Small-on-Large Theory with Applications to Granular Materials and Fluid/Solid Systems

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Abstract. Some mechanics involved in calculating wave speeds in pre-stressed materials of geophysical interest are discussed. Two topics are explored: granular materials, and fluid-solid composite systems. Elasticity in the granular material is dominated by intergranular contact which produces strong nonlinearity. The contact also introduce possible hysteresis and loss of strict macroscopic hyperelasticity. Starting with the Hertz solution for two spheres, these notes develop the theory to the stage where macroscopic elastic properties can be estimated, and the stress dependence of bulk waves calculated. Fluid-solid composite systems, such as a fluid-filled borehole in an elastic solid, are complicated by the ability of the fluid to slip under pre-stress. The interface conditions in the different material descriptions are examined and a procedure is described to determine the modified wave speed using a perturbation integral approach. The example of the fundamental guided mode of the borehole, the tube wave, is described in detail.

1 Introduction

The theory of acoustoelasticity, also known as "small on large", is well developed for hyperelastic materials. The theoretical underpinning began with Toupin and Bernstein (1961), and a good overview can be found in (Thurston, 1984). The focus in these notes is on two classes of problems that do not fit into the standard theory, for differing reasons: either the nature of the internal mechanics (grain to grain contact) or inhomogeneity, in particular the combination of a solid and an inviscid fluid. These problems are of practical interest in understanding the dynamics of materials such as sandstone, and in situations where nonlinear wave behaviour is useful to estimate geophysical properties.

We first look at the mechanics of materials made from granular aggregates, using fundamental mechanics of contact forces (Hertz etc.) to derive a macroscopic elastic model which is not necessarily hyperelastic. We will see how small-on-large theory may still be developed, and the stress dependence of small amplitude waves calculated.

The second class of problems concerns waves in fluid/solid composite materials. The primary issue of concern is how to deal with the slip of the inviscid fluid at the interface with the solid material. Slip presents considerable conceptual difficulty for the small-on-large formulation, but by careful analysis, we will recast the problem into standard format, allowing application to problems of practical interest.

The sequence of topics reflects the above descriptions, although we begin with a quick introduction to acoustoelasticity theory in Section 2. The following four Sections deal exclusively with granular materials. Section 3 considers preliminary concepts for granular media - the fundamental idea used throughout is the response at a contact zone for two spheres pressed together. The mechanics of Hertz contact theory are developed in Section 3, where a simple 1D acoustic model is also described. Generalizations of the Hertz model to tangential forces and other contact models are considered in Section 4. Sections 5 and 6 are devoted to 3D effects in grain packings under confining stress. In Section 5 the finite and incremental elasticity of a random packing of spheres is derived using energy methods. We will see that the existence of a strain energy function depends upon the type of contact. If the tangential contact stiffness is independent of the normal force, then the energy is well defined for all values of the macroscopic strain. Otherwise, the strain energy of the system is load-path dependent. Section 6 discusses acoustoelasticity of granular media in light of the prior finding. The final two Sections are devoted to the more difficult problem of how to formulate a small-on-large theory of fluid/solid systems. The analysis is delicate and focused on the interfacial traction continuity conditions. We show in Section 7 that these are best formulated in the intermediate description. Section 8 applies the theory of Section 7 to calculate the change in wave speed of a mode in a fluid-solid system. The nature of the wave motion in the composite system makes explicit solutions of the small on large equations intractable, but a more general perturbation integral method is described. The method is illustrated for the sample problem of the tube wave in a pressurized borehole.

The presentation is for the most part self contained, with reference to related literature when needed. Section 2 is introductory and recommended reading. After that the reader can continue on with Sections 3 through 6 for granular materials, or may jump directly to the fluid-solid problem in Sections 7 and 8 with no loss in continuity.

2 Acoustoelasticity, or Small-on-Large

2.1 Linearised equations of motion

The acoustoelastic effect is the change in the speed of small amplitude waves in an elastic body caused by a pre-imposed static stress. The effect is relatively small in metals, e.g., $\sim 10^{-5}$/MPa for aluminum, but it is still practical for measuring existing states of stress, such as residual stress. The situation is quite different in geophysical materials, especially those in regions of the earth containing fluid and gaseous pockets. Permeable rocks such as sandstone are characterised by granularity, interstitial contact and force concentration, which leads to strong stress dependence.

Three states and their coordinates need to be distinguished: the natural, $\mathbf{X}$; the initial, $\mathbf{x}$; and the current, $\mathbf{x}'$. For simplicity, suppose that the initial stress and strain are uniform and defined by the static displacements $\mathbf{u}^{(0)}$: $\mathbf{x} = \mathbf{X} + \mathbf{u}^{(0)}$. The acoustoelastic response is measured by the further small on large dynamic displacement $\mathbf{u}^{(1)}$: $\mathbf{x}' = \mathbf{x} + \mathbf{u}^{(1)} = \mathbf{X} + \mathbf{u}$ where $\mathbf{u}$ is the total displacement. Thus,

$$\mathbf{u}^{(0)} = \mathbf{x} - \mathbf{X}, \qquad \mathbf{u}^{(1)} = \mathbf{x}' - \mathbf{x}, \qquad \mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}. \tag{2.1}$$

The objective is to derive the equation of motion for $\mathbf{u}^{(1)}$ in terms of the intermediate description $\mathbf{x}$.

The small-on-large theory is based on the assumptions that

$$|\mathbf{u}^{(1)}| \ll |\mathbf{u}^{(0)}|, \tag{2.2}$$

and that the associated deformation gradients are similarly related. The standard procedure in small-on-large theory is essentially to perform a doubly asymptotic expansion where we are interested in effects $\mathrm{O}(u^{(0)})$ and $\mathrm{O}(u^{(1)})$, but most importantly, $\mathrm{O}(u^{(0)}\, u^{(1)})$. It is the latter which enables us to determine the first derivative of physical parameters, such as wave speeds, as a function of the prestrain.

We start with

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{div}_X\, \mathbf{P}\,, \tag{2.3}$$

where the non-symmetric tensor $\mathbf{P} = (\rho_0/\rho)\, \boldsymbol{\sigma}(\mathbf{F}^{-1})^T$ is the first Piola-Kirchhoff stress tensor or sometimes known as the Lagrangian stress tensor. ρ_0 and ρ are the undeformed (Lagrangian) and current (Eulerian) mass densities, and $\mathbf{F} = \nabla_X \mathbf{x}'$ is of course the finite deformation gradient. Assuming a strain energy (hyperelasticity) U per unit mass, we have $\mathbf{P} = \rho_0\, \partial U/\partial \mathbf{F}$, and since $U = U(\mathbf{E})$ where $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$ is the symmetric finite strain, we have

$$\mathbf{P} = \rho_0\, \mathbf{F} \frac{\partial U}{\partial \mathbf{E}}. \tag{2.4}$$

In the applications of interest here the strain energy is usually given as a power series in the strain:

$$\rho_0 U = \frac{1}{2!}\, C_{ijkl}\, E_{ij}\, E_{kl} + \frac{1}{3!}\, C_{ijklmn}\, E_{ij}\, E_{kl}\, E_{mn} + \cdots\,. \tag{2.5}$$

We will restrict the analysis to this general form of the energy, although other functional forms could be considered. Note that the symmetry of $\mathbf{E}$ implies that the second and third order moduli can be expressed using Voigt's notation: $C_{ijkl} = c_{IJ}$, $C_{ijklmn} = c_{IJK}$, where $I, J, K \in \{1,2,3,4,5,6\}$ with the relationships ij = 11, 22, 33, 23, 31, 12 $\leftrightarrow$ I = 1, 2, 3, 4, 5, 6. Equations (2.4) and (2.5) together imply[1] that

$$P_{ij} = C_{ijkl} \frac{\partial u_k}{\partial X_l} + \frac{1}{2}\, M_{ijklmn} \frac{\partial u_k}{\partial X_l} \frac{\partial u_m}{\partial X_n} + \frac{1}{3}\, M_{ijklmnpq} \frac{\partial u_k}{\partial X_l} \frac{\partial u_m}{\partial X_n} \frac{\partial u_p}{\partial X_q} + \cdots\,, \tag{2.6}$$

where

$$M_{ijklmn} = C_{ijklmn} + C_{ijln}\, \delta_{km} + C_{jnkl}\, \delta_{im} + C_{jlmn}\, \delta_{ik}\,. \tag{2.7}$$

Note that $M_{ijklmn} \neq M_{jiklmn}$, which means that the non-symmetry of $\mathbf{P}$ is a second-order effect.

The initial deformation is in equilibrium, $\sigma^{(0)}_{ij,j} = 0$, where

$$\sigma^{(0)}_{ij} = C_{ijkl} e^{(0)}_{kl}\,, \qquad e^{(0)}_{ij} = \frac{1}{2}\Big(\frac{\partial u^{(0)}_i}{\partial x_j} + \frac{\partial u^{(0)}_j}{\partial x_i}\Big), \tag{2.8}$$

[1] Put $\mathbf{F} = \mathbf{I} + \mathbf{H}$ and expand in $\mathbf{H}$.

i.e., $\sigma^{(0)}$ and $\mathbf{e}^{(0)}$ are the static pre-stress and pre-strain. A linear equation for the additional dynamic disturbance $\mathbf{u}^{(1)}(\mathbf{x}, t)$ follows from eq. (2.3) after the change of variable $\mathbf{X} \to \mathbf{x}$ and the use of the chain rule[2],

$$\rho_0 \frac{\partial^2 u_i^{(1)}}{\partial t^2} = B_{ijkl} \frac{\partial^2 u_k^{(1)}}{\partial x_j \partial x_l}, \tag{2.9}$$

where the effective elastic stiffnesses comprise terms of order zero and one in the applied deformation $u_{m,n}^{(0)} \equiv \partial u_m^{(0)}/\partial x_n$,

$$B_{ijkl} = C_{ijkl} + \delta_{ik} C_{jlqr} u_{q,r}^{(0)} + C_{rjkl} u_{i,r}^{(0)} + C_{irkl} u_{j,r}^{(0)} + C_{ijrl} u_{k,r}^{(0)} + C_{ijkr} u_{l,r}^{(0)} + C_{ijklmn} u_{m,n}^{(0)}. \tag{2.10}$$

The assumption of uniform initial stress and strain implies that the coefficients B_{ijkl} are constants.

Table 1. Stress derivatives of longitudinal and transverse wave speeds in an isotropic solid. $\mathbf{n}$ is the direction of propagation and $\mathbf{m}$ is the polarization $[K = \lambda + \frac{2}{3}\mu,\ E = 2\mu(1+\nu),\ \nu = \lambda/2(\lambda+\mu)]$. From Norris (1998).

Stress	Mode	$\mathbf{n}$	$\mathbf{m}$	$\rho_0(dv^2/dp)_0$
Hydrostatic	longitudinal	arbitrary	$\parallel \mathbf{n}$	$-\frac{(5-3\nu)}{1+\nu} - \frac{1}{3K}(c_{111} + 2c_{112})$
Hydrostatic	transverse	arbitrary	$\perp \mathbf{n}$	$-\frac{3(1-\nu)}{1+\nu} - \frac{1}{3K}(c_{144} + 2c_{166})$
Uniaxial	longitudinal	$\parallel$ stress	$\parallel \mathbf{n}$	$-1 - \frac{4(1-\nu)}{(1+\nu)(1-2\nu)} - \frac{1}{E}(c_{111} - 2\nu c_{112})$
Uniaxial	transverse	$\parallel$ stress	$\perp \mathbf{n}$	$-\frac{2}{1+\nu} + \frac{1}{E}[\nu c_{144} - (1-\nu)c_{166}]$

2.2 Applications: Plane waves in an infinite body

Plane waves Assume a plane wave propagating in the direction of the unit vector $\mathbf{n}$, $\mathbf{u}^{(1)} = \mathbf{m} \sin\omega(t - \mathbf{n}\cdot\mathbf{x}/v)$, where the polarization $\mathbf{m}$ (constant) satisfies, from Eq. (2.10),

$$\rho_0 v^2 \mathbf{m} = \mathbf{\Lambda m}, \qquad \Lambda_{ik} \equiv B_{ijkl} n_j n_l. \tag{2.11}$$

$\rho_0 v^2$ is the eigenvalue of $\mathbf{\Lambda}$ for the eigenvector $\mathbf{m}$.

Isotropy We consider materials that are isotropic in the undeformed state, although the preferred axes of the pre-stress can introduce apparent anisotropy in the plane waves. In general, the eigenvalue equation (2.11) predicts one quasi-longitudinal wave in the direction of $\mathbf{n}$, and two quasi-transverse waves. We now summarise some of these effects.

[2]Thus, $\frac{\partial}{\partial X_j} = \frac{\partial}{\partial x_j} + u_{k,j}^{(0)} \frac{\partial}{\partial x_k} + \ldots$ where the remaining terms are of lower order in $u_{m,n}^{(0)}$.

The number of second and third order moduli, at most 21 and 56, respectively, reduces in the presence of material symmetry. For isotropic solids we may use $\text{tr}\,\mathbf{E}^k$, $k = 1, 2, 3$ as the three invariants, so that the expansion of the strain energy to third order is

$$\rho_0 U = \frac{\lambda}{2}(\text{tr }\mathbf{E})^2 + \mu\,\text{tr }\mathbf{E}^2 + \frac{C}{3}(\text{tr }\mathbf{E})^3 + B(\text{tr }\mathbf{E})\text{ tr }\mathbf{E}^2 + \frac{A}{3}\text{ tr }\mathbf{E}^3 + \cdots . \tag{2.12}$$

Here λ and μ are the Lamé moduli and A, B, C are the third order moduli, in the notation of Landau and Lifshitz (1970). The isotropic moduli have the form

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + 2\mu I_{ijkl} \,, & \text{(2.13a)}\\ C_{ijklmn} &= 2C\delta_{ij}\delta_{kl}\delta_{mn} + 2B(\delta_{ij}I_{klmn} + \delta_{kl}I_{mnij} + \delta_{mn}I_{ijkl}) \\ &\quad + \frac{1}{2}A(\delta_{ik}I_{jlmn} + \delta_{il}I_{jkmn} + \delta_{jk}I_{ilmn} + \delta_{jl}I_{ikmn}) \,. & \text{(2.13b)} \end{aligned}$$

where $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

With no loss in generality let the coordinate axes coincide with the principal axes of static stress $\sigma_{ij}^{(0)} = C_{ijkl}e_{kl}^{(0)}$ and the principal axes of static strain[3] $e_{ij}^{(0)}$.

