Correction: eq. (64b)

Elastic cloaking theory

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Transformation theory is developed for the equations of linear anisotropic elasticity. The transformed equations correspond to non-unique material properties that can be varied for a given transformation by selection of the matrix relating displacements in the two descriptions. This gauge matrix can be chosen to make the transformed density isotropic for any transformation although the stress in the transformed material is not generally symmetric. Symmetric stress is obtained only if the gauge matrix is identical to the transformation matrix, in agreement with Milton et al.[1]. The elastic transformation theory is applied to the case of cylindrical anisotropy. The equations of motion for the transformed material with isotropic density are expressed in Stroh format, suitable for modeling cylindrical elastic cloaking. It is shown that there is a preferred approximate material with symmetric stress that could be a useful candidate for making cylindrical elastic cloaking devices.

I. INTRODUCTION

Interest in cloaking of wave motion has surged with the demonstration of the possibility of practical electromagnetic wave cloaking[2]. The principle underlying the effect is the so-called transformation or change-of-variables method[3,4] in which the material parameters in the physical domain are defined by a spatial transformation. The concept of material properties defined by transformation is not restricted to electromagnetism, and has stimulated interest in applying the same method to other wave fields. The first applications in acoustics were obtained by direct analogy with the electromagnetic case[5–7]. It was quickly realized that the fundamental mathematical identity behind the acoustic transformation is the equivalence[8] of the Laplacian in the original coordinates to a differential operator in the transformed (physical) coordinates, according to $\mathrm{Div}\,\mathrm{Grad}\,f \to J\,\mathrm{div}\,J^{-1}\boldsymbol{F}\boldsymbol{F}^t\,\mathrm{grad}\,f$, where $\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$ is the deformation gradient of the transformation (see §II) and $J = \det\boldsymbol{F}$. This connection, plus the realization that the tensor within the operator can be interpreted as a tensor of inertia means that the homogeneous acoustic wave equation can be mapped to the equation for an inhomogeneous fluid with anisotropic density.

However, the material properties for acoustic cloaking do not have to be identified as a fluid with a single bulk modulus and a tensorial inertia. There is a large degree of freedom in the choice of the cloaking material properties[9,10]: a compressible fluid with anisotropic density is a special case of *pentamode materials*[11,12] with anisotropic inertia. The non-uniqueness of the material properties (for a given transformation) is a feature not found in the electromagnetic case, where, for instance, the tensors of electric permittivity and magnetic permeability are necessarily proportionate for a transformation of the vacuum. The extra freedom in the acoustic case means that either or both of the scalar parameters, density and elastic stiffness (bulk modulus), can become tensorial quantities after the transformation. While most papers on acoustic cloaking have considered materials with scalar stiffness and tensorial inertia, e.g.[5–7,13–18], see[19] for a review, there is no physical reason for such restricted material properties. Cloaking with such materials requires very large total mass[9,20], but the more general class of pentamode materials with anisotropic inertia does not have this constraint. In fact, it is often possible to choose the material properties so that the inertia is isotropic, in which case the total mass is simply the mass of the equivalent undeformed region[10]. This property can be realized if the transformation is a pure stretch, as is the case when there is radial symmetry. This distinction between the cloaking material properties is critical but, judging by the continued emphasis on anisotropic inertia in the literature, does not seem to have been fully appreciated. Apart from[9,10] there have been few studies[21,22] of acoustic cloaking with anisotropic stiffness.

The non-uniqueness of the transformed material properties found in the acoustic theory transfers to elastodynamics. The first study of the transformation of the elastodynamic equations by Milton et al.[1] concluded that the appropriate class of constitutive equations for the transformed material are the Willis equations for material response. The general

form of the Willis equations are[1,23]

$$\operatorname{div}\boldsymbol{\sigma} = \dot{\boldsymbol{p}}, \tag{1a}$$

$$\boldsymbol{\sigma} = \boldsymbol{C}^{\text{eff}} * \boldsymbol{e} + \boldsymbol{S}^{\text{eff}} * \boldsymbol{u}, \tag{1b}$$

$$\boldsymbol{p} = \boldsymbol{S}^{\text{eff}\,\dagger} * \boldsymbol{e} + \boldsymbol{\rho}^{\text{eff}} * \dot{\boldsymbol{u}}, \tag{1c}$$

where $\boldsymbol{e} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t)$, $*$ denotes time convolution and † is the adjoint. The stress in (1b) is symmetric, and the elastic moduli enjoy all of the symmetries for normal elasticity, *viz.* $C^{\text{eff}}_{klij} = C^{\text{eff}}_{ijkl}$ and $C^{\text{eff}}_{jikl} = C^{\text{eff}}_{ijkl}$. Brun et al.[24] considered the transformation of isotropic elasticity in cylindrical coordinates for the particular transformation function used by[3,4] and found that the transformed material properties are those of a material with isotropic inertia and elastic behavior of Cosserat type. The governing equations for Cosserat elastic materials[25] are

$$\operatorname{div}\boldsymbol{\sigma} = \rho^{\text{eff}}\,\ddot{\boldsymbol{u}}, \tag{2a}$$

$$\boldsymbol{\sigma} = \boldsymbol{C}^{\text{eff}}\nabla\boldsymbol{u}, \tag{2b}$$

with elastic moduli satisfying the major symmetry $C^{\text{eff}}_{klij} = C^{\text{eff}}_{ijkl}$ but not the minor symmetry, $C^{\text{eff}}_{jikl} \neq C^{\text{eff}}_{ijkl}$. This implies that the stress is not necessarily symmetric, $\boldsymbol{\sigma}^t \neq \boldsymbol{\sigma}$, and that it depends not only on the strain $\boldsymbol{e}$ (the symmetric part of the displacement gradient) but also upon the local rotation $\boldsymbol{\omega} = \frac{1}{2}(\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^t)$. Not only are the parameters such as $\boldsymbol{C}^{\text{eff}}$ in eqs. (1) and (2) different, the constitutive theories are mutually incompatible: one has symmetric stress, the other a non-symmetric stress. We show in this paper that both theories are possible versions of the transformed elastodynamic equations, and that they are only two from a spectrum of possible constitutive theories. Apart from the two references mentioned[1,24], the only other example of transformation elasticity concerns flexural waves obeying the biharmonic equation[26], which is beyond the realm of the present paper.

The purpose here is to consider the transformation method for elastodynamics, and to describe the range of constitutive theories possible. The starting point is the observation[10] that the extra degrees of freedom noted for the acoustic transformation can be ascribed to the linear relation between the displacement fields in the two coordinate systems. This "gauge" transformation introduces a second matrix or tensor, in addition to the deformation gradient from the change of coordinates. As discussed in[10], the variety of acoustically transformed material properties arises from the freedom in the displacement gauge. The same freedom is also present in the elastic case, and as we will show, it allows one to derive a broader class of constitutive properties than those suggested by Milton et al.[1] and by Brun et al.[24]. The material properties found in these studies correspond to specific choices of the gauge matrix.

Cloaking is achieved with transformations that deform a region in such a way that the mapping is one-to-one everywhere except at a single point, which is mapped into the cloak inner boundary. This is a singular transformation, and in practice, the mapped region would be of finite size, e.g. a small sphere, for which the mapping is everywhere regular. Our objective here is to understand the nature of the material necessary to produce the transformation effect, in particular, what type of constitutive behavior is necessary: such as isotropic or anisotropic inertia.

The outline of the paper is as follows. The notation and setup of the problem are given in §II where the displacement gauge matrix is introduced. The general form of the transformed equations of motion are presented in §III. Constitutive equations resulting from specific forms of the gauge matrix are discussed in §IV, particularly the Willis equations and Cosserat elasticity, which are shown to coincide under certain circumstances. The special case of transformed acoustic materials is discussed in §V. The elastic transformation theory with isotropic density is applied in §VI to radial transformation of cylindrically anisotropic solids. Based on this formulation, an elastic material with isotropic density and standard stress-strain relations is proposed in §VI C as an approximation to the transformed material. A summary of the main results is given in §VII.

II. NOTATION AND SETUP

Two related configurations are considered: the original Ω, and the transformed region ω, also called the physical or current domain. The transformation from Ω to ω is described by the point-wise deformation from $\boldsymbol{X} \in \Omega$ to $\boldsymbol{x} \in \omega$. The symbols ∇, ∇_X and div, Div indicate the gradient and divergence operators in $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively, and the superscript t denotes transpose. The component form of $\operatorname{div}\mathbf{E}$ is $\partial E_i/\partial x_i$ or $\partial E_{ij}/\partial x_i$ when $\mathbf{E}$ is a vector

and a second order tensor-like quantity, respectively. Upper and lower case subscripts $(I, J, \ldots, i, j, \ldots)$ are used to distinguish between the domains, and the summation convention on repeated subscripts is assumed. It is useful to describe the transformation using language of finite deformation in continuum mechanics. Thus, $\boldsymbol{X}$ describes a particle position in the Lagrangian or undeformed configuration, and $\boldsymbol{x}$ is particle location in the Eulerian or deformed physical state. The transformation or mapping is assumed to be one-to-one and invertible. For perfect cloaking the transformation is one-to-many at the single point $\boldsymbol{X} = \boldsymbol{O}$, but this can be avoided by always considering near-cloaks, where, for instance, the single point is replaced by a small hole which is mapped to a much larger hole.

