

Diffuse wave density and directionality in anisotropic solids

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Several general results are derived for diffuse waves in anisotropic solids, including concise expressions for the modal density per unit volume $d(\omega)$, and for the participation factor matrix $\mathbf{G}$. The latter is a second-order tensor which describes the orientational distribution of diffuse wave or reverberant energy, and reduces to the identity $\mathbf{I}$ under isotropy. Calculations of $\mathbf{G}$ for a variety of example materials show significant deviation from $\mathbf{I}$ even under moderate levels of anisotropy.

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I. INTRODUCTION

We consider how material anisotropy effects the directional partition of reverberant or diffuse wave energy. Diffuse waves in solids are the long time response when multiple scattering has equilibrated the energy distribution among modes. Preferential orientation of the root mean square particle velocity does not arise in isotropic materials but is a characteristic of anisotropy. Our objective is to describe this orientation effect and to quantify it in real materials. An ability to determine, directly or by inference, the orientational distribution of kinetic energy density in a solid allows one to essentially “hear” the texture of a crystal. We will demonstrate that the key quantity that needs to be measured is the autocorrelation function, or the Green’s function evaluated at its source. By deriving an explicit formula for the autocorrelation, or the admittance matrix, we can completely describe the directional distribution of the diffuse wave energy.

We introduce two quantities for the description of reverberant energy in the presence of anisotropy: the participation tensor $\mathbf{G}$ and the modal spectral density per unit volume, $d(\omega)$. The two are in fact intimately related as we will see. Under steady state time harmonic conditions the total energy of a body is equally divided between potential and kinetic. The latter is $\frac{1}{2}\omega^2 \int dV \rho |\bar{\mathbf{u}}|^2$ where $|\bar{\mathbf{u}}|$ is the root mean square particle displacement, and assuming a uniform spatial distribution, the total energy is $E = V\rho\omega^2 |\bar{\mathbf{u}}|^2$. This may be inverted to express the mean square displacement. Let $\bar{u}_i = |\bar{\mathbf{u}} \cdot \mathbf{e}_i|$ where $\mathbf{e}_i$, $i=1,2,3$ is an orthonormal triad. Since $\bar{u}_1^2 + \bar{u}_2^2 + \bar{u}_3^2 = |\bar{\mathbf{u}}|^2$ we may write

$$\bar{u}_i^2 = \frac{E}{3V\rho\omega^2} \bar{G}_i, \quad \bar{G}_i = \mathbf{e}_i \cdot \mathbf{G} \cdot \mathbf{e}_i, \quad (1)$$

for $i=1,2,3$ (no sum) where $\mathbf{G}$ is a second-order symmetric tensor satisfying

$$\text{tr } \mathbf{G} = 3. \quad (2)$$

For isotropic materials $\mathbf{G}$ is simply the unit matrix or identity (second-order) tensor. Deviations from this can occur under three general situations: (i) If the field point is near

a surface or boundary. This was considered in detail by Weaver,¹ who found expressions for the components of $\mathbf{G}$ at a free surface in terms of simple integrals, see also Egle.² (ii) By analogy, $\mathbf{G}$ will be influenced by local inhomogeneity in the material, for instance if the field point is close to a rigid inclusion, or a void. We will not discuss this further here. (iii) Material anisotropy can also influence $\mathbf{G}$. Here we consider the simplest case of a field point in a homogeneous material of infinite extent. It is expected that $\mathbf{G}$ displays the symmetries appropriate to the degree of anisotropy. Thus, it is characterized by a single parameter for materials with isotropic and cubic symmetries, and by two or three parameters for materials with lower symmetry.

The spectral density of modes D at frequency ω in a volume V is $D(\omega) = Vd(\omega)$. It can be estimated as $D = \partial N / \partial \omega \approx V\omega^2 / c^3$ by noting the total number of modes scales as $N(\mathbf{k}) \approx Vk^3$ where $k = \omega / c$ is typical wave number. A more precise counting yields, for isotropic bodies, the well-known result³

$$d(\omega) = \frac{\omega^2}{2\pi^2} \left(\frac{2}{c_l^3} + \frac{1}{c_t^3} \right), \quad (3)$$

where c_l and c_t are the longitudinal and transverse elastic wave speeds.

The objective is to derive analogous expressions of $d(\omega)$ and $\mathbf{G}$ for anisotropic elastic materials. This will be achieved by explicit calculation of the admittance tensor $\mathbf{A}$, defined in Sec. II, combined with a general relation between $d(\omega)$, $\mathbf{G}$, and $\mathbf{A}$. The spectral density and the participation tensor in the presence of material anisotropy do not appear to have received much attention. Some work on the related issue of admittance in bounded anisotropic thin plate systems has appeared.⁴ Weaver⁵ considered isotropic plates of finite thickness and infinite lateral extent. Tewary *et al.*⁶ derived an expression for the admittance at the free surface of an anisotropic half space as a double integral. Here the focus is on infinite systems, and the modal density per unit volume in this limit. Finite structures, such as plates both thin and of finite thickness, will be considered in a separate paper.

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TABLE I. The form of the participation tensor $\mathbf{G}$ for the different material symmetries. TI, tet, and trig are abbreviations for transverse isotropy, tetragonal, and trigonal symmetries, respectively. The $\mathbf{e}$ unit vectors are defined by the symmetry, while $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ result from averaging. The positive numbers α , β , and γ are constrained as indicated in order to satisfy Eq. (2).

$\mathbf{G}$	Material symmetry
$\mathbf{I}$	Isotropic, cubic
$\alpha\mathbf{e} \otimes \mathbf{e} + \beta(\mathbf{I} - \mathbf{e} \otimes \mathbf{e})$	TI, tet, trig $\alpha + 2\beta = 3$
$\alpha\mathbf{e}_1 \otimes \mathbf{e}_1 + \beta\mathbf{e}_2 \otimes \mathbf{e}_2 + \gamma\mathbf{e}_3 \otimes \mathbf{e}_3$	Orthotropic $\alpha + \beta + \gamma = 3$
$\alpha\mathbf{e} \otimes \mathbf{e} + \beta\mathbf{a} \otimes \mathbf{a} + \gamma\mathbf{b} \otimes \mathbf{b}$	Monoclinic $\alpha + \beta + \gamma = 3$
$\alpha\mathbf{a} \otimes \mathbf{a} + \beta\mathbf{b} \otimes \mathbf{b} + \gamma\mathbf{c} \otimes \mathbf{c}$	Triclinic $\alpha + \beta + \gamma = 3$

Our principal results are that the modal spectral density per unit volume and the participation tensor are given by

$$d(\omega) = \frac{\omega^2}{2\pi^2} \langle \text{tr } \mathbf{Q}^{-3/2} \rangle, \quad (4a)$$

$$\mathbf{G} = 3 \frac{\langle \mathbf{Q}^{-3/2} \rangle}{\langle \text{tr } \mathbf{Q}^{-3/2} \rangle}, \quad (4b)$$

where $\mathbf{Q}(\mathbf{n})$ is the acoustical or Christoffel tensor for plane waves propagating in the direction $\mathbf{n}$, and $\langle f \rangle$ is the orientation average of a function that depends on the direction,

$$\langle f \rangle \equiv \frac{1}{4\pi} \int_{4\pi} d\Omega(\mathbf{n}) f(\mathbf{n}). \quad (5)$$

In an isotropic solid Eq. (4a) reduces to Eq. (3) and $\mathbf{G}$ is simply the identity $\mathbf{I}$. After deriving Eq. (4), the remainder of the paper will explore its implications, in particular the form of $\mathbf{G}$ is investigated, and the parameters in Table I deduced. It is interesting to note that the material constant that determines the density of states of diffuse waves, $\text{tr} \langle \mathbf{Q}^{-3/2} \rangle$, also defines the Debye temperature Θ of a crystal. Thus (see Chap. 9 of Ref. 17),

$$\Theta = \frac{h}{k} \left(\frac{18\pi^2}{V_a \text{tr} \langle \mathbf{Q}^{-3/2} \rangle} \right)^{1/3}, \quad (6)$$

where h is Planck's constant, k is Boltzmann's constant, and V_a is the volume per atom or lattice site. Fedorov¹⁷ provides a detailed discussion of $\text{tr} \langle \mathbf{Q}^{-3/2} \rangle$ in this context. The emphasis in this paper is on the more general tensor $\langle \mathbf{Q}^{-3/2} \rangle$ although connections with Fedorov's analysis will be mentioned later.

