Acoustic metafluids

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Acoustic metafluids are defined as the class of fluids that allow one domain of fluid to acoustically mimic another, as exemplified by acoustic cloaks. It is shown that the most general class of acoustic metafluids are materials with anisotropic inertia and the elastic properties of what are known as pentamode materials. The derivation uses the notion of finite deformation to define the transformation of one region to another. The main result is found by considering energy density in the original and transformed regions. Properties of acoustic metafluids are discussed, and general conditions are found which ensure that the mapped fluid has isotropic inertia, which potentially opens up the possibility of achieving broadband cloaking.
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I. INTRODUCTION

Ideal acoustic stealth is provided by the acoustic cloak, a shell of material that surrounds the object to be rendered acoustically "invisible." Stealth can also be achieved by "hiding under the carpet,"[1] as shown in Fig. 1. A simpler situation but one that displays the essence of the acoustic stealth problem is depicted in Fig. 2. The common issue is how to make one region of fluid acoustically mimic another region of fluid. The fluids are different as are the domains they occupy; in fact, the mimicking region is typically smaller in size, and it can be viewed as a compacted version of the original.

The subject of this paper is not acoustic cloaks, or carpets, or ways to hide things, but rather the type of material necessary to achieve stealth. We define these materials as *acoustic metafluids*, which as we will see can be considered fluids with microstructure and properties outside those found in nature. The objective is to derive the general class of acoustic metafluids, and in the process show that there is a closed set which can be mapped from one to another. Acoustic metafluids are defined as the class of fluids that (a) acoustically mimic another region as in the examples of Figs. 1 and 2, and (b) can themselves be mimicked by another acoustic metafluid in the same sense. The requirement (b) is important, implying that there is a closed set of acoustic metafluids. The set includes as a special case the "normal" acoustic fluid of uniform density and bulk modulus. Acoustic metafluids can therefore be used to create stealth devices in a normal fluid. But, in addition, acoustic metafluids can provide stealth in any type of acoustic metafluid. The reciprocal nature of these fluids make them a natural generalization of normal acoustic fluids.

The acoustic cloaks that have been investigated to date fall into two categories in terms of the type of acoustic metafluid proposed as cloaking material. Most studies, e.g., Refs. 2–7, consider the cloak to comprise fluid with the normal stress-strain relation but anisotropic inertia, what we call *inertial cloaking*. Particular realization of inertial cloaks are in principle feasible using layers of isotropic normal fluid,[8–11] the layers are introduced in order to achieve a homogenized medium that approximates a fluid with anisotropic inertia. An alternative and more general approach[12,13] is to consider anisotropic inertia combined with anisotropic elasticity. The latter is introduced by generalizing the stress strain relation to include what are known as pentamode[14] elastic materials.[12,15,13] Clearly, the question of how to design and fabricate acoustic metafluids remains open. The focus of this paper is to first characterize the acoustic metafluids as a general type of material. In fact, as will be shown, this class of materials contains broad degrees of freedom, which can significantly aid in future design studies.

The paper is organized as follows. The concept of acoustic metafluids is introduced in Sec. II through two "acoustic mirage" examples. The methods used to find the acoustic metafluid in these examples are simple but not easily generalized. An alternative and far more powerful approach is discussed in Sec. III: the transformation method. This is based on using the change in variables between the coordinates of the two regions combined with differential relations to identify the metafluid properties of the transformed domain. Leonhardt and Philbin[16] provided an instructive review of the transformation method in the context of optics. The transformation method does not, however, define the range of material properties capable of being transformed. This is the central objective of the paper and it is resolved in Sec. IV by considering the energy density in the original and transformed domains. Physical properties of acoustic metafluids are discussed in Sec. V, including the unusual property that the top surface is not horizontal when at rest under gravity. The subset of acoustic metafluids that have isotropic inertia is considered in Sec. VI, and a concluding summary is presented in Sec. VII.

II. ACOUSTIC MIRAGES AND SIMPLE METAFLUIDS

The defining property of an *acoustic metafluid* is its ability to mimic another acoustic fluid that occupies a different domain. The simplest type of acoustic illusion is what may

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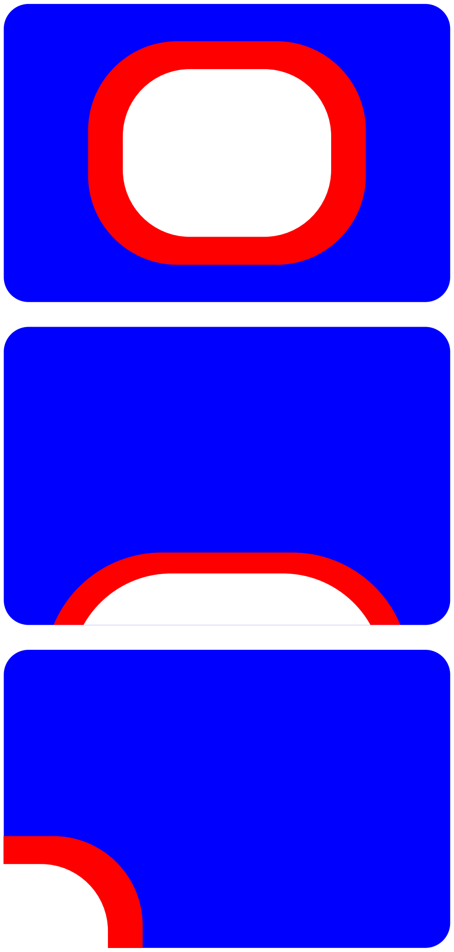

FIG. 1. (Color online) Three ways to acoustically hide something: envelope it with a cloak (top), hide it under a carpet (middle), or hide the object in the corner. In each case the acoustic metafluid in the cloaking layer emulates the acoustic properties of uniform fluid occupying the layer plus the hidden region.

be called an *acoustic mirage* where an observer hears, for example, a reflection from a distant wall, but in reality the echo originates from a closer boundary. Two examples of acoustic mirages are discussed next.

A. 1D mirage

Consider perhaps the simplest configuration imaginable, a one- dimensional (1D) semi-infinite medium. The upper picture in Fig. 2 shows the left end of an acoustic half-space $x \geqslant 0$ with uniform density ρ_0 and bulk modulus κ_0. The wave speed is $c_0 = \sqrt{\kappa_0/\rho_0}$. Now replace the region $0 \leqslant x < b$ with a shorter section $0 < b-a \leqslant x < b$ filled with an acoustic metafluid. The acoustic mirage effect requires that

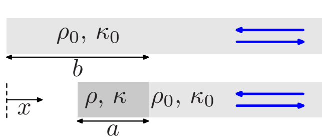

FIG. 2. (Color online) The top shows waves in a semi-infinite medium $x \geqslant 0$. The wave incident from the right reflects from a perfectly reflecting boundary at $x=0$. The lower figure shows the same medium in $x > b$ with the region $0 \leqslant x < b$ replaced by a shorter region of acoustic metafluid. Its properties are such that it produces a perfect acoustic illusion or "mirage."

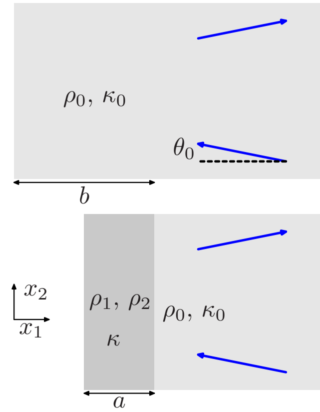

FIG. 3. (Color online) The same as Fig. 2 but now for oblique incidence. The same strategy used for the 1D case is no longer adequate; however, a solution may be found if the metafluid in the layer of thickness a is allowed to have anisotropic inertia defined by the inertias ρ_1 and ρ_2 in the x_1 and x_2 directions.

an observer in $x > b$ hears a response as if the half-space is as shown in the top of Fig. 2. This occurs if the metafluid region is such that (i) no reflection occurs at the interface $x=b$, i.e., the acoustic impedance in the modified region is the same as before; and (ii) the round trip travel time of a wave incident from the right is unchanged. The impedance condition and the travel time requirement ensure that the amplitude and phase of any signal is exactly the same as in the original half-space, and hence the mirage is accomplished.

Let the acoustic metafluid have material properties ρ and κ, with speed $c = \sqrt{\kappa/\rho}$. Conditions (i) and (ii) are satisfied if

$$\rho c = \rho_0 c_0, \qquad \frac{a}{c} = \frac{b}{c_0}, \tag{1}$$

respectively, implying the density and bulk modulus in the shorter region are

$$\rho = \frac{b}{a}\rho_0, \tag{2a}$$

$$\kappa = \frac{a}{b}\kappa_0. \tag{2b}$$

In this example the acoustic metafluid is another acoustic fluid, although with greater density and lesser bulk modulus. Note that the total mass of the metafluid region is unchanged from the original: $\rho_0 b = \rho a$.