The deformation gradient is defined $\boldsymbol{F} = \nabla_X \boldsymbol{x}$ with inverse $\boldsymbol{F}^{-1} = \nabla \boldsymbol{X}$, or in component form $F_{iI} = \partial x_i/\partial X_I$, $F^{-1}_{Ii} = \partial X_I/\partial x_i$. The Jacobian of the deformation is $J = \det \boldsymbol{F} = |\boldsymbol{F}|$, or in terms of volume elements in the two configurations, $J = \mathrm{d}\, v/\,\mathrm{d}\, V$. The polar decomposition implies $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$, where $\boldsymbol{R}$ is proper orthogonal ($\boldsymbol{R}\boldsymbol{R}^t = \boldsymbol{R}^t\boldsymbol{R} = \boldsymbol{I}$, $\det \boldsymbol{R} = 1$) and the left stretch tensor $\boldsymbol{V} \in \mathrm{Sym}^+$ is the positive definite solution of $\boldsymbol{V}^2 = \boldsymbol{B}$ where $\boldsymbol{B}$ $(= \boldsymbol{F}\boldsymbol{F}^t)$ is the left Cauchy-Green or Finger deformation tensor.

The infinitesimal displacement $\boldsymbol{U}(\boldsymbol{X}, t)$ and stress $\boldsymbol{\Sigma}(\boldsymbol{X}, t)$ satisfy the equations of linear elasticity in the original domain:

$$\left.\begin{aligned} \mathrm{Div}\, \boldsymbol{\Sigma} &= \rho_0 \ddot{\boldsymbol{U}}, \\ \boldsymbol{\Sigma} &= \boldsymbol{C}^{(0)} \nabla_X \boldsymbol{U}, \end{aligned}\right\} \quad \text{in } \Omega, \tag{3}$$

where ρ_0 is the (scalar) mass density and the the elements of the elastic stiffness tensor satisfy the full symmetries

$$C^{(0)}_{IJKL} = C^{(0)}_{JIKL}, \quad C^{(0)}_{IJKL} = C^{(0)}_{KLIJ}. \tag{4}$$

The first identity expresses the symmetry of the stress and the second is the consequence of an assumed strain energy density function. The total energy density is the sum of the strain and kinetic energy densities,

$$\mathcal{E}_0 = \mathcal{W}_0 + \mathcal{T}_0 \quad \text{where } \mathcal{W}_0 = \tfrac{1}{2} C^{(0)}_{IJKL}\, U_{J,I} U_{L,K}, \quad \mathcal{T}_0 = \tfrac{1}{2}\rho_0 \dot{\boldsymbol{U}}^t \dot{\boldsymbol{U}}. \tag{5}$$

Particle displacement in the transformed domain is $\boldsymbol{u}(\boldsymbol{x}, t)$. Our objective is to find its governing equations. In order to proceed, we need in addition to the geometrical quantity $\boldsymbol{F}$, a kinematic relation that relates the displacements in the two domains. We assume a linear "gauge" change in the displacement defined by a non-singular matrix $\boldsymbol{A}$ as

$$\boldsymbol{U} = \boldsymbol{A}^t \boldsymbol{u} \qquad (U_I = A_{iI} u_i). \tag{6}$$

According to its definition the matrix $\boldsymbol{A}$ is, like $\boldsymbol{F}$, not a second order tensor because it has one "leg" in both domains. The choice of the transpose, $\boldsymbol{A}^t$ in equation (6), means that $\boldsymbol{A}$ and $\boldsymbol{F}$ are similar objects, although at this stage they are not related.

The arbitrariness in the choice of $\boldsymbol{A}$ is the central theme of this paper. This approach generalizes that of[10] which was restricted to acoustic materials, and of Milton et al.[1] for elasticity. The point of departure with[1] here rests with the assumed independence of the gauge matrix $\boldsymbol{A}$. Milton et al. assume a similar relation between the displacement fields; eq. (6) is identical to eq. (2.2) in[1]; however, the matrix $\boldsymbol{A}$ in[1] is assumed at the outset to be equal to the deformation gradient ($\boldsymbol{A} = \boldsymbol{F}$). We will return to this distinction later. As noted by Milton et al.[1], the relation $\mathrm{d}\, \boldsymbol{X} = \boldsymbol{F}^{-1}\, \mathrm{d}\, \boldsymbol{x}$ might lead one to expect $\boldsymbol{A} = \boldsymbol{F}^{-t}$ by identifying $\mathrm{d}\, \boldsymbol{X}$ and $\mathrm{d}\, \boldsymbol{x}$ with $\boldsymbol{U}$ and $\boldsymbol{u}$, respectively. However, the displacements are not associated with the coordinate transformation, unlike in the theory of finite deformation, and hence $\boldsymbol{F}$ and $\boldsymbol{A}$ are independent. Milton et al.[1] specify $\boldsymbol{A} = \boldsymbol{F}$ on the basis that this is the only choice that guarantees a symmetric stress. We will return to this point later.

III. GENERAL FORM OF THE TRANSFORMED EQUATIONS

Under the transformation and the gauge change the energy density transforms as $\mathcal{E}_0 \to \mathcal{E} = \mathcal{W} + \mathcal{T}$ according to $\mathcal{E}\, \mathrm{d}\, V = \mathcal{E}_0\, \mathrm{d}\, V_0$, so that

$$\mathcal{E} = \mathcal{W} + \mathcal{T} = \tfrac{1}{2} J^{-1} \big\{ C^{(0)}_{IJKL} \left(u_j A_{jJ}\right)_{,i} \left(u_l A_{lL}\right)_{,k} F_{iI} F_{kK} + \rho_0 \dot{u}_i \dot{u}_j A_{iI} A_{jI}. \big\} \tag{7}$$

Hence,

$$\mathcal{W} = \tfrac{1}{2} J^{-1} C^{(0)}_{IJKL}\, F_{iI} F_{kK} \left(u_j A_{jJ}\right)_{,i} \left(u_l A_{lL}\right)_{,k}, \qquad \mathcal{T} = \tfrac{1}{2} \dot{\boldsymbol{u}}^t \boldsymbol{\rho} \dot{\boldsymbol{u}}, \tag{8}$$

where the (symmetric) density tensor is

$$\boldsymbol{\rho} = \boldsymbol{\rho}^t = \rho_0\, J^{-1} \boldsymbol{A} \boldsymbol{A}^t, \tag{9}$$

The equations of motion in the deformed, or current material, are determined by the Euler-Lagrange equations of the Lagrangian density $\mathcal{L} = \mathcal{W} - \mathcal{T}$, as

$$A_{jJ} \left(J^{-1} C^{(0)}_{IJKL}\, F_{iI} (u_l A_{lL})_{,k} F_{kK}\right)_{,i} - \rho_{ij} \ddot{u}_i = 0. \tag{10}$$

Using the identity

$$(J^{-1} F_{iI})_{,i} = 0, \tag{11}$$

this can be written

$$\mathcal{Q}_{ijIJ} \left(J C^{(0)}_{IJKL} \left(u_l \mathcal{Q}_{klKL}\right)_{,k}\right)_{,i} - \rho_{ij} \ddot{u}_i = 0, \tag{12}$$

where the fourth order quantity

$$\mathcal{Q}_{ijIJ} = J^{-1} F_{iI} A_{jJ}, \tag{13}$$

is introduced for later use.

The transformed system (12) is equivalent to the equilibrium equations

$$\sigma_{ij,i} = \dot{p}_j, \tag{14a}$$

and the constitutive relations

$$\sigma_{ij} = C^{\text{eff}}_{ijkl} u_{l,k} + S^{\text{eff}}_{ijl}\, \dot{u}_l, \quad p_l = S^{\text{eff}}_{ijl}\, u_{j,i} + \rho^{\text{eff}}_{jl} \dot{u}_k, \tag{14b}$$

with parameters defined as follows in the Fourier time domain (dependence $e^{-i\omega t}$)

$$C^{\text{eff}}_{ijkl} = J C^{(0)}_{IJKL} \mathcal{Q}_{ijIJ} \mathcal{Q}_{klKL}, \tag{15a}$$

$$S^{\text{eff}}_{ijl} = (-i\omega)^{-1} J C^{(0)}_{IJKL} \mathcal{Q}_{ijIJ} \mathcal{Q}_{klKL,k}, \tag{15b}$$

$$\rho^{\text{eff}}_{jl} = \rho_{jl} + (-i\omega)^{-2} J C^{(0)}_{IJKL} \mathcal{Q}_{ijIJ,i} \mathcal{Q}_{klKL,k}, \tag{15c}$$

where the density ρ_{jl} is given by (9). The elastic moduli and the density satisfy the general symmetries

$$C^{\text{eff}}_{ijkl} = C^{\text{eff}}_{klij}, \quad \rho^{\text{eff}}_{jl} = \rho^{\text{eff}}_{lj}, \tag{16}$$

but not the full symmetries required for the Willis constitutive model (1). Equations (14)-(15) are the fundamental result of the transformation theory. The remainder of the paper is concerned with their simplification and interpretation.