For propagation direction $\mathbf{n} = \mathbf{e}_1$ it follows from (2.11) that the eigenvectors (polarizations) are in the coordinate directions and the corresponding propagation speeds are

$$\begin{aligned} \rho_0 v_l^2 &= \rho_0 c_l^2 + \sigma_{11}^{(0)} + (4\rho_0 c_l^2 + c_{111})e_{11}^{(0)} + c_{112}(e_{22}^{(0)} + e_{33}^{(0)}) \,, & \text{(2.14a)}\\ \rho_0 v_{t2}^2 &= \rho_0 c_t^2 + \sigma_{11}^{(0)} + (2\rho_0 c_t^2 + c_{166})(e_{11}^{(0)} + e_{22}^{(0)}) + c_{144}e_{33}^{(0)} \,, & \text{(2.14b)}\\ \rho_0 v_{t3}^2 &= \rho_0 c_t^2 + \sigma_{11}^{(0)} + (2\rho_0 c_t^2 + c_{166})(e_{11}^{(0)} + e_{33}^{(0)}) + c_{144}e_{22}^{(0)} \,. & \text{(2.14c)} \end{aligned}$$

Here, c_l and c_t are the unperturbed longitudinal and transverse wave speeds: $\rho_0 c_t^2 = \mu$, $\rho_0 c_l^2 = \lambda + 2\mu$.

Pressure derivatives Acoustoelastic measurements are normally performed by varying the applied static stress according to a single parameter p, such as a hydrostatic pressurization, $\boldsymbol{\sigma}^{(0)} = -p\mathbf{I}$, or a uniaxial compression in the direction $\mathbf{t}$, $\boldsymbol{\sigma}^{(0)} = -p\,\mathbf{t}\otimes\mathbf{t}$. Some specific applications of the above formulae are listed in Table 1, which gives the dimensionless derivative $\rho_0(dv^2/dp)_0$ for states of hydrostatic pressurization and uniaxial compression (the subscript 0 on the derivative indicates evaluation at $p = 0$).

Values of the pressure derivatives of velocity are listed in Table 2. The values for rock (Berea sandstone) are noticeably larger than the others, indicating a high degree of nonlinearity. We will see this is a natural consequence once we understand the nonlinear elasticity of granular materials.

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[3]Stress and strain are coaxial for isotropy but not in general.

Table 2. Dimensionless dependence of sound speed on pressure for some common materials. Data compiled from the literature, see Norris (1998).

Material	$\rho_0(dv_l^2/dp)_0$	$\rho_0(dv_t^2/dp)_0$
Pyrex	−8.6	−2.84
Fused Silica	−4.32	−1.42
Water	5.0	0
Steel	7.45	1.46
Polystyrene	11.6	1.57
Aluminum	12.4	2.92
PMMA	15.0	3.0
Cemented glass beads	288	84
Berea sandstone	1628	956

3 Granular materials - preliminaries

3.1 Two spheres in contact

We need two basic results: The normal displacement on the surface of a half-space from a unit point force directed into the solid (the z-direction) is (Timoshenko, 1970)

$$u(x, y, 0) = \frac{1-\nu}{2\pi\mu\sqrt{x^2+y^2}}, \tag{3.1}$$

where μ is the shear modulus and ν Poisson's ratio. The second basic identity is

$$\int_0^{2\pi} \mathrm{d}\,\theta \int_0^a \frac{r'\,\mathrm{d}\,r'\,\sqrt{a^2-r'^2}}{\sqrt{r^2+r'^2-2rr'\cos\theta}} = \frac{\pi^2}{2}\big(a^2 - \frac{r^2}{2}\big), \quad r \le a. \tag{3.2}$$

This is a consequence of a result in potential theory for the interior of an ellipsoid of uniform charge density.

The fundamental solution for the normal force on a traction free half-space is employed to analyze the fundamental problem in the mechanics of granular media. Two spherical particles of radii R_1 and R_2 are pressed together under equal and opposite forces of magnitude N. The spheres just touch at a single point under zero applied load. The problem is to find (i) how far the spheres approach one another, (ii) the size of the contact zone, and (iii) the stress distribution across the region of contact. Each of these will be functions of N, R_1, R_2 and the elastic constants of the spheres.

We first examine the kinematics of the contact zone. Let $z = 0$ be the tangent plane of the unloaded spheres. Radial symmetry about the $z-$axis is assumed, so that all quantities will depend upon x and y through $r = \sqrt{x^2+y^2}$. Several simplifying approximations are necessary. First, the contact region is small in size, with radius a much less that either of the sphere radii: $a \ll$min(R_1, R_2). Accordingly, the surfaces of the two spheres are approximated by parabolas in and near the contact region. Introduce coordinates z_1 and z_2 directed into the respective spheres, Figure 1. The free surfaces of the unloaded spheres are therefore described by[4] $z_I = r^2/(2R_I)$, $I = 1, 2$.

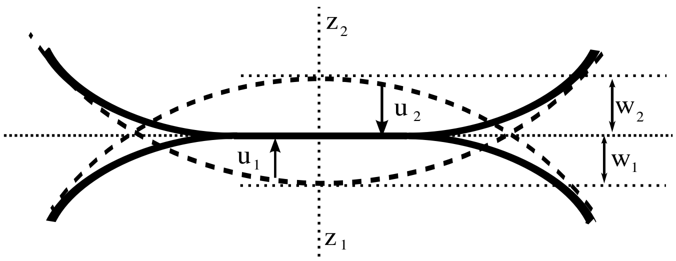

Figure 1. The two spheres under a confining normal force N.

Under the applied load the elastic material deforms and the surfaces in and near the contact region are distorted, becoming flat inside the contact region $0 \le r \le a$. At the same time, the spheres approach one another by w_1 and w_2, defined as the distances that the centers of the spheres move. Since these are far from the contact region, it is clear that the motion of most of each sphere is as a rigid body, with the elastic deformation confined to the contact region and its neighborhood.

Let $u_1(r)$ and $u_2(r)$ denote the distance that the surface of each sphere in the contact zone is moved towards the center of that sphere in the flattening process. Let $q(r) \ge 0$ be the load acting on the contact zone. Since a is much less than either R_1 or R_2, the normal displacements u_1 and u_2 can be approximated by the equivalent surface displacements of the flat surface of a semi-infinite elastic material, subject to a prescribed loading over the circular region $0 \le r \le a$, although at this stage both the loading and the extent of the contact region are unknowns. Thus, according to this local approximation, the fundamental solution (3.1) may be used to represent u_1 and u_2 as integrals of the unknown load distribution over the unknown contact area,

$$u_I(r) = \frac{(1-\nu_I)}{2\pi\mu_I} \int_0^{2\pi} \mathrm{d}\,\theta \int_0^a \frac{r'\,\mathrm{d}\,r'\,q(r')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\,, \qquad I = 1, 2. \tag{3.3}$$

Hence $u_1(r)$ and $u_2(r)$ are proportional.

Consider two points on either side of the contact zone at $r \le a$. Before the compression they were a distance $[z_1(r) + z_2(r)]$ apart. But the pressing causes them to move a total distance towards each other of $(w_1 + w_2)$ for the rigid body motion, minus $[u_1(r) + u_2(r)]$ due to the elastic deformation. Thus,

$$u_1(r) + u_2(r) = w_1 + w_2 - z_1(r) - z_2(r), \quad 0 \le r \le a. \tag{3.4}$$

[4]These can be derived as follows: The surface of sphere 1 is spherical with equation $r^2 + (z_1 - R_1)^2 = R_1^2$, equivalently $z_1 = \frac{r^2}{2R_1} + \frac{z_1^2}{2R_1}$. This can be solved by iteration, $z_1 = \frac{r^2}{2R_1} + \frac{r^4}{8R_1^3} + \ldots$ for $r \ll R_1$.

Let $2w \equiv w_1 + w_2$ represent the total approach of the two spheres, then

$$\frac{4}{\pi C_n} \int_0^{2\pi} \mathrm{d}\,\theta \int_0^a \frac{r' \,\mathrm{d}\, r' \, q(r')}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta}} = 2w - \frac{r^2}{R}\,, \tag{3.5}$$

where C_n (n for normal force) is the elastic modulus that governs the interaction and R is the harmonic average of the sphere radii,

$$C_n = \left(\frac{1-\nu_1}{8\mu_1} + \frac{1-\nu_2}{8\mu_2} \right)^{-1}, \qquad R = \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right)^{-1}. \tag{3.6}$$

The unknowns here are the contact radius a, the half-distance of approach w, and most importantly the distribution of load $q(r)$. Equation (3.5) looks like a formidable integral equation with several unknowns. However, the solution can be found using the identity (3.2), which suggests that we take $q(r)$ proportional to $\sqrt{a^2 - r^2}$:

$$q(r) = \frac{q_0}{a} \sqrt{a^2 - r^2}\,. \tag{3.7}$$

Then equation (3.2) implies

$$\frac{2\pi q_0}{a C_n} \big(a^2 - \frac{r^2}{2} \big) = 2w - \frac{r^2}{R}\,. \tag{3.8}$$

At the same time, the load should have resultant N, i.e.,

$$2\pi \int_0^a r \,\mathrm{d}\, r \, q(r) = N \qquad \Rightarrow \qquad \frac{2}{3} \pi a^2 q_0 = N. \tag{3.9}$$

Equation (3.8) provides two further relations, which follow by equating the coefficients of r^2 and of r^0. Together with (3.9) these are sufficient to find the three unknowns a, w and q_0:

$$a = \left(\frac{3NR}{2C_n} \right)^{1/3}, \quad w = \left(\frac{9N^2}{4RC_n^2} \right)^{1/3}, \quad q_0 = \frac{1}{\pi} \left(\frac{3NC_n^2}{2R^2} \right)^{1/3}. \tag{3.10}$$

Thus, w may be expressed in a form independent of the elastic properties.

$$w = \frac{a^2}{R}. \tag{3.11}$$

That is, the spheres approach twice the distance achieved by slicing off caps of radius a on each.

Note the dependence on the load N,

$$a \sim N^{1/3}, \quad w \sim N^{2/3}, \quad q_0 \sim N^{1/3}. \tag{3.12}$$

In particular, w is a nonlinear function of N, which has important implications for the elastic behavior of materials made of contacting grains. The starting point for later analysis is the nonlinear relation relating the normal force to the approach distance w,

$$N = \frac{2}{3} C_n \, R^{1/2} w^{3/2}. \tag{3.13}$$

Small additional changes are related by

$$\Delta N = C_n a \, \Delta w. \tag{3.14}$$

This linear relation and others like it will be used in Section 5 when we consider packings of spheres. Landau and Lifshitz (1970) consider non-spherical contact for which the bodies are smooth and locally ellipsoidal but do not necessarily have coincident principal axes. They show that the contact zone is elliptical.

3.2 Waves in a 1D chain of spheres in compression

Consider a linear chain of identical spheres or beads of radius R, under a compressive force N. This results in each adjacent pair of beads approaching by total distance $2w$ defined by $(3.10)_2$ where $C_n = 4\mu/(1-\nu)$. The elastic deformation occurs in the region of contact, with most of the bead undeformed but undergoing rigid body motion. We can therefore think of the system as a chain of point masses each of mass $m = \frac{4}{3}\pi R^2 \rho$. Let $u_n(t)$ be the displacement of the n-th sphere from its equilibrium position, then the motion of that bead follows from (3.13) as

$$\ddot{u}_n = A\Big([2w - (u_n - u_{n-1})]^{3/2} - [2w - (u_{n+1} - u_n)]^{3/2} \Big), \qquad A = \frac{2\sqrt{2} C_n}{\pi R^{5/2} \rho}. \tag{3.15}$$

The linear case is defined by $|u_{n+1} - u_n| \ll w$, for which (3.15) reduces to the dynamic equilibrium equation for a 1D system of nearest neighbour masses coupled by linear springs,

$$\ddot{u}_n = 3A\sqrt{\frac{w}{2}}\, \big(u_{n+1} - 2u_n + u_{n-1}\big). \tag{3.16}$$

We consider the continuum limit of lengths (such as wavelengths) much larger than the bead size. Let $u_n(t) \to u(2nR, t)$, then the right member in (3.16) is clearly related to $\partial^2 u / \partial x^2$, i.e.,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{3.17}$$

where $c^2 = A6\sqrt{2w}R^2$. Thus, the speed of linear disturbances is

$$c = \frac{3}{\sqrt{\pi \rho}} \left(\frac{4C_n}{3R} \right)^{1/3} N^{1/6}. \tag{3.18}$$

Coste and Gilles (1999) also consider waves that propagate due to an applied force (impact) in the absence of pre-compression. They show that a solitary-like wave is caused by an impact, and the wave travels with speed $\propto N_m^{1/6}$ where N_m is the maximum force of the solitary wave. The speed is given by (3.18) with $N = 0.303N_m$. Experimental data agreed with these predications.

4 Generalizations: tangential force and other contact models

We are interested in the finite and infinitesimal elasticity of an ensemble of spheres, for which the single pair in contact under normal and tangential loads describes the fundamental mechanics. In this section we consider more realistic force contact models, including the crucial role of tangential (shear) forces.

4.1 Overview of theories for intergranular forces

Several contact models can be considered in the same framework. For simplicity, the spheres have the same radius R, and $2w(\geq 0)$ is the mutual approach along the line joining their centers. The vector $2\mathbf{s}$ describes the relative tangential displacement between the two spheres. Figure 4.1 shows the setup. The force may be decomposed into a normal force N and tangential force $\mathbf{T}$.

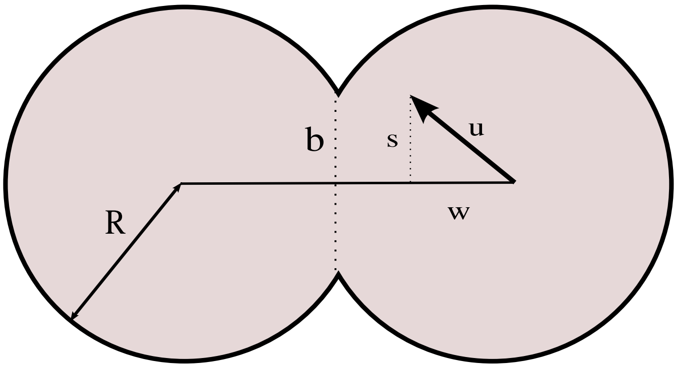

Figure 2. Two spheres in contact. The radius of the initial unstressed circle of contact, b may or may not be zero, depending on the model considered. The center of one sphere is displaced relative to the other by a vector $\mathbf{u}$ which is resolved into component $2w$ along the line of centers, and $2s$ perpendicular to it.

