Elastic moduli approximation of higher symmetry for the acoustical properties of an anisotropic material

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(Received 5 November 2005; revised 19 January 2006; accepted 20 January 2006)

The issue of how to define and determine an optimal acoustical fit to a set of anisotropic elastic constants is addressed. The optimal moduli are defined as those which minimize the mean-squared difference in the acoustical tensors between the given moduli and all possible moduli of a chosen higher material symmetry. The solution is shown to be identical to minimizing a Euclidean distance function, or equivalently, projecting the tensor of elastic stiffness onto the appropriate symmetry. This has implications for how to best select anisotropic constants to acoustically model complex materials. © *2006 Acoustical Society of America.* [DOI: 10.1121/1.2173525]

PACS number(s): 43.35.Cg, 43.20.Jr [YHB] Pages: 2114–2121

I. INTRODUCTION

The theory of wave propagation in anisotropic elastic solids was historically developed for homogeneous crystals with *a priori* known symmetry.[1] The number of elastic moduli are then well defined. However, the symmetry is not always apparent in noncrystalline and generally complex materials, for which measurements can yield as many as 21 anisotropic elastic coefficients. Whether one has some idea of the underlying material symmetry or simply prefers to deal with fewer parameters, the question arises of how to best fit the given set of elastic moduli to, for instance, a transversely isotropic material model. This issue occurs in acoustical measurements of composites composites,[2] and in geophysical applications[3] where laboratory measurements might yield 21 moduli, but seismic modeling requires a higher symmetry, such as transverse isotropy. The purpose of this paper is to provide a simple but unambiguous means to find the reduced set of anisotropic elastic constants that is in a certain sense the best acoustical fit to the given moduli. The optimal material minimizes the mean-square difference in the slowness surfaces of the given moduli and of all possible sets of elastic constants of the chosen symmetry.

The prevailing approach to finding a reduced set of moduli does not invoke acoustical properties, but views the moduli as elements in a vector space which are projected onto the higher elastic material symmetry. This is achieved by defining a Euclidean norm for the moduli $\mathbb{C}$ according to $\|\mathbb{C}\|^2 = C_{ijkl}C_{ijkl}$, where C_{ijkl} are the elements of the stiffness tensor. Lower case Latin suffices take on the values 1, 2, and 3, and the summation convention on repeated indices is assumed here and in Eq. (A1). This provides a natural definition for distance, from which one can find the elastic tensor of a given symmetry nearest to an anisotropic elastic tensor, or equivalently, define a projection appropriate to the higher symmetry. Gazis *et al.*[4] outline the procedure in terms of fourth-order tensors, while more recently Browaeys and Chevrot[3] provide projection matrices for $\mathbb{C}$ expressed as a 21-dimensional vector. Helbig[5] provides a useful overview of the problem from a geophysical perspective, Cavallini[6] examines isotropic projection specifically, and Gangi[7] gives formulas for several other symmetries. One drawback of the approach is that, while it minimizes the Euclidean distance between the original and projected stiffness tensors, it does not provide the analogous closest *compliance* (the inverse of stiffness). In this sense the stiffness projection method, while simple and attractive, is unsatisfactory because it is not invariant under inversion. Alternative procedures based on non-Euclidean norms such as the Riemannian[8] or log-Euclidean[9] metrics do not have this deficiency, and, in principle, provide a unique "projection" regardless of whether one uses the stiffness or the compliance tensor.

An apparently quite different approach is to try to find higher-symmetry moduli which in some way better approximate the acoustic properties of the given moduli. Thus, Fedorov[10] considered the question of what elastically *isotropic* material is the best acoustic fit to a given set of anisotropic moduli. He defined best fit to mean the effective bulk and shear moduli $\{\kappa, \mu\}$ which minimize the mean-square difference between the slowness surfaces of the original anisotropic material and the isotropic material characterized by $\{\kappa, \mu\}$ (density is unaffected). Fedorov obtained explicit expressions for the moduli, Eq. (26.19) of Ref. 10, or in the present notation

$$\kappa = \tfrac{1}{9}C_{iijj}, \quad \mu = \tfrac{1}{10}C_{ijij} - \tfrac{1}{30}C_{iijj}. \tag{1}$$

Fedorov's procedure for finding a suitable set of higher-symmetry moduli is physically appealing, especially as it seeks to approximate acoustical properties. Also, as Fedorov and others[10,4,3] have shown, κ and μ are precisely the isotropic moduli found by the stiffness projection method. However, Fedorov only considered effective isotropic moduli and it does not appear that anyone has attempted to generalize his method to symmetries other than isotropic. The purpose of this paper is to solve what may be termed the generalized Fedorov problem. The solution is found for the acoustically best-fitting moduli of arbitrary symmetry, which is of higher symmetry than the given moduli but lower than isotropy. The

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 0001-4966/2006/119(4)/2114/8/$22.50

central result is that the solution of the generalized Fedorov problem possesses the same crucial property as the solution obtained by Fedorov,[10] that is, the best-fit moduli are identical to those obtained by the stiffness projection method. This result provides a strong physical and acoustical basis for using the Euclidean projection scheme that has been absent until now.

We begin in Sec. II with the definition of elastic tensors and associated notation. The generalized Fedorov problem is introduced and solved in Sec. III. Examples are given in Sec. IV.