B. 2D mirage

Consider the same problem in two dimensional (2D) under oblique wave incidence, Fig. 3. A wave incident at angle θ_0 from the normal has travel time $b/(c_0 \cos\theta_0)$ in the original layer. If the shortened region has wave speed c, then the modified travel time is $a/(c \cos\theta)$, where θ is defined by the Snell–Descartes law, $(1/c)\sin\theta = (1/c_0)\sin\theta_0$. At the same time, the reflectivity of the modified layer is $R = (Z-Z_0)/(Z+Z_0)$, where $Z = \rho c/\cos\theta$. The impedance and travel time conditions are

$$\frac{\rho c}{\cos\theta} = \frac{\rho_0 c_0}{\cos\theta_0}, \quad \frac{a}{c\cos\theta} = \frac{b}{c\cos\theta_0}. \tag{3}$$

Solving for the modified parameters implies κ is again given by Eq. (2b) but the density is now

$$\rho = \frac{b}{a}\frac{\rho_0}{2\cos^2\theta_0}\left[1 + \sqrt{1 - \frac{a^2}{b^2}\sin^2 2\theta_0}\right]. \tag{4}$$

The mirage works only for a single direction of incidence, θ_0, and is therefore unsatisfactory. The underlying problem here is that three conditions need to be met: Snell–Descartes' law, matched impedances, and equal travel times, with only two parameters, ρ and κ. Some additional degree of freedom is required.

1. Anisotropic inertia

One method to resolve this problem is to introduce the notion of anisotropic mass density, see Fig. 3. The density of the metafluid region is no longer a scalar, but becomes a tensor: $\rho \to \boldsymbol{\rho}$. The equation of motion and the constitutive relation in the metafluid are

$$\boldsymbol{\rho}\dot{\mathbf{v}} = -\nabla p, \quad \dot{p} = -\kappa\nabla\cdot\mathbf{v}, \tag{5}$$

where $\mathbf{v}$ is the particle velocity and p is the acoustic pressure. Assuming 2D dependence with constant anisotropic density of the form

$$\boldsymbol{\rho} = \begin{bmatrix}\rho_1 & 0\\ 0 & \rho_2\end{bmatrix} \tag{6}$$

and eliminating $\mathbf{v}$ imply that the pressure satisfies a scalar wave equation

$$\ddot{p} - c_1^2 p_{,11} - c_2^2 p_{,22} = 0, \tag{7}$$

where $c_j = \sqrt{\kappa/\rho_j}$, $j=1,2$. Equations (5) and (7) are discussed in greater detail and generality in Sec. V, but for the moment we cite two results necessary for finding the metafluid in Fig. 3: the phase speed in direction $\mathbf{n}=n_1\mathbf{e}_1+n_2\mathbf{e}_2$ is v, and the associated wave or group velocity vector is $\mathbf{w}$, where

$$v^2 = c_1^2 n_1^2 + c_2^2 n_2^2, \tag{8a}$$

$$\mathbf{w} = v^{-1}(c_1^2 n_1\mathbf{e}_1 + c_2^2 n_2\mathbf{e}_2). \tag{8b}$$

2. Solution of the 2D mirage problem

The travel time is $a/\mathbf{w}\cdot\mathbf{e}_1$, and the impedance is now $Z=\rho_1 v/\cos\theta$, where, referring to Fig. 3, $n_1=\cos\theta$ and $n_2=\sin\theta$. Hence, the conditions for zero reflectivity and equal travel times are

$$\frac{\rho_1 v}{\cos\theta} = \frac{\rho_0 c_0}{\cos\theta_0}, \tag{9a}$$

$$\frac{av}{c_1^2\cos\theta} = \frac{b}{c_0\cos\theta_0}. \tag{9b}$$

Dividing the latter two relations implies κ is given by Eq. (2b). Snell–Descartes' law, $v^{-1}\sin\theta = c_0^{-1}\sin\theta_0$, combined with Eq. (9a), yields that

$$\frac{c_0^2}{v^2} = \frac{\rho_1^2}{\rho_0^2} + \left(1 - \frac{\rho_1^2}{\rho_0^2}\right)\sin^2\theta_0, \tag{10}$$

while Snell–Descartes' law together with Eq. (8a) implies

$$\frac{c_0^2}{v^2} = \frac{c_0^2\rho_1}{\kappa} + \left(1 - \frac{\rho_1}{\rho_2}\right)\sin^2\theta_0. \tag{11}$$

Comparison of Eqs. (10) and (11) implies two identities for ρ_1 and ρ_2. In summary, the three parameters of the modified region are

$$\kappa = \frac{a}{b}\kappa_0, \quad \rho_1 = \frac{b}{a}\rho_0, \quad \rho_2 = \frac{a}{b}\rho_0. \tag{12}$$

These give the desired result: no reflection and the same travel time for any angle of incidence. The metafluid layer faithfully mimics the wave properties of the original layer as observed from exterior vantage points.

3. Anisotropic stiffness

An alternative solution to the quandary raised by Eq. (4) is to keep the density isotropic but to relax the standard isotropic constitutive relation between stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon} = [\nabla\mathbf{u} + (\nabla\mathbf{u})^t]/2$ to allow for material anisotropy. Thus, the standard relation $\boldsymbol{\sigma}=-p\mathbf{I}$ with $p=\kappa\,\mathrm{tr}\,\boldsymbol{\varepsilon}$ is replaced by the stress-strain relation for *pentamode materials*[10]

$$\boldsymbol{\sigma} = \kappa(\mathbf{Q}:\boldsymbol{\varepsilon})\mathbf{Q}, \tag{13a}$$

$$\mathrm{div}\,\mathbf{Q} = 0. \tag{13b}$$

The physical meaning of the symmetric second order tensor $\mathbf{Q}$ is discussed later within the context of a more general constitutive theory. The requirement $\mathrm{div}\,\mathbf{Q}=0$ was first noted by Norris[12] and is discussed in Sec. V. Standard acoustics corresponds to $\mathbf{Q}=\mathbf{I}$.

Rewriting Eq. (13a) as $\boldsymbol{\sigma}=-p\mathbf{Q}$ and using the divergence free property of $\mathbf{Q}$, the equation of motion and the constitutive relation in the metafluid are now

$$\rho\dot{\mathbf{v}} = -\mathbf{Q}\nabla p, \quad \dot{p} = -\kappa\mathbf{Q}:\nabla\mathbf{v}. \tag{14}$$

Eliminating $\mathbf{v}$ yields the scalar wave equation

$$\ddot{p} - \kappa\mathbf{Q}:\nabla(\rho^{-1}\mathbf{Q}\nabla p) = 0. \tag{15}$$

General properties of this equation have been discussed by Norris[12] and will be examined later in Sec. V. For the purpose of the problem in Fig. 3, $\mathbf{Q}$ is assumed constant of the form

$$\mathbf{Q} = \begin{bmatrix}Q_1 & 0\\ 0 & Q_2\end{bmatrix}, \tag{16}$$

then it follows that the phase speed and group velocity in direction $\mathbf{n}$ are

$$v^2 = C_1^2 n_1^2 + C_2^2 n_2^2, \quad \mathbf{w} = v^{-1}(C_1^2 n_1\mathbf{e}_1 + C_2^2 n_2\mathbf{e}_2), \tag{17}$$

where $C_j = Q_j\sqrt{\kappa/\rho}$, $j=1,2$. Proceeding as before, using Snell–Descartes' law and the conditions of equal travel time and matched impedance yields

$$Q_1^2 = \frac{a}{b}\frac{\kappa_0}{\kappa}, \quad Q_2^2 = \frac{b}{a}\frac{\rho}{\rho_0}, \quad Q_1 Q_2 = \frac{b}{a}.$$

Since the important physical quantity is the product of κ with $\mathbf{Q}\otimes\mathbf{Q}$, any one of the three parameters κ, Q_1, and Q_2, may be independently selected. A natural choice is to impose $\operatorname{div}\mathbf{Q}=0$ at the interface, which means that the "traction" vector $\mathbf{Qn}$ is continuous, where $\mathbf{n}$ is the interface normal. In this case $\mathbf{n}=\mathbf{e}_1$ so that $Q_1=1$ and

$$\rho = \frac{b}{a}\rho_0, \quad \kappa = \frac{a}{b}\kappa_0, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{b}{a} \end{bmatrix}. \tag{18}$$

Comparing the alternative metafluids defined by Eqs. (12) and (18) note that in each case the density and the stiffness associated with the normal $\mathbf{e}_1$ direction both equal their 1D values given by Eq. (2). The first metafluid of Eq. (12) has a smaller inertia in the transverse direction $\mathbf{e}_2(\rho_2<\rho_1)$. The second metafluid defined by Eq. (18) has increased stiffness in the transverse direction $(Q_2>Q_1)$. The net effect in each case is an increased phase speed in the transverse direction as compared with the normal direction $(c_2>c_1, C_2>C_1)$.