Our starting point is the incremental relations between the forces and the displacements,

$$\Delta N = D_n(w)\,\Delta w, \qquad \Delta \mathbf{T} = D_t(w)\,\Delta \mathbf{s}, \tag{4.1}$$

where D_n and D_t are contact stiffnesses. These are of the form

$$D_n = C_n\,a_n(w), \qquad D_t = C_t\,a_t(w), \tag{4.2}$$

where C_n and C_t are actual stiffnesses (i.e. with units of pressure)

$$C_n = \frac{4\mu}{1-\nu}, \qquad C_t = \frac{8\mu}{2-\nu}. \tag{4.3}$$

The lengths a_n and a_t depend upon the specific type of contact but do *not* depend upon the material properties of the spheres. Several models are discussed next, and the functional forms for a_n and a_t are summarized in Table 3. All models considered have smooth contacts with reversible slip or rough contacts with no relative slip. Both lengths are defined by the current value of w, but are independent of $\mathbf{s}$. Note also that the properties of each contact, though calculated within the approximations of ordinary linear elasticity theory, lead to nonlinear restoring forces. As we shall see, these nonlinear forces can, in turn, lead to extremely large nonlinear elastic constants for the aggregate media.

Table 3. The lengths a_n and a_t for different contact conditions. Models I, II, and III have two sub-cases, corresponding to: (a) smooth contact with reversible slip; and (b) rough contact with no subsequent slip.

	Contact model	$a_n(w)$	$a_t(w)$ (a)	$a_t(w)$ (b)
I	Hertzian contact	$(Rw)^{\frac{1}{2}}$	0	a_n
II	Initial contact radius b (Digby)	$[R^2w^2 + \frac{b^4}{4})^{\frac{1}{2}} + \frac{b^2}{2}]^{\frac{1}{2}}$	b	a_n
III	Ogival contact $l = R\alpha\frac{\pi}{2}$ (Spence/Goddard)	$(Rw + l^2)^{\frac{1}{2}} - l$	0	a_n
IV	Frictional sliding (Mindlin and Deresiewicz)	$(Rw)^{\frac{1}{2}}$	$(\frac{\theta}{a_n} + \frac{1-\theta}{c})^{-1}$	

All the models in Table 3 have $a_t \le a_n$, with equality for the infinitely rough contacts. When $a_t = a_n$ the ratio D_n/D_t reduces to $C_n/C_t = 1+\nu/(2-2\nu)$, which is approximately 1.17 for rocks ($\nu = 1/4$). This value is not consistent with velocity data[5].

We now discuss the results summarized in Table 3, dealing first with the oblique force constants.

4.2 Oblique loading on a pair of spheres

Consider two spheres forced together under a load N_0, resulting in a circular contact zone of radius a and a normal displacement w as discussed in section 3.1. Let a_0 be the initial contact radius for applied force N_0. An *o*blique compressive force is now applied,

[5]Domenico (1977) reports velocity data for unconsolidated glass beads under pressure. Winkler (1983) demonstrated that Domenico's data yields a value for the ratio D_n/D_t ranging from 1.79 to 3.36, which is consistent with a contact model with $a_t < a_n$. This is one possible justification for a contact model that allows for frictionless sliding in tangential deformation, as in the Digby model IIa for which $a_t = 0$.

resulting in a tangential force $\mathbf{T}$ and total normal force N. The additional force is applied incrementally in such a manner that its line of action is constant, and it lies *outside* the friction cone, that is

$$\frac{\mathrm{d}\,T}{\mathrm{d}\,N} = \beta \geq f, \tag{4.4}$$

where β is constant and f is the coefficient of friction between the surfaces. This type of loading explicitly leads to slip, as we will see, but it also includes the case of zero slippage in the limit as $\beta \to f$. The pertinent parameter is

$$\theta = f/\beta, \qquad 0 \leq \theta \leq 1, \tag{4.5}$$

where the limit $\theta = 1$ corresponds to no slip anywhere within the contact zone. This will also be described as infinite roughness.

As the load changes according to (4.4), the contact zone grows,

$$a = (N/N_0)^{1/3}\, a_0 = (1+\theta L)^{1/3}\, a_0, \tag{4.6}$$

where $L = T/(fN_0)$. At the same time a slip zone in the shape of a circular annulus grows inward[6], so that the adhered region is of radius c where

$$c = \big(1 - T/(fN)\big)^{1/3}\, a = [1-(1-\theta)L]^{1/3}\, a_0. \tag{4.7}$$

Thus, $c \leq a_0$, as expected, with equality only for infinitely rough surfaces.

The change in relative tangential displacement, $2s$, is related to the applied load by the differential relation[7] (Mindlin and Deresiewicz, 1953, eq. (82))

$$\frac{\mathrm{d}\,T}{\mathrm{d}\,s} = C_t \left(\frac{\theta}{a} + \frac{1-\theta}{c}\right)^{-1}. \tag{4.8}$$

There are three limits of this general relation that are of interest.

(i) If the surfaces are frictionless, then $c \equiv 0$ identically (there is no adhered region) and case Ia of Table 3 is obtained.

(ii) If the friction is infinite, or $\theta = 1$, then $\mathbf{T}$ satisfies $(4.1)_2$ with $a_n = a_0$. If we integrate this relation with the initial condition that $N_0 = 0$ (which only makes sense for infinite roughness), we recover (Walton, 1987, eq. $(2.5)_1$)

$$\mathbf{T} = \frac{2}{3}\, C_t\, a\, \mathbf{s}. \tag{4.9}$$

Thus, Walton's model corresponds to homothetic loading from zero confining stress. This is model Ib in Table 3. The more general relation in equation (4.8) allows us to consider different load paths and finite friction. However, we will use (4.9) in practice.

[6]The largest shear stresses occur at the boundary - they are in fact singular (Mindlin, 1949)

[7]Mindlin and Deresiewicz provided an exhaustive analysis of the mechanics near the contact region of two elastic spheres. This work (Mindlin, 1949; Mindlin and Deresiewicz, 1953; Deresiewicz, 1958) extended the classical Hertz analysis to account for tangential forces and oblique loading.

(iii) Alternatively, if the normal load is kept fixed as the tangential force is applied, then $\theta = 0$, and we have instead

$$\mathrm{d}\,\mathbf{T} = C_t\, c\, \mathrm{d}\,\mathbf{s}. \tag{4.10}$$

This is the case considered by Mindlin in equation (103) of (Mindlin, 1949), and is model IV in Table 3.

4.3 Other models for normal loading of a pair of spheres in contact

The models of the previous subsection can be generalized to account for initial contact. Digby (1981) considered the case of a circular zone of radius b of initial contact between the spheres at zero confining force. He showed that the contact zone for a compressive normal force N is a circular region of radius a given by the formula

$$a\,(a^2 - b^2)^{1/2} = Rw. \tag{4.11}$$

This gives the formula for $a_n = a$ in Table 3, for model II.

Another solvable model is that of the *ogival* indenter, composed of a sphere with a conical tip of interior angle $\pi - 2\alpha$, $0 \leq \alpha \ll 1$, pressed into a sphere of the same radius[8]. The contact zone radius is given by (Spence, 1968, Appendix D)

$$a^2 + 2al = Rw, \tag{4.12}$$

where $l = R\alpha\frac{\pi}{2}$. This follows by solving for the normal force required to press the ogival body a distance w against a planar surface of the same material.

4.4 Path dependence

The incremental rule for $\Delta\mathbf{T}$ in (4.1) does not hold for all deformation paths in the $w - \mathbf{s}$ space. Generally, the increment in $\Delta\mathbf{T}$ depends upon whether w is increasing or decreasing, with different rules applicable in each case. However, it can be demonstrated (Johnson and Norris, 1997) that the distinction disappears for the models of Table 3 if the trajectory is self repeating - that is, it retraces itself whenever w is decreasing. We will only consider paths which are self repeating in this sense.

Thus, $\mathbf{T}$ does not possess a unique functional form for models Ib, IIb, IIIb, and IV. Rather, it depends upon the path history of the loading, and is only defined along a given path in the $w-\mathbf{s}$ space. The path dependence vanishes only for constant tangential stiffness. In order to integrate the force-displacement equations for a single contact with path dependence it is necessary to assume some relationship between w and $\mathbf{s} = \mathbf{s}(w)$. We can then rewrite the incremental force relations (4.1) as

$$\mathrm{d}N = C_n\, a_n(w)\, \mathrm{d}w, \qquad \mathrm{d}\mathbf{T} = C_t\, a_t(w)\, \mathrm{d}\mathbf{s}. \tag{4.13}$$

[8] The ogival/spherical contact geometry was discussed by Goddard (1990), and is based on a class of solutions generated by Spence (1968) for frictionless indentation of self-similar shapes on planar surfaces.

The contact forces can be determined by integrating these three equations subject to the initial conditions $N = \mathbf{T} = 0$ at $w = 0$, yielding

$$N = C_n\, A_n(w), \qquad \mathbf{T} = C_t \int_{path} a_t(w)\; d\mathbf{s} \tag{4.14}$$

where A_n is a path independent quantity having the dimension of area:

$$A_n(w) = \int_0^w a_n(\xi)\, \mathrm{d}\,\xi \quad . \tag{4.15}$$

The precise form of $\mathbf{T}$ therefore depends upon the path $\mathbf{s}(w)$. For example, consider a linear relationship,

$$\frac{\mathrm{d}\,\mathbf{s}}{\mathrm{d}\,w} = \text{constant}. \tag{4.16}$$

Equations (4.14) become

$$N = C_n\, A_n(w), \qquad \mathbf{T} = C_t\, a_t^{(1)}(w)\, \mathbf{s}, \tag{4.17}$$

where $a_t^{(1)}$ also has the dimension of a length

$$a_t^{(1)}(w) = \frac{1}{w} \int_0^w a_t(\xi)\, \mathrm{d}\,\xi. \tag{4.18}$$

it should be emphasized that w and $\mathbf{s}$ are *not* independent variables in Eq. (4.17). The constraint (4.16) implies that the spheres approach one another along a constantly directed line. Alternatively, one could assume that the line of action of the force at each contact remains constant. That is,

$$\frac{\mathrm{d}\,\mathbf{T}}{\mathrm{d}\,N} = \text{constant}, \tag{4.19}$$

and the constant can differ from contact to contact. The two constraints (4.16) and (4.19) are equivalent only if the ratio D_t/D_n is constant along the loading path. This is the case for the path dependent models Ib, IIb, and IIIb, but not for IV. We will use (4.16) later for specific examples, but emphasize that the subsequent analysis applies to arbitrary self repeating paths $\mathbf{s} = \mathbf{s}(w)$.

4.5 Elastic energy of a pair of spheres

Before considering the ensemble of grains, we need one last quantity - the stored energy of the pair of compressed spheres. Let $\mathbf{f}$ be the total force exerted by a sphere at a single contact, and let $\mathbf{u}$ be the total displacement of the center of the sphere from its original position. The sphere then moves a further distance d$\mathbf{u}$. The work done by the sphere associated with that contact, assuming it does not rotate, is

$$\mathrm{d}W \;=\; \mathbf{f} \cdot \mathrm{d}\,\mathbf{u} \;=\; N\mathrm{d}w + \mathbf{T} \cdot \mathrm{d}\mathbf{s}. \tag{4.20}$$

Alternatively, the potential energy for a single contact is $2\mathrm{d}W$, when viewed in terms of the two spheres each contributing $\mathrm{d}W$. We speak of a potential energy function for the path dependent models with the understanding that we are restricting our discussion to motion on one specific path. The work done along that path is a conserved quantity; going down the path is the reverse of going up.

For either the path independent models, or assuming (4.16) for the path dependent ones, the explicit nature of N and T in (4.17) and (4.20) implies that

$$W = C_n\, V_n(w) + \frac{1}{2}\, C_t\, a_t^{(2)}(w)\, s^2, \tag{4.21}$$

where V_n is a volume and $a_t^{(2)}$ a length,

$$V_n(w) = \int_0^w \mathrm{d}\,\zeta \int_0^\zeta a_n(\xi)\,\mathrm{d}\,\xi, \qquad a_t^{(2)}(w) = \frac{2}{w^2}\int_0^w \mathrm{d}\,\zeta \int_0^\zeta a_t(\xi)\,\mathrm{d}\,\xi. \tag{4.22}$$

We have set $W = 0$ at the initial point, with no loss in generality. Thus, W is one half of the energy stored in a single contact. The result (4.21) is valid for all contact models, with the understanding that w and $s = |\mathbf{s}|$ are related for the path dependent models.

The length a_n, area A_n, and volume V_n are fundamental quantities, as is the dimensionless quantity $a_n'(w)$ which occurs later. Explicit formulae for $a_n'(w)$, A_n, and V_n are given in Table 4.

Table 4. The quantities a_n', A_n, and V_n for the contact models of Table 3.

Model	$a_n'(w)$	$A_n(w)$	$V_n(w)$
I	$\frac{R}{2a_n} = \frac{1}{2}R^{1/2}w^{-1/2}$	$\frac{2}{3}\,wa_n = \frac{2}{3}R^{1/2}w^{3/2}$	$\frac{4}{15}\,w^2 a_n = \frac{4}{15}R^{1/2}w^{5/2}$
II	$\frac{R(a_n^2-b^2)^{1/2}}{2a_n^2-b^2}$	$\frac{2}{3}\,w\left(a_n + \frac{b^2}{2a_n}\right)$	$\frac{4}{15}\,w^2\left(a_n + \frac{b^2}{a_n}\right) - \frac{(a_n-b)b^4}{15R^2}$
III	$\frac{R}{2(a_n+d)}$	$\frac{2}{3}\,a_n w - \frac{a_n^2 d}{3R}$	$\frac{2}{5}\,\frac{A_n}{R}(a_n+d)^2 - \frac{w^2 d}{10}$

5 Stress-strain relations for a granular medium

5.1 Kinematic assumption for an ensemble of spheres

In order to connect the 2-sphere models to the elastic response of a random packing of spheres, we make a kinematic assumption relating the displacement of a single sphere to the macroscopic deformation gradient $\mathbf{F}$. The displacement of the center of a given sphere is

$$u_i = H_{ij}X_j, \quad \text{or} \quad \mathbf{u} = \mathbf{H}\cdot\mathbf{X}, \tag{5.1}$$

where $\mathbf{F} = \mathbf{I} + \mathbf{H}$ and $\mathbf{X}$ is the position of the center of the sphere. Let $\mathbf{n}$ be the unit vector joining the centers of two contacting spheres, at $\mathbf{X}$ and $\mathbf{X} + 2R\mathbf{n}$. The associated displacements are $\mathbf{H}\cdot\mathbf{X}$ and $\mathbf{H}\cdot\mathbf{X}+2R\mathbf{f}\cdot\mathbf{n}$, and so the components of the relative displacement are

$$w = -\mathbf{n}\cdot\mathbf{H}\cdot\mathbf{n}R, \qquad \mathbf{s} = \mathbf{Q}\cdot\mathbf{H}\cdot\mathbf{n}\,R, \tag{5.2}$$

where $\mathbf{Q}=\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$ projects onto the tangent plane.