II. PRELIMINARIES

A. The elasticity tensor and related notation

The solution of the generalized Fedorov problem is most easily accomplished using tensors, which are reviewed here along with some relevant notation. Boldface lower case Latin quantities indicate three-dimensional vectors, such as the orthonormal basis $\{\mathbf{e}_i, i=1,2,3\}$, open-face symbols such as $\mathbb{C}$ indicate fourth-order elasticity tensors, and boldface capitals, e.g., $\mathbf{A}$, are second-order symmetric tensors, with some exceptions. Components defined relative to the basis vectors allow us to represent arbitrary tensors in terms of the fundamental tensors formed from the basis vectors, thus, $\mathbf{A}=A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$, $\mathbb{C}=C_{ijkl}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l$, where $\otimes$ is the tensor product. The identity tensors of second and fourth order are $\mathbf{I}$ and $\mathbb{I}$, respectively, and we will also use the fourth-order tensor $\mathbb{J}$. These have components

$$I_{ij}=\delta_{ij},\quad I_{ijkl}=\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\quad J_{ijkl}=\tfrac{1}{3}\delta_{ij}\delta_{kl}. \tag{2}$$

Isotropic elasticity tensors can be expressed as a linear combination of $\mathbb{I}$ and $\mathbb{J}$. Using Lamé moduli μ and $\lambda=\kappa-\frac{2}{3}\mu$, for instance,

$$\mathbb{C}=2\mu\mathbb{I}+3\lambda\mathbb{J}. \tag{3}$$

Products of tensors are defined by summation over pairs of indices: $(\mathbb{C}\mathbf{A})_{ij}=C_{ijkl}A_{kl}$ and $(\mathbb{A}\mathbb{B})_{ijkl}=A_{ijpq}B_{pqkl}$. Thus, $\mathbb{I}\mathbf{I}=\mathbb{J}\mathbf{I}=\mathbf{I}$, $\mathbb{I}\mathbb{I}=\mathbb{I}$, $\mathbb{J}\mathbb{J}=\mathbb{J}$, and $\mathbb{I}\mathbb{J}=\mathbb{J}\mathbb{I}=\mathbb{J}$.

The inner product of a pair of tensors of the same order is defined

$$\langle u,v\rangle=\mathrm{tr}(uv), \tag{4}$$

where tr $\mathbf{A}=A_{ii}$ and tr $\mathbb{A}=A_{ijij}$, and the Euclidean norm of a tensor is

$$\|u\|\equiv\langle u,u\rangle^{1/2}. \tag{5}$$

This will be used to compare differences between tensors.

An anisotropic elastic stiffness tensor $\mathbb{C}$ relates stress $\mathbf{T}$ and strain $\mathbf{E}$ according to the generalized form of Hooke's law,

$$\mathbf{T}=\mathbb{C}\mathbf{E}. \tag{6}$$

Stress and strain are symmetric second-order tensors, implying that their components are also symmetric, $T_{ij}=T_{ji}$, $E_{ij}=E_{ji}$, which in turn implies the first two of the following identities:

$$C_{ijkl}=C_{ijlk},\quad C_{ijkl}=C_{jikl},\quad C_{ijkl}=C_{klij}. \tag{7}$$

The third property is a consequence of the existence of a positive strain energy of the form $W=\frac{1}{2}\langle\mathbf{E},\mathbb{C}\mathbf{E}\rangle$, which also constrains the moduli to be positive definite. The squared norm of the elasticity tensor is

$$\|\mathbb{C}\|^2\equiv\langle\mathbb{C},\mathbb{C}\rangle=C_{ijkl}C_{ijkl}. \tag{8}$$

The 21-dimensional space of elasticity tensors can be decomposed as a 15-dimensional space of totally symmetric fourth-order tensors plus a six-dimensional space of asymmetric fourth-order tensors,[11,12] by

$$\mathbb{C}=\mathbb{C}^{(s)}+\mathbb{C}^{(a)}, \tag{9}$$

where

$$C^{(s)}_{ijkl}=\tfrac{1}{3}(C_{ijkl}+C_{ilkj}+C_{ikjl}). \tag{10}$$

The elements of the totally symmetric part satisfy the relations $C^{(s)}_{ijkl}=C^{(s)}_{ikjl}$ in addition to (7). Thus, $C^{(s)}_{ijkl}$ is unchanged under any rearrangement of its indices, and $\mathbb{C}^{(s)}$ has at most 15 independent elements. This can be seen by the explicit representation of the asymmetric part in terms of the remaining six independent elements, which are the components of the symmetric second-order tensor

$$D_{ij}=C_{ijkk}-C_{ikjk}. \tag{11}$$

Thus,

$$C^{(a)}_{ijkl}=\tfrac{1}{3}[2D_{ij}I_{kl}+2D_{kl}I_{ij}-D_{ik}I_{jl}-D_{il}I_{jk}-D_{jk}I_{il}-D_{jl}I_{ik}+D_{mm}(I_{ijkl}-3J_{ijkl})]. \tag{12}$$

An asymmetric tensor satisfies, by definition,[11]

$$C^{(a)}_{ijkl}+C^{(a)}_{ilkj}+C^{(a)}_{ikjl}=0. \tag{13}$$

Note that the symmetric part of an asymmetric tensor is zero, and vice versa, and that the decomposition (9) is orthogonal in the sense of the Euclidean norm, $\|\mathbb{C}\|^2=\|\mathbb{C}^{(s)}\|^2+\|\mathbb{C}^{(a)}\|^2$. The partition of $\mathbb{C}$ as a sum of totally symmetric and traceless asymmetric tensors is the first step in Backus' harmonic decomposition of elasticity tensors,[11] a partition that has proved useful for representations of elastic tensors[13] and has also been used to prove that there are exactly eight distinct elastic symmetries.[14] Finally, for future reference, note that the totally symmetric part of the fourth-order identity is

$$I^{(s)}_{ijkl}=\frac{1}{3}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj})\Leftrightarrow\mathbb{I}^{(s)}=\frac{2}{3}\mathbb{I}+\mathbb{J}. \tag{14}$$

B. The acoustical tensor and the tensor $\mathbb{C}^*$

The acoustical tensor, also known as Christoffel's matrix,[1] arises in the study of plane crested waves with displacement of the form $\mathbf{u}(\mathbf{x},t)=\mathbf{b}h(\mathbf{n}\cdot\mathbf{x}-vt)$, where $\mathbf{b}$ is a fixed unit vector describing the polarization, $\mathbf{n}$ is the phase direction, also a unit vector, v is the phase velocity, and h is an arbitrary but sufficiently smooth function. Substituting this wave form into the equations of motion