The outline of the paper is as follows. The admittance tensor $\mathbf{A}$ is defined and calculated in Sec. II, from which the main result (4) follows. Several alternative representations of the fundamental quantity $\mathbf{Q}^{-3/2}$ are developed in Sec. III. In particular it is shown that $\mathbf{G}$ for transverse isotropy can be evaluated as a single integral. Weak anisotropy is considered in Sec. IV and numerical examples are presented in Sec. V.

II. DERIVATION OF d AND $\mathbf{G}$

A. Admittance tensor

The admittance $\mathbf{A}$ is a second-order tensor defined by the average power radiated by a time harmonic point force $\mathbf{F}$ according to

$$\Pi = \mathbf{F} \cdot \mathbf{A} \cdot \mathbf{F}. \quad (7)$$

Alternatively, $\mathbf{A}$ is equal to the power expended at the source point—which is the more conventional definition of admittance, as the the inverse of drive point impedance. The admittance is clearly related to the autocorrelation of the Green's function, and as such is a special case of the two-point cross correlation of the Green's function.⁷ The important connection for the present purposes is the relation between the radiation from a point force and the diffuse wave density.^{8,9} In the present notation this becomes

$$\mathbf{A} = \frac{\pi}{12\rho} d(\omega) \mathbf{G}. \quad (8)$$

A short derivation of Eq. (8) is given in Appendix A. The admittance of isotropic bodies is simply determined from Eq. (3) and $\mathbf{G} = \mathbf{I}$. Our objective here is to calculate $\mathbf{A}$ for anisotropic solids, and then to use the result to determine $d(\omega)$ and $\mathbf{G}$.

The central result for $\mathbf{A}$ is the following: The second-order symmetric admittance tensor of Eq. (7) that determines the total power radiated to infinity from the point source averaged over a period is

$$\mathbf{A} = \frac{\omega^2}{8\pi\rho} \langle \mathbf{Q}^{-3/2} \rangle, \quad (9)$$

where $\mathbf{Q}(\mathbf{n})$ is the acoustical tensor,

$$Q_{ik}(\mathbf{n}) = c_{ijkl} n_j n_l \text{ with } c_{ijkl} = \frac{1}{\rho} C_{klij}. \quad (10)$$

The elastic moduli (stiffness) C_{ijkl} have the symmetries $C_{ijkl} = C_{klij}$ and $C_{ijkl} = C_{jikl}$, and thus have at most 21 independent elements. Note that $\mathbf{A}$ has dimensions of admittance (inverse impedance). We next derive Eq. (9) by explicitly calculating the admittance for a time harmonic point force.

B. Radiation from a point force

The displacement resulting from a point force $\mathbf{F} \cos \omega t$ at the origin is $\mathbf{u}(\mathbf{x}, t) = \text{Re } \tilde{\mathbf{u}}(\mathbf{x}, \omega) e^{-i\omega t}$ where $\tilde{\mathbf{u}}$ satisfies

$$C_{ijkl} \tilde{u}_{k,jl} + \rho \omega^2 \tilde{u}_i = -F_i \delta(\mathbf{x}), \quad -\infty \leq x_1, x_2, x_3 \leq \infty.$$

Here ρ is the mass density and $\delta(\mathbf{x})$ is the three-dimensional Dirac delta function. The equation of motion may be written

$$\mathbf{Q}(\nabla) \tilde{\mathbf{u}} + \omega^2 \tilde{\mathbf{u}} = -\frac{1}{\rho} \delta(\mathbf{x}) \mathbf{F}, \quad (11)$$

and the problem definition is completed by the requirement that the energy radiates away from the point source.

The solution to Eq. (11) in a solid of infinite extent is well known. For our purpose we will find the following representation from Norris [Ref. 10, Eq. (3.22)] useful for determining the admittance:

$$\begin{aligned}\tilde{\mathbf{u}} = & \frac{1}{8\pi^2\rho|\mathbf{x}|} \oint d\theta(\mathbf{n}) \sum_{j=1}^3 \frac{\mathbf{q}_j \otimes \mathbf{q}_j}{\lambda_j} \mathbf{F} \\ & + \frac{1}{16\pi^2\rho} \int_{4\pi} d\Omega(\mathbf{n}) \sum_{j=1}^3 \frac{ik_j}{\lambda_j} \mathbf{q}_j \otimes \mathbf{q}_j \mathbf{F} e^{ik_j\mathbf{n}\cdot\mathbf{x}}.\end{aligned}\quad (12)$$

Here $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ the eigenvectors of $\mathbf{Q}(\mathbf{n})$, which then has the spectral decomposition

$$\mathbf{Q}(\mathbf{n}) = \lambda_1 \mathbf{q}_1 \otimes \mathbf{q}_1 + \lambda_2 \mathbf{q}_2 \otimes \mathbf{q}_2 + \lambda_3 \mathbf{q}_3 \otimes \mathbf{q}_3. \quad (13)$$

Also, $k_j = \omega/\lambda_j^{1/2}$ are the wave numbers of the three distinct branches of the slowness surface defined by the eigenvectors. The first integral in Eq. (12) is around the unit circle formed by the intersection of the plane $\mathbf{n}\cdot\mathbf{x}=0$ with the unit $\mathbf{n}$ sphere. This is just the static Green's function of elasticity.¹⁰ The important dynamic quantity is the second integral which is evaluated over the sphere $\{|\mathbf{n}|=1\}$. In order to make this more apparent, we rewrite Eq. (12) as

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}|_{\omega=0} + \frac{i\omega}{4\pi\rho} \sum_{j=1}^3 \left\langle e^{ik_j\mathbf{n}\cdot\mathbf{x}} \frac{\mathbf{q}_j \otimes \mathbf{q}_j}{\lambda_j^{3/2}} \right\rangle \mathbf{F}, \quad (14)$$

and note for future reference that the first term on the right-hand side is real valued.

The average power radiated per period is equal to the power expended by the force

$$\Pi = \lim_{x \rightarrow 0} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \cos \omega t \mathbf{F} \cdot \mathbf{v}(\mathbf{0}, t), \quad (15)$$

where $\mathbf{v}(\mathbf{x}, t) = \text{Re}(-i\omega \tilde{\mathbf{u}}(\mathbf{x}, \omega) e^{-i\omega t})$ is the particle velocity. Thus,

$$\Pi = \frac{\omega^2}{8\pi\rho} \sum_{j=1}^3 \left\langle \frac{1}{\lambda_j^{3/2}} (\mathbf{q}_j \cdot \mathbf{F})^2 \right\rangle. \quad (16)$$

The spectral decomposition (13) implies that

$$\lambda_1^{-3/2} \mathbf{q}_1 \otimes \mathbf{q}_1 + \lambda_2^{-3/2} \mathbf{q}_2 \otimes \mathbf{q}_2 + \lambda_3^{-3/2} \mathbf{q}_3 \otimes \mathbf{q}_3 = \mathbf{Q}^{-3/2},$$

which together with Eq. (7) proves the main result (9).

The scalar $d(\omega)$ and the tensor $\mathbf{G}$ are defined such that their product is $12\rho/\pi$ times the admittance $\mathbf{A}$, see Eqs. (3), (4), (8), and (9). This defines d and $\mathbf{G}$ to within a constant, which is determined uniquely by the constraint $\text{tr } \mathbf{G} = 3$. We therefore obtain the general results of Eq. (4). As discussed, d is the generalization of the classical density of states per unit volume, Eq. (3) for isotropic solids, and the participation factor tensor $\mathbf{G}$ describes the directional distribution of the energy at a point. While it is convenient to consider them separately, d and $\mathbf{G}$ are both defined by the averaged tensor $\langle \mathbf{Q}^{-3/2} \rangle$, which will be the focus of the remainder of the paper.

Before considering the properties of d and $\mathbf{G}$ we note that the isotropic modal density of states follows immediately from Eq. (4a). Starting with the acoustical tensor for an isotropic solid,

$$\mathbf{Q}(\mathbf{n}) = c_t^2 \mathbf{n} \otimes \mathbf{n} + c_l^2 (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \quad \text{isotropy}, \quad (17)$$

we have $\mathbf{Q}^{-3/2} = c_l^{-3} \mathbf{n} \otimes \mathbf{n} + c_t^{-3} (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$. Then using the fact that $\langle \mathbf{n} \otimes \mathbf{n} \rangle = \frac{1}{3} \mathbf{I}$ it follows that

$$\langle \mathbf{Q}^{-3/2} \rangle = \frac{1}{3} (c_l^{-3} + 2c_t^{-3}) \mathbf{I}. \quad (18)$$

Hence, the density of states per unit volume is $d = (\omega^2/2\pi^2) \times (c_l^{-3} + 2c_t^{-3})^{-1}$, in agreement with the well known identity (3), and $\mathbf{G} = \mathbf{I}$, as expected.