The total strain energy density per unit volume is

$$U \;=\; \frac{1}{V}\sum_{contacts} W \;=\; \frac{1}{V}\sum_{contacts}\int \mathbf{f}\cdot \mathrm{d}\,\mathbf{u}, \tag{5.3}$$

where the sum is over all contacts on each sphere (each contact is counted twice since W is only half the contact energy), and V is the total volume of the sample. The effective medium approximation assumes that all grains are statistically the same and each may be replaced by its ensemble average. Let N_g be the number of grains in the volume V, and define $\langle\cdots\rangle$ as the average over the distribution of directions $\mathbf{n}$. Therefore,

$$U \;=\; \frac{n(1-\phi)}{V_0}\,\langle W[w(\mathbf{n}),s(\mathbf{n})]\rangle \;=\; \frac{n(1-\phi)}{V_0}\,\Big\langle \int \mathbf{f}\cdot\mathrm{d}\,\mathbf{u}\Big\rangle, \tag{5.4}$$

where n is the number of contacts per grain, V_0 is the volume of a single sphere, and ϕ is the porosity of the sample[9],

$$V_0 = \frac{4}{3}\pi R^3, \quad 1-\phi = \frac{N_g V_0}{V}. \tag{5.5}$$

We furthermore assume an isotropic distribution: $Q(\mathbf{n}) = 1/4\pi$.

A note on rotation The skew symmetric part of the deformation gradient is associated with rigid body rotation, which does not influence the energy on the microscale. In order to make explicit this assumption we rewrite (5.2) as

$$w = -\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R, \qquad \mathbf{s} = \mathbf{Qen}\,R, \tag{5.6}$$

where $\mathbf{e}$ is the symmetric part of the macroscopic linear strain,

$$\mathbf{e} \equiv \frac{1}{2}\left(\mathbf{H}+\mathbf{H}^T\right), \tag{5.7}$$

and we shall write $\mathbf{Qen}$ as a shorthand for the vector $\mathbf{Q}\cdot\mathbf{e}\cdot\mathbf{n}$ (w is unaffected). In summary, within the mean field approximation adopted here only the symmetric part of the macroscopic deformation gradient influences the internal strain energy and so we need consider only symmetric deformations.

We first examine the simpler case where the contacts are not path dependent, and then consider the path dependent models. We will see there is not much formal difference in *some* of the results, but it is worth doing them separately as the distinction is important. The first is hyperelastic, and therefore we can immediately derive second and third order elastic constants - the ingredients of acoustoelasticty. This is not the case for the path dependent models, which we will see leads to a different formulation for acoustoelastic effects.

[9] Strictly speaking, Eqs. (5.5) do not apply to the Digby model II because of the missing spherical caps, nor to the Spence/Goddard model III because the ogive volume is not the same as the sphere's.

5.2 Elastic moduli at finite strain

In the following we consider $\mathbf{e}$ as the finite strain. The deviation from the true finite strain $\mathbf{E}$ is of second order in $\mathbf{e}$ (or $\mathbf{H}$). But, as we will argue shortly when we examine the relative magnitudes of the second and third order moduli, the difference may be safely ignored for our purposes. The deformation energy for the path independent contact models follows directly from Eq. (5.4), using Eqs. (4.21) and (5.6):

$$U = (1-\phi)\frac{n}{V_0}\left[C_n\left\langle V_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\right\rangle + \frac{1}{2}\,C_t\,R^2\left\langle a_t^{(2)}(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\,|\mathbf{Qen}|^2\right\rangle\right]. \tag{5.8}$$

The stress follows immediately from

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \mathbf{e}}, \tag{5.9}$$

as

$$\sigma_{ij} = (1-\phi)\frac{nR}{V_0}\left[C_t a_t\,R\left\langle\frac{1}{2}(n_iQ_{jk}+n_jQ_{ik})e_{kl}n_l\right\rangle - C_n\left\langle A_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)n_in_j\right\rangle\right]. \tag{5.10}$$

This gives the stress at finite strain $\mathbf{e}$ for models Ia, IIa, and IIIa.

If a further *small* strain, $\boldsymbol{\epsilon}$, is superimposed upon $\mathbf{e}$ then the resulting stress can be found by simply substituting $\mathbf{e}+\boldsymbol{\epsilon}$ for $\mathbf{e}$ in (5.10). Let the total stress be $\boldsymbol{\sigma}+\tau$, then τ is a nonlinear function of $\boldsymbol{\epsilon}$, which can be expanded in a Taylor series about $\boldsymbol{\epsilon}=0$. The linear coefficients are, by definition, the coefficients of linear elasticity at the finite strain $\mathbf{e}$ and can be obtained directly by inspection from the strain energy expansion. Thus,

$$U(\mathbf{e}+\boldsymbol{\epsilon}) = U(\mathbf{e}) + \sigma_{ij}\epsilon_{ij} + \frac{1}{2}C^*_{ijkl}\epsilon_{ij}\epsilon_{kl} + \frac{1}{6}C^*_{ijklmn}\epsilon_{ij}\epsilon_{kl}\epsilon_{mn} + \ldots, \tag{5.11}$$

where C^*_{ijkl} and C^*_{ijklmn} are the second and third order elastic moduli, respectively. The * is used to distinguish them from the unstressed moduli of (2.5). They satisfy the usual symmetries associated with a material possessing a strain energy:

$$C^*_{ijkl} = C^*_{jikl}, \qquad C^*_{ijkl} = C^*_{klij}, \tag{5.12a}$$

$$C^*_{ijklmn} = C^*_{jiklmn}, \quad C^*_{ijklmn} = C^*_{klijmn} = C^*_{mnklij}. \tag{5.12b}$$

The first identity is a statement of the symmetry of the stress.

Explicit expansion of the strain energy, using eqs. (5.8), (5.9), and (5.11), yields

$$C^*_{ijkl} = (1-\phi)\frac{nR^2}{V_0}\left[\left\langle\,[C_na_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R) - C_ta_t]\;n_in_jn_kn_l\right\rangle + C_ta_t\left\langle Q_{ijkl}(\mathbf{n})\right\rangle\right], \tag{5.13}$$

where

$$Q_{ijkl}(\mathbf{n}) \equiv \frac{1}{4}\left(\delta_{ik}n_jn_l + \delta_{il}n_jn_k + \delta_{jl}n_in_k + \delta_{jk}n_in_l\right). \tag{5.14}$$

Similarly, the third order constants are given by

$$C^*_{ijklmn} = -(1-\phi)\frac{3n}{4\pi}\,C_n\left\langle a_n'(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\,n_in_jn_kn_ln_mn_n\right\rangle\;. \tag{5.15}$$

The tangential stiffness does not contribute to the third order moduli because it has been assumed to be a Hookean spring.

Scaling and finite strain: Granular media have large TOE Consider the sample subject to a strain of order $\epsilon \ll 1$. The change in the sample dimension is $\Delta V/V = O(\epsilon)$, whereas the relative change in the stiffness of the system is of order $\epsilon\, C^*_{ijklmn}/C^*_{pqrs}$. The relative magnitude of the second and third order moduli follows from equations (5.13) and (5.15) and Tables 3 and 4:

$$\frac{\text{Second order moduli}}{\text{Third order moduli}} = \mathrm{O}\left(\frac{a_n(w)}{Ra_n'(w)}\right) = \mathrm{O}\left(\frac{a_n^2}{R^2}\right) \ll 1\,. \tag{5.16}$$

The relative stiffness change far exceeds the relative volume change, which is a fundamental characteristic of granular media with Hertzian contact forces. It is manifested in the relatively large change of elastic wave speeds as a function of confining stress, which we saw in the data of Table 2 for Berea sandstone, and will discuss below using the contact model.

We have so far omitted any mention of the finite strain tensor, defined as $\mathbf{E} = \mathbf{e} + \frac{1}{2}\mathbf{H}^T\mathbf{H}$. This is normally the fundamental quantity of finite elasticity, in particular, a hyperelastic material is a function of $\mathbf{E}$. The strain energy is assumed to have a power series expansion of symbolic form $U = U_0 + C_2EE + C_3EEE + \ldots$, where C_2, C_3 represent second and third order moduli. Alternatively, expanding in terms of $\mathbf{H}$, we have

$$U = U_0 + C_2ee + C_3eee + \mathrm{O}(C_2eHH,\, C_3eeHH)\,. \tag{5.17}$$

The scaling of the moduli in (5.16) indicates that the correction term, C_2eHH, is negligible in comparison with C_3eee, and hence it is entirely consistent to take the energy as $U = U_0 + C_2ee + C_3eee$, correct to third order in the finite strain. Consequently, there is no necessity to distinguish the finite strain $\mathbf{E}$ from the linear strain $\mathbf{e}$, even for the purpose of discussing nonlinear elastic effects (up to the order considered here).

5.3 Elasticity for path dependent models

A general result for stress There is a simple method for determining the stress which is worth mention. Consider the surface integral of traction×displacement over the exterior surface of the granular medium

$$\int_{\partial V} \mathbf{t}\cdot\mathbf{u}\,\mathrm{d}S = \int_{\partial V} \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{u}\,\mathrm{d}S, \tag{5.18}$$

This can recast as a volume integral over the total volume of grains, V, which in turn can be converted again into the sum of surface integrals of traction×displacement for each grain/sphere. Take $\mathbf{u} = \mathbf{e}\cdot\mathbf{X}$, for constant strain $\mathbf{e}$ (see (5.1)), then we have

$$V\boldsymbol{\sigma} : \mathbf{e} = \sum_{grains} \int_{\partial V_i} \mathbf{t}\cdot\mathbf{u}\,\mathrm{d}S. \tag{5.19}$$

The surface integral of traction on each grain reduces to the contact zones, which can be replaced by the resultant forces. At the same time, the displacement at the contact satisfies the mean field assumption. As a result, using the arbitrary form of $\mathbf{e}$, a symmetric tensor, we obtain

$$\boldsymbol{\sigma} = \frac{R}{2V} \sum_{contacts} (\mathbf{f} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{f}). \tag{5.20}$$

This is an exact relation for point contacts, commonly used to derive stress (Digby, 1981; Walton, 1987).

In order to apply (5.20) to the granular pack, we assume the contact path history is parameterized according to $w = \xi$, $\mathbf{s} = \mathbf{s}(\xi)$. The path may differ from contact to contact and is subject only to the constraints that it is self repeating and the end points are given by Eq. (5.6). Then, using equation (4.14),

$$\mathbf{f} = -C_n\, A_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\,\mathbf{n} + C_t \int_{path} a_t(\xi)\; d\mathbf{s}(\xi)\,, \tag{5.21}$$

and the stress follows accordingly as

$$\boldsymbol{\sigma} = (1-\phi)\frac{nR}{V_0}\left[\frac{C_t}{2}\Big\langle \int_{path} a_t(\xi)\; (\mathbf{n}\otimes \mathrm{d}\,\mathbf{s} + \mathrm{d}\,\mathbf{s}\otimes\mathbf{n})\Big\rangle - C_n\,\big\langle A_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\mathbf{n}\otimes\mathbf{n}\big\rangle\right], \tag{5.22}$$

This is valid for arbitrary deformation paths, subject to the end condition $(5.6)_2$ for $\mathbf{s}$.

As an example, consider the linear $\mathbf{s}$ trajectory of Eq. (4.16), for which (5.22) reduces to

$$\sigma_{ij} = (1-\phi)\frac{nR}{V_0}\left[C_t\, R\big\langle a_t^{(1)}(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\, \frac{1}{2}(n_iQ_{jk}+n_jQ_{ik})e_{kl}n_l\big\rangle - C_n\,\big\langle A_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)n_in_j\big\rangle\right]. \tag{5.23}$$

Note that the stress for the energy models, Eq. (5.10), is a special case of (5.23), and follows from the latter by replacing the variable length $a_t^{(1)}$ with its constant value appropriate to the energy model.

Small on large The path dependent strain energy for a single contact, W of (4.21), is not a function of w and $\mathbf{s}$, and hence cannot be differentiated arbitrarily. It is, however, possible to consider variations in W for arbitrary unconstrained changes in w and $\mathbf{s}$ about their equilibrium values, w_0 and $\mathbf{s}_0$. Let $w = w_0 + w_1$ and $\mathbf{s} = \mathbf{s}_0 + \mathbf{s}_1$, where $|w_1| \ll |w_0|$ and $|\mathbf{s}_1| \ll |\mathbf{s}_0|$, and the extra displacements w_1 and $\mathbf{s}_1$ are unrelated. Using $\mathrm{d}\,W = N(w,\mathbf{s})\,\mathrm{d}\,w + \mathbf{T}(w,\mathbf{s})\cdot\mathrm{d}\,\mathbf{s}$, implies

$$W = W_0(w_0,\mathbf{s}_0) + N_0(w_0,\mathbf{s}_0)\,w_1 + \mathbf{T}_0(w_0,\mathbf{s}_0)\cdot\mathbf{s}_1 + \int_0^{w_1} N_1\,\mathrm{d}\,w_1 + \int_0^{\mathbf{s}_1} \mathbf{T}_1\cdot\mathrm{d}\,\mathbf{s}_1, \tag{5.24}$$

where $N = N_0 + N_1$ and $\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_1$. The values of $W_0, \mathbf{T}_0, N_0$ may depend on the loading history $\mathbf{s}(w)$ to that point. The incremental force equations are, from (4.13),

$$\mathrm{d}\,N_1 = C_n\,a_n(w_0 + w_1)\;\mathrm{d}\,w_1,, \qquad \mathrm{d}\,\mathbf{T}_1 = C_t\,a_t(w_0+w_1)\;\mathrm{d}\,\mathbf{s}_1\,. \tag{5.25}$$

Expanding and integrating gives

$$N_1(w_0, \mathbf{s}_0; w_1, \mathbf{s}_1) = C_n \left(a_n(w_0)\, w_1 + \frac{1}{2} a_n'(w_0)\, w_1^2 + \ldots \right), \tag{5.26a}$$

$$\mathbf{T}_1(w_0, \mathbf{s}_0; w_1, \mathbf{s}_1) = C_t \left(a_t(w_0)\, \mathbf{s}_1 + a_t'(w_0) \int_0^{\mathbf{s}_1} w_1 \mathrm{d}\,\mathbf{s} + \ldots \right). \tag{5.26b}$$

The form of N_1 is unambiguous because of the fact that N is always a function of w. On the contrary, the integral in $\mathbf{T}_1$ depends upon the loading path, or equivalently, upon the functional relationship between w_1 and $\mathbf{s}_1$, if any. This integral is second order in the additional displacements; the first order term $C_t a_t(w_0)\,\mathbf{s}_1$ does not depend on the loading history of $(w_1, \mathbf{s}_1)$. Substituting for N_1 and $\mathbf{T}_1$ into (5.24) we see that the incremental strain energy is defined only up to second order in the incremental displacements, but not to third or higher order, *i.e.*,

$$\begin{aligned} W(w_0 + w_1, \mathbf{s}_0 + \mathbf{s}_1) = {} & W_0(w_0, \mathbf{s}_0) + C_n\, A_n(w_0)\, w_1 + C_t\, a_t^{(1)}(w_0)\, \mathbf{s}_0 \cdot \mathbf{s}_1 \\ & + \frac{1}{2} C_n a_n(w_0)\, w_1^2 + \frac{1}{2} C_t a_t(w_0)\, s_1^2 \\ & + \text{ path dependent third order terms in } (w_1, \mathbf{s}_1) \ . \end{aligned} \tag{5.27}$$

This distinguishes the path dependent models from those with strain energy functions. The concept of second order elasticity is valid for path dependent models but we cannot use or define third order elasticity.