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = \rho \ddot{u}_i, \tag{15}$$

where ρ is the density, implies that it is a solution if and only if $\mathbf{b}$ and ρv^2 are eigenvector and eigenvalue of the second order tensor $\mathbf{Q}$,

$$Q_{ik}(\mathbf{n}) = C_{ijkl} n_j n_l. \tag{16}$$

This definition of the acoustical tensor is not the product of a fourth-order tensor with a second-order tensor. In order to express it in this form, which simplifies the analysis later, introduce the related fourth-order tensor $\mathbb{C}^*$ defined by

$$C^*_{ijkl} = \tfrac{1}{2}(C_{ikjl} + C_{iljk}). \tag{17}$$

Thus,

$$Q_{ij}(\mathbf{n}) = C^*_{ijkl} n_k n_l \Leftrightarrow \mathbf{Q}(\mathbf{n}) = \mathbb{C}^* \mathbf{n} \otimes \mathbf{n}. \tag{18}$$

The operation defined by * is of fundamental importance in solving the generalized Fedorov problem, and therefore some properties are noted. First, * is a linear operator that commutes with taking the symmetric and asymmetric parts of a tensor: $\mathbb{C}^{(s)*} = \mathbb{C}^{*(s)} = \mathbb{C}^{(s)}$ and $\mathbb{C}^{(a)*} = \mathbb{C}^{*(a)} = -\frac{1}{2}\mathbb{C}^{(a)}$. Accordingly, $\mathbb{C}^*$ is partitioned as

$$\mathbb{C}^* = \mathbb{C}^{(s)} - \tfrac{1}{2}\mathbb{C}^{(a)}, \tag{19}$$

and repeating the * operation n times yields $\mathbb{C}^{n*} = \mathbb{C}^{(s)} + (-1/2)^n \mathbb{C}^{(a)}$. Taking $n=2$ implies the identity

$$\mathbb{C} = 2\mathbb{C}^{**} - \mathbb{C}^*, \tag{20}$$

from which $\mathbb{C}$ can be found from $\mathbb{C}^*$. Hence the mapping $\mathbb{C} \leftrightarrow \mathbb{C}^*$ is one-to-one and invertible, that is bijective. This property is important in the inverse problem of determining elastic moduli from acoustic data.[15] Acoustic wave speeds and associated quantities are related primarily to $\mathbb{C}^*$ through the acoustical tensor also known as the Christoffel matrix,[1] and this can be determined uniquely from $\mathbb{C}$ using (20). Decompositions of $\mathbb{C}$ and $\mathbb{C}^*$ into totally symmetric and asymmetric parts are unique, and knowledge of one decomposition implies the other. In particular, $\mathbb{C} = \mathbb{C}^*$ if and only if the asymmetric parts of both are zero. This occurs if the moduli satisfy $C_{ijkk} = C_{ikjk}$, which together with the symmetries (7) are equivalent to the Cauchy relations; see Sec. 4.5 of Musgrave.[1] We note that the operation * is self-adjoint in the sense that the following is true for any pair of elasticity tensors:

$$\langle \mathbb{A}, \mathbb{B}^* \rangle = \langle \mathbb{A}^*, \mathbb{B} \rangle. \tag{21}$$

Elastic moduli are usually defined by the Voigt notation: $C_{ijkl} \equiv c_{IJ}$, where $I,J = 1,2,3,\ldots,6$ and $I = 1,2,3,4,5,6$ correspond to $ij = 11,22,33,23,13,12$, respectively, i.e.,

$$\mathbb{C} \leftrightarrow \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ S & Y & M & & c_{55} & c_{56} \\ & & & & & c_{66} \end{pmatrix}. \tag{22}$$

The components of $\mathbb{C}^*$ are then

$$\mathbb{C}^* \leftrightarrow \mathbf{C}^* = \begin{pmatrix} c_{11} & c_{66} & c_{55} & c_{56} & c_{15} & c_{16} \\ & c_{22} & c_{44} & c_{24} & c_{46} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{45} \\ & & & \frac{1}{2}(c_{44}+c_{23}) & \frac{1}{2}(c_{45}+c_{36}) & \frac{1}{2}(c_{46}+c_{25}) \\ S & Y & M & & \frac{1}{2}(c_{55}+c_{13}) & \frac{1}{2}(c_{56}+c_{14}) \\ & & & & & \frac{1}{2}(c_{66}+c_{12}) \end{pmatrix}. \tag{23}$$

III. FEDOROV'S PROBLEM FOR PARTICULAR SYMMETRIES

A. Definition of the problem

We assume as known the elasticity tensor $\mathbb{C}$ of arbitrary material symmetry with as many as 21 independent components. A particular material symmetry is chosen with prescribed symmetry axes or planes, for instance, transverse isotropy with symmetry axis in the direction $\mathbf{a}$, or cubic symmetry with orthogonal cube axes $\mathbf{a},\mathbf{b},\mathbf{c}$. Fedorov's problem for particular symmetries is to determine the elastic stiffness $\mathbb{C}_{\text{sym}}$ of the chosen material symmetry which is the best fit in the sense that it minimizes the orientation-averaged squared difference of the acoustical tensors. We introduce the acoustical distance function

$$f(\mathbb{C},\mathbb{C}_{\rm sym}) \equiv \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \,\|\mathbf{Q}(\mathbb{C},\mathbf{n}) - \mathbf{Q}(\mathbb{C}_{\rm sym},\mathbf{n})\|^2, \tag{24}$$

with $\mathbf{n}=\sin\theta(\cos\phi\mathbf{e}_1+\sin\phi\mathbf{e}_2)+\cos\theta\mathbf{e}_3$. The same function was considered by Fedorov[10] with $\mathbb{C}_{\rm sym}$ restricted to isotropic elasticity. Thus, substituting $\mathbb{C}_{\rm sym}=\alpha_1\mathbb{I}+\alpha_2\mathbb{J}$ into (24) one obtains, after simplification, a positive-definite quadratic in the two unknowns α_1 and α_2. A simple minimization yields $\alpha_1=2\mu$ and $\alpha_2=3\kappa-2\mu$, where κ and μ are defined in Eq. (1). The symmetry of $\mathbb{C}_{\rm sym}$ can be considered as arbitrary for the general problem addressed here, although Fedorov's isotropic result is recovered as a special case of the general solution discussed in Sec. IV.