The 1D and 2D mirage examples illustrate the general idea of acoustic metafluids as fluids that replicate the wave properties of a transformed region. However, the methods used to find the metafluids are not easily generalized to arbitrary regions. How does one find the metafluid that can, for instance, mimic a full spherical region by a smaller shell? This is the cloaking problem. The key to the generalized procedure are the related notions of transformation and finite deformation, which are introduced next.

III. THE TRANSFORMATION METHOD

A. Preliminaries

Let Ω and ω denote the original and the deformed domains (the regions $0\le X<b$ and $b-a\le x<b$ in the examples of Sec. II. The coordinates in each configuration are $\mathbf{X}$ and $\mathbf{x}$, respectively; the divergence operators are Div and div, and the gradient (nabla) operators are $\bar{\nabla}$ and ∇. The upper and lower case indices indicate components, X_I, x_i and the component form of $\operatorname{div}\mathbf{A}$ is $\partial A_i/\partial x_i = A_{i,i}$ or $\partial A_{iJ}/\partial x_i = A_{iJ,i}$ when $\mathbf{A}$ is a vector and a second order tensorlike quantity, respectively, and repeated indices imply summation (case sensitive). Similarly $\operatorname{Div}\mathbf{B}=B_{\mathrm{Ij},I}$.

The finite deformation or transformation is defined by the mapping $\Omega\to\omega$ according to $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X})$. In the terminology of finite elasticity $\mathbf{X}$ describes a particle position in the Lagrangian or undeformed configuration, and $\mathbf{x}$ is particle location in the Eulerian or deformed physical state. The transformation $\boldsymbol{\chi}$ is assumed to be one-to-one and invertible.[17] The deformation gradient is defined $\mathbf{F}=\bar{\nabla}\mathbf{x}$ with inverse $\mathbf{F}^{-1}=\nabla\mathbf{X}$, or in component form $F_{iI}=\partial x_i/\partial X_I$, $F_{Ii}^{-1}=\partial X_I/\partial x_i$. The Jacobian of the finite deformation is $\Lambda=\det\mathbf{F}=|\mathbf{F}|$, or in terms of volume elements in the two configurations, $\Lambda = dv/dV$. The polar decomposition implies $\mathbf{F}=\mathbf{VR}$, where the rotation $\mathbf{R}$ is proper orthogonal ($\mathbf{RR}^t=\mathbf{R}^t\mathbf{R}=\mathbf{I}$, $\det\mathbf{R}=1$) and the left stretch tensor $\mathbf{V}\in\mathrm{Sym}^+$ is the positive definite solution of $\mathbf{V}^2=\mathbf{FF}^t$. Note for later use the kinematic identities[18]

$$\operatorname{div}(\Lambda^{-1}\mathbf{F}) = 0, \tag{19a}$$

$$\operatorname{Div}(\Lambda\mathbf{F}^{-1}) = 0, \tag{19b}$$

and the expression for the Laplacian in $\mathbf{X}$ in terms of derivatives in $\mathbf{x}$, i.e., the chain rule[8]

$$\bar{\nabla}^2 f = \Lambda\,\operatorname{div}(\Lambda^{-1}\mathbf{V}^2\,\nabla f). \tag{20}$$

B. The transformation method

The undeformed domain Ω is of arbitrary shape and comprises a homogeneous acoustic fluid with density ρ_0 and bulk modulus κ_0. The goal is to mimic the scalar wave equation in Ω,

$$\ddot{p} - (\kappa_0/\rho_0)\bar{\nabla}^2 p = 0, \quad \mathbf{X}\in\Omega, \tag{21}$$

by the wave equation of a metafluid occupying the deformed region ω. The basic result[6,8,19] is that Eq. (21) is exactly replicated in ω by the equation

$$\ddot{p} - \kappa\,\operatorname{div}(\boldsymbol{\rho}^{-1}\,\nabla p) = 0, \quad \mathbf{x}\in\omega, \tag{22}$$

where the bulk modulus and inertia tensor are

$$\kappa = \kappa_0\Lambda, \quad \boldsymbol{\rho} = \rho_0\Lambda\mathbf{V}^{-2}. \tag{23}$$

The equivalence of Eq. (21) with Eqs. (22) and (23) is evident from the differential equality (20). The idea is to use the change in variables from $\mathbf{X}$ to $\mathbf{x}$ to identify the metafluid properties. Equation (23) describes a metafluid with anisotropic inertia and isotropic elasticity. It can be used for modeling acoustic cloaks but has the unavoidable feature that the total effective mass of the cloak is infinite. This problem, discussed by Norris,[12] arises from the singular nature of the finite deformation in a cloak which makes $\Lambda\mathbf{V}^{-2}$ nonintegrable. This type of fluid, which could be called an inertial fluid, appears to be the main candidate considered for acoustic cloaking to date. The major exception is Milton *et al.*[13] who considered fluids with properties of pentamode materials, although as we will discuss in Sec. IV, their findings are of limited use for acoustic cloaking.

1. Pentamode materials

Norris[12] showed that Eq. (20) is a special case of a more general identity:

$$\bar{\nabla}^2 p = \Lambda\mathbf{Q}:\boldsymbol{\nabla}(\Lambda^{-1}\mathbf{Q}^{-1}\mathbf{V}^2\,\boldsymbol{\nabla} p), \tag{24}$$

where $\mathbf{Q}$ is *any* symmetric, invertible, and divergence-free ($\operatorname{div}\mathbf{Q}=0$) second order tensor. The increased degrees of freedom afforded by the arbitrary nature of $\mathbf{Q}$ means that Eq. (21) is equivalent to the generalized scalar wave equation in ω,

$$\ddot{p} - \kappa \mathbf{Q}:\nabla(\boldsymbol{\rho}^{-1}\mathbf{Q}\,\nabla\, p) = 0, \quad \mathbf{x} \in \omega, \tag{25}$$

where the modulus κ and the inertia follow from a comparison of Eqs. (21), (24), and (25) as

$$\kappa = \kappa_0 \Lambda, \quad \boldsymbol{\rho} = \rho_0 \Lambda \mathbf{Q}\mathbf{V}^{-2}\mathbf{Q}. \tag{26}$$

As will become apparent later, these metafluid parameters describe a pentamode material with anisotropic inertia. For the moment we return to the acoustic mirages in light of the general transformation method.

C. Mirages revisited

The 2D mirage problem corresponds to the following finite deformation $x_1 = b - a + ab^{-1}X_1$, $x_2 = X_2$ for $0 \leqslant X_1 < b$, $-\infty < X_2 < \infty$. The deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \frac{a}{b} & 0 \\ 0 & 1 \end{bmatrix}, \tag{27}$$

implying $\mathbf{R} = \mathbf{I}$, $\mathbf{V} = \mathbf{F}$ and $\Lambda = a/b$. Equation (23) therefore implies

$$\kappa = \frac{a}{b}\kappa_0, \quad \boldsymbol{\rho} = \begin{bmatrix} \frac{b}{a}\rho_0 & 0 \\ 0 & \frac{a}{b}\rho_0 \end{bmatrix}. \tag{28}$$

These agree with the parameters found in Sec. II, Eq. (12).

Using the more general formulation of Eqs. (25) and (26) with the arbitrary tensor chosen as $\mathbf{Q} = \Lambda^{-1}\mathbf{V}$ yields the metafluid described by Eq. (18). It is interesting to note that although ρ of Eq. (26) is, in general, anisotropic, it can be made isotropic in this instance by any $\mathbf{Q}$ proportional to $\mathbf{V}$. Keeping in mind the requirement seen above that $Q_1 = 0$, we consider as a second example of Eq. (25) the case $\mathbf{Q} = \Lambda^{-1}\mathbf{V}$. The mirage can then be achieved with material properties

$$\rho = \frac{b}{a}\rho_0, \quad \kappa = \frac{a}{b}\kappa_0, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{b}{a} \end{bmatrix}. \tag{29}$$

This again corresponds to a pentamode material with isotropic inertia, equal to that of Eq. (18). These two examples illustrate the power associated with the arbitrary nature of the divergence-free tensor $\mathbf{Q}$. There appears to be a multiplicative degree of freedom associated with $\mathbf{Q}$ that is absent using anisotropic inertia. As will be evident later, this degree of freedom is related to a *gauge transformation*.

IV. THE MOST GENERAL TYPE OF ACOUSTIC METAFLUID

A. Summary of main result

In order to make it easier for the reader to assimilate, the paper's main result is first presented in the form of a theorem. In the following, $\mathbf{Q}_0$ and $\mathbf{Q}$ are arbitrary symmetric, invertible, and divergence-free second order tensors.