Elastic moduli Consider a departure from the stress/strain state $(\boldsymbol{\sigma}, \mathbf{e})$, such that the total strain and stress are $\mathbf{e} + \boldsymbol{\epsilon}$ and $\boldsymbol{\sigma} + \tau$, respectively, where $|\boldsymbol{\epsilon}| \ll |\mathbf{e}|$ and $|\tau| \ll |\boldsymbol{\sigma}|$. The force at a contact becomes $\mathbf{f} + \mathbf{g}$, where $|\mathbf{g}| \ll |\mathbf{f}|$, and the associated incremental displacement $2\mathbf{v}$ of the pair of spheres relative to one another is given by

$$\mathbf{v} = R\,\boldsymbol{\epsilon} \cdot \mathbf{n}. \tag{5.28}$$

The increment in energy for a single contact is therefore $\mathrm{d}\,W = (\mathbf{f} + \mathbf{g}) \cdot \mathrm{d}\,\mathbf{v}$, implying

$$W = W_0 + \mathbf{f} \cdot \mathbf{v} + \int \mathbf{g} \cdot \mathrm{d}\,\mathbf{v}, \tag{5.29}$$

where W_0 is the (path dependent) energy for $\boldsymbol{\epsilon} = 0$, as given by equations (4.21) and (5.2), for example. The total energy density of the aggregate is

$$U = U_0 + \frac{R}{V} \sum_{contacts} \mathbf{f} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} + \frac{R}{V} \sum_{contacts} \int \mathbf{g} \cdot \mathrm{d}\,\boldsymbol{\epsilon} \cdot \mathbf{n}\,, \tag{5.30}$$

where U_0 is the strain energy at $\boldsymbol{\epsilon} = 0$, given by (5.8).

The arbitrary nature of in $\boldsymbol{\epsilon}$ allows us to express the total stress as

$$\boldsymbol{\sigma} + \boldsymbol{\tau} = \frac{\partial U}{\partial \boldsymbol{\epsilon}}. \tag{5.31}$$

The effective stress at strain $\mathbf{e}$ is therefore

$$\boldsymbol{\sigma} = \left.\frac{\partial U}{\partial \boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon}=0}, \tag{5.32}$$

in agreement with (5.20). The total energy density may be rewritten, from equations (5.30) and (5.32),

$$U = U_0 + \boldsymbol{\sigma} : \boldsymbol{\epsilon} + U_1, \tag{5.33}$$

where U_1 is defined as the ultimate term in (5.30). Finally, the incremental stress is

$$\boldsymbol{\tau} = \frac{\partial U_1}{\partial \boldsymbol{\epsilon}}. \tag{5.34}$$

The incremental behavior depends upon the additional strain and stress in excess of $\mathbf{e}$ and $\boldsymbol{\sigma}$, the crucial quantity being the extra contact force $\mathbf{g}$. It follows from the preceding analysis for the single contact that $\mathbf{g}$ is only path independent to first order in the incremental strain. The associated linear form is

$$\mathbf{g} = -RC_n a_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\,(-\mathbf{n}\cdot\boldsymbol{\epsilon}\cdot\mathbf{n})\,\mathbf{n} + RC_t a_t(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\,\mathbf{Q}\boldsymbol{\epsilon}\mathbf{n}. \tag{5.35}$$

The integral in (5.30) can now be evaluated, yielding

$$U_1 = \frac{1}{2} C^*_{ijkl}\epsilon_{ij}\epsilon_{kl}, \tag{5.36}$$

where the stiffness is

$$\begin{aligned} C^*_{ijkl}(\mathbf{e}) = (1-\phi)\frac{nR^2}{V_0}\Big[&\big\langle\, [C_n a_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R) - C_t a_t(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)]\; n_i n_j n_k n_l \big\rangle \\ &+ \big\langle C_t a_t(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\, Q_{ijkl}(\mathbf{n})\big\rangle\Big]. \end{aligned} \tag{5.37}$$

We emphasize that equation (5.37) is always a well-defined, path independent function of the macroscopic strain, $\mathbf{e}$, even for the path dependent models. Of course, the stress $\boldsymbol{\sigma}$ corresponding to that strain $\mathbf{e}$ may well be path dependent. Note that the moduli C^*_{ijkl} possess the usual symmetries associated with an elastic material, (5.12a).

6 Acoustoelasticity of granular media

The wave speeds for small motion superimposed upon the large strain $\mathbf{e}$ are defined by the effective moduli C^*_{ijkl} and the effective density

$$\rho^* = (1-\phi)\rho, \tag{6.1}$$

where ρ is the granular density. We are interested in the incremental change in speed, Δv, when the strain is changed to $\mathbf{e} + \Delta\mathbf{e}$. The additional strain arises from a static deformation, and need not be proportional to the original, finite strain $\mathbf{e}$.

For the models with energy potentials we can use the standard theory of acoustoelasticity as developed in Section 2. Thus, eqs. (2.9) and (2.10) imply

$$\rho^* \,\Delta v^2 = C^*_{ijklmn}\, n_j n_l m_i m_k \,\Delta F_{mn} + \big(C^*_{ikqr}\Delta F_{qr}\delta_{jl} + C^*_{qjkl}\Delta F_{iq} + C^*_{iqkl}\Delta F_{jq} + C^*_{ijkl}\Delta F_{kq} + C^*_{ijkq}\Delta F_{lq}\big) m_i m_l n_j n_k \,, \tag{6.2}$$

where $\mathbf{\Delta F}$ ($= \mathbf{\Delta H}$) is the incremental deformation gradient and $C^*_{ijklmn}(\mathbf{e})$ are the third order moduli. As in Section 2, the wave or phase normal is $\mathbf{n}$, and the polarization direction is $\mathbf{m}$. Based on Eq. (5.16) we can safely and consistently ignore the terms in (6.2) involving the second order moduli as being of a smaller order than the third order moduli terms for the granular medium. Furthermore, using the symmetries of C^*_{ijklmn} in (5.15) implies,

$$\rho^* \,\Delta v^2 = C^*_{ijklmn}(\mathbf{e})\, n_j n_l m_i m_k \,\Delta e_{mn}, \tag{6.3}$$

that is, the change in speed depends only upon the symmetric strain increment defined in accordance with equation (5.7).

The analogous result for the path dependent models can be obtained by returning to the fundamental relation for the incremental strain energy of a single contact, (5.27). At issue is how the coefficients of w_1^2 and s_1^2 are altered as we change w_0 to $w_0 + \Delta w_0$, and $\mathbf{s}_0$ to $\mathbf{s}_0 + \Delta\mathbf{s}_0$. The change in the terms W_0 and $a^{(1)}(w_0)$ in (5.27) are path dependent, and require that the increment in $\mathbf{s}_0$ be related to that for w_0. However, this path dependence does not effect the terms of interest, *i.e.*, we simply replace the arguments of a_n and a_t with $w_0 + \Delta w_0$ in the quadratic terms. The increment in W involving the quadratic small strain is therefore

$$\Delta W = \ldots + \frac{1}{2}\left(C_n a_n'(w_0)\, w_1^2 + C_t a_t'(w_0)\, s_1^2\right)\, \Delta w_0. \tag{6.4}$$

When we translate this result to the aggregate, it is clear that the change in the wave speed is of the form

$$\rho^* \,\Delta v^2 = B^*_{ijklmn}\,(\mathbf{e}) m_j m_l p_i p_k \,\Delta e_{mn}, \tag{6.5}$$

where B^*_{ijklmn} are simply the derivatives of $C^*_{ijkl}(\mathbf{e})$, given by eq. (5.37),

$$B^*_{ijklmn}(\mathbf{e}) = \frac{\partial C^*_{ijkl}(\mathbf{e})}{\partial e_{mn}}. \tag{6.6}$$

The explicit form follows from equations (5.37) and (6.6),

$$B^*_{ijklmn} = -(1-\phi)\frac{3n}{4\pi}\Big[\big\langle\, [C_n a_n'(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R) - C_t a_t'(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)]\; n_i n_j n_k n_l n_m n_n\big\rangle + C_t\big\langle a_t'(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\, Q_{ijkl} n_m n_n\big\rangle\Big]\,. \tag{6.7}$$

Comparing equations (5.15) and (6.7) we see that

$$B^*_{ijklmn} = C^*_{ijklmn} \;\text{ if and only if } a_t \text{ is constant.} \tag{6.8}$$

In general, the third order tensor B^*_{ijklmn} does not have all the symmetries of the third order elastic moduli C^*_{ijklmn} in (5.12a). Thus,

$$B^*_{jiklmn} = B^*_{ijklmn}, \quad B^*_{klijmn} = B^*_{ijklmn}, \quad B^*_{ijklnm} = B^*_{ijklmn}. \tag{6.9}$$

6.1 Examples: Application of the granular medium theory

Hydrostatic confining pressure Consider a hydrostatic strain, $\mathbf{e} = e\,\mathbf{I}$, $e < 0$. The macroscopic stress $\boldsymbol{\sigma}$ of (5.23) is hydrostatic, $\boldsymbol{\sigma} = -p\,\mathbf{I}$, and the confining pressure is

$$p = (1-\phi)\,\frac{nR}{3V_0}\,C_n A_n(-eR). \tag{6.10}$$

This holds for both the energy models and the path dependent models, because it is independent of the tangential stiffness. The effective moduli are isotropic with two second order and (when applicable) three third order moduli. The Lamé moduli are

$$\left.\begin{array}{l}\lambda^* \\ \\ \mu^*\end{array}\right\} = (1-\phi)\frac{n}{20\pi R} \times \left\{\begin{array}{l} C_n a_n(-eR) - C_t a_t(-eR), \\ \\ C_n a_n(-eR) + \frac{3}{2} C_t a_t(-eR). \end{array}\right. \tag{6.11}$$

The speeds of propagation of small amplitude compressional and shear waves are

$$v_c = \left(\frac{\lambda^* + 2\mu^*}{\rho^*}\right)^{\frac{1}{2}}, \qquad v_s = \left(\frac{\mu^*}{\rho^*}\right)^{\frac{1}{2}}, \tag{6.12}$$

respectively. Confining pressure, rather than dilatation, can be used as the load parameter, $\mathrm{d}\,p/\,\mathrm{d}\,e = -3K^*$, where $K^* \equiv \lambda^* + 2\mu^*/3$ is the effective bulk modulus. Figure 6.1 shows a comparison of pressure dependence with some data.

Departures from linear elasticity for the path dependent models requires the moduli B^*_{ijklmn}, which are of the form

$$B^*_{ijklmn} = B_{ijklmn} + \widetilde{B}\, I_{ijkl}\, \delta_{mn}. \tag{6.13}$$

B_{ijklmn} satisfy all the symmetries of third order elastic moduli, *viz.* eq. (5.12a), and

$$\left.\begin{array}{l} B_{111} \\ \\ B_{112} \\ \\ B_{123} \\ \\ \widetilde{B} \end{array}\right\} = \frac{-(1-\phi)\,n}{140\pi} \times \left\{\begin{array}{l} 15C_n\, a_n'(-eR) - C_t\, a_t'(-eR)\,, \\ \\ 3C_n\, a_n'(-eR) - 3C_t\, a_t'(-eR)\,, \\ \\ C_n\, a_n'(-eR) - C_t\, a_t'(-eR)\,, \\ \\ 7C_t\, a_t'(-eR)\,. \end{array}\right. \tag{6.14}$$

These contain the limiting form of the TOE for the energy models: set $a_t' \to 0$, then $B^*_{ijklmn} \to C^*_{ijklmn}$.

As an example, the changes in longitudinal and transverse wave speeds follow from the general expression (6.5),

$$\rho^*\,\frac{\mathrm{d}\,v_c^2}{\mathrm{d}\,p} = -\frac{(B^*_{111} + 2B^*_{112})}{3K^*}, \qquad \rho^*\,\frac{\mathrm{d}\,v_s^2}{\mathrm{d}\,p} = -\frac{(B^*_{441} + 2B^*_{442})}{3K^*}, \tag{6.15}$$

where $B^*_{441} = (B_{112} - B_{123} + \widetilde{B})/2$, $B^*_{442} = (B_{111} - B_{112} + 2\widetilde{B})/4$.

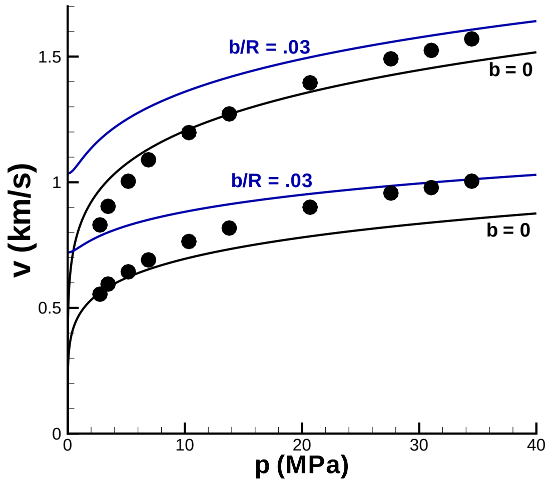

Figure 3. Compressional and shear speeds in unsintered glass beads as a function of confining pressure. The continuous lines are calculated results based on the Digby model of grain-grain contacts. The values $\mu = 24\,GPa$, $\nu = 0.25$, and $\rho_s = 2.42\,gm \cdot cm^{-3}$ are used, appropriate for silica glass. Also $\phi = 0.37$ (Domenico, 1977) and $n = 9$ is assumed. The symbols represent the experimental data of Domenico (1977) which should be compared against the $b = 0$ curves. Based on Norris and Johnson (1997).

