2) and (3) represent the equation of motion of an acoustic medium with

$$\rho' = c\rho_0, \quad \lambda' = c\lambda_0|\zeta'(z)|^2, \tag{8}$$

for constant $c > 0$. These are the correct expressions for the bulk modulus and density in the transformed medium.

The general theory of transformation acoustics allows for the possibility of non-unique expressions for the material parameters in the transformed domain.[2] The material descriptions range from what is known as inertial fluids, with anisotropic density and scalar bulk modulus, to pentamode behavior with elastic constitutive response and generally anisotropic density, plus a spectrum of possibilities in between.[3] In the inertial fluid limit, the transformed bulk modulus and density tensor are (Eq. (2.8) of Ref. 2)

$$\lambda' = \lambda_0 \det \mathbf{F}, \quad \boldsymbol{\rho}' = \rho_0(\det \mathbf{F})(\mathbf{F}\mathbf{F}^T)^{-1}, \tag{9}$$

where $F_{i'i} = \partial x^{i'}/\partial x^i$. At the other extreme, the pentamode material is defined by a density tensor $\boldsymbol{\rho}$ and a fourth order elasticity tensor with components C_{ijkl} (Eq. (4.3) of Ref. 2)

$$C_{ijkl} = \lambda' S_{ij}S_{kl}, \quad \boldsymbol{\rho} = \mathbf{S}\boldsymbol{\rho}'\mathbf{S}, \tag{10}$$

where λ' and $\boldsymbol{\rho}'$ are given by Eq. (9) and $\mathbf{S} = \mathbf{S}^T$ is positive definite satisfying div $\mathbf{S} = 0$, but otherwise arbitrary. For the special case of a conformal transformation, we have det $\mathbf{F} = |\zeta'(z)|^2$, $\mathbf{F}\mathbf{F}^T = |\zeta'(z)|^2\mathbf{I}$, where $\mathbf{I}$ is the identity. Then Eq. (9) reduces to Eq. (8), apart from the factor c which could be incorporated into $\mathbf{F}$, and taking $\mathbf{S} = \mathbf{I}$ Eq. (10) also reduces to Eq. (8), since $C_{ijkl} = \lambda'\delta_{ij}\delta_{kl}$ corresponds to the acoustic fluid of bulk modulus λ'. In conclusion, when the transformation is conformal the parameters are isotropic, with both the inertial and the pentamodal models yielding ρ' and λ' given by Eq. (8).

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