Theorem 1. *The kinetic and strain energy densities in* Ω *of the form*

$$T_0 = \dot{\mathbf{U}}^t \boldsymbol{\rho}_0 \dot{\mathbf{U}}, \quad W_0 = \kappa_0 (\mathbf{Q}_0:\bar{\boldsymbol{\nabla}}\mathbf{U})^2, \tag{30}$$

respectively, are equivalent to the current energy densities in ω:

$$T = \dot{\mathbf{u}}^t \boldsymbol{\rho} \dot{\mathbf{u}}, \quad W = \kappa(\mathbf{Q}:\nabla\mathbf{u})^2, \tag{31}$$

where

$$\kappa = \Lambda \kappa_0, \tag{32a}$$

$$\boldsymbol{\rho} = \Lambda \mathbf{Q}\mathbf{F}^{-t}\mathbf{Q}_0^{-1}\boldsymbol{\rho}_0\mathbf{Q}_0^{-1}\mathbf{F}^{-1}\mathbf{Q}. \tag{32b}$$

Discussion of the implications are given following the proof.

B. Gauge transformation

The energy functions per unit volume in the undeformed configuration, T_0 and W_0, depend on the infinitesimal displacement $\mathbf{U}$ in that configuration. The kinetic energy is defined by the density $\boldsymbol{\rho}_0$, while the strain energy is $W_0 = C_{0IJKL}U_{J,I}U_{L,K}$, where C_{0IJKL} are elements of the stiffness. The density and stiffness possess the symmetries $\boldsymbol{\rho}_{0IJ} = \rho_{0JI}$, $C_{0IJKL} = C_{0KLIJ}$, $C_{0IJKL} = C_{0JIKL}$. The total energy is $E_0 = T_0 + W_0$ and the total energy $E = T + W$ per unit deformed volume is, using $E_0 \mathrm{d}V = E \mathrm{d}v$, simply $E = \Lambda^{-1}E_0$.

Our objective is to find a general class of material parameters $\{\boldsymbol{\rho}_0, \mathbf{C}_0\}$ that maintain the structure of the energy under a general transformation χ. Structure here means that the energy remains quadratic in velocity and strain. In order to achieve the most general form for the transformed energy, introduce a gauge transformation for the displacement. Let

$$\mathbf{U} = \mathbf{G}^t \mathbf{u}, \tag{33}$$

or $U_I = u_i G_{iI}$ in components, where $\mathbf{G}$ is independent of time but can be spatially varying. Thus, using the chain rule $U_{J,I} = U_{J,i}F_{iI}$ yields $E = T + W$, where the kinetic and strain energy densities are

$$T = \dot{\mathbf{u}}^t \boldsymbol{\rho} \dot{\mathbf{u}}, \tag{34a}$$

$$W = \Lambda^{-1} C_{IJKL} F_{iI} F_{kK} (G_{jJ}u_j)_{,i} (G_{lL}u_l)_{,k}, \tag{34b}$$

and the transformed inertia is

$$\boldsymbol{\rho} = \Lambda^{-1}\mathbf{G}\boldsymbol{\rho}_0\mathbf{G}^t. \tag{35}$$

The kinetic energy has the required structure, quadratic in the velocity; the strain energy, however, is not in the desired form. The objective is to obtain a strain energy of standard form

$$W = C_{ijkl}u_{j,i}u_{l,k}, \tag{36}$$

where $\mathbf{C}$ has the usual symmetries: $C_{ijkl} = C_{klij}$, $C_{ijkl} = C_{jikl}$.

The question of how to convert W of Eq. (34b) into the form of Eq. (36) for generally anisotropic elasticity stiffness $\mathbf{C}_0$ will be discussed in a separate paper. The objective here is to find the largest class of metafluids that includes all those previously found.

C. Pentamode to pentamode

In order to proceed assume that the initial stiffness tensor is of pentamode form[15]

$$\mathbf{C}_0 = \kappa_0 \mathbf{Q}_0 \otimes \mathbf{Q}_0, \tag{37a}$$

$$\text{Div } \mathbf{Q}_0 = 0, \tag{37b}$$

that is, $C_{0IJKL}=\kappa_0 Q_{0IJ}Q_{0KL}$, where $\mathbf{Q}_0^t=\mathbf{Q}_0$ is a positive definite symmetric second order tensor. The tensor $\mathbf{Q}_0$ is necessarily divergence-free.[12]

The strain energy density in the physical space after the general deformation and gauge transformation is now

$$W = \kappa_0 \Lambda^{-1}[Q_{0IJ}(u_j G_{jJ})_{,I}]^2. \tag{38}$$

Consider

$$Q_{0IJ}(u_j G_{jJ})_{,I} = Q_{0IJ}G_{jJ}u_{j,I} + Q_{0IJ}G_{jJ,I}u_j. \tag{39}$$

In order to achieve the quadratic structure of Eq. (36) the final term in Eq. (39) must vanish. Since $\mathbf{u}$ is considered arbitrary this in turn implies that $Q_{0IJ}G_{jJ,I}$ must vanish for all j. With no loss in generality let

$$\mathbf{G}^t = \mathbf{Q}_0^{-1}\mathbf{P}, \tag{40}$$

or $G_{jJ}=Q_{0JM}^{-1}P_{Mj}$ in components. Then using the identity for the derivative of a second order tensor, $(\mathbf{A}^{-1})_{,\alpha}=-\mathbf{A}^{-1}\mathbf{A}_{,\alpha}\mathbf{A}^{-1}$, where α can be any parameter, gives

$$\begin{aligned} 0 &= Q_{0IJ}G_{jJ,I} = -Q_{0IJ}Q_{0JK}^{-1}Q_{0KL,I}Q_{0LM}^{-1}P_{MJ} + P_{Ij,I} \\ &= -Q_{0IL,I}Q_{0LM}^{-1}P_{Mj} + P_{Ij,I} = P_{Ij,I}, \end{aligned} \tag{41}$$

where the important property (37b) has been used. Hence, Div $\mathbf{P}=0$. Then using Eq. (39) yields

$$Q_{0IJ}(u_j G_{jJ})_{,I} = P_{Ij}u_{j,I} = P_{Ij}F_{iI}u_{j,i} = \Lambda Q_{ij}u_{j,i}, \tag{42}$$

where the tensor $\mathbf{Q}$ is defined by

$$\mathbf{P} = \Lambda \mathbf{F}^{-1}\mathbf{Q} \Leftrightarrow \mathbf{Q} = \Lambda^{-1}\mathbf{F}\mathbf{P}. \tag{43}$$

The condition Div $\mathbf{P}=0$ becomes, using Eq. (19b),

$$P_{Ij,I} = (\Lambda F_{Ii}^{-1}Q_{ij})_{,I} = \Lambda F_{Ii}^{-1}Q_{ij,I} = \Lambda Q_{ij,i}, \tag{44}$$

implying

$$\text{div } \mathbf{Q} = 0. \tag{45}$$

It has therefore been demonstrated that if the gauge transformation is defined by

$$\mathbf{G}^t = \Lambda \mathbf{Q}_0^{-1}\mathbf{F}^{-1}\mathbf{Q}, \tag{46}$$

where $\mathbf{Q}$ is divergence free in physical coordinates, then the strain energy (38) is $W=\kappa_0\Lambda(Q_{ij}u_{j,i})^2$. This is of the desired form, Eq. (36), with $C_{ijkl}=\kappa Q_{ij}Q_{kl}$, hence completing the proof of Theorem 1.

D. Discussion

1. Equivalence of physical quantities

Theorem 1 states that the pentamode material defined by stiffness κ_0 with stresslike tensor $\mathbf{Q}_0$ and anisotropic inertia $\boldsymbol{\rho}_0$ is converted into another pentamode material with anisotropic inertia. The properties of the new metafluid are defined by the original metafluid and the deformation-gauge pair $\{\mathbf{F},\mathbf{G}\}$, where $\mathbf{F}$ is arbitrary and possibly inhomogeneous, and $\mathbf{G}$ is given by Eq. (46) with $\mathbf{Q}$ symmetric, positive definite, and divergence-free but otherwise completely arbitrary. The special case of a fluid with isotropic stiffness but anisotropic inertia, Eq. (23), is recovered from Theorem 1 when the starting medium is a standard acoustic fluid and $\mathbf{Q}$ is taken to be $\mathbf{I}$.

It is instructive to examine how physical quantities transform: we consider displacement, momentum, and pseudopressure. Eliminating $\mathbf{G}$, it is possible to express the new displacement vector in terms of the original,

$$\mathbf{u} = \Lambda^{-1}\mathbf{Q}^{-1}\mathbf{F}\mathbf{Q}_0\mathbf{U}. \tag{47}$$

Physically, this means that as the metafluid in ω acoustically replicates that in Ω, particle motion in the latter is converted into the mimicked motion defined by Eq. (47).

Define the momentum vectors in the two configurations,

$$\mathbf{m}_0 = \boldsymbol{\rho}_0\dot{\mathbf{U}}, \quad \mathbf{m} = \boldsymbol{\rho}\dot{\mathbf{u}}. \tag{48}$$

Equations (35) and (46) imply that they are related by

$$\mathbf{m} = \mathbf{Q}\mathbf{F}^{-t}\mathbf{Q}_0^{-1}\mathbf{m}_0. \tag{49}$$

The transformation of momentum is similar to that for displacement, Eq. (47), but with the inverse tensor, i.e., $\mathbf{u}=\mathbf{G}^{-t}\mathbf{U}$ while $\mathbf{m}=\Lambda^{-1}\mathbf{G}\mathbf{m}_0$.

