Wave Motion 47 (2010) 183–197

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Effective wavenumbers and reflection coefficients for an elastic medium containing random configurations of cylindrical scatterers

Jean-Marc Conoir [a,*], Andrew N. Norris [b]

[a] Institut Jean Le Rond d'Alembert, UPMC Univ Paris 06, UMR 7190, F-75005 Paris, France
[b] Mechanical and Aerospace Engineering, Rutgers University, Piscataway, NJ 08854, USA

ARTICLE INFO

Article history:
Received 31 March 2009
Received in revised form 25 September 2009
Accepted 29 September 2009
Available online 6 October 2009

Keywords:
Multiple scattering
Elastic coherent waves

ABSTRACT

Propagation of P and SV waves in an elastic solid containing randomly distributed inclusions in a half-space is investigated. The approach is based on a multiple scattering analysis similar to the one proposed by Fikioris and Waterman for scalar waves. The characteristic equation, the solution of which yields the effective wave numbers of coherent elastic waves, is obtained in an explicit form without the use of any renormalisation methods. Two approximations are considered. First, formulae are derived for the effective wave numbers in a dilute random distribution of identical scatterers. These equations generalize the formula obtained by Linton and Martin for scalar coherent waves. Second, the high frequency approximation is compared with the Waterman and Truell approach derived here for elastic waves. The Fikioris and Waterman approach, in contrast with Waterman and Truell's method, shows that P and SV waves are coupled even at relatively low concentration of scatterers. Simple expressions for the reflected coefficients of P and SV waves incident on the interface of the half-space containing randomly distributed inclusions are also derived. These expressions depend on frequency, concentration of scatterers, and the two effective wave numbers of the coherent waves propagating in the elastic multiple scattering medium.

1. Introduction

We consider the problem of elastic wave propagation in heterogeneous solids containing distributions of inhomogeneities. The results have application in geophysical exploration and ultrasonic evaluation of composite materials or biological tissues. Typically, the inhomogeneities can be hard grains, inclusions, micro-cracks, fibers, pores, or contrast agents, for instance. The present work is a contribution in this direction.

The main difficulty is the quantitative description of interactions between many scatterers distributed throughout the heterogeneous solid. In the case of random sets of scatterers this problem cannot be solved exactly, except in very particular cases [25], and in general only approximate solutions are available. These are all based on hypotheses that reduce the many scatterer problem to a problem for one scatterer or one cell. In this way we may focus attention on the coherent wave propagation, which is the statistical average of the dynamics corresponding to all possible configurations of the scatterers. It is well-known that the coherent motion makes each heterogeneous medium appear as a dissipative homogeneous material, and propagation is governed by a complex effective wave number that is frequency dependent. The real part of the wave number is related to the velocity, and the imaginary part represents the attenuation.

* Corresponding author.
E-mail addresses: conoir@lmm.jussieu.fr (J.-M. Conoir), norris@rutgers.edu (A.N. Norris).

0165-2125/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.wavemoti.2009.09.004

Various methods for the analysis of the multiple scattering of elastic waves have been considered in the literature. These include: perturbation methods [14,27]; self consistent methods, which are analogous to the coherent potential approximation [9,15]; and effective medium methods based on self consistent schemes [26,5,13]. The method of asymptotic homogenization [24] was recently considered in the context of fiber-reinforced media where the fibers are arranged on a periodic lattice. All of these approaches have their own advantages and disadvantages.

The method developed here starts from an explicit multiple scattering formulation in which the field scattered from any particular scatterer is expressed as a multipole (far-field) expansion. In the case of scalar waves, this is a classical topic with a large literature. The method was initiated by Foldy for studying the multiple scattering of isotropic point scatterers [11]. The quasi-crystalline approximation (QCA) was subsequently introduced by Lax in order to take into account double scattering [16]. With the use of the QCA, Waterman and Truell derived an expression for the effective wave number of coherent waves that depends on the far-field scattering properties of a single scatterer, in both the forward and backward directions [33]. Waterman and Truell's approach, like that of Twersky [29] which is based on the Foldy's approximation, is known to be valid only for dilute distributions of scatterers. Not long after, Fikioris and Waterman introduced the "hole correction" that allows consideration of higher concentrations of anisotropic finite-size scatterers [10]. Anisotropy refers here to scattering properties that depend on angular orientation, as illustrated for example in a polar radiation diagram. The present approach is based on multipole expansions, and can therefore handle scatterers with non-circular or non spherical shapes. The *T*-matrix method provides a convenient tool to calculate multipole expansions [31].

Numerous investigators have attempted to improve on the original theory [11,33,10] or extend its range of applicability. Recently, Aristégui and Angel [3] showed that Foldy's method also yields predictions of the reflection and transmission of scalar waves by a random distribution of point or line scatterers contained within a slab. Similarly, an extension of Fikioris and Waterman's formalism was developed by Le Bas et al. [17] in order to describe both the reflection and transmission from a slab-fluid region in which elastic cylindrical scatterers are randomly spaced. They show in particular that two different coherent waves can propagate in the fluid slab. Starting from Fikioris and Waterman's approach, Linton and Martin [18] obtained the two-dimensional counterpart of Lloyd and Berry's effective wave number [20,19], taking into account the hole correction and the boundary effect. In the case of immersed fluid cylinders with a sound-speed that is close to that of the surrounding fluid, Martin and Maurel [22] show that Linton and Martin's formula [18] can be derived from Lippman–Schwinger's equation without use of the QCA, thus providing a link between distinct formalisms based on Green functions and multipole expansions. Shear-horizontal elastic waves (SH) are also scalar-type waves, and the effective dynamic properties SH waves in composites made of elastic cylindrical fibers randomly distributed in another elastic solid have been calculated by Aguiar and Angel [2] and Aristégui and Angel [4]. They show that the effective mass density and the effective shear stiffness are complex valued and frequency dependent.

In comparison with the numerous studies of the scalar situation, multiple scattering of elastic waves involving both compressional (P) and shear waves (SV) has received relatively little attention. Varadan et al. [30] and Yang and Mal [34] have considered this problem using multipole expansions. Their analysis is mainly focused on the low-frequency limit (Rayleigh limit) which predicts dynamic effective mechanical properties of particulate composites that are in agreement with Hashin and Rosen's bounds [12]. The Generalized Self Consistent Method (GSCM) [34] is derived by using a self consistent scheme applied to Waterman and Truell's formula [33]. However, this formula, which is valid for scalar waves, is applied to P and SV waves separately [34], and mode conversions between P and SV waves are therefore neglected. In contrast to the GSCM, the theory developed in [30] takes mode conversions into account, but the equations that involve P and SV waves are uncoupled by invoking additional hypotheses above and beyond those for the QCA used in most of the papers previously cited. This is not necessary and it can be avoided by ensuring that the equations faithfully and accurately describe the coupling between the P and SV waves, as we do in this paper. More precisely, we derive equations for P and SV waves in the spirit of the paper by Fikioris and Waterman [10], and effective wave numbers are then obtained in the limit as the scatterer size tends to zero. From this point of view, our results can be considered as a generalization of the work of Linton and Martin [18] for scalar waves to the elastic case. It is shown here that the effective wave numbers, which are predominantly P or SV waves at low concentration of scatterers, depend upon the four far-field scattering functions, including those related to mode conversions. An advantage of our method is that it is valid not only at low-frequency but also at higher frequencies. For this reason we also consider the short-wave limit in some detail. Also, we assume that the scatterers are located in a semi infinite region, a configuration that has received little attention. This allows us to determine the four reflection coefficients related to the solid–solid interface, both in general and in the small concentration and long wavelength limit.

The outline of the paper is as follows. Section 2 contains a derivation of the coupled equations for elastic waves. This is based on the multiple scattering theory of Fikioris and Waterman [10] for acoustic waves. In Section 3 we reformulate the infinite system of equations to derive a single equation for the wavenumber. We present in Section 4 a systematic solution of the coupled equations using asymptotic expansions in the concentration. This generalizes the theory of Linton and Martin [18,19]. The high frequency limit is also derived and compared to the Waterman and Truell approach. Section 5 is devoted to the calculation of reflection coefficients at the interface of the half-space enclosing the random distribution of scatterers. Asymptotic approximations are derived for the reflection coefficients that are valid in the low concentration limit. In Section 6 we derive explicit formulae for low concentration expansions of the effective wavenumbers, formulae which generalize that of Linton and Martin for the scaler case. We begin with the multiple scattering equations for elasticity.

2. Fikioris and Waterman theory for elastic media

The multiple scattering theory of Fikioris and Waterman [10] was derived for acoustic media. Here we develop the theory to consider multiple scattering from identical scatterers in elastic materials.

2.1. Multiple scattering equations

Suppose that time-harmonic P or SV waves are propagating perpendicular to N parallel cylinders located in an elastic solid and that k_L and k_T are the wave numbers of the P and SV waves. We assume the Helmholtz decomposition of the displacement in the form

$$\vec{u} = \vec{\nabla}\psi^L + \vec{\nabla} \times (\psi^T \vec{e}_z), \tag{1}$$

where $\vec{e}_z$ is the unit vector parallel to the cylinders and ψ^L and ψ^T represent potentials for the longitudinal (P) and transverse (SV) components of the waves. Under the influence of the incident waves $\psi^L_{inc}(\vec{r})$ and $\psi^T_{inc}(\vec{r})$, both L and T scattered waves $\psi^L_S(\vec{r};\vec{r}_k)$ and $\psi^T_S(\vec{r};\vec{r}_k)$ are generated by the kth scatterer, so that

$$\psi^L(\vec{r}) = \psi^L_{inc}(\vec{r}) + \sum_{k=1}^{N} \psi^L_S(\vec{r};\vec{r}_k), \tag{2a}$$

$$\psi^T(\vec{r}) = \psi^T_{inc}(\vec{r}) + \sum_{k=1}^{N} \psi^T_S(\vec{r};\vec{r}_k). \tag{2b}$$