III. $\mathbf{Q}^{-3/2}$ AND RELATED QUANTITIES

The key quantity is the tensor $\mathbf{Q}^{-3/2}$ and its directional average. In practice, this may be evaluated numerically without difficulty. It is however useful to examine semiexplicit forms for the tensor, both for general anisotropy and for specific symmetries, particularly the case of transverse isotropy. We begin with two alternative and general formulations based on the spectral properties and the invariants of the acoustical tensor.

A. General representations for arbitrary anisotropy

1. A method based on invariants

Functions of a positive definite tensor can be simplified using the Cayley–Hamilton formula for the tensor, which for $\mathbf{Q}$ is

$$\mathbf{Q}^3 - I_1 \mathbf{Q}^2 + I_2 \mathbf{Q} - I_3 \mathbf{I} = 0. \quad (19)$$

The principal positive invariants of $\mathbf{Q}$ are

$$I_1 = \text{tr } \mathbf{Q}, \quad I_2 = \frac{1}{2} (\text{tr } \mathbf{Q})^2 - \frac{1}{2} \text{tr } \mathbf{Q}^2, \quad I_3 = \det \mathbf{Q}. \quad (20)$$

Based on these fundamental properties, it can be shown that

$$\begin{aligned}\mathbf{Q}^{-3/2} = & [(I_1 I_3 + i_1 i_3 I_2 + i_2 I_3) (\mathbf{Q}^2 - I_1 \mathbf{Q} + I_2 \mathbf{I}) \\ & + i_1 i_3 I_3 (\mathbf{Q} - I_1 \mathbf{I}) - I_3^2 \mathbf{I}] / [(i_1 i_2 - i_3) I_3^2],\end{aligned}\quad (21)$$

where i_1, i_2 , and i_3 are the positive invariants of $\mathbf{Q}^{1/2}$ which can be expressed as functions of the invariants I_1, I_2 , and I_3 , see the following. Details of the derivation of Eq. (21) are given in Appendix B.

The appealing feature of Eq. (21) for $\mathbf{Q}^{-3/2}(\mathbf{n})$ is that it only involves powers of $\mathbf{Q}$, its three invariants, and the additional invariants i_1, i_2 , and i_3 . These are related to I_1, I_2 , and I_3 by^{11,12}

$$i_1^2 - 2i_2 = I_1, \quad i_2^2 - 2i_1 i_3 = I_2, \quad i_3^2 = I_3. \quad (22)$$

The last implies $i_3 = I_3^{1/2}$, while expressions for i_1 and i_2 are given by Hoger and Carlson¹¹ and by Norris.¹² For instance,

$$i_1 = \sqrt{I_1 - \beta + 2\sqrt{I_3/\beta}} + \sqrt{\beta}, \quad (23a)$$

$$i_2 = \sqrt{I_2 - I_3/\beta + 2\sqrt{I_3/\beta}} + \sqrt{I_3/\beta}, \quad (23b)$$

$$i_3 = \sqrt{I_3}, \quad (23c)$$

where β is any eigenvalue of $\mathbf{Q}$, e.g.,

$$\begin{aligned}\beta = & \frac{1}{3} (I_1 + [(\xi + \sqrt{\xi^2 - (I_1^2 - 3I_2)^3}]^{1/3} \\ & + [(\xi - \sqrt{\xi^2 - (I_1^2 - 3I_2)^3}]^{1/3}),\end{aligned}\quad (24a)$$

$$\xi = \frac{1}{2} (2I_1^3 - 9I_1 I_2 + 27I_3). \quad (24b)$$

Note that¹³ $i_1 i_2 - i_3 = \det(i_1 \mathbf{I} - \mathbf{Q}^{1/2}) > 0$.

Taking the trace of Eq. (21) gives

$$\text{tr } \mathbf{Q}^{-3/2} = \frac{(I_1 + i_2)I_2 I_3 + (I_2^2 - 2I_1 I_3)i_1 i_3 - 3I_3^2}{(i_1 i_2 - i_3)I_3^2}. \quad (25)$$

This quantity, when averaged over all orientations, gives the density of states function $d(\omega)$ of Eq. (4a). Hence d can be calculated from the invariants $\mathbf{Q}$ and the derived invariants i_1, i_2, i_3 .

2. A spectral representation

The second form for $\mathbf{Q}^{-3/2}$ is based on the spectral decomposition (17). The latter can be expressed in a form that does not explicitly involve the eigenvectors,

$$\mathbf{Q}^{-3/2} = \lambda_1^{-3/2} \mathbf{N}(\lambda_1) + \lambda_2^{-3/2} \mathbf{N}(\lambda_2) + \lambda_3^{-3/2} \mathbf{N}(\lambda_3). \quad (26)$$

The second-order tensors $\mathbf{N}(\lambda_j)$, which are alternative expressions for the dyadics formed by the eigenvectors, $\mathbf{q}_j \otimes \mathbf{q}_j$, can be expressed in terms of $\mathbf{Q}$ using Sylvester's formula

$$\mathbf{N}(\lambda, \mathbf{n}) = \frac{\lambda \mathbf{Q}^2 + (\lambda - I_1) \lambda \mathbf{Q} + I_3 \mathbf{I}}{\lambda^3 + (\lambda - I_1) \lambda^2 + I_3}. \quad (27)$$

The identity (26) is derived in Appendix B.

Calculation of Eq. (26) requires knowledge of the three eigenvalues, which are zeros of the characteristic polynomial defined by Eq. (19),

$$p(\lambda) = \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3. \quad (28)$$

The eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ can be expressed in terms of the invariants as

$$\left\{ \beta, \frac{1}{2}(I_1 - \beta) \pm \frac{1}{2} \sqrt{(I_1 - \beta)^2 - 4I_3/\beta} \right\}, \quad (29)$$

where β is defined in Eq. (24a). Every¹⁴ derived alternate closed-form expressions based on the trigonometric solution of the characteristic cubic. The alternative version of Eq. (25) is

$$\text{tr } \mathbf{Q}^{-3/2} = \lambda_1^{-3/2} + \lambda_2^{-3/2} + \lambda_3^{-3/2}, \quad (30)$$

which is the starting point for Fedorov's calculation¹⁷ of the trace.

B. Transverse isotropy

Transverse isotropy or hexagonal symmetry is an important class of anisotropy. It occurs in many practical circumstances, whether from layering in the earth to laminated composite materials, or from underlying crystal structure. It is the highest symmetry for which the participation factor tensor is not the identity, since $\mathbf{G} = \mathbf{I}$ under isotropy and cubic material symmetry. We now demonstrate that the evaluation of d and $\mathbf{G}$ may be reduced to the evaluation of two single integrals, one for $\langle \text{tr } \mathbf{Q}^{-3/2} \rangle$ and one for the parameter α that defines $\mathbf{G}$, see Table I.