Note that the transformed stiffness may be expressed in a form similar to the Kelvin representation for the tensor of elastic moduli[27], as

$$\mathbf{C}^{\text{eff}} = \sum_{\alpha=1}^{6} K^{(\alpha)} \mathbf{S}^{(\alpha)} \otimes \mathbf{S}^{(\alpha)}, \quad K^{(\alpha)} = J K_0^{(\alpha)}, \quad \mathbf{S}^{(\alpha)} = J^{-1} \mathbf{F} \mathbf{P}^{(\alpha)} \mathbf{A}^t, \tag{17}$$

where $K_0^{(\alpha)} > 0$ are the Kelvin moduli, $\mathbf{P}^{(\alpha)} \in \text{Sym}$, $\text{tr}\, \mathbf{P}^{(\alpha)} \mathbf{P}^{(\beta)} = \delta_{\alpha\beta}$, are the eigenstrains/eigenstresses, such that the original stiffness has the unique decomposition

$$\mathbf{C}^{(0)} = \sum_{\alpha=1}^{6} K_0^{(\alpha)} \mathbf{P}^{(\alpha)} \otimes \mathbf{P}^{(\alpha)}. \tag{18}$$

The transformed matrices $\mathbf{S}^{(\alpha)}$ do not inherit the orthogonality of the original eigenstrains/eigenstresses $\mathbf{P}^{(\alpha)}$, so that (17) is not the exact Kelvin representation in the transformed coordinates. It does, however, illustrate that the transformed stiffness is positive definite, even though $\mathbf{S}^{(\alpha)}$ are in general not symmetric. The representation (17) is particularly useful in the limiting case of an acoustic fluid in the original domain for which only one of the $K_0^{(\alpha)}$ is non-zero, discussed later.

IV. TRANSFORMED EQUATIONS IN SPECIFIC FORMS

A. Willis equations: $\boldsymbol{A} = \boldsymbol{F}$

The absence of the minor symmetries under the interchange of i and j in C^{eff}_{ijkl} and S^{eff}_{ijl} of (15) implies that the stress is generally not symmetric. Symmetric stress is guaranteed if $\mathcal{Q}_{ijIJ} = \mathcal{Q}_{jiIJ}$ (see eq. (13)), which occurs if the gauge matrix is of the form $\boldsymbol{A} = \zeta \boldsymbol{F}$, for any scalar $\zeta \neq 0$, which may be set to unity with no loss in generality. This recovers the results of Milton et al.[1] that the transformed material is of the Willis form, eq. (1). As noted in[1], this is the only choice for $\boldsymbol{A}$ that yields symmetric stress.

In summary, the governing equations are (14) with material parameters defined by (15) and

$$\mathcal{Q}_{ijIJ} = J^{-1} F_{iI} F_{jJ}. \tag{19}$$

The parameters now display the full symmetries expected of a Willis material:

$$C^{\text{eff}}_{ijkl} = C^{\text{eff}}_{klij}, \quad C^{\text{eff}}_{ijkl} = C^{\text{eff}}_{jikl}, \quad \rho^{\text{eff}}_{jl} = \rho^{\text{eff}}_{lj} \quad S^{\text{eff}}_{ijl} = S^{\text{eff}}_{jil}. \quad . \tag{20}$$

Note that the stiffness tensor is

$$\begin{aligned} C^{\text{eff}}_{ijkl} &= F_{iI} F_{jJ} F_{kK} F_{lL}\, J^{-1} C^{(0)}_{IJKL} = V_{iI} V_{jJ} V_{kK} V_{lL}\, J^{-1} \bar{C}^{(0)}_{IJKL}, \text{ where} \\ \bar{C}^{(0)}_{IJKL} &= R_{IM} R_{JN} R_{PK} R_{LQ}\, C^{(0)}_{MNPQ} \end{aligned} \tag{21}$$

are the original moduli in the rotated frame. The full symmetry of the stiffness tensor also follows immediately from the representation (17) with symmetric $\mathbf{S}^{(\alpha)} = J^{-1} \mathbf{F} \mathbf{P}^{(\alpha)} \mathbf{F}^t$.

B. Cosserat elasticity: $\boldsymbol{A}$ = constant

1. *General form*

The constitutive parameters (15) simplify considerably if the fourth order quantity $\boldsymbol{\mathcal{Q}}$ satisfies

$$\mathcal{Q}_{ijIJ,i} = 0. \tag{22}$$

This differential constraint combined with (11) implies that the gauge $\mathbf{A}$ *must be constant.* In that case the transformed equations of motion become

$$\sigma_{ij,i} = \rho_{ij}\, \ddot{u}_j, \quad \sigma_{ij} = C^{\text{eff}}_{ijkl}\, u_{l,k}, \tag{23}$$

where the effective elastic moduli are defined by (13) and (15a) and the density tensor $\boldsymbol{\rho}$ is given in (9).

Note that the elastic moduli satisfy the symmetry $(16)_1$ associated with the transformed energy density $\mathcal{W} = \frac{1}{2} C^{\text{eff}}_{ijkl} u_{j,i} u_{l,k}$. But C^{eff}_{ijkl} does not satisfy the minor symmetry $(4)_1$ since

$$\begin{aligned} C^{\text{eff}}_{ijkl} - C^{\text{eff}}_{jikl} &= J^{-1} C^{(0)}_{IJKL}\, F_{kK} A_{lL} \big(F_{iI} A_{jJ} - F_{jI} A_{iJ} \big) \\ &= J^{-1} C^{(0)}_{IJKL}\, F_{kK} A_{lL} \big(F_{iI} A_{jJ} - F_{jJ} A_{iI} \big), \end{aligned} \tag{24}$$

which is non-zero in general (note that the second form in (24) uses the minor symmetry $(4)_1$ for the original moduli $C^{(0)}_{IJKL}$). This means that the stress is not necessarily symmetric, $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}^t$, which places the material in the framework of Cosserat elasticity[25]. The number of independent elastic stiffness elements is at most $9(9+1)/2 = 45$ as compared with $6(6+1)/2 = 21$ for normal linear elasticity.

2. *Cosserat elasticity with isotropic density:* $\boldsymbol{A} = \boldsymbol{I}$

Isotropic density can be achieved by taking the constant matrix $\boldsymbol{A}$ proportional to the identity, $\boldsymbol{A} = \zeta \boldsymbol{I}$, with $\zeta = 1$ without loss of generality. In this important case we have $\boldsymbol{\rho} = \rho \boldsymbol{I}$, with

$$\rho = \rho_0\, J^{-1}, \quad C^{\text{eff}}_{ijkl} = J^{-1} C^{(0)}_{IjKl}\, F_{iI} F_{kK}. \tag{25}$$

C. Examples

1. Example 1: SH motion in a plane of material symmetry

The original moduli are assumed to have a plane of symmetry perpendicular to the X_3-axis, and the transformation is assumed to preserve the out of plane coordinate: $x_3 = X_3$. Consider shear horizontal motion $\boldsymbol{U} = (0, 0, U(X_1, X_2, t))$ satisfying the scalar equation

$$\big(C^{(0)}_{A3B3}U_{,B}\big)_{,A} = \rho_0 \ddot{U}, \tag{26}$$

with indices $A, B \in \{1, 2\}$. Under these circumstances, the equation for SH motion in the transformed domain, $\boldsymbol{u} = (0, 0, u(x_1, x_2, t))$, is the same for both the Willis constitutive equations ($\boldsymbol{A} = \boldsymbol{F}$) and the Cosserat model with isotropic density ($\boldsymbol{A} = \boldsymbol{I}$). Thus,

$$\big(C^{\text{eff}}_{\alpha 3\beta 3}u_{,\beta}\big)_{,\alpha} = \rho \ddot{u}, \tag{27}$$

where $\alpha, \beta \in \{1, 2\}$, $\rho = J^{-1}\rho_0$, and $C^{\text{eff}}_{\alpha 3\beta 3} = J^{-1}C^{(0)}_{A3B3}F_{\alpha A}F_{\beta B}$. The equivalence may be expected since the only relevant element of the gauge matrix, A_{33}, is the same for both models ($A_{33} = 1$).

The above conclusion for SH motion in the presence of orthotropic moduli relies only upon the scalar nature of the motion in the original and transformed domains. As such, the SH results also follow from those in §V for fluid acoustics under the standard replacement of fluid density and bulk modulus with inverse shear modulus and inverse solid density, respectively.