Here, the first vector argument $\vec{r}$ specifies the field point of evaluation, while $\vec{r}_k$ is the location of the kth scatterer. The scatters are assumed to be identical in composition and orientation and the properties of a single scatterer are assumed to be known, so that a rule is available that relates the scattered waves $\psi^\alpha_S(\vec{r};\vec{r}_k)$ and the exciting fields $\psi^\alpha_E(\vec{r};\vec{r}_k)$ acting on the kth scatterer ($\alpha = L, T$). This rule defines a linear scattering operator $\mathbf{T}(\vec{r}_k)$ by the relations [31]

$$\psi^L_S(\vec{r};\vec{r}_k) = T^{LL}(\vec{r}_k)\psi^L_E(\vec{r};\vec{r}_k) + T^{TL}(\vec{r}_k)\psi^T_E(\vec{r};\vec{r}_k), \tag{3a}$$

$$\psi^T_S(\vec{r};\vec{r}_k) = T^{LT}(\vec{r}_k)\psi^L_E(\vec{r};\vec{r}_k) + T^{TT}(\vec{r}_k)\psi^T_E(\vec{r};\vec{r}_k), \tag{3b}$$

and

$$\mathbf{T}(\vec{r}_k) = \begin{bmatrix} T^{LL}(\vec{r}_k) & T^{TL}(\vec{r}_k) \\ T^{LT}(\vec{r}_k) & T^{TT}(\vec{r}_k) \end{bmatrix}. \tag{4}$$

If the scatterers are similar but are not all oriented in the same manner then the orientation of scatterers must be assumed to be random. The exciting field acting on the kth scatterer is the sum of the incident waves and the scattered waves from all scatterers other than the kth. It follows that:

$$\psi^L_E(\vec{r},\vec{r}_k) = \psi^L_{inc}(\vec{r}) + \sum_{j\neq k} T^{LL}(\vec{r}_j)\psi^L_E(\vec{r};\vec{r}_j) + \sum_{j\neq k} T^{TL}(\vec{r}_j)\psi^T_E(\vec{r};\vec{r}_j), \tag{5a}$$

$$\psi^T_E(\vec{r},\vec{r}_k) = \psi^T_{inc}(\vec{r}) + \sum_{j\neq k} T^{TT}(\vec{r}_j)\psi^T_E(\vec{r};\vec{r}_j) + \sum_{j\neq k} T^{LT}(\vec{r}_j)\psi^L_E(\vec{r};\vec{r}_j). \tag{5b}$$

Eqs. 2–5 are the multiple scattering equations that generalize those obtained for the scalar case (*cf.* Eqs. (2.9)–(10) in [33]).

In order to derive the equations governing the coherent motion, we use the method initially developed by Foldy [11] and Lax [16] to average over all configurations of cylinders. This method is a very well documented [21,28], and it includes as a special case the quasi-crystalline approximation (QCA). Performing the configurational average transforms Eq. (5) into

$$\left\langle \psi^L_E(\vec{r},\vec{r}_1)\right\rangle = \psi^L_{inc}(\vec{r}) + \int \mathrm{d}\vec{r}_j\, n(\vec{r},\vec{r}_j)\left[T^{LL}(\vec{r}_j)\left\langle\psi^L_E(\vec{r},\vec{r}_j)\right\rangle + T^{TL}(\vec{r}_j)\left\langle\psi^T_E(\vec{r},\vec{r}_j)\right\rangle\right], \tag{6a}$$

$$\left\langle \psi^T_E(\vec{r},\vec{r}_1)\right\rangle = \psi^T_{inc}(\vec{r}) + \int \mathrm{d}\vec{r}_j\, n(\vec{r},\vec{r}_j)\left[T^{TT}(\vec{r}_j)\left\langle\psi^T_E(\vec{r},\vec{r}_j)\right\rangle + T^{LT}(\vec{r}_j)\left\langle\psi^L_E(\vec{r},\vec{r}_j)\right\rangle\right]. \tag{6b}$$

In these equations, $\vec{r}_1$ is the location of one of the cylinders, $\left\langle\psi^\alpha_E(\vec{r},\vec{r}_j)\right\rangle$ ($\alpha = L, T$) are the average coherent fields acting on the jth scatterer, $n(\vec{r},\vec{r}_j)$ the conditional number density of scatterers at $\vec{r}$ if a scatterer is known to be at $\vec{r}_j$, and the integral is taken over the whole surface accessible to scatterers. For a uniform and random array of identical cylinders of constant density n_0 and radius a, the "hole correction" [10] requires

$$n(\vec{r},\vec{r}_j) = \begin{cases} n_0 & \text{for } |\vec{r}-\vec{r}_j| > b, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

with $b > 2a$. Generally, b represents the distance of closest approach between centers of adjacent cylinders. Eq. (6) are the multiple scattering integral equations that generalize the integral equation for acoustic media (*cf.* Eq. (2.1) in [10]). In the same way, the average coherent fields are obtained by performing a configurational average on Eq. (2), yielding

$$\langle\psi^L(\vec{r})\rangle = \psi^L_{inc}(\vec{r}) + n_0\int \mathrm{d}\vec{r}_j\Big[T^{LL}(\vec{r}_j)\langle\psi^L_E(\vec{r},\vec{r}_j)\rangle + T^{TL}(\vec{r}_j)\langle\psi^T_E(\vec{r},\vec{r}_j)\rangle\Big], \tag{8a}$$

$$\langle\psi^T(\vec{r})\rangle = \psi^T_{inc}(\vec{r}) + n_0\int \mathrm{d}\vec{r}_j\Big[T^{TT}(\vec{r}_j)\langle\psi^T_E(\vec{r},\vec{r}_j)\rangle + T^{LT}(\vec{r}_j)\langle\psi^L_E(\vec{r},\vec{r}_j)\rangle\Big]. \tag{8b}$$

It should be noted that Eq. (8) are exact if the observation point $\vec{r}$ is not close to the surface of the volume enclosing the cylinders. Field points at the boundary of the volume require a special treatment, see [18] for details in the acoustic case.

2.2. Modal equations

Our goal in the remainder of this section is to obtain the modal equations, which will be solved in Sections 3 and 4 to give the effective wave numbers. In the following, cylinders are assumed to be randomly distributed in the half-space defined by $S^+ = \{x > 0\}$. The incident wave propagates in the direction of the x-axis, normal to the interface $x = 0$ ($\vec{r} = (x, y)$):

$$\psi^\alpha_{inc}(\vec{r}) = A_\alpha \mathrm{e}^{\mathrm{i}k_\alpha x}, \quad \alpha \in \{L, T\}, \tag{9}$$

with $A_{L,T} = 1$ or 0 depending on the type of the incident wave considered (P or SV). The incident and the effective potentials satisfy the Helmholtz equation

$$\nabla^2\psi^\alpha + k^2_\alpha\psi^\alpha = 0, \quad \alpha \in \{L, T\}. \tag{10}$$

In the same way as $\psi^{L,T}_E(\vec{r},\vec{r}_j)$, the effective potentials $\left\langle\psi^{L,T}_E(\vec{r},\vec{r}_j)\right\rangle$ satisfy the Helmholtz equation and are regular functions at the point $\vec{r}_j$, they can therefore be expressed

$$\left\langle\psi^\alpha_E(\vec{r},\vec{r}_j)\right\rangle = \sum_n A^\alpha_n(\vec{r}_j)J_n(k_\alpha\rho_j)\mathrm{e}^{\mathrm{i}n\theta(\vec{\rho}_j)}, \quad \alpha \in \{L, T\}, \tag{11}$$

with $\vec{\rho}_j = \vec{r} - \vec{r}_j$, $\theta(\vec{\rho}_j) = \arg(\vec{\rho}_j)$ and $\rho_j = |\vec{\rho}_j|$. As usual with the T-matrix approach [31], the transition operators are defined by

$$T^{\alpha\beta}(\vec{r}_j)J_n(k_\alpha\rho_j)\mathrm{e}^{\mathrm{i}n\theta(\vec{\rho}_j)} = T^{\alpha\beta}_n H^{(1)}_n(k_\alpha\rho_j)\ \mathrm{e}^{\mathrm{i}n\theta(\vec{\rho}_j)}, \quad \alpha, \beta \in \{L, T\}, \tag{12}$$

and the corresponding far-field scattering amplitudes of the different interactions are given by

$$T^{\alpha\beta}(\vec{0})\mathrm{e}^{\mathrm{i}k_\alpha x} = \sum_n T^{\alpha\beta}_n H^{(1)}_n(k_\alpha r)\mathrm{e}^{\mathrm{i}n\theta} \underset{(r\to\infty)}{\simeq} \sqrt{\frac{2}{\pi k_\alpha r}}\mathrm{e}^{\mathrm{i}\left(k_\alpha r - \frac{\pi}{4}\right)} f^{\alpha\beta}(\theta), \quad \alpha, \beta \in \{L, T\}, \tag{13}$$

with $\vec{r} = (r\cos\theta, r\sin\theta)$. The far-field scattering functions $f^{\alpha\beta}(\theta)$ are therefore Fourier series with coefficients equal to the modal scattering amplitudes $T^{\alpha\beta}_n$, i.e.,

$$f^{\alpha\beta}(\theta) = \sum_n T^{\alpha\beta}_n \mathrm{e}^{\mathrm{i}n\theta}, \quad \alpha, \beta \in \{L, T\}. \tag{14}$$

Modal coefficients $T^{\alpha\beta}_n$ can be calculated numerically [31,32]. For circular cylinders, they are the components of the T-matrix and satisfy the symmetry relation $T^{\alpha\beta}_{-n} = T^{\alpha\beta}_n$. For non-circular cylinders, they are expressed in terms of the T-matrix components that depend on the orientation of the scatterer, so that in general $T^{\alpha\beta}_{-n} \neq T^{\alpha\beta}_n$.