Transversely isotropic solids have five independent moduli: $c_{11} = c_{22}$, c_{33} , c_{12} , $c_{13} = c_{23}$, $c_{44} = c_{55}$, $c_{66} = \frac{1}{2}(c_{11} - c_{12})$. Let $\mathbf{e}$ be the axis of symmetry. The SH slowness decouples to give

$$\mathbf{Q} = \lambda_3(\mathbf{n} \cdot \mathbf{e}) \mathbf{q}_3 \otimes \mathbf{q}_3 + \mathbf{Q}_\perp, \quad (31)$$

where (Ref. 15, p. 95)

$$\lambda_3(\mathbf{n} \cdot \mathbf{e}) = c_{66} + (c_{44} - c_{66})(\mathbf{n} \cdot \mathbf{e})^2, \quad (32)$$

and $\mathbf{q}_3 = \mathbf{e} \wedge \mathbf{n} / |\mathbf{e} \wedge \mathbf{n}|$. The two-dimensional symmetric tensor $\mathbf{Q}_\perp$ is¹⁵

$$\begin{aligned} \mathbf{Q}_\perp = & [c_{44} + (c_{33} - c_{44})(\mathbf{n} \cdot \mathbf{e})^2] \mathbf{e} \otimes \mathbf{e} \\ & + [c_{11} + (c_{44} - c_{11})(\mathbf{n} \cdot \mathbf{e})^2] \mathbf{d} \otimes \mathbf{d} \\ & + (c_{13} + c_{44}) \mathbf{n} \cdot \mathbf{e} \sqrt{1 - (\mathbf{n} \cdot \mathbf{e})^2} [\mathbf{d} \otimes \mathbf{e} + \mathbf{e} \otimes \mathbf{d}], \end{aligned}$$

where $\mathbf{d} = \mathbf{e} \wedge \mathbf{q}_3$. Replacing $\mathbf{n} \cdot \mathbf{e}$ by the integration parameter ξ , it follows that

$$\langle \lambda_3^{-3/2} \mathbf{q}_3 \otimes \mathbf{q}_3 \rangle = \frac{1}{2} \int_0^1 d\xi \lambda_3^{-3/2}(\xi) \mathbf{I}_\perp, \quad (33)$$

where $\mathbf{I}_\perp$ projects onto the plane perpendicular to $\mathbf{e}$,

$$\mathbf{I}_\perp = \mathbf{I} - \mathbf{e} \otimes \mathbf{e}. \quad (34)$$

It remains to consider the orientational average of $\mathbf{Q}_\perp^{-3/2}$.

The tensor $\mathbf{Q}_\perp$ satisfies a quadratic Cayley–Hamilton equation

$$\mathbf{Q}_\perp^2 - J_1 \mathbf{Q}_\perp + J_2 \mathbf{I}_\perp = 0, \quad (35)$$

with $J_1 = \text{tr } \mathbf{Q}_\perp = \lambda_1 + \lambda_2$ and $J_2 = \det \mathbf{Q}_\perp = \lambda_1 \lambda_2$. Similarly, the Cayley–Hamilton equation for the square root is

$$(\mathbf{Q}_\perp^{1/2})^2 - j_1 \mathbf{Q}_\perp^{1/2} + j_2 \mathbf{I}_\perp = 0, \quad (36)$$

where $j_1 = \text{tr } \mathbf{Q}_\perp^{1/2}$ and $j_2 = \det \mathbf{Q}_\perp^{1/2}$ satisfy $J_1 = j_1^2 - 2j_2$, $J_2 = j_2^2$, and are therefore related to J_1 and J_2 by $j_1 = \sqrt{J_1 + 2\sqrt{J_2}}$, $j_2 = \sqrt{J_2}$. Using Eqs. (35) and (36), respectively, leads to the identities

$$\mathbf{Q}_\perp^{-2} = J_2^{-2} [(J_1^2 - J_2) \mathbf{I}_\perp - J_1 \mathbf{Q}_\perp], \quad (37a)$$

$$\mathbf{Q}_\perp^{1/2} = j_1^{-1} (\mathbf{Q}_\perp + j_2 \mathbf{I}_\perp). \quad (37b)$$

Multiplication of these and further use of Eq. (35) leads to

$$\mathbf{Q}_\perp^{-3/2} = \frac{1}{j_1 j_2 J_2} [(J_1 + j_2)(J_1 \mathbf{I}_\perp - \mathbf{Q}_\perp) - J_2 \mathbf{I}_\perp]. \quad (38)$$

Again using $\xi = \mathbf{n} \cdot \mathbf{e}$, we have

$$\langle \text{tr } \mathbf{Q}^{-3/2} \rangle = \int_0^1 d\xi [J_2^{-3/2} (J_1 - \sqrt{J_2}) \sqrt{J_1 + 2\sqrt{J_2}} + \lambda_3^{-3/2}(\xi)],$$

and from Table I,

$$\alpha = \frac{3}{\langle \text{tr } \mathbf{Q}^{-3/2} \rangle} \int_0^1 d\xi \frac{(J_1 + \sqrt{J_2})(J_1 - \mathbf{e} \cdot \mathbf{Q}_\perp \cdot \mathbf{e}) - J_2}{J_2^{3/2} \sqrt{J_1 + 2\sqrt{J_2}}}.$$

The modal density parameter $\langle \text{tr } \mathbf{Q}^{-3/2} \rangle$ and the scalar α that defines the participation tensor can therefore be expressed as single integrals, which follow from the above-presented results and Eqs. (31)–(33), as

$$\langle \text{tr } \mathbf{Q}^{-3/2} \rangle = \int_0^1 d\xi \left[\frac{(a + b\xi^2 - \sqrt{d + e\xi^2 + f\xi^4})\sqrt{a + b\xi^2 + 2\sqrt{d + e\xi^2 + f\xi^4}}}{(d + e\xi^2 + f\xi^4)^{3/2}} + \frac{1}{[c_{66} + (c_{44} - c_{66})\xi^2]^{3/2}} \right], \quad (39a)$$

$$\alpha = \frac{3}{\langle \text{tr } \mathbf{Q}^{-3/2} \rangle} \int_0^1 d\xi \left[\frac{(c_{11} + c\xi^2)(a + b\xi^2 + \sqrt{d + e\xi^2 + f\xi^4}) - (d + e\xi^2 + f\xi^4)}{(d + e\xi^2 + f\xi^4)^{3/2} \sqrt{a + b\xi^2 + 2\sqrt{d + e\xi^2 + f\xi^4}}} \right], \quad (39b)$$

where

$$a = c_{11} + c_{44}, \quad b = c_{33} - c_{11},$$

$$c = c_{44} - c_{11}, \quad d = c_{11}c_{44},$$

$$e = c_{11}c_{33} - c_{13}^2 - 2c_{44}(c_{11} + c_{13}),$$

$$f = -c_{11}c_{33} + c_{13}^2 + c_{44}(c_{11} + c_{33} + 2c_{13}).$$

IV. WEAK ANISOTROPY

Although the general expressions for the modal density d and the participation tensor $\mathbf{G}$ are not difficult to compute, it is often the case that the medium is to a first approximation isotropic, and appropriate approximations can be made. The state of small or *weak anisotropy* is defined relative to a background isotropic medium, and it is important to select the latter properly. In this section we calculate d and $\mathbf{G}$ in the presence of weak anisotropy. Fedorov¹⁷ provides a detailed analysis of the expansion of $\text{tr}(\mathbf{Q}^{-3/2})$ to arbitrary orders in the perturbation parameter. Our emphasis is more on obtaining estimates of the tensor $\langle \mathbf{Q}^{-3/2} \rangle$, which is not discussed explicitly by Fedorov. We begin with a description of the comparison isotropic moduli and then proceed to calculate the first two terms in a perturbation series for d and $\mathbf{G}$.

A. Background isotropic moduli

Regardless of the level of the anisotropy it is always possible to define a unique set of isotropic moduli which minimize the Euclidean distance between the exact set of moduli and the equivalent isotropic moduli.¹⁶ This procedure is equivalent to requiring that the mean square Euclidean difference in the slowness surfaces is minimal.^{16,17} Thus, let the background isotropic moduli be

$$c_{ijkl}^{(0)} = c_l^2 \delta_{ij} \delta_{kl} + c_t^2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - 2\delta_{ij} \delta_{kl}), \quad (40)$$

where c_l and c_t are the effective longitudinal and transverse wave speeds. These are defined by simultaneously minimizing the quantity $\langle |\mathbf{Q} - \mathbf{Q}_0|^2 \rangle$ with respect to both c_l and c_t , where $\mathbf{Q}_0(\mathbf{n})$ is defined by the moduli $c_{ijkl}^{(0)}$. The unique solution is

$$c_l^2 = \frac{1}{3} \text{tr } \mathbf{C}_l, \quad c_t^2 = \frac{1}{3} \text{tr } \mathbf{C}_t, \quad (41)$$

where the second-order tensors of reduced moduli are

$$\mathbf{C}_{l,ij} = \frac{2}{5} c_{ikjk} + \frac{1}{5} c_{ijkk}, \quad \mathbf{C}_{t,ij} = \frac{3}{10} c_{ikjk} - \frac{1}{10} c_{ijkk}. \quad (42)$$

The background Lamé moduli λ and μ are obtained using $c_l^2 = (\lambda + 2\mu)/\rho$ and $c_t^2 = \mu/\rho$. The elements of $\mathbf{C}_l$ and $\mathbf{C}_t$ follow from

$$c_{ijkk} = \begin{pmatrix} c_{11} + c_{12} + c_{13} & c_{16} + c_{26} + c_{36} & c_{15} + c_{25} + c_{35} \\ c_{16} + c_{26} + c_{36} & c_{12} + c_{22} + c_{23} & c_{14} + c_{24} + c_{34} \\ c_{15} + c_{25} + c_{35} & c_{14} + c_{24} + c_{34} & c_{13} + c_{23} + c_{33} \end{pmatrix},$$

$$c_{ikjk} = \begin{pmatrix} c_{11} + c_{55} + c_{66} & c_{16} + c_{26} + c_{45} & c_{15} + c_{46} + c_{35} \\ c_{16} + c_{26} + c_{45} & c_{22} + c_{44} + c_{66} & c_{24} + c_{34} + c_{56} \\ c_{15} + c_{46} + c_{35} & c_{24} + c_{34} + c_{56} & c_{33} + c_{44} + c_{55} \end{pmatrix}.$$

B. Perturbation analysis

Let

$$c_{ijkl} = c_{ijkl}^{(0)} + \varepsilon c_{ijkl}^{(1)}, \quad (43)$$

where the nondimensional parameter ε is introduced only to simplify the perturbation analysis. In practice ε is set to unity on the assumption that the additional moduli $c_{ijkl} - c_{ijkl}^{(0)}$ are small in comparison with the isotropic background.