2. Example 2: $\boldsymbol{F}$ = constant

When both $\boldsymbol{A}$ and $\boldsymbol{F}$ are constant the Willis equations simplify to those of normal linear elasticity ($S_{ijl} = 0$). At the same time, the constant deformation gradient implies that $\boldsymbol{A} = \boldsymbol{F}$ (=constant) is a permissible choice for the Cosserat medium. The Willis and Cosserat materials are then coincident with density

$$\boldsymbol{\rho} = \boldsymbol{\rho}^t = \rho_0 J^{-1}\, \boldsymbol{B}. \tag{28}$$

and fully symmetric elastic moduli given by (21). Note that the density $\boldsymbol{\rho}$ is anisotropic unless $\boldsymbol{B} = \alpha \boldsymbol{I}$, which means the deformation is a pure expansion, possibly with rotation. But this is a rather trivial case.

Consider an isotropic initial material with original moduli $C^{(0)}_{IJKL} = \lambda_0 \delta_{IJ}\delta_{KL} + \mu_0(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK})$, or equivalently,

$$\boldsymbol{C}^{(0)} = \lambda_0\, \boldsymbol{I} \otimes \boldsymbol{I} + 2\mu_0\, \boldsymbol{I} \boxtimes \boldsymbol{I}, \quad \text{where } \big(\boldsymbol{X} \boxtimes \boldsymbol{X}\big)\boldsymbol{Y} \equiv \frac{1}{2}\boldsymbol{X}\big(\boldsymbol{Y} + \boldsymbol{Y}^t\big)\boldsymbol{X}. \tag{29}$$

The rotated moduli of $(21)_2$ are therefore unchanged, $\bar{\boldsymbol{C}}^{(0)} = \boldsymbol{C}^{(0)}$, and the current density and moduli are

$$\boldsymbol{C} = \lambda\, \boldsymbol{B} \otimes \boldsymbol{B} + 2\mu\, \boldsymbol{B} \boxtimes \boldsymbol{B}, \quad \boldsymbol{\rho} = \rho\, \boldsymbol{B}, \quad \text{where } \{\lambda, \mu, \rho\} = J^{-1}\, \{\lambda_0, \mu_0, \rho_0\}. \tag{30}$$

It is of interest to consider plane wave motion, $\boldsymbol{u} = \boldsymbol{q}\, g(\boldsymbol{n}^t\boldsymbol{x} - vt)$ for unit vector $\boldsymbol{n}$, constant $\boldsymbol{q}$, and $g \in \mathrm{C}^2$. The equation of motion $(23)_1$ implies that the polarization vector $\boldsymbol{q}$ satisfies

$$\big(\boldsymbol{Q} - v^2\boldsymbol{\rho}\big)\, \boldsymbol{q} = 0, \quad \text{where } \boldsymbol{Q} = (\lambda + \mu)(\boldsymbol{B}\boldsymbol{n}) \otimes (\boldsymbol{B}\boldsymbol{n}) + \mu(\boldsymbol{n}^t\boldsymbol{B}\boldsymbol{n})\, \boldsymbol{B}. \tag{31}$$

The solutions of (31) are of two distinct types:

$$\text{longitudinal:} \quad \boldsymbol{q} \parallel \boldsymbol{n}, \quad v = (\boldsymbol{n}^t\boldsymbol{B}\boldsymbol{n})^{1/2}\, c_L, \quad c_L^2 = \frac{\lambda + 2\mu}{\rho} = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \tag{32a}$$

$$\text{quasi-transverse:} \quad \boldsymbol{q} \perp \boldsymbol{B}\boldsymbol{n}, \quad v = (\boldsymbol{n}^t\boldsymbol{B}\boldsymbol{n})^{1/2}\, c_T, \quad c_T^2 = \frac{\mu}{\rho} = \frac{\mu_0}{\rho_0}. \tag{32b}$$

The slowness vector is $\boldsymbol{s} = v^{-1}\boldsymbol{n}$, and the slowness surface is the envelope of the slowness vectors for all propagation directions $\boldsymbol{n}$. The slowness surface is therefore comprised of two sheets which are similar ellipsoids: $\boldsymbol{s}^t\boldsymbol{B}\boldsymbol{s} = c_\alpha^{-2}$, $\alpha = L, T$. Note that the polarizations do not in general form an orthogonal triad.

V. ACOUSTICS AS A SPECIAL CASE

A. General form of transformed equations

We consider the simpler but special case of acoustics in order to further understand the structure of the elastic transformation theory. The acoustic equations are unique in the sense that they are the single example of a pentamode material[11,12] commonly encountered in mechanics. We will demonstrate that the pentamode property introduces a unique degree of freedom not available in the fully elastic situation.

The elastic stiffness of an acoustic fluid is

$$\mathbf{C}_0 = K_0\,\mathbf{I}\otimes\mathbf{I}, \tag{33}$$

which is of pentamode form[11], i.e. the 6×6 Voigt matrix associated with the elements $C^{(0)}_{IJKL} = K_0\delta_{IJ}\delta_{KL}$ has five zero eigenvalues. Equation (12) becomes, using (33),

$$Q_{ij}\left(JK_0\left(u_l Q_{kl}\right)_{,k}\right)_{,i} - \rho_{ij}\ddot{u}_i = 0, \tag{34}$$

with the density tensor given by (9), and

$$\mathbf{Q} = J^{-1}\mathbf{F}\mathbf{A}^t \quad \big(Q_{ij} = J^{-1}F_{iI}A_{jI} = \mathcal{Q}_{ijKK}\big). \tag{35}$$

The general form of the governing equations are again of the form (14), with material parameters

$$C^{\text{eff}}_{ijkl} = K_0 J^{-1} B_{ik}A_{jN}A_{lN}, \tag{36a}$$
$$S^{\text{eff}}_{ijl} = (-i\omega)^{-1}K_0 J^{-1}B_{ik}A_{jN}A_{lN,k}, \tag{36b}$$
$$\rho^{\text{eff}}_{jl} = \rho_{jl} + (-i\omega)^{-2}K_0 J^{-1}B_{ik}A_{jN,i}A_{lN,k}, \tag{36c}$$

recalling that $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^t$. The stress is again not generally symmetric, unless $\boldsymbol{A}$ is proportional to $\boldsymbol{F}$, in which case the transformed material is of Willis form, see §IV A.

B. Cosserat and related forms of the transformed equations

The simplified form of the elasticity in the acoustic fluid implies that the condition (22), which is required to simplify the form of the transformed equations, itself simplifies to the condition that $\mathbf{Q}$ be divergence free:

$$\operatorname{div}\mathbf{Q} = 0 \quad \big(Q_{ij,i} = 0\big). \tag{37}$$

Note that this is a necessary but not sufficient condition for the more general elasticity version (22). Assuming that (37) holds, the transformed equations (34) have the simplified Cosserat structure (23), with

$$\boldsymbol{\rho} = \rho_0 J\,\mathbf{Q}^t\mathbf{B}^{-1}\mathbf{Q}, \quad \mathbf{C}^{\text{eff}} = K_0 J\,\mathbf{Q}\otimes\mathbf{Q}, \quad \operatorname{div}\mathbf{Q} = 0. \tag{38}$$

We note the symmetries $\boldsymbol{\rho} = \boldsymbol{\rho}^t$, $C^{\text{eff}}_{ijkl} = C^{\text{eff}}_{klij}$, but the minor symmetry $C^{\text{eff}}_{ijkl} = C^{\text{eff}}_{jikl}$ does not in general hold unless $\mathbf{Q}$ is symmetric[10]. Thus, the transformed acoustic equations are those of a pentamode material of Cosserat type with anisotropic density. All previous studies of transformation acoustics assumed *a priori* that the transformed materials must have symmetric stress. The present results show that the more general structure of the transformed properties is that of a material with stress not necessarily symmetric. The pentamode structure of $\mathbf{C}^{\text{eff}}$ implies that the equations of motion (23) can be expressed in a form that is clearly related to acoustics[9] by using a scalar "pseudo-pressure" p and "bulk modulus" $K = K_0 J$,

$$\boldsymbol{\rho}\dot{\mathbf{v}} = -\mathbf{Q}\nabla p, \quad \dot{p} = -K\operatorname{tr}(\mathbf{Q}\nabla\mathbf{v}). \tag{39}$$

The condition (37) can be achieved, as in the elastic case, with constant gauge $\boldsymbol{A}$. For instance, taking $\boldsymbol{A} = \boldsymbol{I}$, yields material properties (see (25))

$$\rho = \rho_0 J^{-1}, \quad \mathbf{C}^{\text{eff}} = K_0 J^{-1}\,\mathbf{F}\otimes\mathbf{F}. \tag{40}$$

This describes a material with isotropic density of general pentamode/Cosserat form. That is, the stiffness is pentamode (a single nonzero eigenstiffness[27]) and a single eigenstress of generally non-symmetric form, hence Cosserat. As discussed in[9], isotropic density with symmetric stress can be achieved if $\operatorname{div} h\mathbf{V} = 0$ for some function $h(\mathbf{x})$. One important case is when the deformation is a pure stretch $\mathbf{F} = \mathbf{V}$ (with $h = J^{-1}$, see eq. (11)), for which the transformed material is pure pentamode with isotropic density:

$$\boldsymbol{\rho} = \rho_0 J^{-1}\,\mathbf{I}, \quad \mathbf{C} = K_0 J^{-1}\,\mathbf{V}\otimes\mathbf{V}, \quad \text{pure stretch.} \tag{41}$$

More general conditions under which pure pentamode material with isotropic density can be achieved are discussed in[10].