The integral over directions in (24) can be removed using

$$\frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} = \frac{1}{5}\mathbb{I}^{(s)}, \tag{25}$$

where $\mathbb{I}^{(s)}$ is defined in Eq. (14). The identity follows by noting that the integral must be a totally symmetric isotropic fourth-order tensor of the form $a\mathbb{I}^{(s)}$. Taking the trace of both sides and using $\mathrm{tr}\,\mathbb{I}^{(s)}=I^{(s)}_{ijij}=5$, $\mathrm{tr}(\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n})=1$, gives $a=1/5$. Thus, since

$$\mathbf{Q}(\mathbb{C},\mathbf{n}) = \mathbb{C}^*\mathbf{n}\otimes\mathbf{n}, \quad \mathbf{Q}(\mathbb{C}_{\rm sym},\mathbf{n}) = \mathbb{C}^*_{\rm sym}\mathbf{n}\otimes\mathbf{n}, \tag{26}$$

the distance function reduces to

$$f = \tfrac{1}{5}\langle(\mathbb{C}^*-\mathbb{C}^*_{\rm sym}),\, \mathbb{I}^{(s)}(\mathbb{C}^*-\mathbb{C}^*_{\rm sym})\rangle. \tag{27}$$

Define the modified inner product for elasticity tensors

$$\langle\mathbb{A},\mathbb{B}\rangle_a \equiv \mathrm{tr}(\mathbb{I}^{(s)}\mathbb{A}\mathbb{B}) = \langle\mathbb{A},\mathbb{I}^{(s)}\mathbb{B}\rangle, \tag{28}$$

and norm

$$\|\mathbb{A}\|_a = \langle\mathbb{A},\mathbb{A}\rangle_a^{1/2}. \tag{29}$$

Then, Fedorov's problem for particular symmetries amounts to

$$\text{Fedorov} \Leftrightarrow \text{minimize}\|\mathbb{C}^*-\mathbb{C}^*_{\rm sym}\|_a. \tag{30}$$

The reason the problem is expressed in this form is to make the connection with the notion of projection onto the chosen elastic symmetry, or equivalently of finding the elastic tensor of the chosen symmetry nearest to the given elasticity $\mathbb{C}$. As mentioned in the Introduction, this question has been addressed by several authors and has an explicit solution. The issue is to find the elastic tensor $\mathbb{C}_{\rm sym}$ which minimizes the Euclidean distance function d, where

$$d(\mathbb{C},\mathbb{C}_{\rm sym}) \equiv \|\mathbb{C}-\mathbb{C}_{\rm sym}\|. \tag{31}$$

Comparing Eqs. (30) and (31), the two problems appear tantalizingly similar. It should be realized that Fedorov's problem involves $\mathbb{C}^*$, not $\mathbb{C}$, and that the norms are distinct. However, it will be proved that the problems share the same solution,

$$\{\mathbb{C}_{\rm sym}:\min_{\mathbb{C}_{\rm sym}}\|\mathbb{C}^*-\mathbb{C}^*_{\rm sym}\|_a\} \equiv \{\mathbb{C}_{\rm sym}:\min_{\mathbb{C}_{\rm sym}}\|\mathbb{C}-\mathbb{C}_{\rm sym}\|\}, \tag{32}$$

which gives the central result of this paper, i.e.,

$$\text{Fedorov} \Leftrightarrow \text{Euclidean projection}. \tag{33}$$

This equivalence enables us to provide an explicit solution to the generalized Fedorov problem, e.g., using the methods of Gazis *et al.*,[4] Browaeys and Chevrot,[3] or others. The remainder of this section develops a proof by construction of the solution of the Fedorov problem, which is shown to be identical to the Euclidean projection. Some further concepts and notation are required and introduced next.

B. Basis tensors

The solution uses a fundamental decomposition of the chosen material symmetry using basis tensors. These form a vector space for the symmetry in the sense that any elasticity tensor of that symmetry may be expressed uniquely in terms of N linearly independent tensors $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_N$, where $2\leqslant N\leqslant 13$ is the dimension of the vector space for the material symmetry. For instance, isotropic elasticity tensors are of the form $\mathbb{C}_{\rm iso}=\alpha_1\mathbb{I}+\alpha_2\mathbb{J}$, $\alpha_1,\alpha_2>0$. The procedure is analogous for other material symmetries, cubic, transversely isotropic, etc., and is described in detail by Walpole,[16] who provides expressions for the base tensors of the various symmetries. Thus, $N=2,3,5,9$ for isotropy, cubic symmetry, transverse isotropy, and orthorhombic symmetry, respectively. $N=13$ corresponds to monoclinic, which is the lowest symmetry apart from triclinic (technically $N=21$), which is no symmetry. The precise form of the basis tensors is irrelevant here (examples are given in Sec. IV, full details are in Ref. 16, and Kunin[17] develops a similar tensorial decomposition); all that is required is that they be linearly independent, and consequently any tensor with the desired symmetry can be written

$$\mathbb{C}_{\rm sym} = \sum_{i=1}^{N} \beta_i \mathbb{V}_i. \tag{34}$$

The coefficients follow by taking inner products with the basis tensors. Let $\mathbf{\Lambda}$ be the $N\times N$ symmetric matrix with elements

$$\Lambda_{ij} \equiv \langle\mathbb{V}_i,\mathbb{V}_j\rangle. \tag{35}$$

$\mathbf{\Lambda}$ is invertible by virtue of the linear independence of the basis tensors, and therefore

$$\beta_i = \sum_{j=1}^{N} \Lambda^{-1}_{ij}\langle\mathbb{C}_{\rm sym},\mathbb{V}_j\rangle. \tag{36}$$

Conversely, Eq. (34) describes all possible tensors with the given symmetry, and in particular, $\mathbb{C}_{\rm sym}=0$ if and only if $\beta_i=0$, $i=1,2,\ldots,N$.