We seek expansions in powers of the small parameter ε . The key quantity $\mathbf{Q}^{-3/2}$ will be determined as the product of $\mathbf{Q}^{-2}$ and $\mathbf{Q}^{1/2}$. Based on Eq. (43), the acoustical tensor is

$$\mathbf{Q} = \mathbf{Q}_0 + \varepsilon \mathbf{Q}_1, \quad (44)$$

and simple perturbation gives

$$\mathbf{Q}^{-2} = \mathbf{Q}_0^{-2} - \varepsilon (\mathbf{Q}_0^{-2} \mathbf{Q}_1 \mathbf{Q}_0^{-1} + \mathbf{Q}_0^{-1} \mathbf{Q}_1 \mathbf{Q}_0^{-2}) + O(\varepsilon^2).$$

Let

$$\mathbf{Q}^{1/2} = \mathbf{Q}_0^{1/2} + \varepsilon \mathbf{S}_1 + O(\varepsilon^2),$$

then $\mathbf{S}_1$ satisfies

$$\mathbf{Q}_0^{1/2} \mathbf{S}_1 + \mathbf{S}_1 \mathbf{Q}_0^{1/2} = \mathbf{Q}_1. \quad (45)$$

In order to calculate $\mathbf{Q}^{-2}$ and also the square root of $\mathbf{Q}$, we now use the fact that the leading order moduli $c_{ijkl}^{(0)}$ are isotropic. The explicit form of $\mathbf{Q}_0^{1/2}$ follows from Eq. (17) and the identity

$$\mathbf{Q}_0^m = c_l^{2m} \mathbf{n} \otimes \mathbf{n} + c_t^{2m} \mathbf{P}, \quad (46)$$

where m is any real number and $\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$. Equation (45) can be solved by observing that $\mathbf{Q}_1$ may be partitioned $\mathbf{Q}_1 = (\mathbf{Q}_1^{(1)} + \mathbf{Q}_1^{(2)} + \mathbf{Q}_1^{(3)})$ where $\mathbf{Q}_1^{(1)} = \mathbf{n} \cdot \mathbf{Q}_1 \cdot \mathbf{n} \mathbf{n} \otimes \mathbf{n}$, $\mathbf{Q}_1^{(2)} = \mathbf{P} \mathbf{Q}_1 \mathbf{P}$, and $\mathbf{Q}_1^{(3)} = \mathbf{P} \mathbf{Q}_1 \cdot \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{P} \mathbf{Q}_1 \cdot \mathbf{n}$. Assuming a solution of

the form $\mathbf{S}_1 = p_1 \mathbf{Q}_1^{(1)} + p_2 \mathbf{Q}_1^{(2)} + p_3 \mathbf{Q}_1^{(3)}$, the coefficients can be determined easily from Eq. (45), i.e.,

$$\mathbf{S}_1 = \frac{1}{2c_l} \mathbf{Q}_1^{(1)} + \frac{1}{2c_t} \mathbf{Q}_1^{(2)} + \frac{1}{c_l + c_t} \mathbf{Q}_1^{(3)}. \quad (47)$$

Combining the asymptotic expansions for $\mathbf{Q}^{-2}$ and $\mathbf{Q}^{1/2}$ gives

$$\mathbf{Q}^{-3/2} = \mathbf{Q}_0^{-3/2} + \varepsilon \mathbf{V}_1 + O(\varepsilon^2), \quad (48)$$

where

$$\begin{aligned} \mathbf{V}_1 &= \mathbf{Q}_0^{-2} \mathbf{S}_1 - \mathbf{Q}_0^{-2} \mathbf{Q}_1 \mathbf{Q}_0^{-1/2} - \mathbf{Q}_0^{-1} \mathbf{Q}_1 \mathbf{Q}_0^{-3/2} \\ &= -\frac{3}{2c_l^5} \mathbf{Q}_1 - \left[\frac{(c_l^2 + c_t^2 + c_l c_t)}{c_l^3 c_t^3 (c_l + c_t)} - \frac{3}{2c_l^5} \right] \\ &\quad \times [\mathbf{Q}_1 \cdot \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{Q}_1 \cdot \mathbf{n}] \\ &\quad + \left[2 \frac{(c_l^2 + c_t^2 + c_l c_t)}{c_l^3 c_t^3 (c_l + c_t)} - \frac{3}{2c_l^5} - \frac{3}{2c_t^5} \right] \\ &\quad \times (\mathbf{n} \cdot \mathbf{Q}_1 \cdot \mathbf{n}) \mathbf{n} \otimes \mathbf{n}. \end{aligned}$$

The orientational average $\langle \mathbf{Q}^{-3/2} \rangle$ can then be effected using the identities

$$\begin{aligned} \langle n_i n_j n_k n_l \rangle &= \frac{1}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \equiv K_{ijkl}, \\ \langle n_i n_j n_k n_l n_p n_q \rangle &= \frac{1}{7} (\delta_{ij} K_{klpq} + \delta_{ik} K_{jlpq} + \delta_{il} K_{kj pq} + \delta_{ip} K_{kljq} \\ &\quad + \delta_{iq} K_{klpj}). \end{aligned}$$

The resulting expressions for $\langle \mathbf{Q}^{-3/2} \rangle$ is

$$\begin{aligned} \langle \mathbf{Q}^{-3/2} \rangle_{ij} &= \frac{1}{3} \left(\frac{2}{c_l^3} + \frac{1}{c_t^3} \right) \delta_{ij} + \varepsilon \left\{ -\frac{1}{2c_l^5} c_{ikjk}^{(1)} - \frac{2}{15} \right. \\ &\quad \times \left[\frac{(c_l^2 + c_t^2 + c_l c_t)}{c_l^3 c_t^3 (c_l + c_t)} - \frac{3}{2c_l^5} \right] (c_{ijkk}^{(1)} + 2c_{ikjk}^{(1)}) \\ &\quad + \frac{1}{105} \left[2 \frac{(c_l^2 + c_t^2 + c_l c_t)}{c_l^3 c_t^3 (c_l + c_t)} - \frac{3}{2c_l^5} - \frac{3}{2c_t^5} \right] \\ &\quad \times [\delta_{ij} (c_{kkll}^{(1)} + 2c_{klkl}^{(1)}) + 4(c_{ijkk}^{(1)} + 2c_{ikjk}^{(1)})] \Big\} \\ &\quad + O(\varepsilon^2). \end{aligned}$$

We note that both $c_{ijij}^{(1)}$ and $c_{ijij}^{(1)}$ vanish by virtue of the choice of the background isotropic moduli. This implies that the trace of $\langle \mathbf{Q}^{-3/2} \rangle$ differs from the isotropic approximant only at the second order of anisotropic perturbation,

$$\text{tr} \langle \mathbf{Q}^{-3/2} \rangle = \frac{2}{c_l^3} + \frac{1}{c_t^3} + O(\varepsilon^2). \quad (49)$$

This is in agreement with Fedorov,¹⁷ who also provides explicit forms for the higher order terms; for instance, the expansion for cubic crystals up to fourth order in the perturbation is given by Eqs. (50.12)–(50.14) of Ref. 17. The leading order approximation of Eq. (49) when combined with the identity (4b), gives