Substituting Eqs. (11) and (12) into Eq. (6) gives

$$\sum_n [A^L_n(\vec{r}_1) - \mathrm{i}^n A_L \mathrm{e}^{\mathrm{i}k_L x_1}]J_n(k_L\rho_1)\mathrm{e}^{\mathrm{i}n\theta(\vec{\rho}_1)} = \sum_n T^{LL}_n \int_{S^+} \mathrm{d}\vec{r}_j\, n(\vec{r}_j,\vec{r}_1)A^L_n(\vec{r}_j)H^{(1)}_n(k_L\rho_j)\mathrm{e}^{\mathrm{i}n\theta(\vec{\rho}_j)}$$
$$+ \sum_n T^{TL}_n \int_{S^+} \mathrm{d}\vec{r}_j\, n(\vec{r}_j,\vec{r}_1)A^T_n(\vec{r}_j)H^{(1)}_n(k_L\rho_j)\mathrm{e}^{\mathrm{i}n\theta(\vec{\rho}_j)}, \tag{15a}$$

$$(L \leftrightarrow T). \tag{15b}$$

The symbol $(L \leftrightarrow T)$ means the same equation(s) as the previous one(s) but with the L and T indices permuted. In order to write series of (15) as a function of the coordinates centered on $\vec{r}_1 = (x_1, y_1)$, we use the change of variables $\theta(\vec{\rho}_j) = \theta(-\vec{\rho}_j) \pm \pi$ with the addition theorem [1]

$$H^{(1)}_n(k_\alpha\rho_j)\mathrm{e}^{\mathrm{i}n\theta(-\vec{\rho}_j)} = \sum_m (-1)^m \mathrm{e}^{\mathrm{i}(n-m)\theta(\vec{r}_{j1})}H^{(1)}_{n-m}(k_\alpha r_{j1})J_m(k_\alpha\rho_1)\mathrm{e}^{\mathrm{i}m\theta(\vec{\rho}_1)}, \tag{16}$$

where $\vec{r}_{j1} = \vec{r}_j - \vec{r}_1$, $\theta(\vec{r}_{j1}) = \arg(\vec{r}_{j1})$ and $r_{j1} = |\vec{r}_{j1}|$. The condition $|\vec{\rho}_1| < |\vec{r}_{j1}|$ must hold in order that the waves scattered from $\vec{r}_j$ are expressed with the use of Eq. (16) in the vicinity of $\vec{r}_1$. It then follows that:

$$A_n^L(\vec{r}_1) = \sum_p (-1)^p T_{n+p}^{LL} \int_{S^+} \mathrm{d}\vec{r}_j\, n(\vec{r}_j, \vec{r}_1) A_{n+p}^L(\vec{r}_j) H_p^{(1)}(k_L r_{j1}) \mathrm{e}^{ip\theta(\vec{r}_{j1})} + \sum_p (-1)^p T_{n+p}^{TL} \int_{S^+} \mathrm{d}\vec{r}_j\, n(\vec{r}_j, \vec{r}_1) A_{n+p}^T(\vec{r}_j) H_p^{(1)}(k_L r_{j1}) \mathrm{e}^{ip\theta(\vec{r}_{j1})} + i^n A_L \mathrm{e}^{ik_L x_1}, \tag{17a}$$

$$(L \leftrightarrow T). \tag{17b}$$

We seek coherent waves that propagate in the equivalent homogeneous medium, which consists in assuming that solutions of Eq. (17) may be written in the form

$$A_n^L(\vec{r}_j) = i^n A_n^L \mathrm{e}^{i\vec{\xi}.\vec{r}_j} + i^n B_n^L \mathrm{e}^{i\vec{\xi}'.\vec{r}_j} = i^n A_n^L \mathrm{e}^{i\xi x_j} + i^n B_n^L \mathrm{e}^{i\xi' x_j}, \tag{18a}$$

$$A_n^T(\vec{r}_j) = i^n A_n^T \mathrm{e}^{i\vec{\xi}.\vec{r}_j} + i^n B_n^T \mathrm{e}^{i\vec{\xi}'.\vec{r}_j} = i^n A_n^T \mathrm{e}^{i\xi x_j} + i^n B_n^T \mathrm{e}^{i\xi' x_j}. \tag{18b}$$

Here $\vec{\xi} = (\xi, 0)$ and $\vec{\xi}' = (\xi', 0)$ are the effective wave vectors of coherent waves that propagate in the direction of the x-axis, and the coefficients A_n^L and B_n^L are at this stage unknown. Here, two coherent waves with ξ and ξ' as wave numbers are assumed to propagate, which is a natural hypothesis for scarce concentrations of scatterers. In such situations the homogeneous medium looks like an elastic medium in which the two waves that propagate are predominantly P or SV waves (see Section 4.1). In the following, the concentration is supposed to be low enough that only two coherent waves propagate. It is worthwhile to note that a search for solutions of Eq. (17) in the following form:

$$A_n^L(\vec{r}_j) = i^n A_n^L\ \mathrm{e}^{i\vec{\xi}.\vec{r}_j} = i^n A_n^L \mathrm{e}^{i\xi x_j}, \tag{19a}$$

$$A_n^T(\vec{r}_j) = i^n A_n^T \mathrm{e}^{i\vec{\xi}'.\vec{r}_j} = i^n A_n^T \mathrm{e}^{i\xi' x_j}, \tag{19b}$$

leads to an ill-conditioned problem (the number of unknowns does not equal the number of equations available). Solutions of the form (19) were considered in [30], although in the context of uncoupled integral equations.

Inserting the *ansatz* (18) into Eq. (17) yields

$$A_n^L \mathrm{e}^{i\vec{\xi}.\vec{r}_1} + B_n^L \mathrm{e}^{i\vec{\xi}'.\vec{r}_1} = A_L \mathrm{e}^{ik_L x_1} + \sum_p i^{-p} \left[T_{n+p}^{LL} A_{n+p}^L I_p^L(\xi) + T_{n+p}^{LL} B_{n+p}^L I_p^L(\xi') + T_{n+p}^{TL} A_{n+p}^T I_p^L(\xi) + T_{n+p}^{TL} B_{n+p}^T I_p^L(\xi') \right], \tag{20a}$$

$$(L \leftrightarrow T). \tag{20b}$$

with

$$I_p^\alpha(\zeta) = \int_{S^+} \mathrm{d}\vec{r}_j\, n(\vec{r}_j, \vec{r}_1) H_p^{(1)}(k_\alpha r_{j1}) \mathrm{e}^{ip\theta(\vec{r}_{j1})} \mathrm{e}^{i\vec{\zeta}.\vec{r}_j}, \quad \zeta \in \{\xi, \xi'\},\ \alpha \in \{L, T\}. \tag{21}$$

Taking the hole correction (7) into account, the integral (21) may be written

$$I_p^\alpha(\zeta) = n_0 \mathrm{e}^{i\vec{\zeta}.\vec{r}_1} \int_{S_b} \mathrm{d}\vec{r}_{j1}\, \phi_p(k_\alpha r_{j1}) \mathrm{e}^{i\vec{\zeta}.\vec{r}_{j1}}, \quad \zeta \in \{\xi, \xi'\},\ \alpha \in \{L, T\}, \tag{22}$$

where S_b is the entirety of S^+ with the exclusion of the hole $|\vec{r}_{j1}| < b$ centered at $\vec{r}_1$, and ϕ_p are the cylindrical wave functions with respect to the same origin, with

$$\phi_p(k_\alpha r_{j1}) = H_p^{(1)}(k_\alpha r_{j1}) \mathrm{e}^{ip\theta(\vec{r}_{j1})}, \quad \alpha \in \{L, T\}. \tag{23}$$

The integral in Eq. (22) has been previously calculated in [17,18], with the result

$$I_p^\alpha(\zeta) = i^p \left[\frac{2n_0\pi}{\zeta^2 - k_\alpha^2} N_p^\alpha(\zeta) \mathrm{e}^{i\zeta x_1} + \frac{2in_0}{k_\alpha(\zeta - k_\alpha)} \mathrm{e}^{ik_\alpha x_1} \right], \quad \zeta \in \{\xi, \xi'\},\ \alpha \in \{L, T\}, \tag{24}$$

where

$$N_p^\alpha(\zeta) = \zeta b J_p'(\zeta b) H_p^{(1)}(k_\alpha b) - k_\alpha b J_p(\zeta b) H_p^{(1)'}(k_\alpha b), \quad \zeta \in \{\xi, \xi'\},\ \alpha \in \{L, T\}. \tag{25}$$

As a consequence Eq. (20) become

$$A_n^L \mathrm{e}^{i\xi x_1} + B_n^L \mathrm{e}^{i\xi' x_1} = A_L \mathrm{e}^{ik_L x_1} + \mathrm{e}^{i\xi x_1} \frac{2\pi n_0}{\xi^2 - k_L^2} \sum_p \left[T_{n+p}^{LL} A_{n+p}^L + T_{n+p}^{TL} A_{n+p}^T \right] N_p^L(\xi) + \mathrm{e}^{i\xi' x_1} \frac{2\pi n_0}{\xi'^2 - k_L^2} \sum_p [T_{n+p}^{LL} B_{n+p}^L + T_{n+p}^{TL} B_{n+p}^T] N_p^L(\xi') + \mathrm{e}^{ik_L x_1} \frac{2in_0}{k_L(\xi - k_L)} \sum_p \left[T_{n+p}^{LL} A_{n+p}^L + T_{n+p}^{TL} A_{n+p}^T \right] + \mathrm{e}^{ik_L x_1} \frac{2in_0}{k_L(\xi' - k_L)} \sum_p [T_{n+p}^{LL} B_{n+p}^L + T_{n+p}^{TL} B_{n+p}^T], \tag{26a}$$

$$(L \leftrightarrow T). \tag{26b}$$

2.2.1. The generalized Fikioris and Waterman equations

Eq. (26) are satisfied whatever the value of x_1 if the coefficients of $e^{i\xi x_1}$, $e^{i\xi' x_1}$, $e^{ik_L x_1}$ and $e^{ik_T x_1}$ are all zero. Equating the coefficients of $e^{i\xi x_1}$ and $e^{i\xi' x_1}$ to zero gives what is known as the Lorentz–Lorenz law. We obtain

$$A_n^L - \frac{2\pi n_0}{\xi^2 - k_L^2}\sum_p \left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right)N_{n-p}^L(\xi) = 0, \quad (L \leftrightarrow T), \tag{27a}$$

$$B_n^L - \frac{2\pi n_0}{\xi'^2 - k_L^2}\sum_p \left(T_p^{LL}B_p^L + T_p^{TL}B_p^T\right)N_{n-p}^L(\xi') = 0, \quad (L \leftrightarrow T). \tag{27b}$$

Eq. (27) are the desired modal equations that generalize those of Fikioris and Waterman. They provide two identical homogeneous systems of linear algebraic equations which involve either the unknowns $\left\{A_p^L, A_p^T\right\}$ with ξ or $\left\{B_p^L, B_p^T\right\}$ with ξ'. The existence of nontrivial solutions of the homogeneous system determines the effective wave numbers ξ and ξ'. This is the goal of the next section.

2.2.2. The extinction theorem for elastic waves

Equating the coefficients of $e^{ik_L x_1}$ and $e^{ik_T x_1}$ in (26) to zero corresponds to what is known as the extinction theorem. In this case it becomes

$$A_L + \frac{2in_0}{k_L(\xi - k_L)}\sum_p \left[T_p^{LL}A_p^L + T_p^{TL}A_p^T\right] + \frac{2in_0}{k_L(\xi' - k_L)}\sum_p \left[T_p^{LL}B_p^L + T_p^{TL}B_p^T\right] = 0, \tag{28a}$$

$$(L \leftrightarrow T). \tag{28b}$$

The extinction theorem is useful for calculating reflection coefficients for the waves reflected at the interface $x = 0$, see Section 5.