C. The elastic projection

It helps to first derive the solution that minimizes the Euclidean distance function of (31). By expressing the unknown projected solution in the form (34), it follows that the minimum of $\|\mathbb{C}-\mathbb{C}_{\rm sym}\|^2$ is determined by setting to zero the derivatives with respect to β_i, which gives the system of simultaneous equations

$$\sum_{j=1}^{N} \langle \mathbb{V}_i, \mathbb{V}_j \rangle \beta_j = \langle \mathbb{V}_i, \mathbb{C} \rangle, \quad i = 1,2, \ldots, N. \tag{37}$$

The inner products are the elements of the invertible matrix $\mathbf{\Lambda}$, and so

$$\mathbb{C}_{\text{sym}} = \sum_{i,j=1}^{N} \langle \mathbb{C}, \mathbb{V}_i \rangle \Lambda_{ij}^{-1} \mathbb{V}_j. \tag{38}$$

Furthermore, the distance function at the optimal $\mathbb{C}_{\text{sym}}$ satisfies

$$\|\mathbb{C} - \mathbb{C}_{\text{sym}}\|^2 = \|\mathbb{C}\|^2 - \|\mathbb{C}_{\text{sym}}\|^2, \tag{39}$$

as expected for a projection using the Euclidean norm. This is essentially the method used by Arts *et al.*[18] Helbig[5] describes the procedure as the 'optimum approximation' of an arbitrary elasticity tensor by projection of a 21-dimensional vector onto a subspace of fewer dimensions, and Browaeys and Chevrot[3] list the explicit forms of the 21D projection operators for various symmetries.

Define the projection operator $\mathcal{P}_{\text{sym}}$ which maps $\mathbb{C}$ onto the chosen symmetry,

$$\mathcal{P}_{\text{sym}} \mathbb{C} \equiv \sum_{i,j=1}^{N} \langle \mathbb{C}, \mathbb{V}_i \rangle \Lambda_{ij}^{-1} \mathbb{V}_j. \tag{40}$$

Equations (38)–(40) imply that $\mathcal{P}_{\text{sym}}\mathbb{C}$ is the Euclidean projection, also equal to the closest elasticity tensor of the chosen symmetry to $\mathbb{C}$. We note the following important property:

$$\mathcal{P}_{\text{sym}} \mathbb{C}^* = (\mathcal{P}_{\text{sym}} \mathbb{C})^*. \tag{41}$$

In other words, the operation * commutes with the projection operator. This is not surprising if one considers that the * operation is a linear mapping on the symmetric and asymmetric parts of $\mathbb{C}$, and therefore * maintains the material symmetry of $\mathbb{C}$. However, a more detailed proof of the identity (41) is provided in the Appendix.

D. Solution of the generalized Fedorov problem

We now calculate the optimal $\mathbb{C}_{\text{sym}}$ for the generalized Fedorov problem, and show that it is equivalent to the moduli from the Euclidean norm. The starting point this time is to express the unknown $\mathbb{C}^*_{\text{sym}}$ (rather than $\mathbb{C}_{\text{sym}}$) in terms of the basis tensors,

$$\mathbb{C}^*_{\text{sym}} = \sum_{i=1}^{N} \alpha_i \mathbb{V}_i. \tag{42}$$

This is justified by the fact that all tensors of the given symmetry are linear combinations of the basis tensors. Furthermore, the coefficients α_i are related to those in Eq. (34) by the $N \times N$ matrix $\mathbf{P}$ introduced in the Appendix. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote the $N \times 1$ arrays with elements α_i and α_i; then,

$$\boldsymbol{\beta} = \mathbf{P}^t \boldsymbol{\alpha} \Leftrightarrow \boldsymbol{\alpha} = 2\mathbf{P}^t \boldsymbol{\beta} - \boldsymbol{\beta}. \tag{43}$$

Equation (30) implies that the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$ satisfy the system of N equations,

$$\sum_{j=1}^{N} \langle \mathbb{V}_i, \mathbb{V}_j \rangle_a \alpha_j = \langle \mathbb{V}_i, \mathbb{C}^* \rangle_a, \quad i = 1,2, \ldots, N, \tag{44}$$

or, in matrix format,

$$\mathbf{S}\mathbf{\Lambda}\boldsymbol{\alpha} = \mathbf{S}\boldsymbol{\gamma}. \tag{45}$$

Here, $\boldsymbol{\gamma}$ with elements γ_i and the $N \times N$ matrix $\mathbf{S}$ are defined by

$$\gamma_i = \langle \mathbb{V}_i, \mathbb{C}^* \rangle, \tag{46a}$$

$$\sum_{j=1}^{N} S_{ij} \mathbb{V}_j = \tfrac{1}{2}(\mathbb{I}^{(s)} \mathbb{V}_i + \mathbb{V}_i \mathbb{I}^{(s)}) \equiv \mathbb{U}_i. \tag{46b}$$