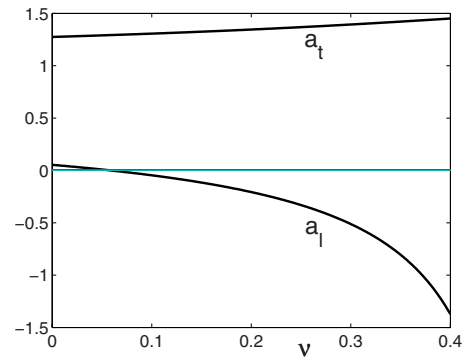


FIG. 1. (Color online) The nondimensional parameters a_l and a_t as a function of the Poisson's ratio ν .

$$\begin{aligned} G_{ij} &= \delta_{ij} - \varepsilon \left(\frac{2}{c_l^3} + \frac{1}{c_t^3} \right)^{-1} \left\{ \frac{3}{2c_l^5} c_{ikjk}^{(1)} + \frac{3}{35} \left[2 \frac{(c_l^2 + c_t^2 + c_l c_t)}{c_l^3 c_t^3 (c_l + c_t)} \right. \right. \\ &\quad \left. \left. + \frac{2}{c_l^5} - \frac{5}{c_t^5} \right] \times (c_{ijkk}^{(1)} + 2c_{ikjk}^{(1)}) \right\} + O(\varepsilon^2). \end{aligned}$$

Ignoring terms of order ε^2 and then setting $\varepsilon \rightarrow 1$ yields the leading order approximation to the participation tensor as

$$\mathbf{G} \approx \mathbf{I} + a_l (\mathbf{I} - c_l^{-2} \mathbf{C}_l) + a_t (\mathbf{I} - c_t^{-2} \mathbf{C}_t), \quad (50)$$

where the nondimensional coefficients are

$$a_l = \frac{6}{7(2 + \kappa^{-3})} \left(\frac{1}{\kappa^3} + \frac{1}{\kappa} - \frac{1}{\kappa + 1} + 1 - \frac{3}{4} \kappa^2 \right), \quad (51a)$$

$$a_t = \frac{3}{2 + \kappa^{-3}}, \quad (51b)$$

and

$$\kappa \equiv \frac{c_l}{c_t}. \quad (52)$$

Figure 1 shows a_l and a_t as functions of the Poisson's ratio ν , using $\kappa^2 = 2(1 - \nu)/(1 + 2\nu)$. Note that $1.27 \dots < a_t < 3/2$ for $0 < \nu < 1/2$ while $a_l \approx -\frac{9}{28}(1 - 2\nu)^{-1}$ as $\nu \rightarrow 1/2$.

C. Transversely isotropic materials

As an example of the general perturbation approach, we consider the particular case of TI materials. We take the axis of symmetry ($\mathbf{e}$ in Sec. III) in the 3 direction so that

$$\begin{aligned} c_{ijkk} &= \begin{pmatrix} c_{11} + c_{12} + c_{13} & 0 & 0 \\ 0 & c_{11} + c_{12} + c_{13} & 0 \\ 0 & 0 & c_{33} + 2c_{13} \end{pmatrix}, \\ c_{ikjk} &= \begin{pmatrix} c_{11} + c_{44} + c_{66} & 0 & 0 \\ 0 & c_{11} + c_{44} + c_{66} & 0 \\ 0 & 0 & c_{33} + 2c_{44} \end{pmatrix}, \end{aligned}$$

where $c_{66} = \frac{1}{2}(c_{11} - c_{12})$. The wave speeds in the background isotropic medium are then

$$c_l^2 = \frac{1}{15}(8c_{11} + 3c_{33} + 4c_{13} + 8c_{44}), \quad (53a)$$

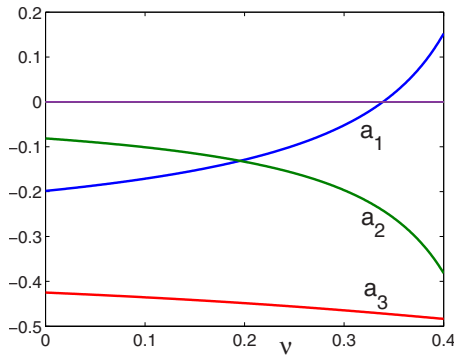


FIG. 2. (Color online) The nondimensional parameters a_1 , a_2 , and a_3 for weak transverse isotropy as a function of the Poisson's ratio ν of the background medium.

$$c_t^2 = \frac{1}{30}(2c_{11} + 2c_{33} - 4c_{13} + 12c_{44} + 10c_{66}). \quad (53b)$$

According to Table I the participation tensor is defined by a single parameter, α , which to leading order is unity. Let

$$\alpha = 1 - 2\beta, \quad (54)$$

so that

$$\mathbf{G} = \begin{pmatrix} 1 + \beta & 0 & 0 \\ 0 & 1 + \beta & 0 \\ 0 & 0 & 1 - 2\beta \end{pmatrix}. \quad (55)$$

Applying the general perturbation theory we find that the leading order correction to the isotropic participation tensor is given by

$$\begin{aligned} \beta = & \frac{a_l}{15c_t^2}(-4c_{11} + 3c_{33} + c_{13} + 2c_{44}) \\ & + \frac{a_t}{30c_t^2}(-c_{11} + 2c_{33} - c_{13} + 3c_{44} - 5c_{66}), \end{aligned} \quad (56)$$

where a_l and a_t are defined in Eq. (51a).

Thomsen's anisotropy parameters¹⁸ ϵ, γ, δ provide a means to characterize weakly anisotropic TI materials. The parameters are defined $\epsilon = (c_{11} - c_{33})/(2c_{33})$, $\delta = [(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2]/[2c_{33}(c_{33} - c_{44})]$, $\gamma = (c_{66} - c_{44})/(2c_{44})$, and are commonly used in geophysical applications to describe rock properties. The correction term β can be expressed in terms of the Thomsen parameters as

$$\beta \approx a_1\epsilon + a_2\delta + a_3\gamma, \quad (57)$$

where the coefficients a_1 , a_2 , and a_3 are (see Fig. 2)

$$a_1 = -\frac{8a_l}{15} - \frac{\kappa^2 a_t}{15}, \quad a_2 = \frac{a_l}{15} - \frac{\kappa^2 a_t}{30}, \quad a_3 = -\frac{a_t}{3}. \quad (58)$$

V. EXAMPLES AND DISCUSSION

The participation matrix was computed for many anisotropic solids. Table II summarizes the results for a selection of materials with anisotropy ranging from weak to strong. Table II provides the numerical values of diagonal elements of $\mathbf{G}$ (there are no off-diagonal elements for the symmetries considered). In each case the elements sum to three, $G_{11} + G_{22} + G_{33} = 3$, although the individual numbers can differ markedly from unity.

In order to quantify the level of anisotropy, Table II also shows the number *dist*. This is a nondimensional positive measure of the degree of anisotropy of a set of anisotropic elastic constants. *dist* is chosen here as the log-Euclidean distance or length from isotropy,^{19,20} although other measures are possible, see Norris¹⁹ for a comparative discussion. The log-Euclidean distance has the advantage that it is invariant regardless of whether the compliance or stiffness tensor are considered. We use *dist* as a convenient and simple measure of the degree of anisotropy. Appendix C provides a little more detail on its exact definition, including a short MATLAB script to compute *dist*.

Large deviations from the isotropic participation tensor are apparent. Consider the ratio R of the largest to smallest

TABLE II. The participation matrix $\mathbf{G}$ for a variety of anisotropic materials. Sym denotes material symmetry: transversely isotropic (TI), tetragonal (Tet), or orthotropic (Orth). The Frobenius ($p=2$) norm is used to compare $\mathbf{G}$ with the isotropic result ($\mathbf{I}$) and with the perturbation approximation $\tilde{\mathbf{G}}$ defined by Eq. (50). *dist* is a nondimensional and invariant measure of the anisotropy (Ref. 19), equal to zero for isotropy. *dist* ≥ 1 signifies considerable anisotropy.