2.2.3. The Waterman and Truell method

Waterman and Truell's approach is an alternative method relevant at very low concentration of cylinders [3] and simpler than the more general approach of Fikioris and Waterman represented by the system (27). The Waterman and Truell approximation assumes a pair correlation function with the following property [33,18]:

$$n(\vec{r}, \vec{r}_1) = \begin{cases} n_0 & \text{for } |x - x_1| > \eta, \\ 0 & \text{otherwise}, \end{cases} \tag{29}$$

for $\eta \to 0$ with $\vec{r} = (x, y)$ and $\vec{r}_1 = (x_1, y_1)$. In this limit Eq. (21) are improper integrals in the sense of Cauchy principal value, which may be calculated as in [17,18]. We find

$$A_n^L - \frac{2n_0}{ik_L}\sum_p \left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right)\left[\frac{1}{\xi - k_L} - \frac{(-1)^{n+p}}{\xi + k_L}\right] = 0, \tag{30a}$$

$$B_n^L - \frac{2n_0}{ik_L}\sum_p \left(T_p^{LL}B_p^L + T_p^{TL}B_p^T\right)\left[\frac{1}{\xi' - k_L} - \frac{(-1)^{n+p}}{\xi' + k_L}\right] = 0, \tag{30b}$$

$$(L \leftrightarrow T), \tag{30c}$$

which are the modal equations that generalize those of Waterman and Truell, Eq. (27) in [33]. The extinction theorem (28) is unchanged.

3. Matrix form of the modal equations

In this section we focus on reformulating either of the identical systems of Eq. (27) with the objective of determining a single scalar equation for the wavenumbers. With no loss in generality we consider the first system, which in full is as follows:

$$A_n^L - \frac{2n_0\pi}{\xi^2 - k_L^2}\sum_p \left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right)N_{n-p}^L(\xi) = 0, \tag{31a}$$

$$A_n^T - \frac{2n_0\pi}{\xi^2 - k_T^2}\sum_p \left(T_p^{LT}A_p^L + T_p^{TT}A_p^T\right)N_{n-p}^T(\xi) = 0. \tag{31b}$$

The goal is a simplified equation for the wavenumbers ξ and ξ'.

We begin by rewriting the doubly infinite set of equations in matrix form. The unknown amplitudes are represented by vectors $\mathbf{a}_\alpha$, defined by $\mathbf{a}_\alpha = (\ldots, A_{-1}^\alpha, A_0^\alpha, A_1^\alpha, \ldots)^t$, $\alpha \in \{L, T\}$, so that the amplitude A_p^α is in the pth position of the infinitely

long column vector. Introduce the constant vector $\mathbf{e} = (\ldots, 1, 1, 1, \ldots)^t$, and the infinite square matrices $\overline{\mathbf{Q}}^\alpha$ and $\mathbf{T}^{\alpha\beta}$ with elements

$$\overline{Q}^\alpha_{mn} = \frac{\frac{i\pi}{2} N^\alpha_{m-n}(\xi) - 1}{\xi^2 - k_\alpha^2}, \quad T^{\alpha\beta}_{mn} = T^{\alpha\beta}_n \delta_{mn}, \quad \alpha, \beta \in \{L, T\}. \tag{32}$$

Then Eq. (31) can be expressed

$$\mathbf{a}_L - \epsilon\left(\overline{\mathbf{Q}}^L + \frac{\mathbf{e}\mathbf{e}^t}{y_L}\right)(\mathbf{T}^{LL}\mathbf{a}_L + \mathbf{T}^{TL}\mathbf{a}_T) = \mathbf{0}, \tag{33a}$$

$$\mathbf{a}_T - \epsilon\left(\overline{\mathbf{Q}}^T + \frac{\mathbf{e}\mathbf{e}^t}{y_T}\right)(\mathbf{T}^{LT}\mathbf{a}_L + \mathbf{T}^{TT}\mathbf{a}_T) = \mathbf{0}, \tag{33b}$$

where the scalars ϵ and y_α are

$$\epsilon = -i4n_0, \tag{34a}$$

$$y_\alpha = \xi^2 - k_\alpha^2, \quad \alpha \in \{L, T\}. \tag{34b}$$

The pair of Eq. (33) are combined to form a single equation involving doubly infinite vectors and matrices

$$\left\{\mathscr{I} - \epsilon\overline{\mathbf{Q}}\mathbf{T} - \epsilon\frac{\mathbf{e}_L\mathbf{e}_L^t}{y_L}\mathbf{T} - \epsilon\frac{\mathbf{e}_T\mathbf{e}_T^t}{y_T}\mathbf{T}\right\}\mathbf{a} = \mathbf{0}, \tag{35}$$

with vectors

$$\mathbf{e}_L = \begin{pmatrix}\mathbf{e}\\ \mathbf{0}\end{pmatrix}, \quad \mathbf{e}_T = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}\end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix}\mathbf{a}_L\\ \mathbf{a}_T\end{pmatrix}, \tag{36}$$

and matrices

$$\mathscr{I} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \quad \overline{\mathbf{Q}} = \begin{bmatrix}\overline{\mathbf{Q}}^L & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{Q}}^T\end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix}\mathbf{T}^{LL} & \mathbf{T}^{TL}\\ \mathbf{T}^{LT} & \mathbf{T}^{TT}\end{bmatrix}. \tag{37}$$

The system (35) can be replaced by an equivalent equation of simpler form. In order to achieve this we multiply from the left by $\mathbf{T}^{1/2}$ and rearrange the terms,

$$\left\{\mathscr{I} - \epsilon\mathbf{Q} - \frac{\epsilon}{y_L}\mathbf{f}_L\mathbf{f}_L^t - \frac{\epsilon}{y_T}\mathbf{f}_T\mathbf{f}_T^t\right\}\mathbf{u} = \mathbf{0}, \tag{38}$$

where the doubly infinite vectors are

$$\mathbf{f}_L = \mathbf{T}^{1/2}\mathbf{e}_L, \quad \mathbf{f}_T = \mathbf{T}^{1/2}\mathbf{e}_T, \quad \mathbf{u} = \mathbf{T}^{1/2}\mathbf{a}, \tag{39}$$

and the sole matrix, apart from the identity, is

$$\mathbf{Q} = \mathbf{T}^{1/2}\overline{\mathbf{Q}}\mathbf{T}^{1/2}. \tag{40}$$

The wavenumbers must clearly satisfy the condition that $\det \boldsymbol{P} = 0$ where $\boldsymbol{P}$ is the infinite matrix premultiplying $\boldsymbol{u}$ in Eq. (38). This approach was adopted by Varadan et al. [30], although the infinite system of equations they considered are not the same as the present system (39). While $\det \boldsymbol{P} = 0$ is a sufficient condition for determining the coherent wavenumbers, it is not always necessary, particularly in the limit of small concentration, which is equivalent to the limit of small ϵ. We now develop a necessary condition that is far simpler in form, and which, as we note below in Section 4.1, is both necessary and sufficient for small ϵ. We first rewrite (38) in the equivalent form

$$\left\{\mathscr{I} - \epsilon(\mathscr{I} - \epsilon\mathbf{Q})^{-1}\left(\frac{\mathbf{f}_L\mathbf{f}_L^t}{y_L} + \frac{\mathbf{f}_T\mathbf{f}_T^t}{y_T}\right)\right\}\mathbf{u} = \mathbf{0}. \tag{41}$$

Taking the inner product of (41) with $\mathbf{f}_L^t$ and $\mathbf{f}_T^t$ yields a system of two equations for the scalar quantities $\mathbf{f}_\alpha^t\mathbf{u}$, $\alpha \in \{L, T\}$,

$$\begin{bmatrix}1 - \epsilon\frac{M_{LL}}{y_L} & -\epsilon\frac{M_{TL}}{y_T}\\ -\epsilon\frac{M_{LT}}{y_L} & 1 - \epsilon\frac{M_{TT}}{y_T}\end{bmatrix}\begin{pmatrix}\mathbf{f}_L^t\mathbf{u}\\ \mathbf{f}_T^t\mathbf{u}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{42}$$

The four matrix elements are given by

$$M_{\alpha\beta} = \mathbf{f}_\alpha^t(\mathscr{I} - \epsilon\mathbf{Q})^{-1}\mathbf{f}_\beta, \quad \alpha, \beta \in \{L, T\}. \tag{43}$$

Taking the determinant of the 2×2 matrix yields the desired equation for ξ:

$$(y_L - \epsilon M_{LL})(y_T - \epsilon M_{TT}) - \epsilon^2 M_{LT}M_{TL} = 0, \tag{44}$$

or more explicitly

$$\left(\xi^2 - k_L^2 - \epsilon M_{LL}\right)\left(\xi^2 - k_T^2 - \epsilon M_{TT}\right) - \epsilon^2 M_{LT} M_{TL} = 0. \tag{45}$$

Eq. (45) is the fundamental equation for determining the coherent wavenumbers.

4. Asymptotic solutions of the wavenumber equation

4.1. Preliminary observations

Before considering solutions to Eq. (45) we note that the matrix elements can be expressed

$$M_{\alpha\beta} = \mathbf{e}_\alpha^t \mathbf{T}^{1/2^t} (\mathscr{I} - \epsilon \mathbf{T}^{1/2^t} \overline{\mathbf{Q}} \mathbf{T}^{1/2})^{-1} \mathbf{T}^{1/2} \mathbf{e}_\beta = \mathbf{e}_\alpha^t [\mathbf{T}^{-1} - \epsilon \overline{\mathbf{Q}}]^{-1} \mathbf{e}_\beta, \quad \alpha, \beta \in \{L, T\}. \tag{46}$$

This indicates that the elements can be calculated without evaluation of the square root matrix $\mathbf{T}^{1/2}$. The elements M_{LL} depend upon ξ, but the form of the wavenumber Eq. (45) suggests a natural asymptotic expansion in the parameter ϵ. Thus, using $(\mathscr{I} - \epsilon \mathbf{Q})^{-1} = \mathscr{I} + \epsilon \mathbf{Q} + \epsilon^2 \mathbf{Q}^2 + \cdots$, Eq. (43) implies

$$\begin{aligned} M_{\alpha\beta} &= \mathbf{f}_\alpha^t \mathbf{f}_\beta + \epsilon \mathbf{f}_\alpha^t \mathbf{Q} \mathbf{f}_\beta + \epsilon^2 \mathbf{f}_\alpha^t \mathbf{Q}^2 \mathbf{f}_\beta + \cdots \\ &= \mathbf{e}_\alpha^t \mathbf{T} \mathbf{e}_\beta + \epsilon \mathbf{e}_\alpha^t \mathbf{T} \overline{\mathbf{Q}} \mathbf{T} \mathbf{e}_\beta + \epsilon^2 \mathbf{e}_\alpha^t \mathbf{T} \overline{\mathbf{Q}} \mathbf{T} \overline{\mathbf{Q}} \mathbf{T} \mathbf{e}_\beta + \cdots, \quad \alpha, \beta \in \{L, T\}. \end{aligned} \tag{47}$$

Based on this expansion, the modal equation Eq. (45) has the following leading order expansion in n_0:

$$1 + \frac{4in_0}{\xi^2 - k_L^2} f^{LL}(0) + \frac{4in_0}{\xi^2 - k_T^2} f^{TT}(0) = 0. \tag{48}$$

This indicates that even at low concentration the effective wave numbers which are solutions of Eq. (44) depend on a coupling between P and SV waves. It should also be noted that Eq. (48) is an algebraic equation of order 2 with regard to ξ^2. Consequently, as expected at low concentration, there are two solutions that propagate in the direction of the x-axis (the other ones propagate in the opposite direction).