Condition (37) may also be satisfied by non-constant gauge matrices. For instance, $\mathbf{A} = J\mathbf{F}^{-t}$ gives $\mathbf{Q} = \mathbf{I}$ which clearly satisfies (37). In this case the transformed medium has the properties

$$\boldsymbol{\rho} = \rho_0 J\,\mathbf{B}^{-1}, \quad \mathbf{C}^{\text{eff}} = K_0 J\,\mathbf{I}\otimes\mathbf{I}. \tag{42}$$

This corresponds to a fluid with isotropic (hydrostatic) stress, bulk modulus $K = K_0 J$, and anisotropic density $\boldsymbol{\rho}$. This type of material was the first to be considered for acoustic cloaking[5–7] yet it is remarkable that it does not possess a generalization to elasticity. Rather, it is made possible by the simple structure of the second order quantity $\boldsymbol{Q}$ as compared with its fourth order elasticity analog $\boldsymbol{\mathcal{Q}}$. The condition (22) for $\boldsymbol{\mathcal{Q}}$ is only satisfied by constant $\mathbf{A}$, but the analogous condition (37) for $\boldsymbol{Q}$ has at least one non-constant solution for $\boldsymbol{A}$. The one noted here, $\mathbf{A} = J\mathbf{F}^{-t}$, yields the class of acoustic cloaking materials that use anisotropic inertia as the active mechanism.

The transformed acoustic material can also be understood as the special case in which five of the Kelvin moduli $K_0^{(\alpha)}$ vanish (see (18)), say $\alpha = 2, 3, \ldots, 6$, and the remaining single eigenvector is the identity, $\mathbf{P}^{(1)} = \mathbf{I}$. The transformed material has a single non-zero eigen-stiffnesses with associated eigen-tensor (see eq. (35))

$$\mathbf{S}^{(1)} = J^{-1}\mathbf{F}\mathbf{A}^t. \tag{43}$$

The inertial transformed material (42) results from the choice of $\mathbf{A}$ that makes $\mathbf{S}^{(1)} = \mathbf{I}$, that is $\mathbf{A} = J\mathbf{F}^{-t}$.

VI. APPLICATIONS IN CYLINDRICAL ELASTICITY

The non-uniqueness in the form of the transformed equations of elasticity provides the designer of elastic cloaking devices with a wide variety of possible materials from which to choose. These range from materials with Willis constitutive behavior (1), to Cosserat materials (25), each special cases of the general constitutive relations (14)-(15). The same non-uniqueness exist for acoustic cloaking, where designs based on inertial cloaking on the one hand, and pentamode materials on the other, reflect the choice of completely different material properties. One aspect common to all of these materials is their exotic nature. In the absence of available materials with exactly the right properties it is reasonable to ask what can be achieved using "normal" materials, i.e. those with isotropic density and standard linear elastic response, including a symmetric stress.

With that goal in mind we examine in this section cloaking in the context of the Cosserat materials described by (25). This class of metamaterials is chosen as the starting point because of its property of isotropic density and the fact that the constitutive behavior is local. That is, the stress depends on the displacement gradient alone, and not on displacement as in (14)-(15). The goal is to find if there is a material with symmetric stress that provides an optimal, in some sense, approximation. We consider transformations in cylindrical coordinates, starting with the formulation of the Cosserat constitutive equations in cylindrical coordinates. Thus, for the remainder of the paper $\mathbf{A}$ is assumed to be a constant, and we use $\boldsymbol{C}$ instead of $\boldsymbol{C}^{\text{eff}}$.

A. General theory for Cosserat materials

1. *Cosserat notation in cylindrical coordinates*

The cylindrical coordinates are referred to by the indices $1, 2, 3$ for r, θ, z, respectively. The usual Voigt notation, which means $\{1, 2, 3, 4, 5, 6\} = \{11, 22, 33, 23, 31, 12\}$, is augmented with three additional indices to describe Cosserat

elasticity: $\{\bar{4},\bar{5},\bar{6}\}=\{32,13,21\}$, so that the elastic stiffness tensor becomes in the 9-index Voigt-Cosserat notation,

$$\boldsymbol{C}=\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{1\bar{4}} & c_{15} & c_{1\bar{5}} & c_{16} & c_{1\bar{6}} \\ & c_{22} & c_{23} & c_{24} & c_{2\bar{4}} & c_{25} & c_{2\bar{5}} & c_{26} & c_{2\bar{6}} \\ & & c_{33} & c_{34} & c_{3\bar{4}} & c_{35} & c_{3\bar{5}} & c_{36} & c_{3\bar{6}} \\ & & & c_{44} & c_{4\bar{4}} & c_{45} & c_{4\bar{5}} & c_{46} & c_{4\bar{6}} \\ & & & & c_{\bar{4}\bar{4}} & c_{\bar{4}5} & c_{\bar{4}\bar{5}} & c_{\bar{4}6} & c_{\bar{4}\bar{6}} \\ & & & & & c_{55} & c_{5\bar{5}} & c_{56} & c_{5\bar{6}} \\ & & & & & & c_{\bar{5}\bar{5}} & c_{\bar{5}6} & c_{\bar{5}\bar{6}} \\ & S & Y & M & & & & c_{66} & c_{6\bar{6}} \\ & & & & & & & & c_{\bar{6}\bar{6}} \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \bar{4} \\ 5 \\ \bar{5} \\ 6 \\ \bar{6} \end{pmatrix}. \tag{44}$$

Following[28], the traction vectors $\mathbf{t}_i=\mathbf{t}_i(\mathbf{x},t)$, $i=r,\theta,z$, are defined by the orthonormal basis vectors $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ of the cylindrical coordinates $\{r,\theta,z\}$ according to $\mathbf{t}_i=\mathbf{e}_i\boldsymbol{\sigma}$ $(i=r,\theta,z)$, where $\boldsymbol{\sigma}(\mathbf{x},t)$ is the stress, and a comma denotes partial differentiation. With the same basis vectors, and assuming the summation convention on repeated indices, the elements of stress are $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$ where $\boldsymbol{\varepsilon}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^t)$ is the strain, $C_{ijkl}=C_{ijkl}(\mathbf{x})$ are elements of the fourth order (anisotropic) elastic stiffness tensor. The traction vectors are[29]

$$\begin{pmatrix} \mathbf{t}_r \\ \mathbf{t}_\theta \\ \mathbf{t}_z \end{pmatrix} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} & \mathbf{P} \\ \mathbf{R}^t & \mathbf{T} & \mathbf{S} \\ \mathbf{P}^t & \mathbf{S}^t & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{,\,r} \\ \frac{1}{r}(\mathbf{u}_{,\,\theta}+\mathbf{K}\,\mathbf{u}) \\ \mathbf{u}_{,\,z} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{Q} & \mathbf{R} & \mathbf{P} \\ \mathbf{R}^t & \mathbf{T} & \mathbf{S} \\ \mathbf{P}^t & \mathbf{S}^t & \mathbf{M} \end{pmatrix} = \begin{pmatrix} (e_re_r) & (e_re_\theta) & (e_re_z) \\ & (e_\theta e_\theta) & (e_\theta e_z) \\ & & (e_ze_z) \end{pmatrix},$$

where $\mathbf{K}=\mathbf{e}_\theta\otimes\mathbf{e}_r-\mathbf{e}_r\otimes\mathbf{e}_\theta$, and in notation similar to that of[30], the matrix (ab) has components $(ab)_{jl}=a_iC_{ijkl}b_k$ for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. The explicit form of the various matrices is apparent with the use of Voigt's notation

$$\mathbf{Q}=\begin{pmatrix} c_{11} & c_{16} & c_{1\bar{5}} \\ c_{16} & c_{66} & c_{\bar{5}6} \\ c_{1\bar{5}} & c_{\bar{5}6} & c_{\bar{5}\bar{5}} \end{pmatrix}, \quad \mathbf{T}=\begin{pmatrix} c_{\bar{6}\bar{6}} & c_{2\bar{6}} & c_{4\bar{6}} \\ c_{2\bar{6}} & c_{22} & c_{24} \\ c_{4\bar{6}} & c_{24} & c_{44} \end{pmatrix}, \quad \mathbf{M}=\begin{pmatrix} c_{55} & c_{\bar{4}5} & c_{35} \\ c_{\bar{4}5} & c_{\bar{4}\bar{4}} & c_{3\bar{4}} \\ c_{35} & c_{3\bar{4}} & c_{33} \end{pmatrix},$$

$$\mathbf{S}=\begin{pmatrix} c_{5\bar{6}} & c_{\bar{4}\bar{6}} & c_{3\bar{6}} \\ c_{25} & c_{2\bar{4}} & c_{23} \\ c_{45} & c_{4\bar{4}} & c_{34} \end{pmatrix}, \quad \mathbf{P}=\begin{pmatrix} c_{15} & c_{1\bar{4}} & c_{13} \\ c_{56} & c_{\bar{4}6} & c_{36} \\ c_{5\bar{5}} & c_{\bar{4}\bar{5}} & c_{3\bar{5}} \end{pmatrix}, \quad \mathbf{R}=\begin{pmatrix} c_{1\bar{6}} & c_{12} & c_{14} \\ c_{6\bar{6}} & c_{26} & c_{46} \\ c_{\bar{5}\bar{6}} & c_{2\bar{5}} & c_{4\bar{5}} \end{pmatrix}. \tag{45}$$

Note that the six elements in each of the symmetric matrices $\mathbf{Q}$, $\mathbf{T}$, $\mathbf{M}$ and the nine in each of $\mathbf{S}$, $\mathbf{P}$, $\mathbf{R}$, are independent. That is, there is one-to-one correspondence between the elements of the six matrices and the 45 independent components of the Cosserat elasticity tensor (44).