Stress induced anisotropy The directional averages in, e.g., (5.37), are not isotropic if the applied stress/strain is not hydrostatic. For instance, if

$$e_{ij} = e_0\delta_{ij} + e_3\delta_{i3}\delta_{j3}, \tag{6.16}$$

then C^*_{ijkl} have the symmetries of a transversely isotropic material with axes in the $3-$direction. A further complication is that the directional averages are no longer as explicit as before. For instance, C^*_{3333} involves

$$\big\langle a_n(-\mathbf{n}\cdot\mathbf{e}\cdot\mathbf{n}R)\,n_3^4\big\rangle = \frac{1}{2}\int_0^\pi \mathrm{d}\,\theta \sin\theta\, a_n(-e_0R - e_3R\cos^2\theta)\,\cos^4\theta\,. \tag{6.17}$$

This and the other integrals are performed in (Johnson et al., 1998) where comparisons with experimental data are also presented showing the expected anisotropy.

7 Acoustoelasticity of solid/fluid systems

We consider the effects of initial deformation on waves in a heterogeneous body comprising constituents that are either inviscid fluid or elastic solid. The inviscid nature of the fluid presents subtle and interesting effects. For instance, an applied static deformation causes material particles to slip relative to one another, with the result that material particles originally side-by-side become distant relatives. The interfaces between the distinct fluid and solid regions therefore represent possible *material slip* surfaces, *i.e.*, surfaces across which the material deformation is not necessarily continuous. In order to fix ideas, we will consider in detail the small-on-large theory for the speed of a tube wave[10] in the presence of prestress.

Acoustoelasticity of fluid/solid systems is complicated not only by the slip effect at interfaces, but also the spatially inhomogeneous nature of the solution. The prestress/prestrain is not the same in the fluid and solid regions, and the wave solution is not a simple plane wave. The tube wave, for instance, decays radially to zero in the solid surrounding the fluid filled bore. The inhomogeneity introduces difficulties that are surmountable within the theory of acoustoelasticity by using a regular perturbation approach. We consider this in Section 8 after we have dealt with the more technical and physically interesting issues of slip.

7.1 Interface notation

Three configurations are again distinguished: the reference, intermediate and current. The intermediate configuration is obtained by the static deformation of each point according to $\mathbf{X} \to \mathbf{x}(\mathbf{X})$. This mapping is not necessarily continuous, and gives rise to the possibility of interfaces across which material particles slip relative to one another. A slip interface is defined by a surface in the reference configuration $\{\mathcal{L}_0 : f(\mathbf{X}) = 0\}$, such that $\mathcal{L}_0$ forms a bounding surface between the disjoint sets of points on either side. The slip condition means that

$$\lim_{\epsilon\downarrow 0} \mathbf{x}(\mathbf{X} - \epsilon\mathbf{N}^{(1)}) \neq \lim_{\epsilon\downarrow 0} \mathbf{x}(\mathbf{X} - \epsilon\mathbf{N}^{(2)}), \qquad \mathbf{X} \in \mathcal{L}_0, \tag{7.1}$$

where $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ are unit normals to $\mathcal{L}_0$. The points on either side of $\mathcal{L}_0$ define material surfaces for the respective regions in the reference configuration. We assume that the static deformation maintains these material surfaces as bounding surfaces, *i.e.*, that the limits on (7.1), though distinct, are both elements of the same deformed interface $\widetilde{\mathcal{L}}$. The static deformation then forms *two* distinct bijective mappings between $\mathcal{L}_0(\mathbf{X})$ and $\widetilde{\mathcal{L}}(\mathbf{x})$, defined by the unequal members in (7.1). Our interest here is in systems where the slip interface separates an inviscid compressible fluid from an elastic solid. Define $\mathcal{L}_{0f}$ and $\mathcal{L}_{0s}$ as the fluid and solid surfaces that coincide with $\mathcal{L}_0$. We define $\widetilde{\mathcal{L}}_f$ and $\widetilde{\mathcal{L}}_s$ by analogy, and note the identities $\mathcal{L}_0 = \mathcal{L}_{0f} \cup \mathcal{L}_{0s}$ and $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_f \cup \widetilde{\mathcal{L}}_s$. An important consequence of the slip is that $\widetilde{\mathcal{L}}_f$ *does not* coincide with $\widetilde{\mathcal{L}}_s$.

Consider, for example, a pressurized borehole in which a column of fluid surrounded by solid is compressed in a piston-like manner in the static deformation. In this case $\mathcal{L}_{0f}$ and $\mathcal{L}_{0s}$ are defined by the cylindrical fluid/solid interface. The solid undergoes no strain in the axial direction under the static deformation, hence the axial extents of $\widetilde{\mathcal{L}}_s$ and $\mathcal{L}_{0s}$ are the same. However, the axial length of $\widetilde{\mathcal{L}}_f$ is diminished compared with $\mathcal{L}_{0f}$

[10] A tube wave is simply the lowest order acoustic mode for a wave propagating down an infinitely long fluid filled circular hole in an isotropic solid.

because of the smaller fluid surface meeting the solid after compression, so that $\widetilde{\mathcal{L}}_f \subset \widetilde{\mathcal{L}}_s$ in this particular case. Obviously, $\widetilde{\mathcal{L}}_s \subset \widetilde{\mathcal{L}}_f$ would hold if the fluid were to undergo an expansion (negative compression).

The basic measure of stress is now taken as the total Piola-Kirchhoff stress tensor $\mathbf{T}$, sometimes known as the nominal stress and the transpose $\mathbf{T}^T$ is the Piola-Kirchhoff stress of the first kind, *i.e.*, the stress $\mathbf{P}$ in Section 2. The total stress can be expressed in the intermediate and current coordinates as $\widetilde{\mathbf{T}}$ and $\boldsymbol{\sigma}$, respectively, with

$$\widetilde{T}_{\alpha j} = (\tilde{\rho}/\rho_0)\, x_{\alpha,L}\, T_{Lj}, \qquad \sigma_{ij} = (\rho/\rho_0)\, x'_{i,L}\, T_{Lj}\,, \tag{7.2}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ is the Cauchy stress tensor. By definition $\rho_0\,\mathrm{d}\,V_0 = \tilde{\rho}\,\mathrm{d}\,\widetilde{V} = \rho\,\mathrm{d}\,V$, and hence the densities are related by $\rho_0 = \tilde{\rho}\,\det(x_{\alpha,L}) = \rho\det(x'_{i,L})$. The point form of the equations of motion in the three descriptions are,

$$T_{Lj,L}(\mathbf{X}) = \rho_0\, u_{j,tt}, \qquad \mathbf{X} \in V_0, \tag{7.3a}$$
$$\widetilde{T}_{\alpha j,\alpha}(\mathbf{x}) = \tilde{\rho}\, u_{j,tt}, \qquad \mathbf{x} \in \widetilde{V}, \tag{7.3b}$$
$$\sigma_{ij,i}(\mathbf{x}') = \rho\, u_{j,tt}, \qquad \mathbf{x}' \in V. \tag{7.3c}$$

The different notations for the three configurations are summarized in Table 5.

Table 5. Parameters in the three descriptions.

Description		Position	Volume/Surface	Density	Disps./Stresses
Reference	$\mathcal{R}_0$	$\mathbf{X}$	V_0 , S_0, $\mathbf{N}$, $\mathcal{L}_0$	ρ_0	u_j, T_{Mj}, $T^{(0)}_{Mj}$, $T^{(1)}_{Mj}$
Intermediate	$\widetilde{\mathcal{R}}$	$\mathbf{x}$	$\widetilde{V}$, $\widetilde{S}$, $\widetilde{\mathbf{N}}$, $\widetilde{\mathcal{L}}$	$\tilde{\rho}$	u_j, $\widetilde{T}_{\alpha j}$, $\widetilde{T}^{(0)}_{\alpha j}$, $\widetilde{T}^{(1)}_{\alpha j}$
Current	$\mathcal{R}$	$\mathbf{x}'$	V, S, $\mathbf{n}$, $\mathcal{L}$	ρ	u_j, σ_{ij} , $\sigma^{(0)}_{ij}$, $\sigma^{(1)}_{ij}$

The force acting across the material element of area $\mathrm{d}\,S_0$ with unit normal $\mathbf{N}$, is the same as the force on the current element of area $\mathrm{d}\,S$ with unit normal $\mathbf{n}$, where the current and reference directions and areas are related by

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,S_0} = (\det\mathbf{F})\,\left|\mathbf{N}\cdot\mathbf{F}^{-1}\right|, \qquad \mathbf{n} = \frac{\mathbf{N}\cdot\mathbf{F}^{-1}}{|\mathbf{N}\cdot\mathbf{F}^{-1}|}, \tag{7.4}$$

where $\mathbf{F}$ is the deformation gradient with components $F_{iK} = x'_{i,K}$. Similarly,

$$\mathbf{n}\cdot\boldsymbol{\sigma}\;\mathrm{d}\,S = \widetilde{\mathbf{N}}\cdot\widetilde{\mathbf{T}}\;\mathrm{d}\,\widetilde{S} = \mathbf{N}\cdot\mathbf{T}\;\mathrm{d}\,S_0\,, \tag{7.5}$$

where S_0, $\widetilde{S}$, and S refer to the same element of material surface. The traction $\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}$ is continuous across any interfaces in the medium, viewed as a function of current coordinates. However, the traction is *not* necessarily a continuous function of the reference coordinates.

7.2 Acoustoelasticity assumptions

The current state of the medium results from additional small acoustic disturbances in which material particles deform according to the $\mathbf{x} \rightarrow \mathbf{x}'$. We again distinguish the static and dynamic displacement fields, $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$, defined by eq. (2.1) which again partition the total displacement: $\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}$. Small-on-large stresses are defined by

$$T_{Mj}(\mathbf{X}) = T^{(0)}_{Mj}(\mathbf{X}) + T^{(1)}_{Mj}(\mathbf{X}) + \ldots, \tag{7.6a}$$

$$\widetilde{T}_{\alpha j}(\mathbf{x}) = \widetilde{T}^{(0)}_{\alpha j}(\mathbf{x}) + \widetilde{T}^{(1)}_{\alpha j}(\mathbf{x}) + \ldots, \tag{7.6b}$$

$$\sigma_{ij}(\mathbf{x}') = \sigma^{(0)}_{ij}(\mathbf{x}') + \sigma^{(1)}_{ij}(\mathbf{x}') + \ldots. \tag{7.6c}$$

The extra stresses $T^{(1)}_{Mj}$, $\widetilde{T}^{(1)}_{\alpha j}$, and $\sigma^{(1)}_{ij}$ are by assumption linear in the small strains. The prestress is in equilibrium, and therefore satisfies the equilibrium equations in any of the three coordinate systems,

$$T^{(0)}_{Mj,M} = \widetilde{T}^{(0)}_{\alpha j,\alpha} = \sigma^{(0)}_{ij,i} = 0. \tag{7.7}$$

7.3 Interface conditions

Consider an element of surface, with area $\mathrm{d}\,S_0$, $\mathrm{d}\,\widetilde{S}$, or $\mathrm{d}\,S$ in the configuration $\mathcal{R}_0$, $\widetilde{\mathcal{R}}$, or $\mathcal{R}$, respectively. The associated outward normals are defined by the unit vectors in Table 5. The relative surface areas are

$$\frac{\mathrm{d}\,\widetilde{S}}{\mathrm{d}\,S} = 1 - E^{(1)}_{\perp} + \ldots, \qquad \frac{\mathrm{d}\,S_0}{\mathrm{d}\,S} = 1 - E^{(0)}_{\perp} - E^{(1)}_{\perp} + \ldots, \tag{7.8}$$

where

$$E^{(0)}_{\perp} = u^{(0)}_{L,L} - N_K N_L\, u^{(0)}_{K,L}, \qquad E^{(1)}_{\perp} = u^{(1)}_{\alpha,\alpha} - \widetilde{N}_\alpha \widetilde{N}_\beta\, u^{(1)}_{\alpha,\beta}. \tag{7.9}$$

The total traction per unit current area can now be expressed in the alternative descriptions as

$$\mathbf{n} \cdot (\boldsymbol{\sigma}^{(0)} + \boldsymbol{\sigma}^{(1)}) = (1 - E^{(1)}_{\perp})\, \widetilde{\mathbf{N}} \cdot (\widetilde{\mathbf{T}}^{(0)} + \widetilde{\mathbf{T}}^{(1)})\,. \tag{7.10}$$

The traction expressions (7.10) are well-known although it is usually assumed that the surfaces remain bonded, and hence surface elements transform in the same manner on either side of a material interface. This assumption is *not* valid here. There is no slippage in bonded solid materials, and therefore the area changes $\mathrm{d}\,\widetilde{S}/\,\mathrm{d}\,S$ and $\mathrm{d}\,S_0/\,\mathrm{d}\,S$ in these identities are continuous across the interface. However, it is the lack of continuity of these functions in the presence of slip that is the key to our problem. The possibility of jumps in these quantities must be taken into account. At the same time, the slip of particles at the interface makes pointwise conditions difficult to formulate in reference and intermediate coordinates, because the traction condition must be applied at the same current point on either side of the interface.

Continuity of the total traction vector implies, using (7.10) and expanding subject to the asymptotic small-on-large procedure, that the intermediate stress tensor satisfies the continuity condition

$$\Big\{ \widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)} + \widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(1)} - E^{(1)}_{\perp}\, \widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)} \Big\}_{\mathcal{L}} = \mathbf{0}. \tag{7.11}$$

The notation $\{f\}_{\mathcal{L}}$ means the difference in the quantity f evaluated at adjacent points $\mathbf{x}' \pm \epsilon\mathbf{n}$, $\epsilon \to 0$, on either side of the interface $\mathcal{L}$. The jump condition (7.11) involves quantities at neighbouring current positions, not at neighbouring intermediate positions. The transformation from one to the other requires evaluating (7.11) at the different points $\mathbf{x} = \mathbf{x}' - \mathbf{u}^{(1)}$ for the same $\mathbf{x}'$ but different $\mathbf{u}^{(1)}$ on either side of $\mathcal{L}$. Split $\mathbf{u}^{(1)}$ into normal and tangential parts according to $\mathbf{u}^{(1)} = \mathbf{u}_n^{(1)} + \mathbf{u}_\perp^{(1)}$, so that

$$\mathbf{x} = (\mathbf{x}' - \mathbf{u}_n^{(1)}) - \mathbf{u}_\perp^{(1)}. \tag{7.12}$$

This permits us to convert the jump condition (7.11) to one defined on $\widetilde{\mathcal{L}}$. In so doing, the bracketed term on the right of equation (7.12) is considered constant, and the quantities in (7.11) have to be evaluated at points in $\widetilde{\mathcal{R}}$ separated by $\left\{-\mathbf{u}_\perp^{(1)}\right\}_{\mathcal{L}}$. This can only effect the first term in (7.11) because the changes in the other terms are negligible. Thus,

$$\left\{\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)}\right\}_{\mathcal{L}} \to \left\{\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)} - \mathbf{u}_\perp^{(1)} \cdot \nabla\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)}\right\}_{\widetilde{\mathcal{L}}} = \left\{-\mathbf{u}_\perp^{(1)} \cdot \nabla\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)}\right\}_{\widetilde{\mathcal{L}}}, \tag{7.13}$$

where we use the fact that the prestress is in equilibrium,

$$\left\{\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)}\right\}_{\widetilde{\mathcal{L}}} = \mathbf{0}. \tag{7.14}$$

The desired, converted form of (7.11) is therefore

$$\left\{\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(1)} - E_\perp^{(1)}\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)} - \mathbf{u}_\perp^{(1)} \cdot \nabla\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(0)}\right\}_{\widetilde{\mathcal{L}}} = \mathbf{0}. \tag{7.15}$$

The general continuity condition (7.15) is our main result; it applies to any interface conditions, whether bonded or not. The analogous jump condition across $\mathcal{L}_0$ is rather more complicated (Norris et al., 1994)

$$\left\{\mathbf{N} \cdot \mathbf{T}^{(1)} - E_\perp^{(1)}\mathbf{N} \cdot \mathbf{T}^{(0)} - \mathbf{u}_\perp^{(1)} \cdot \nabla\mathbf{N} \cdot \mathbf{T}^{(0)} - E_\perp^{(0)}\,\mathbf{N} \cdot \mathbf{T}^{(1)} - \mathbf{u}_\perp^{(0)} \cdot \nabla\mathbf{N} \cdot \mathbf{T}^{(1)}\right\}_{\mathcal{L}_0} = \mathbf{0}. \tag{7.16}$$

Both (7.15) and (7.16) reduce to the simple, standard forms

$$\left\{\widetilde{\mathbf{N}} \cdot \widetilde{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}} = \mathbf{0}, \qquad \left\{\mathbf{N} \cdot \mathbf{T}^{(1)}\right\}_{\mathcal{L}_0} = \mathbf{0}, \tag{7.17}$$

when the interface is bonded.