Material	Sym	G_{11}	G_{22}	G_{33}	$ \mathbf{G} - \mathbf{I} $	$ \mathbf{G} - \tilde{\mathbf{G}} $	<i>dist</i>
Beryllium ^a	TI	1.05	1.05	0.89	0.13	0.00	0.22
Sulphur ^a	Orth	0.95	1.32	0.73	0.42	0.11	0.95
Cadmium ^a	TI	0.73	0.73	1.55	0.67	0.10	1.02
Barium titanate ^b	Tet	0.81	0.81	1.39	0.48	0.01	1.11
Rochelle salt ^a	Orth	1.38	0.65	0.97	0.52	0.09	1.16
Zinc ^a	TI	0.71	0.71	1.58	0.71	0.14	1.17
Graphite/Epoxy ^c	TI	1.38	1.38	0.25	0.92	0.81	2.35
Tellurium dioxide ^d	Tet	1.30	1.30	0.40	0.74	0.72	2.87
Mercurous iodide ^d	Tet	1.37	1.37	0.26	0.91	0.14	3.02
Spruce ^a	Orth	1.35	1.63	0.02	1.22	1.30	5.59

Elastic moduli from Ref. 15.

From Ref. 21.

From Ref. 22.

From Ref. 23.

element of $\mathbf{G}$. Even for small to moderate anisotropy, such as cadmium we see that $R=G_{33}/G_{11}>2$. The ratio becomes much larger for the more anisotropic materials considered. Spruce is included because of its enormous ratio, $R\approx 80$. These ratios can be compared with the results for the relative partition of the diffuse wave energy at the free surface of an isotropic solid. If $\mathbf{e}_3$ is the normal to the surface, then the calculations of Weaver⁸ indicate that $1\leq G_{33}/G_{11}<1.25$ where the lower (upper) bound is reached as ν approaches $1/2$ (0). The upper bound ≈ 1.25 is approximate and based on Fig. 3 of Ref. 8.

The numbers in Table II indicate that the perturbation approximation is adequate for small anisotropy. This can be characterized loosely as $0<\text{dist}\leq 1$, and strong anisotropy is $\text{dist}\geq 2$, roughly. The examples in Table II suggest that the weak anisotropy approximation is not useful in the presence of strong anisotropy. This is evident from the fact that the errors $|\mathbf{G}-\mathbf{I}|$ and $|\mathbf{G}-\tilde{\mathbf{G}}|$ are of the same order of magnitude for the strongly anisotropic materials, whereas $|\mathbf{G}-\tilde{\mathbf{G}}|$ is much less than $|\mathbf{G}-\mathbf{I}|$ for weak anisotropy.

We note that for all materials considered the numerical calculations show Eq. (49) underestimating $\text{tr}(\mathbf{Q}^{-3/2})$. However, the more refined perturbation expansion of $\text{tr}(\mathbf{Q}^{-3/2})$ by Fedorov¹⁷ suggests that this is not a universal result.

The dependence of $\mathbf{G}$ and $d(\omega)$ on the moduli is obviously complicated by virtue of the averages required in Eq. (4). However, the formula (50) for $\mathbf{G}$ for weak anisotropy illustrates the dependence more explicitly. The form of the matrices $\mathbf{C}_l$ and $\mathbf{C}_t$ imply that only 12 combinations of the 21 independent anisotropic moduli enter into the first term in the perturbation expansion. For orthotropic materials, with 9 independent moduli, this number reduces to 6, and the matrices $\mathbf{C}_l$ and $\mathbf{C}_t$ are then diagonal. In the case of weak TI only two combinations of moduli influence $\mathbf{G}$, see Eq. (56).

The nondimensional tensor $\mathbf{G}$ also has important implications for radiation from a point source. The connection follows from the relation (8) between $\mathbf{G}$ and $\mathbf{A}$, combined with the correspondence between the drive point admittance tensor and the radiation efficiency in Eq. (7). Thus, the direction in which a force must be applied to most efficiently radiate power is the principal direction of $\mathbf{G}$ with the largest element. Conversely, the least amount of power is radiated if the force is directed along the principal direction with the smallest element. For instance, Table II indicates that a point force of given magnitude will radiate most power in cadmium if the force is directed along the axis of hexagonal symmetry. The situation is reversed for aligned graphite/epoxy, where forcing along the fiber direction produces the least amount of total radiated power.

The inverse problem of determining anisotropy from measurements of $\mathbf{G}$ is clearly ill-posed. However, possible measurement could be advantageous in particular circumstances. Consider for instance, three-component measurement of the displacement downhole in a borehole environment. Assuming the frequency is such that the wavelengths are large compared with the bore radius, the three-component data are sufficient to compute the autocorrelation

and hence $\mathbf{G}$. The principal directions of $\mathbf{G}$ and the relative magnitude of its diagonal elements provide significant information about the local geostatigraphy and formation properties.

VI. CONCLUSION

We have derived general formulas for diffuse waves in anisotropic solids. The main results are concise expressions for the modal density per unit volume and frequency, $d(\omega)$ of Eq. (4a), and the participation tensor $\mathbf{G}$ of Eq. (4b). The latter is a material constant with one or two independent constants, and with principal axes dictated by the material symmetry. In the absence of symmetry the participation tensor defines principal axes for diffuse wave energy distribution, and for radiation efficiency. Calculation of $d(\omega)$ and $\mathbf{G}$ requires, in general, averaging over the surface of the unit sphere. Single integrals suffice for transverse isotropy, with the important quantities given in Eq. (39). In the case of weak anisotropy, a perturbation scheme produces explicit formulas, Eqs. (49) and (50). The main quantity in all cases is the second-order averaged tensor $\langle \mathbf{Q}^{-3/2} \rangle$. We have illustrated the results through calculations for several materials. These display the main effects that would occur in all anisotropic solids. In particular, the deviation $\mathbf{G}$ from the unit identity tensor can be significant. Ratios of 2 or more for the relative magnitude of diffuse wave energy in different directions in crystals can occur under moderate levels of anisotropy, with far larger ratios possible in realistic materials.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF EQ. (8)

We use an argument based on a modal representation⁸ for the solution to the point force problem,

$$\left(\rho \frac{\partial^2}{\partial t^2} - L\right)\mathbf{u} = \mathbf{F}\delta(\mathbf{x} - \mathbf{x}_0)\cos \omega t, \quad (\text{A1})$$

where L is a second-order differential operator. The resulting velocity $\mathbf{v} = \partial \mathbf{u} / \partial t$ may be found by standard means as

$$\mathbf{v} = \frac{1}{\rho} \text{Re} \sum_m \frac{-i\omega \mathbf{F} \cdot \mathbf{u}_m(\mathbf{x}_0)\mathbf{u}_m(\mathbf{x})}{\omega_m^2 - \omega^2 - i0} e^{-i\omega t},$$

where the modes $\mathbf{u}_m(\mathbf{x})e^{-i\omega_m t}$ are solutions of the homogeneous equation (A1), with the properties

$$\delta(\mathbf{x} - \mathbf{x}_0)\mathbf{I} = \sum_m \mathbf{u}_m(\mathbf{x})\mathbf{u}_m(\mathbf{x}_0),$$

$$\int_V d\mathbf{x} \mathbf{u}_m(\mathbf{x}) \cdot \mathbf{u}_m(\mathbf{x}) = 1.$$

The power output averaged over a cycle is therefore

$$\begin{aligned}\Pi(\mathbf{x}_0, \omega) &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \cos \omega t \mathbf{F} \cdot \mathbf{v}(\mathbf{x}_0, t) \\ &= \frac{1}{2\rho} \sum_m [\mathbf{F} \cdot \mathbf{u}_m(\mathbf{x}_0)]^2 \operatorname{Re} \frac{-i\omega}{\omega_m^2 - \omega^2 - i0}. \quad (\text{A2})\end{aligned}$$

The strict nondissipative limit of $\operatorname{Re}[-i\omega(\omega_m^2 - \omega^2 - i0)^{-1}]$ is $\pi\omega\delta(\omega_m^2 - \omega^2) = \frac{1}{2}\pi\delta(\omega_m - \omega)$ where δ is the Dirac delta function. However, modal overlap in the presence of nonzero dissipation spreads the influence over many modes. The effect is to make $\operatorname{Re}[-i\omega(\omega_m^2 - \omega^2 - i0)^{-1}] \rightarrow \frac{1}{2}\pi f(\omega_m - \omega)$ where $f(\nu)$ is smooth with bounded support in $\nu \in \{-\Omega, \Omega\}$, say, and unit sum:

$$\sum_{\omega_m}' f(\omega_m - \omega) = 1. \quad (\text{A3})$$

Here $\sum_{\omega_m}'$ indicates the sum over modal frequencies $\omega_m \in \{\omega - \Omega, \omega + \Omega\}$. Using the density of modes, $Vd(\omega_m)$, to replace the sum over modes in Eq. (A2) by a sum over modal frequencies, gives

$$\Pi(\mathbf{x}_0, \omega) = \frac{\pi V}{4\rho} \sum_{\omega_m}' d(\omega_m) f(\omega_m - \omega) [\mathbf{F} \cdot \mathbf{u}_m(\mathbf{x}_0)]^2. \quad (\text{A4})$$