Stress in the two configurations is defined by Hooke's law in each:

$$\boldsymbol{\sigma}_0 = \mathbf{C}_0 : \bar{\nabla}\mathbf{U}, \quad \boldsymbol{\sigma} = \mathbf{C} : \nabla\mathbf{u}, \tag{50}$$

where $\mathbf{C}_0$ and $\mathbf{C}$ are the fourth order elasticity tensors,

$$\mathbf{C}_0 = \kappa_0\mathbf{Q}_0 \otimes \mathbf{Q}_0, \quad \mathbf{C} = \kappa\mathbf{Q} \otimes \mathbf{Q}, \tag{51}$$

that is, $C_{ijkl}=\kappa Q_{ij}Q_{kl}$, etc. Using Eqs. (37), (42), and (51) yields

$$\boldsymbol{\sigma}_0 = -p\mathbf{Q}_0, \tag{52a}$$

$$\boldsymbol{\sigma} = -p\mathbf{Q}, \tag{52b}$$

where p is the same in each configuration,

$$p = -\kappa\mathbf{Q} : \nabla\mathbf{u} = -\kappa_0\mathbf{Q}_0 : \bar{\nabla}\mathbf{U}. \tag{53}$$

The quantity p is similar to pressure, and can be exactly identified as such when $\mathbf{Q}$ is diagonal, but it is not pressure in the usual meaning. For this reason it is called the pseudopressure. It is interesting to compare the equal values of p in Ω and ω with the more complicated relations (47) and (49) for the displacement and momentum.

2. Equations of motion

The equations of motion can be derived as the Euler–Lagrange equation for the Lagrangian $T-W$. A succinct form is as follows, in terms of the the momentum density $\mathbf{m}$ and the stress tensor $\boldsymbol{\sigma}$:

$$\dot{\mathbf{m}} = \nabla\boldsymbol{\sigma}. \tag{54}$$

The constitutive relation may be expressed as an equation for the pseudopressure p,[12]

$$\ddot{p} = -\kappa \mathbf{Q}:\nabla \ddot{\mathbf{u}}, \tag{55}$$

while Eqs. (53), (48), and (54) imply that the acceleration is

$$\ddot{\mathbf{u}} = -\boldsymbol{\rho}^{-1}\mathbf{Q}\,\nabla\, p. \tag{56}$$

Eliminating p between the last two equations implies that the displacement satisfies

$$\kappa \mathbf{Q}\mathbf{Q}:\nabla \ddot{\mathbf{u}} - \boldsymbol{\rho}\ddot{\mathbf{u}} = 0. \tag{57}$$

This is, as expected, the elastodynamic equation for a pentamode material with anisotropic inertia. Alternatively, eliminating $\ddot{\mathbf{u}}$ between Eqs. (55) and (56) yields a scalar wave equation for the pseudopressure,

$$\ddot{p} - \kappa \mathbf{Q}:\nabla(\boldsymbol{\rho}^{-1}\mathbf{Q}\,\nabla\, p) = 0. \tag{58}$$

This clearly reduces to the standard acoustic wave equation when $\mathbf{Q}=\mathbf{I}$ and $\boldsymbol{\rho}$ is isotropic.

3. Relation to the findings of Milton et al.

The present findings appear to contradict those of Milton *et al.*[13] who found the negative result that it is not in general possible to find a metafluid that replicates a standard acoustic medium after arbitrary finite deformation. However, their result is based on the assumption that $\mathbf{G}\equiv\mathbf{F}$ [their Eq. (2.2)]. Equation (46) implies that $\mathbf{Q}$ must then be

$$\mathbf{Q} = \Lambda^{-1}\mathbf{F}\mathbf{Q}_0\mathbf{F}^t. \tag{59}$$

Using Eqs. (19a) and (37b) yields $Q_{ij,i}=\Lambda^{-1}Q_{0IJ}F_{jJ,I}$. Hence, this particular $\mathbf{Q}$ can only satisfy the requirement (45) that $\mathrm{div}\,\mathbf{Q}=0$ if

$$Q_{0IJ}\frac{\partial^2 x_i}{\partial X_I \partial X_J} = 0. \tag{60}$$

Milton *et al.*[13] considered $\mathbf{Q}_0$ isotropic (diagonal), in which case Eq. (60) means that the only permissible finite deformations are harmonic, i.e., those for which $\bar{\nabla}^2\mathbf{x}=0$. In short, the negative findings of Milton *et al.*[13] are a consequence of constraining the gauge to $\mathbf{G}\equiv\mathbf{F}$, which in turn severely restricts the realizability of metafluids except under very limited types of transformation deformation. The main difference in the present analysis is the inclusion of the general gauge transformation which enables us to find metafluids under arbitrary deformation.

V. PROPERTIES OF ACOUSTIC METAFLUIDS

The primary result of the paper, summarized in Theorem 1, states that the class of acoustic metafluids is defined by the most general type of pentamode material with elastic stiffness $\kappa\mathbf{Q}\otimes\mathbf{Q}$ where $\mathrm{div}\,\mathbf{Q}=0$, and anisotropic inertia $\boldsymbol{\rho}$. We now examine some of the unusual physical properties, dynamic and static, to be expected in acoustic metafluids. Some of the dynamic properties have been discussed by Norris,[12] but apart from Milton and Cherkaev[15] no discussion of static effects has been given.

A. Dynamic properties: plane waves

Consider plane wave solutions for displacement of the form $\mathbf{u}(\mathbf{x},t)=\mathbf{q}e^{ik(\mathbf{n}\cdot\mathbf{x}-vt)}$, for $|\mathbf{n}|=1$ and constants $\mathbf{q}$, k, and v, and uniform metafluid properties. Nontrivial solutions satisfying the equations of motion (57) require

$$(\kappa(\mathbf{Q}\mathbf{n})\otimes(\mathbf{Q}\mathbf{n}) - \boldsymbol{\rho}v^2)\mathbf{q} = 0. \tag{61}$$

The acoustical or Christoffel[20] tensor $\kappa(\mathbf{Q}\mathbf{n})\otimes(\mathbf{Q}\mathbf{n})$ is rank one and it follows that of the three possible solutions for v^2, only one is not zero, the quasilongitudinal solution

$$v^2 = \kappa\mathbf{n}\cdot\mathbf{Q}\boldsymbol{\rho}^{-1}\mathbf{Q}\mathbf{n}, \quad \mathbf{q} = \boldsymbol{\rho}^{-1}\mathbf{Q}\mathbf{n}. \tag{62}$$

The slowness surface is an ellipsoid. The energy flux velocity[20] (or wave velocity or group velocity) is

$$\mathbf{w} = v^{-1}\kappa\mathbf{Q}\boldsymbol{\rho}^{-1}\mathbf{Q}\mathbf{n}. \tag{63}$$

$\mathbf{w}$ is in the direction $\mathbf{Q}\mathbf{q}$, and satisfies $\mathbf{w}\cdot\mathbf{n}=v$, a well known relation for generally anisotropic solids with isotropic density.

B. Static properties

1. Five easy modes

The static properties of acoustic metafluids are just as interesting, if not more so. Hooke's law (52b) is

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \tag{64}$$

where $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^t$ is strain and the stiffness $\mathbf{C}$ is defined by Eq. (50b). The strain energy is $W=\kappa(\mathbf{Q}:\boldsymbol{\varepsilon})^2$. Note that $\mathbf{C}$ is not invertible in the usual sense of fourth order elasticity tensors. If considered as a 6×6 matrix mapping strain to stress then the stiffness is rank one: it has only one nonzero eigenvalue. This means that there are five independent strains, each of which will produce zero stress and zero strain energy, hence the name *pentamode*.[15] The five "easy" strains are easily identified in terms of the principal directions and eigenvalues of $\mathbf{Q}$. Let

$$\mathbf{Q} = \lambda_1\mathbf{q}_1\mathbf{q}_1 + \lambda_2\mathbf{q}_2\mathbf{q}_2 + \lambda_3\mathbf{q}_3\mathbf{q}_3, \tag{65}$$

where $\{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}$ is an orthonormal triad. Three of the easy strains are pure shear: $\mathbf{q}_i\mathbf{q}_j+\mathbf{q}_j\mathbf{q}_i$, $i\neq j$ and the other two are $\lambda_3\mathbf{q}_2\mathbf{q}_2-\lambda_2\mathbf{q}_3\mathbf{q}_3$ and $\lambda_1\mathbf{q}_2\mathbf{q}_2-\lambda_2\mathbf{q}_1\mathbf{q}_1$. Any other zero-energy strain is a linear combination of these. Note that there is no relation analogous to Eq. (64) for strain in terms of stress because only the single "component" $\mathbf{Q}:\boldsymbol{\varepsilon}$ is relevant, i.e., energetic.