The same procedure use in deriving (7.15) can be applied immediately to give the analogous jump condition in the intermediate description for displacement:

$$\left\{\widetilde{\mathbf{N}} \cdot \mathbf{u}^{(1)} - E_\perp^{(1)}\widetilde{\mathbf{N}} \cdot \mathbf{u}^{(0)} - \mathbf{u}_\perp^{(1)} \cdot \nabla\widetilde{\mathbf{N}} \cdot \mathbf{u}^{(0)}\right\}_{\widetilde{\mathcal{L}}} = \mathbf{0}. \tag{7.18}$$

The corresponding condition in the original description follows from (7.16).

Next, we will see how the jump conditions can be incorporated into acoustoelasticity calculations of practical interest.

7.4 Reformulation of jump conditions in quasi-divergent form

The dynamical equations for the small motion follow from eqs. (7.3) and (7.6),

$$T^{(1)}_{Lj,L}(\mathbf{X}) = \rho_0\, u^{(1)}_{j,tt}, \qquad \mathbf{X} \in V_0, \tag{7.19a}$$

$$\widetilde{T}^{(1)}_{\alpha j,\alpha}(\mathbf{x}) = \tilde{\rho}\, u^{(1)}_{j,tt}, \qquad \mathbf{x} \in \widetilde{V}, \tag{7.19b}$$

$$\sigma^{(1)}_{ij,i}(\mathbf{x}') = \rho\, u^{(1)}_{j,tt}, \qquad \mathbf{x}' \in V. \tag{7.19c}$$

The equations in current coordinates lead to a natural or divergence formulation, in the sense that the interface conditions can be found by integrating the "diverged" stress, giving $\{\mathbf{n}\cdot\boldsymbol{\sigma}^{(1)}\}_{\mathcal{L}} = \mathbf{0}$. However, the position of the interface $\mathcal{L}$ is not known *a priori*, which makes this formulation of little use for acoustoelasticity. The other two formulations are not in divergence form, because the interface conditions (7.15) and (7.16) for stress and the associated displacement conditions are clearly unrelated to the differential equations of motion.

We now show how the equations (7.4b) in intermediate coordinates can be rewritten in what we call *quasi-divergent form.* Our purpose is to use the small-on-large equations in the same manner as for homogeneous solids, viz. to calculate the change in velocity of waves. The specific example in the next Section deals with the guided borehole wave, but other types are possible, e.g. the Scholte wave at a flat fluid-solid interface. The method employs an energy-type argument for a quadratic form in the unperturbed wave solution. The reader is advised to skip forward to peruse the integral perturbation approach. The crucial ingredient in the formulation is the ability to integrate across surfaces, which is automatic if the bulk equations are in divergent form and, most importantly, if the jump conditions are in accord with the equations of motion. The latter is not the case for our formulation, eqs. (7.15) and (7.18).

There is, however, a way to simplify these jump conditions. The integrand (the quadratic form) comprises solutions that are time harmonic and also sinusoidal in space. Terms that are "in-phase" therefore contribute, while those that are out of phase produce no energy when averaged over a cycle. Thus, if can reformulate the jump conditions into forms that leave only out-of-phase terms on the interface, then these can be ignored.

The objective then is to identify terms in the stress condition eq. (7.15) that are in- or out-of-phase with the leading term: $\left\{\widetilde{\mathbf{N}}\cdot\widetilde{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}}$. The phase is relative to the stress $\widetilde{\mathbf{T}}^{(1)}$, which comprises first spatial derivatives of displacement. It is therefore in-phase with $E^{(1)}_{\perp}$ and out of phase with $\mathbf{u}^{(1)}$. We therefore concentrate on the middle term in eq. (7.15). Similarly, the middle term in eq. (7.18) is out of phase with the leading, first term. Consequently, we only need to pay attention to the third term in eq. (7.15).

We make two assumptions on the prestress/prestrain:

(a) The fluid pressure $p^{(0)}$ is constant.

(b) The normal displacement at the interface $\widetilde{\mathbf{N}}\cdot\mathbf{u}^{(0)}$ is constant.

The first is always true for a fluid/solid composite medium in which the fluid phase is connected. A fluid saturated porous medium serves as a good example. The second assumption is a bit more restrictive, as it limits the types of interfaces. However, it includes the borehole, and the flat interface (Scholte waves) among others, and it could probably be relaxed to apply to other configurations. Condition (b) immediately implies that the out of phase third term in eq. (7.15) vanishes, and hence

$$\left\{\widetilde{\mathbf{N}}\cdot\mathbf{u}^{(1)}\right\} = \text{out of phase terms.} \tag{7.20}$$

We write this as

$$\left\{\widetilde{\mathbf{N}}\cdot\mathbf{u}^{(1)}\right\} := \mathbf{0}. \tag{7.21}$$

We therefore concentrate on the stress jump condition.

The homogeneous prestress in the fluid is

$$\widetilde{\mathbf{T}}^{(0)} = -p^{(0)}\,\mathbf{I}, \qquad \text{in the fluid.} \tag{7.22}$$

The inviscid nature of the fluid means that the initial traction on the interface is everywhere a normal stress of $-p^{(0)}$. Consider the modified stress

$$\widehat{\mathbf{T}}^{(1)} = \widetilde{\mathbf{T}}^{(1)} + p^{(0)}\,\left(\mathbf{I}\,\mathrm{div}\,\mathbf{u}^{(1)} - (\nabla\mathbf{u}^{(1)})^T\right). \tag{7.23}$$

This addition to $\widetilde{\mathbf{T}}^{(1)}$ is divergence free, and hence the equations of motion in intermediate coordinates, (7.4b) may be equally well written

$$\widehat{T}^{(1)}_{\alpha j,\alpha}(\mathbf{x}) = \tilde{\rho}\,u^{(1)}_{j,tt}, \qquad \mathbf{x}\in\widetilde{V}. \tag{7.24}$$

We will now demonstrate that the interface conditions (7.15) can be expressed as

$$\left\{\widetilde{\mathbf{N}}\cdot\widehat{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}} := \mathbf{0}. \tag{7.25}$$

The jump condition (7.15) may be rewritten, using (7.22) and (7.23), as

$$\left\{\widetilde{\mathbf{N}}\cdot\widehat{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}} + p^{(0)}\,\left\{(\mathbf{u}^{(1)}_{\perp}\cdot\nabla)\widetilde{\mathbf{N}}\right\}_{\widetilde{\mathcal{L}}} + p^{(0)}\,\left\{E^{(1)}_{\perp}\widetilde{\mathbf{N}} - \widetilde{\mathbf{N}}\,\mathrm{div}\,\mathbf{u}^{(1)} + \widetilde{\mathbf{N}}\cdot(\nabla\mathbf{u}^{(1)})^T\right\}_{\widetilde{\mathcal{L}}} = \mathbf{0}. \tag{7.26}$$

The differential operator $(\mathbf{u}^{(1)}_{\perp}\cdot\nabla)$ in the middle term of the right member is the same as $(\mathbf{u}^{(1)}.\nabla_{\perp})$ where $\nabla_{\perp}$ represents the in-surface derivatives. The final term in (7.26) can be simplified using this operator, to give

$$\left\{\widetilde{\mathbf{N}}\cdot\widehat{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}} + p^{(0)}\,\left\{(\mathbf{u}^{(1)}\cdot\nabla_{\perp})\widetilde{\mathbf{N}}\right\}_{\widetilde{\mathcal{L}}} + p^{(0)}\,\left\{\widetilde{\mathbf{N}}\cdot(\nabla_{\perp}\mathbf{u}^{(1)})^T\right\}_{\widetilde{\mathcal{L}}} = \mathbf{0}. \tag{7.27}$$

The second term is out of phase with the first, and can be ignored. The third term can be split into a sum of two terms, one in phase and one out of phase. Retaining the former and ignoring the latter gives

$$\left\{\widetilde{\mathbf{N}}\cdot\widehat{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}} + p^{(0)}\,\nabla_{\perp}\left\{\widetilde{\mathbf{N}}\cdot\mathbf{u}^{(1)}\right\}_{\widetilde{\mathcal{L}}} := \mathbf{0}. \tag{7.28}$$

The second term of the right member is the surface gradient of the jump (7.21). The latter identity in and of itself is not sufficient reason to ignore this term, since it is in phase with the stress jump $\left\{\widetilde{\mathbf{N}}\cdot\widehat{\mathbf{T}}^{(1)}\right\}_{\widetilde{\mathcal{L}}}$. However, the original equation for the displacement jump, (7.18), implies that the second term in (7.28) is of order ${p^{(0)}}^2\mathbf{u}^{(1)}$, and ignorable on that account. Hence, we have proved the equivalence of (7.15) and (7.25).

Small-on-large constitutive relations The linearized constitutive equations for the perturbed stresses in the intermediate description are,

$$\widetilde{T}^{(1)}_{\alpha j} = \widetilde{C}_{\alpha j \beta\gamma}\, u^{(1)}_{\gamma,\beta}, \tag{7.29}$$

where

$$\begin{aligned}\widetilde{C}_{\alpha j\beta\gamma} &= C_{\alpha j\beta\gamma} + T^{(0)}_{\alpha\beta}\,\delta_{j\gamma} - C_{\alpha j\beta\gamma}\, u^{(1)}_{\delta,\delta} + C_{\alpha j\beta\gamma Pl}\, u^{(1)}_{l,P} \\ &\quad + u^{(1)}_{\alpha,P}\, C_{Pj\beta\gamma} + u^{(1)}_{j,P}\, C_{\alpha P\beta\gamma} + u^{(1)}_{\beta,P}\, C_{\alpha jP\gamma} + u^{(1)}_{\gamma,P}\, C_{\alpha j\beta P}\,.\end{aligned} \tag{7.30}$$

These follow from, for instance, eq. (2.9) with the change of density $\rho_0 \rightarrow \tilde{\rho}$. The modified stress is then

$$\widehat{T}^{(1)}_{\alpha j} = (C_{\alpha j\beta\gamma} + \Delta C_{\alpha j\beta\gamma})\, u^{(1)}_{\gamma,\beta}, \tag{7.31}$$

where the change in stiffness follows from equations (7.23), (7.29), and (7.30), as

$$\begin{aligned}\Delta C_{ijkl} &= \widetilde{C}_{ijkl} + p^{(0)}\,(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) - C_{ijkl} \\ &= p^{(0)}\,(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) - C_{ijkl}\, u^{(0)}_{m,m} + C_{ikmn}\, u^{(0)}_{m,n}\delta_{jl} + C_{ijklmn}\, u^{(0)}_{m,n} \\ &\quad + C_{pjkl}\, u^{(0)}_{i,p} + C_{ipkl}\, u^{(0)}_{j,p} + C_{ijpl}\, u^{(0)}_{k,p} + C_{pjkp}\, u^{(0)}_{l,p}\,.\end{aligned} \tag{7.32}$$

We are now ready to consider acoustoelastic problems in composite fluid/solid systems.

8 Perturbation theory for small-on-large

The undeformed mode of the solid/fluid system is defined by the time dependent displacement field $\mathbf{u}^m$ and the deformed mode by $\mathbf{u}$. The original mode is time-harmonic with radial frequency ω^m, and the perturbed mode has frequency ω. The equations of motion for both modes can be formulated in Lagrangian, intermediate, or current coordinates. However, the current coordinates have the unavoidable difficulty that the interface position depends upon the solution $\mathbf{u}^{(1)}$. The Lagrangian coordinates do not have this problem, but the interface conditions present other serious issues. Furthermore, there is a conceptual difficulty dealing with Lagrangian coordinates for heterogeneous media: the "speed" of a small-on-large wave is not a useful concept in Lagrangian coordinates, as the case of a pressurized borehole demonstrates. In that case the axial deformation of the solid and inviscid fluid are different, so a wave speed defined relative to the original length of a fluid column is not the same as a wave speed referred to the solid.

We therefore perform the analysis entirely in intermediate coordinates, using the quasi-divergent formulation of the previous Section. This permits us to use standard methods from mode perturbation theory. For simplicity we drop the notation for the intermediate configuration ($\tilde{\mathbf{u}} \rightarrow \mathbf{u}$ etc.) and consider it as the current configuration, in the spirit of perturbation theory.

8.1 The perturbation integral

The idea is to consider the wave solution as a mode of the system. This is unambiguous in finite systems, and can be extended to infinite systems by considering solutions with fixed wave numbers. Our objective is the (modal) frequency, from which changes in wave speeds can be calculated.