The tensors $\mathbb{U}_i$, $i = 1,2, \ldots, N$ form a linearly independent set of basis tensors for the given symmetry. This may be seen by assuming the contrary, i.e., that there is a set of nonzero coefficients a_i such that

$$\sum_{i=1}^{N} a_i \mathbb{U}_i = 0. \tag{47}$$

Let $\mathbb{A}$ be the nonzero tensor,

$$\mathbb{A} = \sum_{i=1}^{N} a_i \mathbb{V}_i. \tag{48}$$

Then, Eq. (47) requires that

$$\tfrac{2}{3}\mathbb{A} + \tfrac{1}{2}(\mathbb{A}\mathbf{I}) \otimes \mathbf{I} + \tfrac{1}{2}\mathbf{I} \otimes (\mathbb{A}\mathbf{I}) = 0. \tag{49}$$

Multiplication by $\mathbf{I}$ implies

$$\tfrac{13}{6}\mathbb{A}\mathbf{I} + \tfrac{3}{2}\langle \mathbb{J}, \mathbb{A} \rangle \mathbf{I} = 0, \tag{50}$$

and taking the inner product with $\mathbf{I}$ gives

$$11 \langle \mathbb{J}, \mathbb{A} \rangle = 0. \tag{51}$$

Therefore,

$$\mathbb{A} = 0, \tag{52}$$

and the $\mathbb{U}_i$ tensors form a linearly independent basis for the symmetry. In particular, the tensors $\mathbb{V}_i$ can be expressed in terms of this alternate basis, and so the matrix $\mathbf{S}$ is invertible. Hence,

$$\boldsymbol{\alpha} = \mathbf{\Lambda}^{-1} \boldsymbol{\gamma}. \tag{53}$$

We are now in a position to determine the optimal $\mathbb{C}^*_{\text{sym}}$ and hence $\mathbb{C}_{\text{sym}}$. Equations (42) and (53), along with (40), imply

$$\mathbb{C}^*_{\text{sym}} = \sum_{i,j=1}^{N} \langle \mathbb{C}^*, \mathbb{V}_i \rangle \Lambda_{ij}^{-1} \mathbb{V}_j = \mathcal{P}_{\text{sym}} \mathbb{C}^*. \tag{54}$$

Using the fundamental property of the projection operator (41), the optimal elasticity as determined by (54) is seen to be exactly the same as the Euclidean projection, i.e., of Eq. (38). This completes the proof of the main result, the equivalence (33).

IV. EXAMPLES

The general procedure for projection is illustrated for several symmetries, and an example application is discussed in this section.

A. Basis functions for isotropic, cubic, and hexagonal materials

The tensor decomposition procedure is described for the three highest symmetries: isotropic, cubic, and transversely isotropic. The fundamental matrices $\mathbf{\Lambda}$ and $\mathbf{P}$ are given explicitly.

1. Isotropic approximation (*N*=2)

Let the basis tensors be $\mathbb{V}_1=\mathbb{J}$, $\mathbb{V}_2=\mathbb{K}\equiv\mathbb{I}-\mathbb{J}$. Then, $\mathbf{\Lambda}$ is a 2×2 diagonal matrix, $\mathbf{\Lambda}=\mathrm{diag}(1,5)$, and the optimal moduli are

$$\mathbb{C}_{\mathrm{iso}}=3\kappa\mathbb{J}+2\mu\mathbb{K}, \tag{55}$$

where

$$\kappa=\tfrac{1}{3}\langle\mathbb{J},\mathbb{C}\rangle,\quad \mu=\tfrac{1}{10}\langle\mathbb{K},\mathbb{C}\rangle. \tag{56}$$

This is precisely Fedorov's original result,[10] Eq. (1). The * operation is defined by the matrix $\mathbf{P}$ of Eq. (A1), which is

$$\mathbf{P}=\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{5}{3} & \frac{1}{6}\end{pmatrix}. \tag{57}$$

2. Cubic materials (*N*=3)

Let $\mathbf{a},\mathbf{b},\mathbf{c}$ be the cube axes, and select as basis tensors[16] $\mathbb{V}_1=\mathbb{J}$, $\mathbb{V}_2=\mathbb{L}\equiv\mathbb{I}-\mathbb{H}$, and $\mathbb{V}_3=\mathbb{M}\equiv\mathbb{H}-\mathbb{J}$, where

$$\mathbb{H}=\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}+\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}\otimes\mathbf{c}. \tag{58}$$

Then, $\mathbf{\Lambda}=\mathrm{diag}(1,3,2)$ and the optimal moduli of cubic symmetry are

$$\mathbb{C}_{\mathrm{cub}}=3\kappa'\mathbb{J}+2\mu'\mathbb{L}+2\eta\mathbb{M}, \tag{59}$$

where

$$\kappa'=\tfrac{1}{3}\langle\mathbb{J},\mathbb{C}\rangle,\quad \mu'=\tfrac{1}{6}\langle\mathbb{L},\mathbb{C}\rangle\quad \eta=\tfrac{1}{4}\langle\mathbb{M},\mathbb{C}\rangle. \tag{60}$$

It is assumed here that the axes of the cubic material are known; otherwise, a numerical search must be performed to find the axes which give the closest, i.e., largest, projection. This additional step is discussed in the numerical example below. Also,

$$\mathbf{P}=\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ 1 & \frac{1}{2} & -\frac{1}{2}\\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}. \tag{61}$$

3. Transverse isotropy/hexagonal symmetry (*N*=5)

Let $\mathbf{a}$ be the direction of the symmetry axis, and define[16]

$$\mathbb{V}_1=\mathbf{A}\otimes\mathbf{A}, \tag{62}$$

$$\mathbb{V}_2=\tfrac{1}{2}\mathbf{B}\otimes\mathbf{B},\quad \mathbb{V}_3=\tfrac{1}{2}(\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A}),\quad \mathbb{V}_4=2\mathbb{V}_3^*,\quad \mathbb{V}_5=2\mathbb{V}_2^*-\mathbb{V}_2,$$

where $\mathbf{A}=\mathbf{a}\otimes\mathbf{a}$ and $\mathbf{B}=\mathbf{I}-\mathbf{A}$. Then, $\mathbf{\Lambda}=\mathrm{diag}(1,1,1,2,2)$ and the optimal transverse isotropy (TI) moduli are given by

$$\mathbb{C}_{\mathrm{TI}}=\sum_{i=1}^{3}\langle\mathbb{C},\mathbb{V}_i\rangle\mathbb{V}_i+\tfrac{1}{2}\sum_{i=4}^{5}\langle\mathbb{C},\mathbb{V}_i\rangle\mathbb{V}_i. \tag{63}$$