It is possible to write $\boldsymbol{\sigma}$ in the form (52) where $p=-\kappa\mathbf{Q}:\boldsymbol{\varepsilon}$. Under static load in the absence of body force choose κ such that $p=$const, or equivalently, Eq. (37b). The relevant strain component is then $\mathbf{Q}:\boldsymbol{\varepsilon}=-\kappa^{-1}p$ and the surface tractions supporting the body in equilibrium are $\mathbf{t}=\boldsymbol{\sigma}\cdot\mathbf{n}=-p\mathbf{Q}\cdot\mathbf{n}$. Figure 4 illustrates the tractions required to maintain a block of metafluid in static equilibrium. Note that the traction vectors act obliquely to the surface, implying that shear forces are necessary. Furthermore, the tractions are not of uniform magnitude. These properties are to be compared with a normal acoustic fluid which can be maintained in static equilibrium by constant hydrostatic pressure.

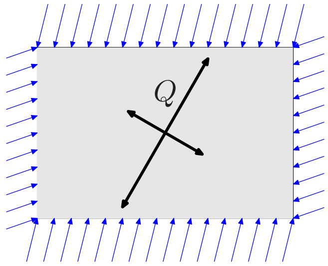

FIG. 4. (Color online) A rectangular block of metafluid is in static equilibrium under the action of surface tractions as shown. The two orthogonal arrows inside the rectangle indicate the principal directions of $\mathbf{Q}$ (30° from horizontal and vertical) and the relative magnitude of its eigenvalues (2:1). The equispaced arrows faithfully represent the surface loads.

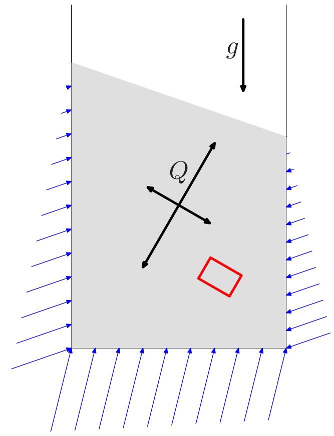

FIG. 5. (Color online) The same metafluid of Fig. 4 with isotropic density is in equilibrium under gravity. The upper surface is traction-free but nonhorizontal, an essential feature of metafluids. For this particular metafluid the top surface makes an angle of 19.11° with the horizontal. Also, the tractions on the horizontal lower boundary are inhomogeneous although parallel. The small rectangle is discussed in Fig. 6.

2. div $\mathbf{Q}=0$

The *gedanken* experiment of Fig. 4 also implies that $\mathbf{Q}$ has to be divergence-free. Thus, imagine a smoothly varying but inhomogeneous metafluid $\mathbf{C}=\kappa\mathbf{Q}\otimes\mathbf{Q}$ in static equilibrium under the traction $\mathbf{t}=-p\mathbf{Q}\cdot\mathbf{n}$ for constant p. The divergence theorem then implies div $\mathbf{Q}=0$ everywhere in the interior. This argument is a bit simplistic, but it provides the basis for a more rigorous proof.[12] Thus, stress in the metafluid must be of the form $-f\bar{\mathbf{Q}}$ where $\bar{\mathbf{Q}}$ is a scalar multiple of $\mathbf{Q}$. Local equilibrium requires div $f\bar{\mathbf{Q}}=0$, or $\nabla \ln f=-\bar{\mathbf{Q}}^{-1}$ div $\bar{\mathbf{Q}}$. This can be integrated to find f to within a constant. Now define $\mathbf{Q}=f\bar{\mathbf{Q}}$, and note that the tractions must be of the form $\mathbf{t}=-p\mathbf{Q}\cdot\mathbf{n}$ for constant p. The normalized $\mathbf{Q}$ is divergence-free.

3. Nonhorizontal free surface

Consider the same metafluid in equilibrium under a body force, e.g., gravity. Assuming the inertia is isotropic (see the comments about inertia at zero frequency in Sec. VI),

$$\text{div}\,\boldsymbol{\sigma}+\rho\mathbf{g}=0. \tag{66}$$

Use Eq. (52b) with div $\mathbf{Q}=0$ and the invertibility of $\mathbf{Q}$ implies

$$\nabla p=\rho\mathbf{Q}^{-1}\mathbf{g}. \tag{67}$$

For constant $\mathbf{Q}^{-1}\rho\mathbf{g}$ this can be integrated to give an explicit form for the pseudopressure,

$$p=(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{Q}^{-1}\rho\mathbf{g}, \tag{68}$$

where $\mathbf{x}_0$ is any point lying on the surface of zero pressure. Unlike normal fluids, the surface where $p=0$ does not have to be horizontal, see Fig. 5. The pseudopressure increases in the direction of $\mathbf{g}$, as in normal fluids. However, it is possible that p varies in the plane $\mathbf{x}\cdot\mathbf{g}=0$. For instance, the traction along the lower surface in Fig. 5 decreases in magnitude from left to right.

VI. METAFLUIDS WITH ISOTROPIC DENSITY

A. Necessary constraints on the finite deformation

The most practical case of interest is of course where the initial properties are those of a standard acoustic fluid with isotropic density and isotropic stress. The circumstances under which the mapped inertia $\boldsymbol{\rho}$ is also isotropic are now investigated. Acoustic metafluids with isotropic inertia are an important subset since it can be argued that achieving anisotropic inertia could be more difficult than the anisotropic elasticity. Indeed, the very concept of anisotropic inertia is meaningless at zero frequency, unlike anisotropic stiffness.

Assuming $\boldsymbol{\rho}_0=\rho_0\mathbf{I}$ and $\mathbf{Q}_0=\mathbf{I}$ then the current density becomes, using Eq. (32b) and the fact that $\mathbf{Q}$ is symmetric,

$$\boldsymbol{\rho}=\rho_0\Lambda\mathbf{Q}\mathbf{V}^{-2}\mathbf{Q}. \tag{69}$$

If $\boldsymbol{\rho}=\rho\mathbf{I}$ then Eq. (69) implies $\mathbf{Q}$ must be of the form

$$\mathbf{Q}=\rho^{1/2}(\rho_0\Lambda)^{-1/2}\mathbf{V}. \tag{70}$$

Thus, $\mathbf{Q}$ is proportional to the stretch tensor $\mathbf{V}$ and the coefficient of proportionality defines the current density.

It is not in general possible to choose $\mathbf{Q}$ in the form

$$\mathbf{Q}=\alpha\mathbf{V}, \tag{71}$$

where $\alpha\neq 0$. Certainly, $\mathbf{Q}$ of Eq. (71) is symmetric and invertible but not necessarily divergence-free. The latter condition requires div$(\alpha\mathbf{V})=0$ which in turn may be expressed $\nabla\ln\alpha=-\mathbf{V}^{-1}$ div $\mathbf{V}$. The necessary and sufficient condition that $\mathbf{V}^{-1}$ div $\mathbf{V}$ is the gradient of a scalar function, and hence α can be found which makes $\mathbf{Q}$ of Eq. (71) possible, is that $\mathbf{V}$ satisfy

$$\text{curl}\,\mathbf{V}^{-1}\,\text{div}\,\mathbf{V}=0. \tag{72}$$

This condition is not very useful. It does, however, indicate that the possibility of achieving isotropic $\boldsymbol{\rho}$ depends on the underlying finite deformation; there is a subset of general deformations that can yield isotropic inertia. The deformation gradient $\mathbf{F}$ has nine independent elements, $\mathbf{V}$ has six, and $\mathbf{R}$ has three. The condition (72) is therefore a differential constraint on six parameters. We now demonstrate an alternative statement of the condition div$(\alpha\mathbf{V})=0$ in terms of the rotation $\mathbf{R}$. This will turn out to be more useful, leading to general forms of potential deformation gradients.

Substituting $\mathbf{F}=\alpha^{-1}\mathbf{Q}\mathbf{R}$ into the identity (19a) and using Eq. (45) imply that

$$Q_{ij}(\Lambda^{-1}\alpha^{-1}R_{jK})_{,i}=0. \tag{73}$$

Using $\mathbf{Q}=\alpha\mathbf{F}\mathbf{R}^t$ and the identity $\mathbf{F}^t\nabla=\bar{\nabla}$ along with $\alpha\neq 0$ yield

$$R_{jM}(\Lambda^{-1}\alpha^{-1}R_{jK})_{,M}=0. \tag{74}$$

Then using the identity $(R_{jK}R_{jM})_{,M}=0$, Eq. (74) yields

$$\beta_{,K}=R_{jK}R_{jM,M}\leftrightarrow\bar{\nabla}\beta=\mathbf{R}^t\,\mathrm{Div}\,\mathbf{R}^t, \tag{75}$$

where $\beta=-\ln(\Lambda\alpha)$.