Consider the perturbed and unperturbed equations of motion as, respectively,

$$\sigma_{ji,j}(\mathbf{x}) + \rho\omega^2\, u_i(\mathbf{x}) = 0, \tag{8.1a}$$

$$\sigma^m_{ji,j}(\mathbf{x}) + \rho_0(\omega^m)^2\, u^m_i(\mathbf{x}) = 0. \tag{8.1b}$$

Contract the first with $\mathbf{u}^m$ and the second with $\mathbf{u}$, and integrate the difference over an arbitrary volume V, yielding

$$\int_V [\rho\omega^2 - \rho_0(\omega^m)^2] u_j u^m_j \, \mathrm{d}V = \int_V \left(\sigma^m_{ji,j}\, u_i - \sigma_{ji,j}\, u^m_i\right) \mathrm{d}V. \tag{8.2}$$

Integrate by parts, using the fact that the equations are in quasi-divergence form which permits us to integrate "through" interfaces, and ignore remainder terms that are out-of-phase and have zero average. This gives

$$\int_V [\rho\omega^2 - \rho_0(\omega^m)^2] u_j u^m_j \, \mathrm{d}V = -\int_V \left(\sigma^m_{ji}\, u_{i,j} - \sigma_{ji}\, u^m_{i,j}\right) \mathrm{d}V. \tag{8.3}$$

We can now estimate the shift in modal frequency, using

$$\rho = \rho_0 + \Delta\rho, \qquad \mathbf{C} = \mathbf{C}_0 + \Delta\mathbf{C}, \qquad \omega = \omega^m + \Delta\omega, \tag{8.4}$$

where $\Delta\rho = -u^{(0)}_{k,k}\,\rho_0$ is the change in density associated with the prestrain, and $\Delta\mathbf{C}$ are the incremental elastic moduli - defined by the prestress. Substituting into (8.3), making the approximation $\mathbf{u} \approx \mathbf{u}^m$, and retaining only the terms linear in the deviations, gives

$$2\omega^m\,\Delta\omega = \frac{\int_V \Delta C_{ijkl}\, u^m_{i,j} u^m_{k,l}\,\mathrm{d}V}{\int_V \rho_0\, u^m_q u^m_q\,\mathrm{d}V} - (\omega^m)^2\,\frac{\int_V \Delta\rho\, u^m_i u^m_i\,\mathrm{d}V}{\int_V \rho_0\, u^m_q u^m_q\,\mathrm{d}V}. \tag{8.5}$$

Finally, the change in phase speed associated with the material change is

$$\frac{\Delta v}{v} = \frac{\Delta\omega}{\omega^m}. \tag{8.6}$$

Fluid-solid media The second-and third-order moduli (Kostek et al., 1993) of an inviscid fluid are[11]

$$C_{ijkl} = A_f\,\delta_{ij}\,\delta_{kl}, \tag{8.7a}$$

$$C_{ijklmn} = (A_f - B_f)\,\delta_{ij}\,\delta_{kl}\,\delta_{mn} - 2A_f\left[\delta_{ij} I_{klmn} + \delta_{kl} I_{ijmn} + \delta_{mn} I_{ijkl}\right], \tag{8.7b}$$

where A_f and B_f are the usual linear and nonlinear moduli of a fluid with $A_f = \rho_f c^2$ and c the acoustic sound speed. The static prestrain in the fluid must be both uniform (homogeneous) and symmetric, with the stress hydrostatic of pressure $p^{(0)}$. Thus, we may take the prestrain as $u^{(0)}_{i,j} = -p^{(0)}/(3A_f)\delta_{ij}$. The change in the fluid density is

[11] With $I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$.

$\Delta \rho_f = (p^{(0)}/A_f)\,\rho_f$, and the perturbation in A_f follows from equation (7.30) as $\Delta A_f = (1 + B_f/A_f)p^{(0)}$.

In applying equations (8.5) and (8.6) to fluid-solid media we need only consider the integrals over the solid portion V_s, because the effects of the fluid perturbations enter through the dependence of v on ρ_f and A_f. Hence,

$$\frac{\Delta v}{v} = \frac{p^{(0)}}{v}\left[\frac{\rho_f}{A_f}\frac{\partial v}{\partial \rho_f} + \left(1 + \frac{B_f}{A_f}\right)\frac{\partial v}{\partial A_f}\right] + \frac{\int_{V_s} \Delta C_{ijkl}\, u^m_{i,j} u^m_{k,l}\,\mathrm{d}V}{2(\omega^m)^2 \int_V \rho_0\, u^m_q u^m_q\,\mathrm{d}V} - \frac{\int_{V_s} \Delta\rho\, u^m_i u^m_i\,\mathrm{d}V}{2\int_V \rho_0\, u^m_q u^m_q\,\mathrm{d}V}. \tag{8.8}$$

The general formula (8.8) can be used to calculate the change in speed of a Scholte wave[12] Norris and Sinha (1995). We next illustrate its application to a wave of considerable utility in the oil exploration and drilling industry, the tube wave.

8.2 Example: The tube wave

Consider a circular fluid-filled borehole of radius a in an isotropic elastic formation. The initial stress is caused by an applied pressure $p^{(0)}$ in the borehole fluid, which induces an inhomogeneous deformation in the solid. We will use (8.8) to find the change in the tube wave speed due to the pressure.

The unperturbed wave is defined by the low frequency behavior of the Stoneley wave mode (White, 1983). The pressure in the fluid and the strain in the solid are, respectively,

$$p^m = p^{(1)}\,\cos\left[\omega^m(t - \frac{x_3}{v})\right], \qquad 0 \le r < a, \tag{8.9a}$$

$$\mathbf{e}^m = p^{(1)}\,\frac{a^2}{2\mu r^2}\,\mathbf{e}\,\cos\left[\omega^m(t - \frac{x_3}{v})\right], \qquad a < r < \infty, \tag{8.9b}$$

where x_3 direction is the borehole axis, $r^2 = x_1^2 + x_2^2$ and

$$e_{\alpha\beta} = \delta_{\alpha\beta} - 2\,r^{-2}\,x_\alpha x_\beta\,, \qquad \alpha,\beta = 1,2; \qquad e_{3k} = e_{k3} = 0. \tag{8.10}$$

$p^{(1)}$ is the tube wave pressure, and μ the shear modulus of the isotropic formation (solid medium). The unperturbed speed is v,

$$\frac{1}{v^2} = \rho_f\,(\frac{1}{A_f} + \frac{1}{\mu}). \tag{8.11}$$

The frequency is arbitrary, as long as the condition $\omega a/v \ll 1$ is met (which justifies the quasistatic approximation).

Terms that are in-phase with the bore pressure have the same cos dependence as (8.9a), and terms that are out-of-phase have sin instead. The product of in- and out-of-phase terms has zero average over one wavelength. Also, the problem depends only upon the modal behavior in the $x_1 - x_2$ plane, and we therefore drop the explicit dependence upon x_3 and t.

[12] A Scholte wave travels along a flat fluid/solid interface, and reduces to the Rayleigh wave in the absence of fluid.

The static strain caused by the applied pressure, $p^{(0)}$ has the form

$$\mathbf{e}^{(0)} = p^{(0)}\,\frac{a^2}{2\mu r^2}\,\mathbf{e}, \qquad a < r < \infty. \tag{8.12}$$

Thus, $e_{kk}^{(0)} = u_{k,k}^{(0)} = 0$, implying that the mass density of the formation is unchanged under the prestress. The dependence of the tube wave speed on the applied pressure follows from (8.8) and (8.11) as

$$\frac{\Delta v}{v} = -\frac{p^{(0)}}{2A_f} + \frac{p^{(0)}}{2A_f}\left(\frac{1 + B_f/A_f}{1 + A_f/\mu}\right) + \frac{\int_{V_s} \Delta C_{ijkl}\, u_{i,j}^m u_{k,l}^m \,\mathrm{d}\,V}{2(\omega^m)^2 \int_V \rho_0\, u_q^m u_q^m \,\mathrm{d}\,V}. \tag{8.13}$$

The integrand can be simplified, using equation (7.32) and the symmetries of the elastic moduli, to give

$$\Delta C_{ijkl}\, u_{i,j}^m\, u_{k,l}^m = p^{(1)^2}\,\frac{a^4}{4\mu^2 r^4}\,\left[5\,C_{\alpha\alpha kl}\,e_{kl} \,+\, C_{ijklmn}\,e_{ij}e_{kl}e_{mn} - 2p^{(0)}\right], \quad \text{in } \ V_s, \tag{8.14}$$

where the repeated Greek suffices indicate summation over $\alpha = 1$ and 2. Substituting the moduli for isotropic elasticity, which involves two second-order and three third-order elastic moduli, it turns out that the terms involving C_{ijkl} and C_{ijklmn} are individually zero independent of the five moduli. The volumetric integrals in (8.13) should be understood as time averaged over one cycle, or alternatively, as averages over one axial wavelength, and they can be replaced by integrals in any cross plane. Thus,

$$\int_{V_s} \Delta C_{ijkl}\, u_{i,j}^m u_{k,l}^m \,\mathrm{d}\,V \to -p^{(0)}\, p^{(1)^2}\,\frac{a^4}{\mu^2}\,2\pi \int_a^\infty \frac{r\,\mathrm{d}\,r}{2r^4} \;= -p^{(0)}\, p^{(1)^2}\,\frac{\pi a^2}{\mu^2}. \tag{8.15}$$

The integral in the denominator of (8.13) is dominated, in the quasistatic limit, by the integral of $(u_3^m)^2$ over the fluid volume, because all other displacements are $\mathrm{O}(\omega)$ in magnitude relative to the axial displacement in the bore fluid. Furthermore, the axial velocity is approximated by the relation $u_{3,t}^m = (\rho_f v)^{-1} p^m$, appropriate to a medium with acoustic impedance $\rho_f v$. Hence,

$$(\omega^m)^2 \int_V \rho_0\, u_q^m u_q^m \,\mathrm{d}\,V \to \int_{V_f} \rho_f\,(u_{3,t}^m)^2 \,\mathrm{d}\,V = p^{(1)^2}\,\frac{\pi a^2}{\rho_f v^2}. \tag{8.16}$$

Combining equations (8.11), (8.13), (8.15), and (8.16), we find

$$\frac{\Delta v}{v} = \frac{p^{(0)}}{2A_f(1 + A_f/\mu)}\left(\frac{B_f}{A_f} - \frac{A_f}{\mu} - \frac{A_f^2}{2\mu^2}\right), \tag{8.17}$$

or,

$$\rho_f\;\frac{\mathrm{d}\,v^2}{\mathrm{d}\,p}\bigg|_{p=0} = \left(\frac{B_f}{A_f} + \frac{1}{2}\right)\frac{v^4}{c^4} \,-\frac{1}{2}\,. \tag{8.18}$$

Equation (8.18) agrees with Johnson et al. (1994). They considered the possibility of variable elastic properties within the formation, but their general formula collapses to

(8.18) for a uniform formation. When the formation is rigid ($\mu \to \infty$), then $v = c$ and (8.18) reduces to the well known equation for the pressure dependence of an acoustic wave in an infinite fluid. Equation (8.18) indicates that the fluid nonlinearity parameter B_f/A_f, which is positive for fluids, is diminished by the presence of the formation. However, the nonlinearity parameter is magnified when the formation properties are not uniform (Johnson et al., 1994).

It is remarkable that eq. (8.18) is completely independent of the nonlinear parameters of the solid[13]. It may be checked that if one uses the "standard" form of the small-on-large theory that ignores the slip at the fluid solid interface, then the solid TOE appear in the equation for Δv. The simple form of eq. (8.18) is a consequence of the more elaborate, but correct, form of small-on-large theory.

Bibliography

C. Coste and B. Gilles. On the validity of Hertz contact law for granular material. *European Physical Journal B*, 7(1):155–168, 1999.

H. Deresiewicz. Stress-strain relations for a simple model of a granular medium. *J. Appl. Mech.*, 25:402–406, 1958.

P. J. Digby. The effective elastic moduli of porous granular rock. *J. Appl. Mech.*, 48: 803–808, 1981.

S. N. Domenico. Elastic properties of unconsolidated porous sand reservoirs. *Geophysics*, 42:1339–1368, 1977.

J. D. Goddard. Nonlinear elasticity and pressure-dependent wave speeds in granular media. *Proc. R. Soc. Lond. A*, 430:105–131, 1990.

D. L. Johnson and A. N. Norris. Rough elastic spheres in contact: Memory effects and the transverse force. *J. Mech. Phys. Solids*, 45:1025–1036, 1997.

D. L. Johnson, S. Kostek, and A. N. Norris. Nonlinear tube waves. *J. Acoust. Soc. Am.*, 96:1829–1843, 1994.

D. L. Johnson, L. M. Schwartz, D. Elata, J. G. Berryman, B. Hornby, and A. N. Norris. Linear and nonlinear elasticity of granular media: Stress-induced anisotropy of a random sphere pack. *J. Appl. Mech.*, 65:380–388, 1998.

S. Kostek, B. Sinha, and A. N. Norris. Third order elastic moduli for an inviscid fluid. *J. Acoust. Soc. Am.*, 94:3014–3017, 1993.

L. D. Landau and E. M. Lifshitz. *Theory of Elasticity.* Pergamon Press, 1970.

R. D. Mindlin. Compliance of elastic bodies in contact. *J. Appl. Mech.*, 71:259–268, 1949.

R. D. Mindlin and H. Deresiewicz. Elastic spheres in contact under varying oblique forces. *J. Appl. Mech.*, 75:327–344, 1953.

A. N. Norris. Finite-amplitude waves in solids. In M. F. Hamilton and D. T. Blackstock, editors, *Nonlinear Acoustics*, pages 263–277, New York, 1998. Academic Press.

A. N. Norris and D. L. Johnson. Nonlinear elasticity of granular media. *J. Appl. Mech.*, 64:39–49, 1997.

[13] This feature disappears for a non-uniform formation and in fact, radial variation of the formation moduli can lead to the solid TOE dominating (Johnson et al., 1994).

A. N. Norris and B. Sinha. The speed of a wave along a fluid/solid interface in the presence of anisotropy and prestress. *J. Acoust. Soc. Am.*, 98:1147–1154, 1995.

A. N. Norris, B. Sinha, and S. Kostek. Acoustoelasticity of solid/fluid composite systems. *Geophys. J. Int.*, 118:439–446, 1994.

D. A. Spence. Self similar solutions to adhesive contact problems with incremental load. *Proc. R. Soc. Lond. A*, 305:55–80, 1968.

R. N. Thurston. Waves in solids. In C. Truesdell, editor, *Mechanics of Solids*, volume 4, pages 109–308, Berlin, 1984. Springer-Verlag.

S. Timoshenko. *Theory of Elasticity*. McGraw-Hill, New York, 1970.

R. A. Toupin and B. Bernstein. Sound waves in deformed perfectly elastic materials. acoustoelastic effect. *J. Acoust. Soc. Am.*, 33:216–225, 1961.

K. Walton. The effective elastic moduli of a random packing of spheres. *J. Mech. Phys. Solids*, 35:213–226, 1987.

J. E. White. *Underground Sound*. Elsevier, Amsterdam, 1983.