We now make the assumption that the support of $f(\nu)$ is small enough that the modal density function, $d(\omega_m)$, may be replaced by $d(\omega)$. This is perfectly reasonable based on known forms for $d(\omega)$, e.g., Eq. (3). At the same time, we assume that the support of $f(\nu)$ is sufficiently large that we may use the equipartition of energy among modes to make the replacement [see Eq. (1)]

$$V \sum_{\omega_m}' f(\omega_m - \omega) \mathbf{u}_m \otimes \mathbf{u}_m \rightarrow V \frac{\rho \omega^2}{E} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} = \frac{1}{3} \mathbf{G}. \quad (\text{A5})$$

Hence,

$$\Pi(\mathbf{x}_0, \omega) = \frac{\pi}{12\rho} d(\omega) \mathbf{F} \cdot \mathbf{G} \cdot \mathbf{F}, \quad (\text{A6})$$

and since $\mathbf{F}$ is arbitrary, the admittance $\mathbf{A}$ follows from the definition of Π in Eq. (7). This completes the derivation of the identity (8).

APPENDIX B: DERIVATION OF EQS. (21) AND (26)

The Cayley–Hamilton relation for $\mathbf{Q}$ is $p(\mathbf{Q})=0$, where p is the characteristic cubic polynomial defined in Eq. (28), and $I_1(\mathbf{n})$, $I_2(\mathbf{n})$, $I_3(\mathbf{n})$ are the invariants defined in Eq. (20). Thus,

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1,$$

$$I_3 = \lambda_1\lambda_2\lambda_3,$$

and since $\lambda_\alpha = v_\alpha^2$, it follows that the invariants are all positive, $I_1 > 0$, $I_2 > 0$ and $I_3 > 0$. Multiplying Eq. (19) by $\mathbf{Q}^{-1}$ and $\mathbf{Q}^{-2}$ yields equations for the same quantities:

$$\mathbf{Q}^{-1} = I_3^{-1} \mathbf{Q}^2 - I_1 I_3^{-1} \mathbf{Q} + I_2 I_3^{-1} \mathbf{I}, \quad (\text{B1a})$$

$$\mathbf{Q}^{-2} = I_3^{-1} \mathbf{Q} - I_1 I_3^{-1} \mathbf{I} + I_2 I_3^{-1} \mathbf{Q}^{-1}. \quad (\text{B1b})$$

Eliminating $\mathbf{Q}^{-1}$ gives an equation for $\mathbf{Q}^{-2}$:

$$\mathbf{Q}^{-2} = I_3^{-2} [I_2 \mathbf{Q}^2 - (I_1 I_2 - I_3) \mathbf{Q} + (I_2^2 - I_1 I_3) \mathbf{I}].$$

We next derive a similar type of equation for $\mathbf{Q}^{1/2}$ using a method due to Hoger and Carlson.¹¹ The product of this with $\mathbf{Q}^{-2}$, combined with the Cayley–Hamilton equation (19) yields the desired relation (21).

First we note the general expression

$$(\mathbf{Q} - \lambda \mathbf{I})^{-1} = \frac{1}{p(\lambda)} [-\mathbf{Q}^2 + (I_1 - \lambda) \mathbf{Q} - (\lambda^2 - I_1 \lambda + I_2) \mathbf{I}], \quad (\text{B2})$$

where p is the characteristic polynomial for $\mathbf{Q}$, from Eq. (28). The identity (B2) may be checked by direct multiplication and use of Eq. (19). The square root tensor $\mathbf{R} \equiv \mathbf{Q}^{1/2}$ satisfies the Cayley–Hamilton equation

$$\mathbf{R}^3 - i_1 \mathbf{R}^2 + i_2 \mathbf{R} - i_3 \mathbf{I} = 0, \quad (\text{B3})$$

where i_1 , i_2 , and i_3 are related to the invariants of $\mathbf{Q}$ by

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1 i_3, \quad I_3 = i_3^2. \quad (\text{B4})$$

Explicit formulas for i_1 , i_2 , and i_3 are given in Eq. (23a). Rearranging Eq. (B3) as $\mathbf{R}(\mathbf{R}^2 + i_2 \mathbf{I}) = i_1 \mathbf{R}^2 + i_3 \mathbf{I}$ and using $\mathbf{R}^2 = \mathbf{Q}$ gives

$$\mathbf{R} = (i_1 \mathbf{Q} + i_3 \mathbf{I})(\mathbf{Q} + i_2 \mathbf{I})^{-1}. \quad (\text{B5})$$

Application of Eq. (B2) along with some simplifications using Eq. (B4), such as $p(-i_2) = -(i_3 - i_1 i_2)^2$, yields

$$\mathbf{Q}^{1/2} = (i_3 - i_1 i_2)^{-1} [\mathbf{Q}^2 + (i_2 - i_1^2) \mathbf{Q} - i_1 i_3 \mathbf{I}]. \quad (\text{B6})$$

Combining Eqs. (B2) and (B6) gives Eq. (21). Alternatively,

$$\mathbf{Q}^{-3/2} = a \mathbf{Q}^2 + b \mathbf{Q} + c \mathbf{I}, \quad (\text{B7})$$

where

$$\begin{aligned}a &= \frac{I_3(i_2 - i_1^2) - I_2 i_1 i_3}{I_3^2(i_3 - i_1 i_2)}, \\ b &= \frac{I_1 I_3(i_1^2 - i_2) + (I_1 I_2 - I_3) i_1 i_3}{I_3^2(i_3 - i_1 i_2)}, \\ c &= \frac{I_3^2 + I_2 I_3(i_2 - i_1^2) + (I_1 I_3 - I_2^2) i_1 i_3}{I_3^2(i_3 - i_1 i_2)}. \quad (\text{B8})\end{aligned}$$

The second form (26) for $\mathbf{Q}^{-3/2}$ is based on the identity (17). The tensor products of eigenvectors for λ_i satisfy

$$\begin{aligned}\mathbf{q}_i \otimes \mathbf{q}_i &= \frac{(\mathbf{Q} - \lambda_j \mathbf{I})(\mathbf{Q} - \lambda_k \mathbf{I})}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}, \\ i &\neq j \neq k \neq i \quad (\text{no sum}).\end{aligned}$$

This follows, for example, by eliminating the other two tensor products using the spectral expressions for $\mathbf{I}$, $\mathbf{Q}$, and $\mathbf{Q}^2$. The dependence on λ_j and λ_k can be removed in favor of λ_i and the invariants I_1 and I_3 , and hence Eq. (27). Note that the latter can be expressed

$$N(\lambda, \mathbf{n}) = \frac{1}{\lambda p'(\lambda)} [\lambda \mathbf{Q}^2 + (\lambda - I_1) \lambda \mathbf{Q} + I_3 \mathbf{I}], \quad (\text{B9})$$

where $p'(x)$ is the derivative of the characteristic polynomial. This indicates that the general expression (27) is invalid at double roots where the slowness surface exhibits degeneracy, and proper limits are required. The possibility of such points does not present a practical impediment to numerical integration.

APPENDIX C: THE LOG-EUCLIDEAN DISTANCE

The procedure¹⁹ is to first calculate an effective isotropic set of moduli analogous to $c_{ijkl}^{(0)}$ of Eq. (40) but for the matrix logarithm of the six-dimensional Voigt matrix of moduli C_{IJ} . Some matrix factors are required to convert from the Voigt notation. The following MATLAB lines compute dist if C is the 6×6 Voigt matrix.

$$J = 1/3 * [1 \ 1 \ 1 \ 0 \ 0 \ 0]' * [1 \ 1 \ 1 \ 0 \ 0 \ 0];$$

$$K = \text{eye}(6) - J;$$

$$T = \text{diag}([1 \ 1 \ 1 \ \text{sqrt}(2) * [1 \ 1 \ 1]]);$$

$$L = \text{logm}(T * C * T);$$

$$\text{dist} = \text{norm}(\text{logm}(J * \exp(\text{trace}(J * L)) + K * \exp(1/5 * \text{trace}(K * L))) - L, 'fro');$$

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