The necessary and sufficient condition that Eq.(75) can be integrated to find β is $\bar{\nabla}\wedge\bar{\nabla}\beta=0$, or using Eq. (75)

$$\mathrm{Curl}\,\mathbf{R}^t\,\mathrm{Div}\,\mathbf{R}^t=0. \tag{76}$$

The integrability condition (76) is in general not satisfied by $\mathbf{R}$, except in trivial cases. Norris[12] noted that isotropic density can be obtained if $\mathbf{R}=\mathbf{I}$. This corresponds to $\beta=$const, and it can be realized more generally if $\mathbf{R}$ is constant. Hence,

Lemma 1. *If the rotation* $\mathbf{R}$ *is constant then a normal acoustic fluid can be mapped to a unique metafluid with isotropic inertia:*

$$E_0=\kappa_0(\mathrm{Div}\,\mathbf{U})^2+\rho_0\dot{\mathbf{U}}\cdot\dot{\mathbf{U}}\quad in\ \Omega, \tag{77}$$

is equivalent to the current energy density

$$E=\lambda(\mathbf{V}:\nabla\mathbf{u})^2+\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\quad in\ \omega, \tag{78}$$

where

$$\lambda=\Lambda^{-1}\kappa_0,\quad \rho=\Lambda^{-1}\rho_0. \tag{79}$$

The total mass of the deformed region ω *is the same as the total mass contained in* Ω.

The parameter λ is used to distinguish it from $\kappa=\Lambda\kappa_0$, because in this case $\mathbf{Q}=\Lambda^{-1}\mathbf{V}$. Also, the displacement fields are related simply by $\mathbf{u}=\mathbf{R}\mathbf{U}$.

As an example of a deformation satisfying Lemma 1: $\mathbf{x}=f(\mathbf{X}\cdot\mathbf{A}\mathbf{X})\mathbf{A}\mathbf{X}$ for any constant positive definite symmetric $\mathbf{A}$. This type of finite deformation includes the important cases of radially symmetric cloaks. Thus, Norris[12] showed that radially symmetric cloaks can be achieved using pentamode materials with isotropic inertia.

B. General condition on the rotation

The results so far indicate that isotropic inertia is achievable for transformation deformations with constant rotation. We would, however, like to understand the broader implications of Eq. (76). The rotation can be expressed in Euler form

$$\mathbf{R}=\exp(\theta\,\mathrm{axt}\,\mathbf{a}), \tag{80}$$

where θ is the angle of rotation, the unit vector $\mathbf{a}$ is the rotation axis, and the axial tensor axt($\mathbf{a}$) is a skew symmetric tensor defined by $\mathrm{axt}(\mathbf{a})\mathbf{b}=\mathbf{a}\wedge\mathbf{b}$. The vector $\theta\mathbf{a}$ encapsulates the three independent parameters in $\mathbf{R}$. The integrability condition (76) is now replaced with a more explicit one in terms of $\theta(\mathbf{X})$ and $\mathbf{a}(\mathbf{X})$. It is shown in the Appendix that

$$\mathbf{R}^t\,\mathrm{Div}\,\mathbf{R}^t=\mathbf{a}\wedge\bar{\nabla}\theta+\mathbf{Z}, \tag{81}$$

where the vector $\mathbf{Z}$ follows from Eq. (A6). In particular, $\mathbf{Z}$ vanishes if the axis of rotation $\mathbf{a}$ is constant. In general, for arbitrary spatial dependence, Eq. (81) implies that the integrability condition (76) is equivalent to the following constraint on the rotation parameters:

$$\mathbf{a}\bar{\nabla}^2\theta-(\mathbf{a}\cdot\bar{\nabla})\bar{\nabla}\theta+(\bar{\nabla}\theta\cdot\bar{\nabla})\mathbf{a}-(\bar{\nabla}\cdot\mathbf{a})\bar{\nabla}\theta+\mathrm{Curl}\,\mathbf{Z}=0. \tag{82}$$

In summary,

Lemma 2. *If the rotation* $\mathbf{R}$ *satisfies* Eq. (76) *or equivalently, if* θ *and* $\mathbf{a}$ *satisfy the condition* (82), *then a normal acoustic fluid can be mapped to a unique metafluid with isotropic inertia according to* Eqs. (77) and (78) *with*

$$\lambda=\Lambda^{-1}e^{-2\beta}\kappa_0,\quad \rho=\Lambda^{-1}e^{-2\beta}\rho_0, \tag{83}$$

where the function β *is defined by* Eq. (75).

C. Simplification in 2D

The integrability condition (76) simplifies for the important general configuration of 2D spatial dependence. In this case $\mathbf{a}$ is constant, $\theta=\hat{\theta}(\mathbf{X}_\perp)$ where $\mathbf{X}_\perp\cdot\mathbf{a}=0$. Equation (82) then reduces to

$$\bar{\nabla}^2\hat{\theta}=0, \tag{84}$$

1. Example

Consider finite deformations with inhomogeneous rotation

$$\theta=\theta_0+\gamma X_1,\quad \mathbf{a}=\mathbf{e}_3, \tag{85}$$

for constants θ_0 and γ. This satisfies Eq. (84) and therefore Eq. (75) can be integrated. The metafluid in ω has isotropic density and pentamode stiffness given by Lemma 2, where $\beta=\gamma(X_2-X_{02})$. The constants θ_0 and X_{02} may be set to zero, with no loss in generality.

As an example of a deformation that has rotation of the form (85), consider deformation of the region $\Omega=[-\pi/2\gamma,\pi/2\gamma]\times[0,L]\times\mathbb{R}$ according to

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{bmatrix}A_{11}&A_{12}&0\\A_{12}&A_{22}&0\\0&0&\alpha\end{bmatrix}\begin{pmatrix}\gamma^{-1}\sin\gamma X_1e^{-\gamma X_2}\\\gamma^{-1}(1-\cos\gamma X_1e^{-\gamma X_2})\\X_3\end{pmatrix}, \tag{86}$$

where $\alpha>0$ and the 2×2 symmetric matrix $\mathbf{A}$ with elements A_{ij} is positive definite. The deformation gradient is $\mathbf{F}=\mathbf{V}(X_2)\mathbf{R}(X_1)$ with $\mathbf{V}=\mathbf{A}e^{-\gamma X_2}+\alpha\mathbf{e}_3\mathbf{e}_3$, and

$$\mathbf{R}=\begin{bmatrix}\cos\gamma X_1&-\sin\gamma X_1&0\\\sin\gamma X_1&\cos\gamma X_1&0\\0&0&1\end{bmatrix}. \tag{87}$$

The mapped metafluid is defined by the energy density in ω

$$E = \frac{1}{\alpha\,\det\mathbf{A}}[\kappa_0[(\mathbf{A}:\nabla\mathbf{u})^2 e^{-2\gamma X_2} + (\alpha u_{3,3})^2] + \rho_0 \dot{\mathbf{u}}\cdot\dot{\mathbf{u}}].$$

In particular, it has isotropic inertia.

This example is not directly applicable in modeling a complete acoustic cloak. However, it opens up the possibility of patching together metafluids with different local properties, each with isotropic inertia so that the entire cloak has isotropic mass density.

VII. SUMMARY AND CONCLUSION

Whether it is the simple 1D acoustic mirage of Fig. 2 or a three-dimensional acoustic cloak, we have seen how acoustic stealth can be achieved using the concept of domain transformation. The fluid in the transformed region exactly replicates the acoustical properties of the original domain. The most general class of material that describes both the mimic and the mimicked fluids is defined as an acoustic metafluid. A general procedure for mapping/transforming one acoustic metafluid to another has been described in this paper.

The results, particularly Theorem 1 in Sec. IV, show that acoustic metafluids are characterized by as few as two parameters $(\rho, \sqrt{\kappa})$ and as many as 12 $(\boldsymbol{\rho}, \sqrt{\kappa}\mathbf{Q})$. This broad class of materials can be described as pentamode materials with anisotropic inertia. It includes the restricted set of fluids with anisotropic inertia and isotropic stress $(\mathbf{Q}=\mathbf{I})$.

The arbitrary nature of the divergence-free tensor $\mathbf{Q}$ adds an enormous amount of latitude to the stealth problem. It may be selected in some circumstances to guarantee isotropic inertia in cloaking materials, examples of which are given elsewhere.[12] In this paper we have derived and described the most general conditions required for $\boldsymbol{\rho}$ to be isotropic. The conditions have been phrased in terms of the rotation part of the deformation, $\mathbf{R}$. If this is a constant then the cloaking metafluid is defined by Lemma 1. Otherwise the condition is Eq. (76) with the metafluid given by Lemma 2. The importance of being able to use metafluids with isotropic inertia should not be underestimated. Apart from the fact that it resolves questions of infinite total effective mass[12] isotropic inertia removes frequency bandwidth issues that would be an intrinsic drawback in materials based on anisotropic inertia.