2. Cylindrically anisotropic materials

We consider materials with no axial dependence whose density and the elasticity tensor may depend upon r, i.e. $\rho=\rho(r)$ and $C_{ijkl}=C_{ijkl}(r)$. We seek solutions in the form of time-harmonic cylindrical waves of azimuthal order $n=0,1,2,\ldots$ and axial wavenumber k_z, such that the displacement-traction 6-vector is of the form

$$\begin{pmatrix} \mathbf{u}(r,\theta,z,t) \\ ir\mathbf{t}_r(r,\theta,z,t) \end{pmatrix} = \boldsymbol{\eta}(r)\,\mathrm{e}^{i(n\theta+k_zz-\omega t)}, \tag{46}$$

where $\boldsymbol{\eta}$ depends only on the radial coordinate r. Accordingly, the governing equations of motion reduce to an ordinary differential equation for this 6×1 vector[29]:

$$\frac{\mathrm{d}\,\boldsymbol{\eta}}{\mathrm{d}\,r}-\frac{i}{r}\,\mathbf{G}(r)\boldsymbol{\eta}(r)=0, \tag{47}$$

with 6×6 system matrix $\mathbf{G}$, where

$$
\begin{aligned}
i\mathbf{G}(r) = &\begin{pmatrix} -\mathbf{Q}^{-1}\widetilde{\mathbf{R}} & -i\mathbf{Q}^{-1} \\ i\big(\widetilde{\mathbf{T}} - \widetilde{\mathbf{R}}^{+}\mathbf{Q}^{-1}\widetilde{\mathbf{R}} - \rho\omega^2 r^2\big) & \widetilde{\mathbf{R}}^{+}\mathbf{Q}^{-1} \end{pmatrix} \\
&+ ik_z r \begin{pmatrix} -\mathbf{Q}^{-1}\mathbf{P} & \mathbf{0} \\ i\big[\mathbf{P}^t\mathbf{Q}^{-1}\widetilde{\mathbf{R}} - \widetilde{\mathbf{S}} - \big(\mathbf{P}^t\mathbf{Q}^{-1}\widetilde{\mathbf{R}} - \widetilde{\mathbf{S}}\big)^{+} + ik_z r\big(\mathbf{P}^t\mathbf{Q}^{-1}\mathbf{P} - \mathbf{M}\big)\big] & -\mathbf{P}^t\mathbf{Q}^{-1} \end{pmatrix}, \\
&\widetilde{\mathbf{R}} = \mathbf{R}\boldsymbol{\kappa}, \qquad \widetilde{\mathbf{S}} = \boldsymbol{\kappa}\mathbf{S}, \qquad \widetilde{\mathbf{T}} = \boldsymbol{\kappa}^{+}\mathbf{T}\boldsymbol{\kappa} = \widetilde{\mathbf{T}}^{+}, \qquad \boldsymbol{\kappa} = \mathbf{K} + in\mathbf{I} = -\boldsymbol{\kappa}^{+}.
\end{aligned}
\tag{48}
$$

Note the symmetry $\mathbf{G} = \mathbf{J}\mathbf{G}^{+}\mathbf{J}$ for real-valued material constants and ω, k_z, where $\mathbf{J}$ has block structure with zero submatrices on the diagonal and off-diagonal identity matrices. This hermiticity-like property has important physical consequences such as conservation of energy[28].

3. Transformation in cylindrical coordinates

Let $R = (X_1^2 + X_2^2)^{1/2}$, $r = (x_1^2 + x_2^2)^{1/2}$, and consider the reverse deformation $R = R(r), X_3 = x_3$. The deformation gradient is then

$$
\boldsymbol{F} = \alpha\boldsymbol{I}_r + \beta\boldsymbol{I}_\theta + \boldsymbol{I}_z, \quad \text{with} \quad \alpha = \frac{\mathrm{d}\,r}{\mathrm{d}\,R}, \quad \beta = \frac{r}{R}, \tag{49}
$$

where $\boldsymbol{I}_r = \boldsymbol{e}_r \otimes \boldsymbol{e}_r$, $\boldsymbol{I}_\theta = \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta$, $\boldsymbol{I}_z = \boldsymbol{e}_z \otimes \boldsymbol{e}_z$. Taking the (assumed constant) gauge matrix as $\boldsymbol{A} = \boldsymbol{I}$ gives isotropic density, $\boldsymbol{\rho} = \rho\,\boldsymbol{I}$ and Cosserat elastic stiffness $\boldsymbol{C}$, where according to eq. (25),

$$
\rho = \frac{1}{\alpha\beta}\rho_0, \tag{50a}
$$

$$
\boldsymbol{C} = \frac{1}{\alpha\beta}\begin{pmatrix}
\alpha^2 c_{11}^{(0)} & \alpha\beta c_{12}^{(0)} & \alpha c_{13}^{(0)} & \alpha\beta c_{14}^{(0)} & \alpha c_{14}^{(0)} & \alpha c_{15}^{(0)} & \alpha^2 c_{15}^{(0)} & \alpha^2 c_{16}^{(0)} & \alpha\beta c_{16}^{(0)} \\
 & \beta^2 c_{22}^{(0)} & \beta c_{23}^{(0)} & \beta^2 c_{24}^{(0)} & \beta c_{24}^{(0)} & \beta c_{25}^{(0)} & \alpha\beta c_{25}^{(0)} & \alpha\beta c_{26}^{(0)} & \beta^2 c_{26}^{(0)} \\
 & & c_{33}^{(0)} & \beta c_{34}^{(0)} & c_{34}^{(0)} & c_{35}^{(0)} & \alpha c_{35}^{(0)} & \alpha c_{36}^{(0)} & \beta c_{36}^{(0)} \\
 & & & \beta^2 c_{44}^{(0)} & \beta c_{44}^{(0)} & \beta c_{45}^{(0)} & \alpha\beta c_{45}^{(0)} & \alpha\beta c_{46}^{(0)} & \beta^2 c_{46}^{(0)} \\
 & & & & c_{44}^{(0)} & c_{45}^{(0)} & \alpha c_{45}^{(0)} & \alpha c_{46}^{(0)} & \beta c_{46}^{(0)} \\
 & & & & & c_{55}^{(0)} & \alpha c_{55}^{(0)} & \alpha c_{56}^{(0)} & \beta c_{56}^{(0)} \\
 & & & & & & \alpha^2 c_{55}^{(0)} & \alpha^2 c_{56}^{(0)} & \alpha\beta c_{56}^{(0)} \\
 & S & Y & M & & & & \alpha^2 c_{66}^{(0)} & \alpha\beta c_{66}^{(0)} \\
 & & & & & & & & \beta^2 c_{66}^{(0)}
\end{pmatrix}. \tag{50b}
$$

Equivalently, the six matrices of (45) that are in one-to-one correspondence with $\boldsymbol{C}$ are

$$
\begin{aligned}
&\mathbf{Q} = \frac{\alpha}{\beta}\mathbf{Q}^{(0)}, \quad \mathbf{T} = \frac{\beta}{\alpha}\mathbf{T}^{(0)}, \quad \mathbf{M} = \frac{1}{\alpha\beta}\mathbf{M}^{(0)}, \\
&\mathbf{S} = \frac{1}{\alpha}\mathbf{S}^{(0)}, \quad \mathbf{P} = \frac{1}{\beta}\mathbf{P}^{(0)}, \quad \mathbf{R} = \mathbf{R}^{(0)}.
\end{aligned}
\tag{51}
$$

These results are consistent with the system equation (47) in the current coordinates and the analogous equation in the original coordinates,

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\,R} = \frac{i}{R}\,\mathbf{G}^{(0)}(R)\,\boldsymbol{\eta}, \tag{52}$$

with $\mathbf{G}^{(0)}(R)$ defined in the same way as (48) using $\mathbf{Q}^{(0)}$, ..., $\mathbf{R}^{(0)}$. Note that the transformed version of (52) may be obtained directly by multiplication with the factor $\mathrm{d}\,R/\,\mathrm{d}\,r$,

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\,r} = \frac{i}{R}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\,\mathbf{G}^{(0)}(R)\,\boldsymbol{\eta}. \tag{53}$$

Comparison of eqs. (47) and (53) implies that the transformed system matrix is

$$\mathbf{G}(r) = \frac{r}{R}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}\,\mathbf{G}^{(0)}(R). \tag{54}$$

This relation, combined with the block structure of $\mathbf{G}$ in (48) and the analogous form for $\mathbf{G}^{(0)}$, implies the matrix relations (51).