Also,

$$\mathbf{P}=\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & \frac{1}{2} & 0\\ 0 & 1 & 0 & 0 & 0\end{pmatrix}. \tag{64}$$

B. Application to acoustically measured data

Let us assume that the material has cubic symmetry but the cube axes orientations are unknown. The effective cubic moduli are, in coordinates coincident with the cube axes,

$$\mathbf{C}_{\mathrm{cub}}=\begin{pmatrix}c_{11}^c & c_{12}^c & c_{12}^c & 0 & 0 & 0\\ c_{12}^c & c_{11}^c & c_{12}^c & 0 & 0 & 0\\ c_{12}^c & c_{12}^c & c_{11}^c & 0 & 0 & 0\\ 0 & 0 & 0 & c_{66}^c & 0 & 0\\ 0 & 0 & 0 & 0 & c_{66}^c & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}^c\end{pmatrix}, \tag{65}$$

where the three constants can be expressed in terms of the bulk modulus and the two shear moduli,

$$c_{11}^c=\kappa'+\tfrac{4}{3}\eta,\quad c_{12}^c=\kappa'-\tfrac{2}{3}\eta,\quad c_{66}^c=\mu'. \tag{66}$$

The effective bulk modulus of (60) is an isotropic invariant that is independent of the cube axes orientation,

$$\kappa'=\tfrac{1}{9}C_{iijj}=\tfrac{1}{9}(c_{11}+c_{22}+c_{33}+2c_{12}+2c_{23}+2c_{31}). \tag{67}$$

Similarly, the combination $(3\mu'+2\eta)/5=\mu$, the isotropic shear modulus implying

$$6\mu'+4\eta=C_{ijij}-\tfrac{1}{3}C_{iijj}=\tfrac{2}{3}(c_{11}+c_{22}+c_{33}-c_{12}-c_{23}-c_{31})+2(c_{44}+c_{55}+c_{66}). \tag{68}$$

Only one of the three cubic parameters depends upon the orientation of the axes, and it follows by considering the squared length of the projected elastic tensor,

$$\|\mathbb{C}_{\rm cub}\|^2 = 9\kappa'^2 + 12\mu'^2 + 8\eta^2 = 9\kappa'^2 + \tfrac{1}{5}(6\mu' + 4\eta)^2 + \tfrac{6}{5}(2\mu' - 2\eta)^2. \quad (69)$$

The first two terms are independent of the cube axes orientation, and therefore the closest cubic projection maximizes the final term. Note that

$$2\mu' - 2\eta = \tfrac{1}{3}C_{ijij} + \tfrac{1}{6}C_{iijj} - \tfrac{5}{6}\langle \mathbb{C}, \mathbb{H}\rangle = \tfrac{5}{2}\kappa' + 2\mu' + \tfrac{4}{3}\eta - \tfrac{5}{6}\langle \mathbb{C}, \mathbb{H}\rangle. \quad (70)$$

The moduli κ', μ', and η must all be positive in order for the material to have positive-definite strain energy. Therefore, $(2\mu' - 2\eta)^2$ is maximum when $\langle \mathbb{C}, \mathbb{H}\rangle$ is minimum. The latter is also a positive quantity, which may be expressed as

$$\langle \mathbb{C}, \mathbb{H}\rangle = C_{ijkl}H_{ijkl} = c'_{11} + c'_{22} + c'_{33}, \quad (71)$$

where c'_{IJ} are the elements of $\mathbb{C}$ of (74) in the coordinate system coincident with the cube axes.

We can also write

$$\langle \mathbb{C}, \mathbb{H}\rangle = \tilde{\mathbf{a}}^t\mathbf{C}\tilde{\mathbf{a}} + \tilde{\mathbf{b}}^t\mathbf{C}\tilde{\mathbf{b}} + \tilde{\mathbf{c}}^t\mathbf{C}\tilde{\mathbf{c}}, \quad (72)$$

where

$$\tilde{\mathbf{a}}^t = (a_1^2, a_2^2, a_3^2, 2a_2a_3, 2a_3a_1, 2a_1a_2), \text{ etc.} \quad (73)$$

The minimum can be found by numerically searching over all possible orientations using Euler angles to transform from the fixed vector basis to the cube axes. The rotated (cube) axes are simply the columns of the 3×3 transformation matrix $\mathbf{R} \in SO(3)$.

As a specific application we reconsider the elastic moduli obtained from ultrasonic measurements by Francois *et al.*[2] In the notation of Eq. (22), the raw data for the stiffness tensor are

$$\mathbf{C} = \begin{pmatrix} 243 & 136 & 135 & 22 & 52 & -17 \\ 136 & 239 & 137 & -28 & 11 & 16 \\ 135 & 137 & 233 & 29 & -49 & 3 \\ 22 & -28 & 29 & 133 & -10 & -4 \\ 52 & 11 & -49 & -10 & 119 & -2 \\ -17 & 16 & 3 & -4 & -2 & 130 \end{pmatrix} \text{(GPa)}. \quad (74)$$

The element c_{53} is given as 49 in Ref. 2, which appears to be a typographical error.