This paper also describes, for the first time, some of the unusual physical features of acoustic metafluids. Strange effects are to be expected in static equilibrium, as illustrated in Figs. 4 and 5. These properties can be best understood through realization of acoustic metafluids, and a first step in that direction is provided by the type of microstructure depicted in Fig. 6. The macroscopic homogenized equations governing the microstructure are assumed in this paper to be those of normal elasticity. It is also possible that the large contrasts required in acoustic metafluids could be modeled with more sophisticated constitutive theories, such as nonlocal models or theories involving higher order gradients. There is considerable progress to be made in the modeling, design, and ultimate fabrication of acoustic metafluids.

In addition to the degrees of freedom associated with the tensor $\mathbf{Q}$, the properties of metafluids depend on the finite

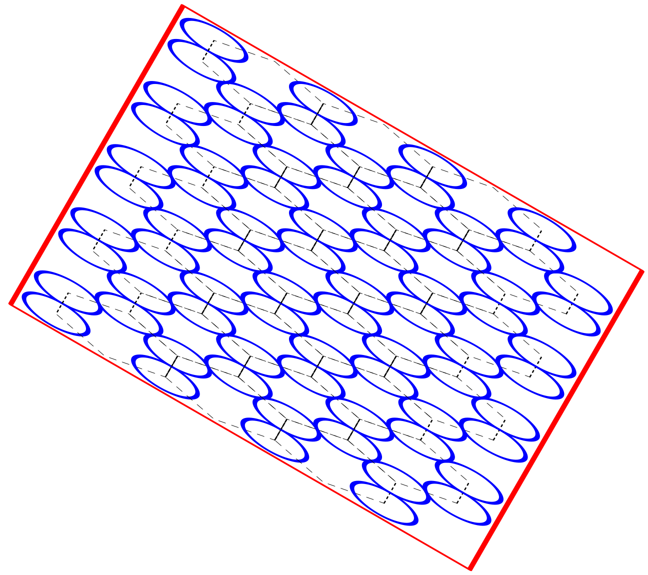

FIG. 6. (Color online) This shows a possible microstructure for a microscopic rectangular region of the metafluid in Figs. 4 and 5, see the latter. The sides of the rectangle are aligned with the principal axes of $\mathbf{Q}$. The metafluid is a pentamode material comprising a regular array of small beads such that each is in lubricated point contact with its three neighbors (2D). The dashed lines indicate the directions of the forces acting between the small oval-shaped beads. Although the structure as shown is unstable under shear, a realistic metafluid might contain some stabilizing mechanisms to enhance its rigidity. Details will be provided in a forthcoming paper.

deformation through $\mathbf{V}$. Even in the simple example of the 1D mirage, one could arrive at the lower picture in Fig. 2 through different finite deformations. This raises the question of how to best choose the nonunique deformation gradient $\mathbf{F}$. The present results indicate some strategies for choosing $\mathbf{F}$ to ensure the cloak inertia has isotropic mass, and the cloaking properties are in effect determined by the elastic pentamodal material. Li and Pendry[1] considered other optimal choices for the finite deformation. Combined with the enormous freedom afforded by the arbitrary nature of $\mathbf{Q}$, there are clearly many optimization strategies to be considered.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF EQUATION (81)

Equation (80) can be written in Euler–Rodrigues form.[21]

$$\mathbf{R} = \mathbf{I} + \sin\theta\mathbf{S} + (1-\cos\theta)\mathbf{S}^2, \tag{A1}$$

where

$$\mathbf{S} \equiv \operatorname{axt}(\mathbf{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{A2}$$

Equation (A1) can be used, for instance, to find $\theta = \cos^{-1}\left[\frac{1}{2}(1-\operatorname{tr}\mathbf{R})\right]$ and hence $\mathbf{a}$ from $\mathbf{S}=(\mathbf{R}-\mathbf{R}^t)/(2\sin\theta)$. For the sake of notational brevity, in the remainder of the Appendix $c=\cos\theta$ and $s=\sin\theta$.

Explicit differentiation of Eq. (A1) yields

$$R_{iJ,J}=(cS_{iJ}-sP_{iJ})\theta_{,J}+sS_{iJ,J}-(1-c)P_{iJ,J}, \tag{A3}$$

where $\mathbf{P}=-\mathbf{S}^2=\mathbf{I}-\mathbf{aa}$. Noting that $S_{iJ,J}=-(\bar{\nabla}\wedge\mathbf{a})_i$ and $P_{iJ,J}=-a_i\bar{\nabla}\cdot\mathbf{a}-\mathbf{a}\cdot\bar{\nabla}a_i$, implies

$$\begin{aligned}\text{Div}\,\mathbf{R}^t &= c\mathbf{a}\wedge\bar{\nabla}\theta-s\bar{\nabla}\wedge\mathbf{a}-s(\mathbf{I}-\mathbf{aa})\cdot\bar{\nabla}\theta+(1-c)\\ &\quad\times[\mathbf{a}(\bar{\nabla}\cdot\mathbf{a})+(\mathbf{a}\cdot\bar{\nabla})\mathbf{a}].\end{aligned} \tag{A4}$$

Multiplying by $\mathbf{R}^t$ using Eq. (A1) gives after some elimination and simplification,

$$\mathbf{R}^t\,\text{Div}\,\mathbf{R}^t=\mathbf{a}\wedge\bar{\nabla}\theta+\mathbf{Z}, \tag{A5}$$

where

$$\begin{aligned}\mathbf{Z}=&-s[c\mathbf{I}+(1-c)\mathbf{aa}]\cdot\bar{\nabla}\wedge\mathbf{a}+(1-c)\mathbf{a}(\bar{\nabla}\cdot\mathbf{a})-(1-c)[\mathbf{I}\\ &-(1-c)\mathbf{aa}]\cdot(\mathbf{a}\cdot\bar{\nabla})\mathbf{a}-s(1-c)\mathbf{a}\wedge(\mathbf{a}\cdot\bar{\nabla})\mathbf{a}.\end{aligned} \tag{A6}$$

[1] J. Li and J. B. Pendry, "Hiding under the carpet: A new strategy for cloaking," Phys. Rev. Lett. **101**, 203901 (2008).
[2] S. A. Cummer and D. Schurig, "One path to acoustic cloaking," New J. Phys. **9**, 45 (2007).
[3] H. Chen and C. T. Chan, "Acoustic cloaking in three dimensions using acoustic metamaterials," Appl. Phys. Lett. **91**(18), 183518 (2007).
[4] L.-W. Cai and J. Sánchez-Dehesa, "Analysis of Cummer-Schurig acoustic cloaking," New J. Phys. **9**, 450 (2007).
[5] S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, J. Pendry, M. Rahm, and A. Starr, "Scattering theory derivation of a 3D acoustic cloaking shell," Phys. Rev. Lett. **100**, 024301 (2008).
[6] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, "Comment on "Scattering theory derivation of a 3D acoustic cloaking shell," e-print arXiv:0801.3279v1.
[7] A. N. Norris, "Acoustic cloaking in 2D and 3D using finite mass," e-print arXiv:0802.0701.
[8] Y. Cheng, F. Yang, J. Y. Xu, and X. J. Liu, "A multilayer structured acoustic cloak with homogeneous isotropic materials," Appl. Phys. Lett. **92**, 151913 (2008).
[9] D. Torrent and J. Sánchez-Dehesa, "Anisotropic mass density by two-dimensional acoustic metamaterials," New J. Phys. **10**, 023004 (2008).
[10] D. Torrent and J. Sánchez-Dehesa, "Acoustic cloaking in two dimensions: A feasible approach," New J. Phys. **10**, 063015 (2008).
[11] H.-Y. Chen, T. Yang, X.-D. Luo, and H.-R. Ma, "The impedance-matched reduced acoustic cloaking with realizable mass and its layered design," Chin. Phys. Lett. **25**, 3696–3699 (2008).
[12] A. N. Norris, "Acoustic cloaking theory," Proc. R. Soc. London, Ser. A **464**, 2411–2434 (2008).
[13] G. W. Milton, M. Briane, and J. R. Willis, "On cloaking for elasticity and physical equations with a transformation invariant form," New J. Phys. **8**, 248–267 (2006).
[14] The name pentamode is based on the defining property that the material supports five easy modes of infinitesimal strain. Sec. V for details.
[15] G. W. Milton and A. V. Cherkaev, "Which elasticity tensors are realizable?," J. Eng. Mater. Technol. **117**, 483–493 (1995).
[16] U. Leonhardt and T. G. Philbin, "Transformation optics and the geometry of light," e-print arXiv:0805.4778.
[17] The bijective property of the mapping does not extend to acoustic cloaks, where there is a single point in Ω mapped to a surface in ω, see Ref. 12 for details.
[18] R. W. Ogden, *Non-Linear Elastic Deformations* (Dover, New York, 1997).
[19] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, "Full-wave invisibility of active devices at all frequencies," Commun. Math. Phys. **275**, 749–789 (2007).
[20] M. J. P. Musgrave, *Crystal Acoustics* (Acoustical Society of America, New York 2003).
[21] A. N. Norris, "Euler-Rodrigues and Cayley formulas for rotation of elasticity tensors," Math. Mech. Solids **3**, 243–260 (2008).