B. Example: a cylindrically orthotropic material

1. *Transformed elastic moduli*

The initial material is assumed to be cylindrically orthotropic, with stiffness in the usual Voigt notation ($\{1,2,3,4,5,6\} \leftrightarrow \{11,22,33,23,31,12\}$) given by

$$\boldsymbol{C}^{(0)} = \begin{pmatrix} c_{11}^{(0)} & c_{12}^{(0)} & c_{13}^{(0)} & 0 & 0 & 0 \\ c_{12}^{(0)} & c_{22}^{(0)} & c_{23}^{(0)} & 0 & 0 & 0 \\ c_{13}^{(0)} & c_{23}^{(0)} & c_{33}^{(0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^{(0)} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^{(0)} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^{(0)} \end{pmatrix}. \tag{55}$$

Hence, using (50b), the transformed elastic stiffness tensor becomes in the 9-index Cosserat notation,

$$\boldsymbol{C} = \frac{1}{\alpha\beta}\begin{pmatrix} \alpha^2 c_{11}^{(0)} & \alpha\beta c_{12}^{(0)} & \alpha c_{13}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \beta^2 c_{22}^{(0)} & \beta c_{23}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & c_{33}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \beta^2 c_{44}^{(0)} & \beta c_{44}^{(0)} & 0 & 0 & 0 & 0 \\ & & & & c_{44}^{(0)} & 0 & 0 & 0 & 0 \\ & & & & & c_{55}^{(0)} & \alpha c_{55}^{(0)} & 0 & 0 \\ & & & & & & \alpha^2 c_{55}^{(0)} & 0 & 0 \\ & S & Y & M & & & & \alpha^2 c_{66}^{(0)} & \alpha\beta c_{66}^{(0)} \\ & & & & & & & & \beta^2 c_{66}^{(0)} \end{pmatrix}. \tag{56}$$

2. *SH motion*

The equation of motion in the undeformed coordinates for $\boldsymbol{U} = (0,0,U(r,\theta,t))$,

$$\frac{1}{R}\left(R c_{55}^{(0)} U_{,R}\right)_{,R} + \frac{1}{R^2}\left(c_{44}^{(0)} U_{,\theta}\right)_{,\theta} - \rho_0 \ddot{U} = 0, \tag{57}$$

transforms to the following equation for $\boldsymbol{u} = (0,0,u(r,\theta,t))$,

$$\frac{1}{r}(r c_{\bar{5}\bar{5}} u_{,r})_{,r} + \frac{1}{r^2}(c_{44} u_{,\theta})_{,\theta} - \rho \ddot{u} = 0, \tag{58}$$

where

$$\rho = \rho_0 \frac{R}{r}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}, \quad c_{44} = c_{44}^{(0)}\,\frac{r}{R}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}, \quad c_{\bar{5}\bar{5}} = c_{55}^{(0)}\,\frac{R}{r}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}. \tag{59}$$

This is an example of the general result in §IV C 1 for SH motion in a plane of symmetry .

3. In-plane motion

The displacement is assumed to have the form $\boldsymbol{u} = (u_r(r,\theta,t), u_\theta(r,\theta,t), 0)$. The density is again isotropic given by eq. $(59)_1$ and the relevant moduli are, from eq. (50b),

$$c_{11} = c_{11}^{(0)}\frac{R}{r}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}, \quad c_{22} = c_{22}^{(0)}\frac{r}{R}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}, \quad c_{12} = c_{12}^{(0)}, \tag{60a}$$

$$c_{66} = c_{66}^{(0)}\frac{R}{r}\frac{\mathrm{d}\,r}{\mathrm{d}\,R}, \quad c_{\bar{6}\bar{6}} = c_{66}^{(0)}\frac{r}{R}\frac{\mathrm{d}\,R}{\mathrm{d}\,r}, \quad c_{6\bar{6}} = c_{66}^{(0)}. \tag{60b}$$

For isotropic initial stiffness tensor, these expressions agree with the Cosserat elastic moduli found by[24], where only the particular transformation of[3,4] was considered, i.e., $R = r_1(r-r_0)/(r_1-r_0)$ for $r \in (r_0, r_1]$.

C. A symmetric approximation for $k_z = 0$

The example in §VI B achieves cloaking of in-plane elastic wave motion using elastic moduli of Cosserat form, (60), generalizing the findings of[24]. The existence of three distinct in-plane shear moduli, c_{1212}, c_{2121}, $c_{1221} = c_{2112}$, is a property of the Cosserat model, but one that is difficult if not impossible to realize in practice. We now show that there is a preferred approximation with only a single shear modulus.

Consider the transformed system matrix $\mathbf{G}$ with $k_z = 0$ for the initial cylindrically orthotropic material (55), which follows from (54) as

$$i\mathbf{G} = \frac{\beta}{\alpha}\begin{pmatrix} -\begin{pmatrix} 0 & {c_{11}^{(0)}}^{-1}c_{12}^{(0)} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\boldsymbol{\kappa} & -i\begin{pmatrix} {c_{11}^{(0)}}^{-1} & 0 & 0 \\ 0 & {c_{66}^{(0)}}^{-1} & 0 \\ 0 & 0 & {c_{55}^{(0)}}^{-1} \end{pmatrix} \\ i\boldsymbol{\kappa}^{+}\begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^{(0)} - {c_{11}^{(0)}}^{-1}{c_{12}^{(0)}}^2 & 0 \\ 0 & 0 & c_{44}^{(0)} \end{pmatrix}\boldsymbol{\kappa} - i\omega^2 R^2\rho_0\boldsymbol{I} & -\boldsymbol{\kappa}\begin{pmatrix} 0 & 1 & 0 \\ {c_{11}^{(0)}}^{-1}c_{12}^{(0)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}. \tag{61}$$

Let $\overline{C}_{ijkl}$ be a set of moduli in the transformed coordinates, corresponding to a normal elastic solid (symmetric stress) of cylindrically orthotropic symmetry, with moduli (in the standard Voigt notation),

$$\overline{\boldsymbol{C}} = \begin{pmatrix} \overline{c}_{11} & \overline{c}_{12} & \overline{c}_{13} & 0 & 0 & 0 \\ \overline{c}_{12} & \overline{c}_{22} & \overline{c}_{23} & 0 & 0 & 0 \\ \overline{c}_{13} & \overline{c}_{23} & \overline{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{c}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{c}_{66} \end{pmatrix}. \tag{62}$$

Assuming the density is isotropic and equal to the transformed density of eq. (50a), the system matrix associated with the stiffness $\overline{C}_{ijkl}$ becomes

$$i\overline{\mathbf{G}} = \begin{pmatrix} -\begin{pmatrix} 0 & {\overline{c}_{11}}^{-1}\overline{c}_{12} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\boldsymbol{\kappa} & -i\begin{pmatrix} {\overline{c}_{11}}^{-1} & 0 & 0 \\ 0 & {\overline{c}_{66}}^{-1} & 0 \\ 0 & 0 & {\overline{c}_{55}}^{-1} \end{pmatrix} \\ i\boldsymbol{\kappa}^{+}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{c}_{22} - {\overline{c}_{11}}^{-1}{\overline{c}_{12}}^2 & 0 \\ 0 & 0 & \overline{c}_{44} \end{pmatrix}\boldsymbol{\kappa} - i\omega^2 r^2\rho\boldsymbol{I} & -\boldsymbol{\kappa}\begin{pmatrix} 0 & 1 & 0 \\ {\overline{c}_{11}}^{-1}\overline{c}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}. \tag{63}$$

Comparison of (61) and (63) suggests the identification

$$\overline{c}_{11} = \frac{\alpha}{\beta} c^{(0)}_{11}, \quad \overline{c}_{22} = \frac{\beta}{\alpha} c^{(0)}_{22}, \quad \overline{c}_{12} = c^{(0)}_{12}, \tag{64a}$$

$$\overline{c}_{44} = \frac{\beta}{\alpha} c^{(0)}_{44}, \quad \overline{c}_{55} = \frac{\beta}{\alpha} c^{(0)}_{55}, \quad \overline{c}_{66} = \frac{\alpha}{\beta} c^{(0)}_{66}. \tag{64b}$$

55 should be alpha/beta, the same as for 66.