Satisfaction of the single Eq. (45) is clearly a necessary condition for the infinite system of homogeneous Eq. (35). An alternative procedure to solving the latter for small ϵ is to simultaneously seek both the wavenumber ξ and the null vector $\mathbf{a}$ as asymptotic series in ϵ. We do not present the details here, but it can be shown that the leading order solution yields $\mathbf{a} = \mathbf{e}_\alpha + \mathrm{O}(\epsilon)$ for $\alpha = L$ or T. This indicates that the effective wave solution can be characterized as either a quasi-L or a quasi-T wave. Furthermore, it may be shown that the solution for the wavenumber obtained by proceeding with the asymptotic series in ϵ is equivalent to that of the simplified Eq. (45). Hence, the latter is both necessary and sufficient for developing the asymptotic wavenumber solution.

The remainder of this section considers asymptotic expansions of the solutions, valid in different limits: low concentration, low-frequency, and high frequency, respectively. We begin with the small concentration expansion.

4.2. Asymptotic expansion at low concentrations

Rather than working with the wavenumber directly it is more convenient to expand the solutions of Eq. (44) about one of the two leading order solutions $y_\alpha = 0,\ \alpha \in \{L, T\}$. We choose to expand about the P-wave root $y_L = 0$, although all of the results below apply to the other solution under the interchange $L \leftrightarrow T$. At low concentration $|\epsilon| \ll 1$ and we therefore assume a formal asymptotic expansion in ϵ as follows:

$$y_L = \epsilon y_L^{(1)} + \epsilon^2 y_L^{(2)} + \epsilon^3 y_L^{(3)} + \cdots. \tag{49}$$

Substituting y_L from (49) into Eq. (44) and noting that $y_T = y_L + k_L^2 - k_T^2$, we obtain

$$\left(y_L^{(1)} - M_{LL} + \epsilon y_L^{(2)} + \epsilon^3 y_L^{(3)} + \ldots\right)\left(k_L^2 - k_T^2 + \epsilon(y_L^{(1)} - M_{TT}) + \epsilon^2 y_L^{(2)} + \ldots\right) - \epsilon M_{LT} M_{TL} = 0. \tag{50}$$

The coefficients in the expansion (49) follow by taking derivatives of Eq. (50) with respect to ϵ at $\epsilon = 0$.

4.2.1. $O(\epsilon^0)$

The leading order term is found by setting $\epsilon = 0$, yielding

$$y_L^{(1)} - M_{LL}|_{\epsilon=0} = 0. \tag{51}$$

Hence,

$$y_L^{(1)} = \mathbf{e}^t \mathbf{T}^{LL} \mathbf{e} = f^{LL}(0). \tag{52}$$

4.2.2. $O(\epsilon^1)$

At the next order Eq. (50) implies

$$\left(k_L^2 - k_T^2\right)\left(y_L^{(2)} - \left.\frac{\mathrm{d}M_{LL}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\right) - M_{LT}M_{TL}|_{\epsilon=0} = 0. \tag{53}$$

The derivative can be found using the expansion (47), resulting in

$$y_L^{(2)} = \mathbf{e}^t\mathbf{T}^{LL}\overline{\mathbf{Q}}^L|_{\xi=k_L}\mathbf{T}^{LL}\mathbf{e} + \mathbf{e}^t\mathbf{T}^{TL}\overline{\mathbf{Q}}^T|_{\xi=k_L}\mathbf{T}^{LT}\mathbf{e} - \frac{\mathbf{e}^t\mathbf{T}^{TL}\mathbf{e}\,\mathbf{e}^t\mathbf{T}^{LT}\mathbf{e}}{k_T^2 - k_L^2}. \tag{54}$$

The quantities $\overline{\mathbf{Q}}^L$ and $\overline{\mathbf{Q}}^T$ in (54) are evaluated at $\xi = k_L$. Using the definition of $\overline{\mathbf{Q}}^T$ in Eq. (32) we have

$$y_L^{(2)} = \mathbf{e}^t\mathbf{T}^{LL}\overline{\mathbf{Q}}_0^L\mathbf{T}^{LL}\mathbf{e} + \frac{i\pi}{2(k_L^2 - k_T^2)}\mathbf{e}^t\mathbf{T}^{TL}\bar{\mathbf{N}}^T(k_L)\mathbf{T}^{LT}\mathbf{e}, \tag{55}$$

where the square matrices $\bar{\mathbf{N}}^\alpha$, $\overline{\mathbf{Q}}_0^\alpha$ and for later use, $\overline{\mathbf{Q}}_0^{\alpha'}$, are defined

$$\overline{N}_{mn}^\alpha(\xi) = N_{m-n}^\alpha(\xi), \quad \overline{\mathbf{Q}}_0^\alpha = \overline{\mathbf{Q}}^\alpha(k_\alpha), \quad \overline{\mathbf{Q}}_0^{\alpha'} = \overline{\mathbf{Q}}^{\alpha'}(k_\alpha), \quad \alpha \in \{L, T\}. \tag{56}$$

Expanding the function $N_p^\alpha(\xi)$ for small $(\xi - k_\alpha)$ it is straightforward to derive

$$\overline{Q}_{0mn}^\alpha = D_{m-n}^{(0)}(k_\alpha), \quad \overline{Q}_{0mn}^{\alpha'} = D_{m-n}^{(1)}(k_\alpha), \tag{57}$$

where

$$D_p^{(0)}(k) = -\frac{i\pi}{4}\left[((kb)^2 - p^2)J_p(kb)H_p^{(1)}(kb) + (kb)^2 J_p'(kb)H_p^{(1)'}(kb)\right], \tag{58a}$$

$$D_p^{(1)}(k) = -\frac{1}{2}D_p^{(0)}(k) + \frac{p^2}{8} - \frac{(kb)^2}{8}\left[1 + i2\pi(kb)^2 J_p(kb)H_p^{(1)}(kb)\right]. \tag{58b}$$

4.2.3. $O(\epsilon^2)$

Taking the second derivative of (50) and setting $\epsilon = 0$ yields

$$\left(k_L^2 - k_T^2\right)\left(2y_L^{(3)} - \left.\frac{\mathrm{d}^2M_{LL}}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0}\right) + 2\left[y_L^{(2)} - \left.\frac{\mathrm{d}M_{LL}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\right]\left[y_L^{(1)} - M_{TT}|_{\epsilon=0}\right] - \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}M_{LT}M_{TL}\right|_{\epsilon=0} = 0. \tag{59}$$

The derivatives are again evaluated using the expansion (47). The second derivative requires some care, since the terms involving $\overline{\mathbf{Q}}$ on the right hand side of (47) are themselves dependent upon ϵ. This is the same as for the acoustic case; in fact, in the acoustic case only the second derivative terms is needed, all the remaining terms in (59) are unique to the elastic problem. Thus, using results from the acoustic case we have

$$\begin{aligned} y_L^{(3)} = {} & \mathbf{e}_L^t\mathbf{T}\overline{\mathbf{Q}}\mathbf{T}\overline{\mathbf{Q}}\mathbf{T}\mathbf{e}_L + f^{LL}(0)\mathbf{e}_L^t\mathbf{T}\overline{\mathbf{Q}}'\mathbf{T}\mathbf{e}_L + \frac{(f^{TT}(0) - f^{LL}(0))}{(k_L^2 - k_T^2)^2}\mathbf{e}_L^t\mathbf{T}\mathbf{e}_T\,\mathbf{e}_T^t\mathbf{T}\mathbf{e}_L \\ & + \frac{1}{2(k_L^2 - k_T^2)}\left[(\mathbf{e}_L^t\mathbf{T}\mathbf{e}_T)\,\mathbf{e}_T^t\mathbf{T}\overline{\mathbf{Q}}\mathbf{T}\mathbf{e}_L + (\mathbf{e}_T^t\mathbf{T}\mathbf{e}_L)\,\mathbf{e}_L^t\mathbf{T}\overline{\mathbf{Q}}\mathbf{T}\mathbf{e}_T\right], \end{aligned} \tag{60}$$

where the doubly infinite square matrices $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}'$ are evaluated at $\xi = k_L$. Higher order terms in the expansion (49) can be determined using the same procedure.

4.2.4. Wavenumber expansion to $O(n_0^2)$

Combining the above expressions for $y_L^{(1)}$ and $y_L^{(2)}$, the quasi-longitudinal wavenumber expansion up to second order in the concentration is

$$\xi^2 = k_L^2 + d_1^L n_0 + d_2^L n_0^2 + \ldots, \tag{61}$$

where

$$d_1^L = -4if^{LL}(0), \tag{62a}$$

$$d_2^L = -\frac{16}{k_L^2}\sum_{m,n} D_{m-n}^{(0)}(k_L)T_m^{LL}T_n^{LL} - \frac{8i\pi}{k_L^2 - k_T^2}\sum_{p,q} N_{p-q}^T(k_L)T_p^{TL}T_q^{LT}. \tag{62b}$$

The third order term follows in a straightforward manner from $y_L^{(3)}$ but is too long to warrant including here. The expansion for the quasi-transverse wavenumber follows by the interchange $L \leftrightarrow T$.