We find that

$$\mathbf{C}_{\rm cub} = \begin{pmatrix} 213.4 & 148.5 & 148.5 & 0 & 0 & 0 \\ 148.5 & 213.4 & 148.5 & 0 & 0 & 0 \\ 148.5 & 148.5 & 213.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 139.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 139.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 139.8 \end{pmatrix} \text{(GPa)}, \quad (75)$$

in agreement with Francois *et al.*[2] The rotation from the coordinate system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of (74) can be expressed as a rotation about a single axis $\mathbf{n}$ through angle θ using Euler's theorem.[19] The angle and axis follow from $2\cos\theta = tr\mathbf{R} - 1$ and $2\sin\theta\mathbf{n} = \mathbf{skew}\,\mathbf{R}$, where $\mathbf{skew}\,\mathbf{Y} = -\epsilon_{ijk}Y_{ij}\mathbf{e}_k$, and ϵ_{ijk} is the third-order alternating tensor. Thus, we find $\theta = 96°$ and $\mathbf{n} = (0.18, 0.06, 0.98)$, and the full set of moduli in the rotated frame is

$$\mathbf{C}' = \begin{pmatrix} 228 & 141 & 148 & -5 & -1 & -2 \\ 141 & 209 & 156 & -6 & 23 & -1 \\ 148 & 156 & 203 & -7 & -2 & 4 \\ -5 & -6 & -7 & 144 & 12 & -3 \\ -1 & 23 & -2 & 12 & 139 & 11 \\ -2 & -1 & 4 & -3 & 11 & 136 \end{pmatrix} \text{(GPa)}. \quad (76)$$

This set of cube axes is unique within rotation under the group of transformations congruent with cubic symmetry. In this preferred coordinate system, the projection onto the cubic moduli in this frame is simply

$$c^c_{11} = \tfrac{1}{3}(c'_{11} + c'_{22} + c'_{33}) = 213.4 \text{ (GPa)},$$

$$c^c_{12} = \tfrac{1}{3}(c'_{12} + c'_{23} + c'_{31}) = 148.5,$$

$$c^c_{66} = \tfrac{1}{3}(c'_{44} + c'_{55} + c'_{66}) = 139.8. \quad (77)$$

V. CONCLUSION

The main result of this paper is the proof that the Euclidean projection of anisotropic elastic constants onto a higher material is identical to minimizing the mean-square difference of the slowness surfaces. This provides a well-grounded acoustical basis for using the Euclidean projection as a natural way to simplify ultrasonic or acoustic data. The equivalence generalizes the original result of Fedorov for the best isotropic acoustical fit to a given set of anisotropic moduli.

ACKNOWLEDGMENT

Discussions with Dr. M. Moakher have benefited this paper and are appreciated.

APPENDIX: THE MATRIX P AND PROOF OF EQ. (41)

The tensor $\mathbb{V}_i^*$ can be expressed using Eq. (19) in terms of the totally symmetric and asymmetric parts of $\mathbb{V}_i$. Both $\mathbb{V}_i^{(s)}$ and $\mathbb{V}_i^{(a)}$ possess the same material symmetry as $\mathbb{V}_i$, and hence the symmetry is inherited by $\mathbb{V}_i^*$. Since the $\mathbb{V}_i$'s themselves form a basis for the material symmetry, it follows that $\mathbb{V}_i^*$ can be written as a linear combination of them. Let $\mathbf{P}$ be the $N\times N$ matrix which defines the * operation in terms of the basis tensors, that is,

$$\mathbb{V}_i^* = \sum_{j=1}^{N} P_{ij}\mathbb{V}_j. \tag{A1}$$

It follows from Eq. (20) that

$$\mathbb{V}_i = -\mathbb{V}_i^* + 2\sum_{j=1}^{N} P_{ij}\mathbb{V}_j^*, \tag{A2}$$

and hence the inverse of $\mathbf{P}$ is

$$\mathbf{P}^{-1} = 2\mathbf{P} - \mathbf{I}_{(N)}, \tag{A3}$$

where $\mathbf{I}_{(N)}$ is the $N\times N$ identity.

Consider the identity

$$\langle \mathbb{V}_i^*, \mathbb{V}_j\rangle = \langle \mathbb{V}_i, \mathbb{V}_j^*\rangle, \tag{A4}$$

which follows from (21). Using (A1) to eliminate $\mathbb{V}_i^*$ and $\mathbb{V}_j^*$ from the left and right members, Eq. (A4) implies

$$\mathbf{P}\mathbf{\Lambda} = \mathbf{\Lambda}\mathbf{P}^t. \tag{A5}$$

It is worth noting the very specific nature of the matrix $\mathbf{P}$ that is required to satisfy Eq. (A5). This matrix essentially defines the * operator, from which the totally symmetric and asymmetric parts of a tensor can be found in terms of the basis tensors. Some examples of $\mathbf{P}$ are given in Sec. IV.

We now turn to the proof of Eq. (41). Starting with the definition of $\mathcal{P}_{\text{sym}}$ in (40), we have

$$\begin{aligned}\mathcal{P}_{\text{sym}}\mathbb{C}^* &= \sum_{i,j=1}^{N} \langle \mathbb{C}^*, \mathbb{V}_i\rangle \Lambda_{ij}^{-1}\mathbb{V}_j = \sum_{i,j=1}^{N} \langle \mathbb{C}, \mathbb{V}_i^*\rangle \Lambda_{ij}^{-1}\mathbb{V}_j \\ &= \sum_{i,j=1}^{N} \langle \mathbb{C}, \mathbb{V}_i\rangle X_{ij}\mathbb{V}_j^*,\end{aligned} \tag{A6}$$

where (21) has been used in the second line, and the matrix $\mathbf{X}$ is defined as

$$\mathbf{X} = \mathbf{P}^t\mathbf{\Lambda}^{-1}\mathbf{P}^{-1}. \tag{A7}$$

Comparison with (A5) gives

$$\mathbf{X} = \mathbf{\Lambda}^{-1}, \tag{A8}$$

and substituting from (A8) into (A6) and comparing it with (38) implies the fundamental property (41).

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