This set of moduli has the property that they correspond to a normal elastic material, as compared with the Cosserat material required for the exact solution. Comparing the latter, given by (56), with the proposed moduli (64) shows that

$$\overline{c}_{11} = c_{11}, \quad \overline{c}_{22} = c_{22}, \quad \overline{c}_{12} = c_{12}, \quad \overline{c}_{44} = c_{44}, \quad \overline{c}_{55} = c_{\bar{5}\bar{5}}, \quad \overline{c}_{66} = c_{66}. \tag{65}$$

Let $\overline{\boldsymbol{\eta}}(r)$ be the solution for the approximate but symmetric moduli (62),

$$\frac{\mathrm{d}\,\overline{\boldsymbol{\eta}}}{\mathrm{d}\,r} - \frac{i}{r}\,\overline{\mathbf{G}}(r)\overline{\boldsymbol{\eta}}(r) = 0. \tag{66}$$

It is evident that the exact and approximate systems (47) and (66) have identical SH solutions. The approximate solution defined by the moduli $\overline{C}_{ijkl}$ possesses a further interesting property related to in-plane wave motion. Thus, the difference between the system matrices of the exact transformed medium and that of the approximate material follows as

$$\frac{i}{r}\mathbf{G} - \frac{i}{r}\overline{\mathbf{G}} = f(r)\mathbf{D} \equiv \boldsymbol{\Delta}, \tag{67}$$

$$\text{where} \quad f(r) = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{1}{r}, \quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & -\mathbf{D}_1^{+} \end{pmatrix}, \quad \mathbf{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -in & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{68}$$

The difference is independent of the material properties, a function of only the transformation function through $f(r)$ and of the azimuthal index n through $\mathbf{D}$ which is rank two and has the property $\mathbf{D}^k = \mathbf{D}$ or $\mathbf{D}^2$ for any odd or even k. This suggests that the choice (64) of the approximate material parameters is in some sense preferred over others. Some consequences are discussed next.

The matricant solutions $\mathbf{M}(r, r_1)$ and $\overline{\mathbf{M}}(r, r_1)$ of, respectively, the exact and approximate systems (47) and (66), are defined[31] such that $\eta = \mathbf{M}\mathbf{c}$ and $\overline{\eta} = \overline{\mathbf{M}}\mathbf{c}$ where $\mathbf{c}$ is the arbitrary initial data vector at $r = r_1$. The matricants therefore satisfy differential equations similar to the state vectors with initial conditions $\mathbf{M}(r_1, r_1) = \mathbf{I}$, $\overline{\mathbf{M}}(r_1, r_1) = \mathbf{I}$. According to[31] (§3 of Ch.VII), the matricant (47) and (66) on an interval $[r_1, r_2]$ are related as $\mathbf{M} = \overline{\mathbf{M}}\mathbf{P}$ through the matrix $\mathbf{P}$ satisfying $\mathbf{P}' = \overline{\mathbf{M}}^{-1}\boldsymbol{\Delta}\overline{\mathbf{M}}\mathbf{P}$. Noting that $\mathbf{R}' = \boldsymbol{\Delta}\mathbf{R}$ with initial condition $\mathbf{R}(r_1, r_1) = \mathbf{I}$ has a simple explicit solution

$$\mathbf{R}^{\pm 1} = \mathbf{I} + \mathrm{diag}\left(\left[\left(\frac{rR_1}{Rr_1}\right)^{\pm 1} - 1 \right]\mathbf{D}_1,\ \left[\left(\frac{rR_1}{Rr_1}\right)^{\mp 1} - 1 \right]\mathbf{D}_1^{+} \right), \tag{69}$$

it follows that

$$\mathbf{M} = \overline{\mathbf{M}}\mathbf{R}\mathbf{T} \quad \text{where} \quad \mathbf{T}' = \boldsymbol{\Delta}_1\mathbf{T}, \quad \mathbf{T}(r_1, r_1) = \mathbf{I}, \tag{70}$$

and $\boldsymbol{\Delta}_1 = (\overline{\mathbf{M}}\mathbf{R})^{-1}(\boldsymbol{\Delta}\overline{\mathbf{M}} - \overline{\mathbf{M}}\boldsymbol{\Delta})\mathbf{R}$. The latter vanishes at $r = r_1$, which suggests that the convergent Peano series $\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_1 + \mathbf{T}_2 + \ldots$ with $\mathbf{T}_0 = \mathbf{I}$, $\mathbf{T}_j' = \boldsymbol{\Delta}_1\mathbf{T}_{j-1}$, $j = 1, 2, \ldots$, forms a natural and regular perturbation solution.

Note that in cylindrical cloaks the original domain $[R_1 < r_1, R_2 = r_2]$ is mapped to the smaller one $[r_1, r_2]$ so that the elements of $\mathbf{R}$ in (69) are sign definite.

VII. DISCUSSION

A complete transformation theory has been developed for elasticity. The material properties after transformation of the elastodynamic equations are given by eqs. (13) through (15). The constitutive parameters depend on both the

transformation and gauge matrices, $\boldsymbol{F}$ and $\boldsymbol{A}$, and do not necessarily have symmetric stress. It was shown in §IV that *a priori* requiring stress to be symmetric implies that the material must be of Willis form (1), with $\boldsymbol{A} = \boldsymbol{F}$ as Milton et al.[1] found. The emphasis here has been on exploring the consequences of relaxing the constraint of symmetric stress. There are several reasons for doing so. First is the fact that the transformation of the acoustic equation in its simplest form, i.e. by identifying an inertial tensor $\boldsymbol{\rho} = J^{-1}\boldsymbol{F}\boldsymbol{F}^t$ from the differential identity $\mathrm{Div}\,\mathrm{Grad}\, f \rightarrow J\,\mathrm{div}\, J^{-1}\boldsymbol{F}\boldsymbol{F}^t\,\mathrm{grad}\, f$, does not follow from the transformed Willis material, even though the inertial fluid has symmetric stress. Other types of transformed acoustic fluids are possible (e.g. pentamode materials), again with symmetric stress and not contained within the framework of eqs. (1). A second and more practical reason for considering the general material as defined by eqs. (13) through (15) with $\boldsymbol{F}$ and $\boldsymbol{A}$ distinct is to broaden the class of materials available for design of elastodynamic cloaks.

Allowing $\boldsymbol{A}$ to be independent of $\boldsymbol{F}$ leads to constitutive models that differ markedly from the Willis material model. In this paper we have emphasized solutions corresponding to time-independent material parameters obtained when $\boldsymbol{A}$ is assumed constant (see eqs. (9), (13) and (15a)),

$$\boldsymbol{\rho} = \rho_0\, J^{-1}\boldsymbol{A}\boldsymbol{A}^t, \quad C_{ijkl} = J^{-1}C^{(0)}_{IJKL}\, F_{iI}A_{jJ}F_{kK}A_{lL}. \tag{71}$$

These transformed quantities correspond to a material with anisotropic density tensor and stress-strain relation of Cosserat type (non-symmetric stress). Setting $\boldsymbol{A} = \boldsymbol{I}$ ensures that the transformed density is isotropic (see eq. (25))

$$\rho = J^{-1}\,\rho_0, \quad C^{\mathrm{eff}}_{ijkl} = J^{-1}C^{(0)}_{IjKl}\, F_{iI}F_{kK}. \tag{72}$$

However, there is no general choice of the gauge matrix $\boldsymbol{A}$ that will make the stress symmetric for a given transformation. This feature distinguishes the elastic transformation problem from the acoustic case, for which it is always possible to achieve symmetric, even isotropic (hydrostatic), stress. If the transformation is homogeneous, corresponding to constant $\boldsymbol{F}$, it is possible to make the elastic stress symmetric, although at the price of anisotropic inertia (see eq. (28)).

Materials displaying non-symmetric stress of the type necessary to achieve elastodynamic cloaking while difficult to envisage, are not ruled out. Effective moduli with the major symmetry $C_{ijkl} = C_{klij}$ that do not display the minor symmetry $C_{ijkl} = C_{jikl}$ are found in the theory of incremental motion superimposed on finite deformation[32]. The similarity with the transformation problem is intriguing: small-on-large motion in the presence of finite prestress corresponds to a transformation of an actual material via a deformation. The deformation in the small-on-large theory is however quite distinct from $\boldsymbol{F}$ in the present context. The formal equivalence of the constitutive parameters (72) with the density and moduli for incremental motion after finite prestress offers the possibility for achieving Cosserat elasticity of the desired form. The crucial quantity is the finite (hyperelastic) strain energy function of the material, which after prestress should have the desired Cosserat incremental moduli. Future work will examine this connection and the types of strain energy functions required.

Another approach is to seek materials with normal elastic behavior that approximate, in some sense, the Cosserat material. Preliminary work in this direction has been considered here. The general theory has been applied to the case of cylindrical anisotropy for arbitrary radial transformation $R \rightarrow r$. The equations of motion for the transformed Cosserat material have been expressed in Stroh format, eq. (47), suitable for numerical implementation. The material required for cloaking of in-plane elastic waves is of Cosserat type with isotropic density. A normal elastic material with density (50a) and elastic moduli defined by eqs. (62) and (64) appears to provide a natural approximation. The properties of this type of approximate material is the subject for planned further study, analytical and numerical.

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