4.3. Long wavelength limit

The long wavelength regime corresponds to small $k_{L,T}b$. As $k_\alpha b \to 0$ we have

$$N_p^\alpha(\xi) \simeq \frac{2}{i\pi}\left(\frac{\xi}{k_\alpha}\right)^{|p|}, \quad \alpha \in \{L,T\}. \tag{63}$$

In the same limit, it follows from its definition in Eqs. (32) and (56) that the elements of the matrix $\overline{\mathbf{Q}}_0^\alpha$ become

$$\overline{Q}_{0mn}^\alpha = \frac{1}{2}|m-n|. \tag{64}$$

The long wavelength limit of the low concentration expansion (61) becomes, up to $O(n_0^2)$,

$$\xi^2 = k_L^2 + \delta_1^L n_0 + \delta_2^L n_0^2 + \ldots, \tag{65}$$

where

$$\delta_1^L = -4if^{LL}(0), \tag{66a}$$

$$\delta_2^L = -\frac{8}{k_L^2}\sum_{m,n}|m-n|T_m^{LL}T_n^{LL} - \frac{16}{k_L^2-k_T^2}\sum_{m,n}\left(\frac{k_L}{k_T}\right)^{|m-n|}T_m^{LT}T_n^{TL}. \tag{66b}$$

Note that the expansion for the wavenumber is independent of b in this limit. Again, the quasi-transverse wavenumber expansion follows from the interchange $L \leftrightarrow T$.

4.4. High $k_{L,T}b$ limit and the Waterman and Truell formula for elastic media

As mentioned in the Introduction, the method we have developed is also valid at high frequency. In the following it is assumed that $k_{L,T}b \to \infty$. This means either the frequency increases, i.e. $k_{L,T}a \to \infty$ with $b = O(1)$, or the separation length increases, $b \to \infty$ with $k_{L,T}a = O(1)$. The high frequency limit $k_{L,T}a \to \infty$ corresponds to wavelengths shorter than the radius a of cylinders. This situation has been studied in [6] which deals with ultrasonic characterization of thermal damage in concrete. However, since b can be related to the concentration c $(0 < c < 1)$ of cylinders [34] by

$$c = \frac{n_0\pi a^2}{n_0\pi b^2} = \frac{a^2}{b^2}, \tag{67}$$

it follows that the limit $b \to \infty$ with $k_{L,T}a = O(1)$ corresponds to a dilute medium $(c \to 0)$. This is one reason why it is of interest to compare the high $k_{L,T}b$ limit to the Waterman and Truell approach.

In the limit as $k_{L,T}b \to \infty$ the leading order asymptotic approximation of $N_p^\alpha(\xi)$ is [1]

$$N_p^\alpha(\xi) \simeq \frac{-1}{\pi k_\alpha}\sqrt{\frac{k_\alpha}{\xi}}[(-1)^p(k_\alpha-\xi)\ \mathrm{e}^{i\xi b} + i(k_\alpha+\xi)\mathrm{e}^{-i\xi b}]\mathrm{e}^{ik_\alpha b}, \quad \alpha = \{L,T\}, \tag{68}$$

It follows that the coupled systems of infinite equations (31) become in this limit

$$A_n^L + P_{LT} + (-1)^n Q_{LT} = 0, \tag{69a}$$

$$(L \leftrightarrow T), \tag{69b}$$

with

$$P_{LT} = P_L\sum_p\left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right), \tag{70a}$$

$$Q_{LT} = Q_L\sum_p(-1)^p\left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right), \tag{70b}$$

$$(L \leftrightarrow T), \tag{70c}$$

and

$$P_\alpha = \frac{2n_0}{ik_\alpha}\sqrt{\frac{k_\alpha}{\xi}}\frac{\mathrm{e}^{i(k_\alpha-\xi)b}}{k_\alpha-\xi}, \quad Q_\alpha = -\frac{2n_0}{k_\alpha}\sqrt{\frac{k_\alpha}{\xi}}\frac{\mathrm{e}^{i(k_\alpha+\xi)b}}{k_\alpha+\xi}, \quad \alpha = \{L,T\}. \tag{71}$$

The structure of Eq. (69) implies the identities

$$A_{-n}^{L,T} = A_n^{L,T} \quad \text{and} \quad A_{n+2}^{L,T} = A_n^{L,T}. \tag{72}$$

Consequently, the problem reduces from calculating an infinite set of unknowns to one with eight unknowns: P_{LT}, P_{TL}, Q_{LT}, Q_{TL} and $A_{0,1}^{L,T}$ satisfying a system of eight homogeneous linear equations. Note that although $P_{LT},\ldots,Q_{TL}$ can

be expressed in terms of $A_{0,1}^{L,T}$, the calculations are simpler with $P_{LT},\ldots,Q_{TL}$ considered as unknowns. The first four equations are obtained by setting $n=0$ and $n=1$ in Eq. (69). Then, we perform an iteration on Eq. (69) using the identities (72), with the result

$$A_n^L - [f^{LL}(0)P_L + (-1)^n f^{LL}(\pi)Q_L]P_{LT} - [f^{LL}(\pi)P_L + (-1)^n f^{LL}(0)Q_L]Q_{LT} - [f^{TL}(0)P_L + (-1)^n f^{TL}(\pi)Q_L]P_{TL} - [f^{TL}(\pi)P_L + (-1)^n f^{TL}(0)Q_L]Q_{TL} = 0, \tag{73a}$$

$$(L \leftrightarrow T). \tag{73b}$$

Four equations are obtained from Eq. (73) by considering the two possibilities for $(-1)^n$, corresponding to $n=0$ and $n=1$. The second set of four equations follow from Eq. (70) combined with the identities (72). Eliminating $A_{0,1}^{L,T}$ from the eight equations results in the following four equations for the four unknowns $P_{LT}, P_{TL}, Q_{LT}, Q_{TL}$,

$$[1+f^{LL}(0)P_L + f^{LL}(\pi)Q_L]P_{LT} + [1+f^{LL}(\pi)P_L + f^{LL}(0)Q_L]Q_{LT} + [f^{TL}(0)P_L + f^{TL}(\pi)Q_L]P_{TL} + [f^{TL}(\pi)P_L + f^{TL}(0)Q_L]Q_{TL} = 0, \tag{74a}$$

$$[1+f^{LL}(0)P_L - f^{LL}(\pi)Q_L]P_{LT} - [1-f^{LL}(\pi)P_L + f^{LL}(0)Q_L]Q_{LT} + [f^{TL}(0)P_L - f^{TL}(\pi)Q_L]P_{TL} + [f^{TL}(\pi)P_L - f^{TL}(0)Q_L]Q_{TL} = 0, \tag{74b}$$

$$(L \leftrightarrow T). \tag{74c}$$

The homogeneous linear system of Eq. (74) has nontrivial solutions if the associated determinant vanishes. Thus, the modal equation at high frequency is

$$\det\begin{bmatrix} 1+f^{LL}(0)P_L & f^{LL}(\pi)P_L & f^{TL}(0)P_L & f^{TL}(\pi)P_L \\ f^{LL}(\pi)Q_L & 1+f^{LL}(0)Q_L & f^{TL}(\pi)Q_L & f^{TL}(0)Q_L \\ f^{LT}(0)P_T & f^{LT}(\pi)P_T & 1+f^{TT}(0)P_T & f^{TT}(\pi)P_T \\ f^{LT}(\pi)Q_T & f^{LT}(0)Q_T & f^{TT}(\pi)Q_T & 1+f^{TT}(0)Q_T \end{bmatrix} = 0. \tag{75}$$

The mode converted forward scattering and back-scattering amplitudes, $f^{LT,TL}(0)$ and $f^{LT,TL}(\pi)$, respectively, are identically zero if the fundamental scatterer has sufficient geometrical symmetry. This is the case for circular cylinders, and occurs generally for cylinders with reflection symmetry about the x-axis. When $f^{LT,TL}(0) = f^{LT,TL}(\pi) = 0$, instead of the determinant of Eq. (75), the condition for satisfaction of the four Eq. (74) becomes two simpler equations:

$$[1+f^{LL}(0)P_L][1+f^{LL}(0)Q_L] - [f^{LL}(\pi)]^2 P_L Q_L = 0, \tag{76a}$$

$$[1+f^{TT}(0)P_T][1+f^{TT}(0)Q_T] - [f^{TT}(\pi)]^2 P_T Q_T = 0. \tag{76b}$$

These provide uncoupled modal equations for the P and SV waves, in contrast to what happens with the Fikioris and Waterman approach.

Starting from Eq. (30) and using the same process as before, we can easily verify that the Waterman and Truell's modal equation for elastic media is once again given by Eq. (75) where now $P_{L,T}$ and $Q_{L,T}$ are defined by

$$P_\alpha = \frac{2n_0}{ik_\alpha}\frac{1}{k_\alpha - \xi}, \quad Q_\alpha = \frac{2n_0}{ik_\alpha}\frac{1}{k_\alpha + \xi}, \quad \alpha = \{L,T\}. \tag{77}$$

So, the modal equation in the limit $k_{L,T}b \to \infty$ is very similar but fundamentally different from that of Waterman and Truell. For circular cylinders, Eq. (76) combined with (77) are nothing else than the Waterman and Truell formula [33] for acoustic waves applied to P and SV waves separately.

5. Reflection coefficients

5.1. Reflection coefficients for an incident P wave

We consider a single type of wave incident on the half-space $x>0$, specifically a P wave, so that $A_L = 1$ and $A_T = 0$. The fields $\langle\psi^L(\vec{r})\rangle$ and $\langle\psi^T(\vec{r})\rangle$ in the domain $\{x<0\}$ then correspond to the P and SV reflected waves respectively. Consequently, the reflection coefficients R^{LL} and R^{LT} can be defined as follows:

$$\langle\psi^\alpha(\vec{r})\rangle = R^{L\alpha}\,\mathrm{e}^{-ik_\alpha x}, \quad (x<0), \quad \alpha = \{L,T\}. \tag{78}$$

Inserting Eqs. (11) and (18) into Eq. (8), and taking into account Eq. (12), we get

$$\langle\psi^L(\vec{r})\rangle = n_0\sum_p i^{-p}\left[\left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right)J_p^L(\xi) + \left(T_p^{LL}B_p^L + T_p^{TL}B_p^T\right)J_p^L(\xi')\right], \tag{79a}$$

$$\langle\psi^T(\vec{r})\rangle = n_0\sum_p i^{-p}\left[\left(T_p^{LT}A_p^L + T_p^{TT}A_p^T\right)J_p^T(\xi) + \left(T_p^{LT}B_p^L + T_p^{TT}B_p^T\right)J_p^T(\xi')\right], \tag{79b}$$

$$(L \leftrightarrow T), \tag{79c}$$

with ($\vec{r}_{j1} = (x_{j1}, y_{j1})$)

$$J_p^\alpha(\zeta) = n_0 e^{i\zeta x} \int_{x_{j1}>-x} d\vec{r}_{j1}\, \phi_p(k_\alpha r_{j1}) e^{i\zeta x_{j1}}, \quad \zeta \in \{\xi, \xi'\}, \ \ \alpha \in \{L, T\}. \tag{80}$$

The integrals in Eq. (80) are calculated in [17,18], with the result that

$$J_p^\alpha(\zeta) = i^{-p} \frac{2in_0}{k_\alpha(\zeta + k_\alpha)} e^{-ik_\alpha x}, \quad \zeta \in \{\xi, \xi'\}, \ \ \alpha \in \{L, T\}, \tag{81}$$

and consequently

$$R^{LL} = \frac{2in_0}{k_L} \sum_p (-1)^p \left(\frac{T_p^{LL} A_p^L + T_p^{TL} A_p^T}{\xi + k_L} + \frac{T_p^{LL} B_p^L + T_p^{TL} B_p^T}{\xi' + k_L} \right), \tag{82a}$$

$$R^{LT} = \frac{2in_0}{k_T} \sum_p (-1)^p \left(\frac{T_p^{LT} A_p^L + T_p^{TT} A_p^T}{\xi + k_T} + \frac{T_p^{LT} B_p^L + T_p^{TT} B_p^T}{\xi' + k_T} \right). \tag{82b}$$

As expected, R^{LL} and R^{LT} depend on the infinite sets of coefficients $A_p^{L,T}$ and $B_p^{L,T}$. These coefficients follow from Eq. (28), which we rewrite in matrix format:

$$\epsilon \frac{(\xi + k_L)}{2k_L y_L} \mathbf{e}_L^t \mathbf{T}\mathbf{a} + \epsilon \frac{(\xi' + k_L)}{2k_L y'_L} \mathbf{e}_L^t \mathbf{T}\mathbf{b} = 1 \quad (= A_L), \tag{83a}$$

$$\epsilon \frac{(\xi + k_T)}{2k_T y_T} \mathbf{e}_T^t \mathbf{T}\mathbf{a} + \epsilon \frac{(\xi' + k_T)}{2k_T y'_T} \mathbf{e}_T^t \mathbf{T}\mathbf{b} = 0 \quad (= A_T), \tag{83b}$$

where the notation is the same as in Section 3, with the additional quantities $\mathbf{b}^t = (\mathbf{b}_L^t, \mathbf{b}_T^t), \mathbf{b}_\alpha = (\ldots, B_{-1}^\alpha, B_0^\alpha, B_1^\alpha, \ldots)^t$, and $y'_\alpha = \xi'^2 - k_\alpha^2, \ \alpha \in \{L, T\}$. The infinite vectors $\mathbf{a}$ and $\mathbf{b}$ are null vectors of (35) corresponding to the quasi-P and -SV roots of Eq. (45), respectively. To be specific, $\{\xi, \mathbf{a}\}$ solve (35) and $\{\xi', \mathbf{b}\}$ satisfy

$$\left\{ \mathscr{I} - \epsilon \overline{\mathbf{Q}}\mathbf{T} - \epsilon \frac{\mathbf{e}_L \mathbf{e}_L^t}{y'_L} \mathbf{T} - \epsilon \frac{\mathbf{e}_T \mathbf{e}_T^t}{y'_T} \mathbf{T} \right\} \mathbf{b} = 0, \tag{84}$$

which together determine the null vectors up to scalar multiples. Finally, Eq. (83) fix the amplitudes of the null vectors in terms of the incident wave amplitudes ($A_L = 1, A_T = 0$).

Once the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$ are determined, the reflection coefficients follow from (82), which can be recast in matrix form as

$$R^{LL} = \epsilon \frac{(k_L - \xi)}{2k_L y_L} \mathbf{e}_L^t \mathbf{J}\mathbf{T}\mathbf{a} + \epsilon \frac{(k_L - \xi')}{2k_L y'_L} \mathbf{e}_L^t \mathbf{J}\mathbf{T}\mathbf{b}, \tag{85a}$$

$$R^{LT} = \epsilon \frac{(k_T - \xi)}{2k_T y_T} \mathbf{e}_T^t \mathbf{J}\mathbf{T}\mathbf{a} + \epsilon \frac{(k_T - \xi')}{2k_T y'_T} \mathbf{e}_T^t \mathbf{J}\mathbf{T}\mathbf{b}, \tag{85b}$$

where $\mathbf{J} = \mathrm{diag}(\ldots, -1, 1, -1, 1, \ldots)$ with $J_{00} = 1$. The reflection coefficients R^{TT} and R^{TL} are obtained in the same same way with $A_L = 0$ and $A_T = 1$ in Eq. (83).

5.2. Asymptotic approximation

As in Section 3, we consider the low concentration asymptotic approximation. Starting with the *ansatz*

$$\mathbf{a} = \mathbf{a}^{(0)} + \epsilon \mathbf{a}^{(1)} + \ldots, \tag{86a}$$

$$\mathbf{b} = \mathbf{b}^{(0)} + \epsilon \mathbf{b}^{(1)} + \ldots, \tag{86b}$$

and using the result from Section 4 that $y_L = \epsilon \mathbf{e}_L^t \mathbf{T}\mathbf{e}_L + \mathrm{O}(\epsilon^2)$ we find that, to leading order, $\mathbf{a}^{(0)} = A_L \mathbf{e}_L$. Similarly, $y'_T = \epsilon \mathbf{e}_T^t \mathbf{T}\mathbf{e}_T + \ \mathrm{O}(\epsilon^2)$ and hence $\mathbf{b}^{(0)} = A_T \mathbf{e}_T = 0$. Eq. (85) then become

$$R^{LL} = -\epsilon \frac{f^{LL}(\pi)}{4k_L^2} + \mathrm{O}(\epsilon^2), \tag{87a}$$

$$R^{LT} = -\epsilon \frac{f^{LT}(\pi)}{2k_T(k_L + k_T)} + \mathrm{O}(\epsilon^2). \tag{87b}$$

The case of SV wave incidence ($A_L = 0, \ A_T = 1$) can be treated in the same way. Combined with the previous results, we find that the leading order approximations to the reflection coefficients are:

$$R^{\alpha\beta} = \frac{2in_0 f^{\alpha\beta}(\pi)}{(k_\alpha + k_\beta)k_\beta} + \mathrm{O}(n_0^2), \quad \alpha, \beta \in \{L, T\}. \tag{88}$$

This result shows that reflected waves do not depend on the effective wave numbers at this order, and helps explain why it can be difficult to detect reflected coherent waves. Experiments usually deal with transmitted coherent waves [8].

6. Generalization of the Linton and Martin formula

The aim of this section is to generalize the acoustics formula of Linton and Martin [18] to elasticity. The advantage of the Linton and Martin formula is that it expresses the wavenumber at low concentration in terms of the far-field scattering function only, rather than the T-matrix elements. In the present context of elastodynamics, this requires that we express the two series in (66b) as integrals of the far-field scattering functions. The analogous acoustic problem involves only the first series in (66b), which was calculated in [18] as

$$-\frac{8}{k_L^2}\sum_{m,n}|m-n|T_m^{LL}T_n^{LL}=\frac{8}{\pi k_L^2}\int_0^{2\pi}\mathrm{d}\theta\cot\left(\frac{\theta}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}[f^{LL}(\theta)]^2. \tag{89}$$

This integral is, however, convergent only if $f^{LL\prime}(0)=0$. That is the case for certain scatterers, including circular cylinders, where $T_{-p}^{LL}=T_p^{LL}\Rightarrow f^{LL\prime}(0)=0$. But the formula is not correct for non-circular cylinders and must therefore be modified. The resolution of this problem is found in the following identity, which follows from Eq. (14):

$$f^{LL}(\theta)f^{LL}(-\theta)=\sum_{m,n}T_n^{LL}T_m^{LL}\cos(n-m)\theta. \tag{90}$$

Using this we derive an identity similar to Eq. (83) of [18] that does not rely upon the assumption $T_{-p}^{LL}=T_p^{LL}$. Thus, instead of (89) we have the more general identity

$$-\frac{8}{k_L^2}\sum_{m,n}|m-n|T_m^{LL}T_n^{LL}=\frac{8}{\pi k_L^2}\int_0^{2\pi}\mathrm{d}\theta\cot\left(\frac{\theta}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}[f^{LL}(\theta)f^{LL}(-\theta)]. \tag{91}$$

Regarding the second series in (66b) and its dual for the quasi-SV wavenumber, we consider

$$S_L\equiv\sum_{m,n}\left(\frac{k_L}{k_T}\right)^{|m-n|}T_m^{LT}T_n^{TL}=\frac{1}{2}\sum_{m,n}\left(T_m^{LT}T_n^{TL}+T_m^{TL}T_n^{LT}\right)\kappa^{-|m-n|}, \tag{92a}$$

$$S_T\equiv\sum_{m,n}\left(\frac{k_T}{k_L}\right)^{|m-n|}T_m^{TL}T_n^{LT}=\frac{1}{2}\sum_{m,n}\left(T_m^{LT}T_n^{TL}+T_m^{TL}T_n^{LT}\right)\kappa^{|m-n|}, \tag{92b}$$

where $\kappa=k_T/k_L>1$. Although it might appear that the series for S_T is divergent, it is not. Based on the original result of Muller [23] for the Helmholtz equation, later extended to elasticity [7], it follows that the far-field scattering functions are necessarily entire functions of the angular argument θ considered as a complex variable. In the present context this implies that the Fourier coefficients of the far-field scattering functions decay in such a manner that S_T series is convergent for any finite κ. This analytic property is apparent by considering, for example, the far-field scattering functions for a circular object.

Our main result is that the two sums can be simplified as follows:

Lemma 1

$$S_L=\frac{(\kappa^2-1)}{2\pi}\int_0^{\pi}\frac{\mathrm{d}\theta G_{LT}(\theta)}{1-2\kappa\cos\theta+\kappa^2}, \tag{93}$$

and

$$S_L+S_T=G_{LT}(i\log\kappa)=\lim_{N\to\infty}\left\{\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta G_{LT}(\theta)\left(\frac{\kappa\cos(N-1)\theta-\cos N\theta}{1-2\kappa\cos\theta+\kappa^2}\right)\kappa^N\right\}, \tag{94}$$

where

$$G_{LT}(\theta)=f^{LT}(\theta)f^{TL}(-\theta)+f^{TL}(\theta)f^{LT}(-\theta)=G_{LT}(-\theta). \tag{95}$$

Note that the limit in (94) does not commute with the integral. This is a consequence of the properties of the far-field scattering functions mentioned before.

Proof. Starting from the definition in Eq. (95) and using Eq. (14), we have

$$G_{LT}(\theta)=\sum_{m,n}\left[T_m^{LT}T_n^{TL}+T_m^{TL}T_n^{LT}\right]\cos(m-n)\theta. \tag{96}$$

Consider the function

$$g(\theta)=\sum_n\kappa^{-|n|}\cos n\theta\iff\kappa^{-|n|}=\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta g(\theta)\cos n\theta. \tag{97}$$

Hence, Eq. (96) implies that

$$S_L = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\theta g(\theta) G_{LT}(\theta). \tag{98}$$

Performing the sum $(97)_1$ yields

$$g(\theta) = \frac{\kappa^2 - 1}{1 - 2\kappa\cos\theta + \kappa^2}, \tag{99}$$

from which the result (93) follows. It remains to prove (94).

The first identity in (94) follows directly from (96) and

$$G_{LT}(i\log\kappa) = \frac{1}{2}\sum_{m,n}\left[T_m^{LT}T_n^{TL} + T_m^{TL}T_n^{LT}\right](\kappa^{m-n} + \kappa^{n-m}). \tag{100}$$

Regarding the second identity in (94), note that replacing κ with κ^{-1} in Eq. $(97)_1$ gives a divergent series. However, the following finite series is perfectly well defined for any positive integer N,

$$h_N(\theta) = \sum_{|n|<N}\kappa^{|n|}\cos n\theta \iff \kappa^{|n|} = \frac{1}{\pi}\int_0^{\pi}\mathrm{d}\theta h_N(\theta)\cos n\theta, \quad |n| < N, \tag{101}$$

and therefore

$$\sum_{\substack{m,n\\ |m-n|<N}}\frac{1}{2}\left(T_m^{LT}T_n^{TL} + T_m^{TL}T_n^{LT}\right)\kappa^{|m-n|} = \frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\theta h_N(\theta)G_{LT}(\theta). \tag{102}$$

Using $1 + \kappa + \kappa^2 + \ldots + \kappa^{N-1} = (1-\kappa^N)/(1-\kappa)$ gives

$$h_N(\theta) = -g(\theta) + 2\kappa^N\left(\frac{\kappa\cos(N-1)\theta - \cos N\theta}{1 - 2\kappa\cos\theta + \kappa^2}\right), \tag{103}$$

which implies the second identity in (94). The integral formula (94) provides a formal procedure to evaluate the far-field scattering functions at an argument that is imaginary, such as $\pm i\log\kappa$ in this case. In practice, it is advisable to use the Fourier series, or perhaps a truncated version of the same if there is any numerical noise present. This is equivalent to a singular value decomposition (SVD) of the far-field scattering operators, which is always necessary in any inverse scattering algorithm. □

Combining the simplified expressions for S_L and S_T with the expansion (65), we arrive at our main results:

Theorem 1. *The long wavelength limit of the low concentration expansions for the quasi-P and -SV wavenumbers are, to* $O(n_0^2)$,

$$\xi^2 = k_L^2 - 4in_0 f^{LL}(0) + \frac{8n_0^2}{\pi k_L^2}\int_0^{2\pi}\mathrm{d}\theta\cot\left(\frac{\theta}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}[f^{LL}(\theta)f^{LL}(-\theta)] + \frac{8n_0^2}{\pi}\int_0^{\pi}\mathrm{d}\theta\frac{f^{LT}(\theta)f^{TL}(-\theta) + f^{TL}(\theta)f^{LT}(-\theta)}{k_T^2 - 2k_Lk_T\cos\theta + k_L^2}, \tag{104}$$

and

$$\begin{aligned}\xi'^2 = {} & k_T^2 - 4in_0 f^{TT}(0) + \frac{8n_0^2}{\pi k_T^2}\int_0^{2\pi}\mathrm{d}\theta\cot\left(\frac{\theta}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}[f^{TT}(\theta)f^{TT}(-\theta)] + \frac{8n_0^2}{\pi}\int_0^{\pi}\mathrm{d}\theta\frac{f^{LT}(\theta)f^{TL}(-\theta) + f^{TL}(\theta)f^{LT}(-\theta)}{k_T^2 - 2k_Lk_T\cos\theta + k_L^2}\\ & - \frac{16n_0^2}{k_T^2 - k_L^2}[f^{LT}(i\log\kappa)f^{TL}(-i\log\kappa) + f^{TL}(i\log\kappa)f^{LT}(-i\log\kappa)].\end{aligned} \tag{105}$$

These formulae generalize the identity found by Linton and Martin [18] for acoustic (scalar) waves. The acoustic equation can be derived from (104) by eliminating the SV wave contributions associated with the index T, giving

$$\xi^2 = k_L^2 - 4in_0 f^{LL}(0) + \frac{8n_0^2}{\pi k_L^2}\int_0^{2\pi}\mathrm{d}\theta\cot\left(\frac{\theta}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}[f^{LL}(\theta)f^{LL}(-\theta)]. \tag{106}$$

References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1974.
[2] A.R. Aguiar, Y.C. Angel, Antiplane coherent scattering from a slab containing a random distribution of cavities, Proc. Roy. Soc. A 456 (2004) (2000) 2883–2909.
[3] C. Aristégui, Y.C. Angel, New results for isotropic point scatterers: foldy revisited, Wave Motion 36 (2002) 383–399.
[4] C. Aristégui, Y.C. Angel, Effective material properties for shear-horizontal acoustic waves in fiber composites, Phys. Rev. E 75 (5) (2007) 056607.
[5] P.G.J. Bussink, P.L. Iske, J. Oortwijn, G.L.M.M. Verbist, Self-consistent analysis of elastic wave propagation in two dimensional matrix-inclusion composites, J. Mech. Phys. Solids 43 (10) (1995) 1673–1690.

[6] J.F. Chaix, V. Garnier, G. Corneloup, Ultrasonic wave propagation in heterogeneous solid media: theoretical analysis and experimental validation, Ultrasonics 44 (2006) 200–210.
[7] A. Charalambopoulos, K. Kiriaki, Characterization of functions as radiation patterns in linear elasticity, Math. Meth. Appl. Sci. 15 (8) (1992) 547–558.
[8] A. Derode, V. Mamou, A. Tourin, Influence of correlations between scatterers on the attenuation of the coherent wave in a random medium, Phys. Rev. E 74 (2006) 036606.
[9] A.J. Devaney, Multiple scattering for discrete, elastic random media, J. Acoust. Soc. Am. 21 (11) (1980) 2603–2611.
[10] J.G. Fikioris, P.C. Waterman, Multiple scattering of waves. II. Hole corrections in the scalar case, J. Math. Phys. 5 (10) (1964) 1413–1420.
[11] L.L. Foldy, The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers, Phys. Rev. 67 (1945) 107–119.
[12] Z. Hashin, W.R. Rosen, The elastic moduli of fiber-reinforced materials, J. Appl. Mech. ASME 31 (1964) 223–232.
[13] S.K. Kanaun, V.M. Levin, F.J. Sabina, Propagation of elastic waves in composites with random set of spherical inclusions (effective medium approach), Wave Motion 40 (2004) 69–88.
[14] F.C. Karal, J.B. Keller, Elastic electromagnetic and other waves in random medium, J. Math. Phys. 5 (10) (1964) 537–547.
[15] J.Y. Kim, J.G. Ih, B.H. Lee, Dispersion of elastic waves in random particulate composites, J. Acoust. Soc. Am. 97 (3) (1995) 1380–1388.
[16] M. Lax, Multiple scattering of waves. II. The effective field in dense systems, Phys. Rev. 85 (4) (1952) 621–629.
[17] P.Y. Le Bas, F. Luppé, J.M. Conoir, Reflection and transmission by randomly spaced elastic cylinders in a fluid slab-like region, J. Acoust. Soc. Am. 117 (3) (2005) 1088–1097.
[18] C.M. Linton, P.A. Martin, Multiple scattering by random configurations of circular cylinders: second-order corrections for the effective wavenumber, J. Acoust. Soc. Am. 117 (6) (2005) 3413–3423.
[19] C.M. Linton, P.A. Martin, Multiple scattering by multiple spheres: a new proof of the lloyd-berry formula for the effective wavenumber, SIAM J. Appl. Math. 66 (5) (2006) 1649–1668.
[20] P. Lloyd, M.V. Berry, Wave propagation through an assembly of spheres. iv. Relation between different scattering theories, Proc. Phys. Soc. Lond. 91 (1967) 678–688.
[21] P.A. Martin, Multiple Scattering: Interaction of Time-harmonic Waves with N Obstacles, Cambridge, New York, 2006.
[22] P.A. Martin, A. Maurel, Multiple scattering by random configurations of circular cylinders: weak scattering without closure assumptions, Wave Motion 45 (2008) 865–880.
[23] C. Muller, Radiation patterns and radiation fields, J. Rat. Mech. Anal. 4 (1955) 235–246.
[24] W.J. Parnell, I.D. Abrahams, Homogenization for wave propagation in periodic-reinforced media with complex microstructure. I-Theory, J. Mech. Phys. Solids 56 (2008) 2521–2540.
[25] S. Robert, J.M. Conoir, Reflection and transmission process from a slab-like region containing a random distribution of cylindrical scatterers in an elastic matrix, Acta Acust Unit Acust 93 (1) (2007) 1–12.
[26] F.J. Sabina, J.R. Willis, A simple self-consistent analysis of wave propagation in particulate composites, Wave Motion 10 (1988) 127–142.
[27] F.E. Stanke, G.S. Kino, A unified theory for elastic wave propagation in polycrystalline materials, J. Acoust. Soc. Am. 75 (5) (1983) 665–681.
[28] L. Tsang, J.A. Kong, Scattering of Electromagnetic Waves – Advanced Topics, John Wiley & Sons, 2001.
[29] V. Twersky, On scattering of waves by random distributions. I. Free scatterers formalism, J. Math. Phys. 3 (4) (1962) 700–715.
[30] V.K. Varadan, Y. Ma, V.V. Varadan, Multiple scattering of compressional and shear waves by fiber-reinforced composite materials, J. Acoust. Soc. Am. 80 (1) (1986) 333–339.
[31] V.K. Varadan, V.V. Varadan, Acoustic Electromagnetic and Elastic Waves Scattering-Focused on the *T*-Matrix Approach, Pergamon Press, New-York, 1980.
[32] N.D. Veksler, J.L. Izbicki, J.M. Conoir, Elastic wave scattering by a cylindrical shell, Wave Motion (1999).
[33] P.C. Waterman, R. Truell, Multiple scattering of waves, J. Math. Phys. 2 (4) (1961) 512–537.
[34] R. Yang, A.K. Mal, Multiple scattering of elastic waves in a fiber-reinforced composite, J. Mech. Phys. Solids 42 (1994) 1945–1968.