THERMOELASTIC RELAXATION IN ELASTIC STRUCTURES, WITH APPLICATIONS TO THIN PLATES

by A. N. NORRIS

(*Mechanical and Aerospace Engineering, Rutgers University Piscataway,* NJ 08854-8058, *USA*)

and D. M. PHOTIADIS

(*Naval Research Laboratory, Code* 7130, *Washington*, DC 20375-5350, *USA*)

[Received 29 March 2004. Revise 25 October 2004]

Summary

A new result enables direct calculation of thermoelastic damping in vibrating elastic solids. The mechanism for energy loss is thermal diffusion caused by inhomogeneous deformation, flexure in thin plates. The general result is combined with the Kirchhoff assumption to obtain a new equation for the flexural vibration of thin plates incorporating thermoelastic loss as a damping term. The thermal relaxation loss is inhomogeneous and depends upon the local state of vibrating flexure, specifically, the principal curvatures at a given point on the plate. Thermal loss is zero at points where the principal curvatures are equal and opposite, that is, saddle shaped or pure anticlastic deformation. Conversely, loss is maximal at points where the curvatures are equal, that is, synclastic or spherical flexure. The influence of modal curvature on the thermoelastic damping is described through a modal participation factor. The effect of transverse thermal diffusion on plane wave propagation is also examined. It is shown that transverse diffusion effects are always small provided the plate thickness is far greater than the thermal phonon mean free path, a requirement for the validity of the classical theory of heat transport. These results generalize Zener's theory of thermoelastic loss in beams and are useful in predicting mode widths in micro- and nano-electromechanics systems oscillators.

1. Introduction

High-Q resonators are central to the development of new devices and applications that include radio frequency filters, next generation magnetic reasonance imaging systems, and torque magnetometers. Silicon based micro- and nano-electromechanical systems (MEMS/NEMS) oscillators are the candidates of choice, and include free-standing planar devices, such as double paddle oscillators (DPOs), and micro-cantilevers. As the oscillators shrink in size, it has been found that the Q achieved is orders of magnitude smaller than expected based on classical fundamental loss mechanisms. Many mechanisms have been proposed, including surface loss (**1** to **3**) that increases with the surface-to-volume ratio. However, under controlled conditions with minimal surface defects and adsorbates, measurements on silicon DPOs have shown that room temperature losses are adequately described by thermoelastic relaxation, while unexplained mechanisms operate at lower temperatures. Interestingly, the mode of vibration of DPOs is designed to be primarily torsional with very little flexure (and hence no thermoelastic coupling). However, as demonstrated by Photiadis

Q. Jl Mech. Appl. Math. (2005) 58 (1), 143–163
doi:10.1093/qjmamj/hbi002

and his co-workers (**4** to **7**) it is precisely the small amount of flexural motion that accounts for loss in these supposedly torsional oscillators.

The purpose of this paper is to provide a consistent theory for predicting intrinsic dissipation arising from thermoelasticity in elastic structures. Particular attention will be given to flexural motion of thin plates. This work is a step towards understanding the fundamental limits of dissipation in small structures such as NEM and MEM devices.

We are concerned with determining the thermomechanical loss of elastic modes; for example, the flexural mode of a rectangular thin plate. A useful point of departure is the classic theory of Zener (**8**) for anelastic thermoelastic damping. The key to this approach is the assumption, implicit in Zener's work, that there is little relative difference between the isentropic (unrelaxed) and isothermal (relaxed) mechanical responses, and hence the mechanical and thermal problems are essentially decoupled. Since the thermoelasticity is weak, the transition from the instantaneous or unrelaxed system to the relaxed state can be viewed as a quasistatic thermal process, governed by the standard equations for thermal diffusion, although now in the presence of an inhomogeneous deformation.

Energy loss in a mechanical oscillator is measured in terms of the quality factor, defined as $Q = 2\pi\mathcal{E}_0/\Delta\mathcal{E}$, where $\mathcal{E}_0$ is the mechanical energy of the oscillator and $\Delta\mathcal{E}$ is the energy loss per cycle. The quality factor for a lightly damped single degree of freedom system with non-dimensional damping $\zeta \ll 1$ is $Q = 1/(2\zeta)$, and by assumption we only consider systems with light damping, or $Q \gg 1$. The relation between Q and $\alpha_{\rm at}$, the attenuation per unit length of a propagating wave of frequency ω, is $Q = \omega/(2\alpha_{\rm at}v)$, where v is the speed of energy propagation, also equivalent to the group velocity. This identity can be derived by assuming the energy is a quadratic function of the field variables, so that energy decays with distance d as $e^{-2\alpha_{\rm at}d}$. The distance travelled in one period is $d = 2\pi v/\omega$, and hence the fractional decrease in energy per period is $1 - e^{-4\pi\alpha_{\rm at}v/\omega} \approx 4\pi\alpha_{\rm at}v/\omega$ from which the relation for Q follows.

Thermoelastic loss can be most simply viewed as a relaxation mechanism with a single relaxation time τ. The generic frequency dependent quality factor $Q(\omega)$ for a relaxation mechanism is

$$Q^{-1} = \frac{\Delta c}{c_0}\frac{\omega\tau}{1+\omega^2\tau^2}, \tag{1.1}$$

where Δc is the (relatively) instantaneous increase in elastic modulus, $c_0 \to c_0 + \Delta c$, caused by the process under consideration. The change in elasticity is well known for thermoelasticity, and τ has been estimated for several configurations. Thus, Zener (**8**) showed for flexure of a beam that

$$Q^{-1} = \frac{E\alpha^2 T}{C_p}\frac{\omega\tau_0}{1+\omega^2\tau_0^2}, \tag{1.2}$$

where E is the isothermal Young's modulus, T is the absolute temperature, α is the volume coefficient of thermal expansion, and C_p is the heat capacity at constant stress. The characteristic relaxation time is $\tau_0 = h^2C_p/(\pi^2K)$, where h is the thickness and K the thermal conductivity. In fact, as Zener first demonstrated (**9**), the simple expression (1.2) is the leading term in an infinite series which is well approximated by the single term (see (4.6) and Appendix B). Zener's method was derived in the context of scalar problems, where the strain, for instance, involves a single component. An important example is the flexure of a beam or reed, as considered originally by Zener, and later by others, for example, (**10**). The present work generalizes Zener's method to consider general elastic deformation. This includes the inhomogeneous deformation associated with modes in thin plates and other structures.

Since the original work of Zener numerous papers have appeared on thermal relaxation in the context of coupled thermoelasticity. In a pair of papers (**11**, **12**) Alblas provided a rigorous formulation using continuum thermomechanics and linear elasticity theory for isotropic materials. He derived detailed and explicit solutions for the thermoelastic damping in vibrating beams, including the circular rod and the rectangular beam. The result for the latter was derived separately by Lifshitz and Roukes (**13**), although Alblas's solution is the more general of the two. These analyses are compared with the present formulation later (see Appendix B). Kinra and Milligan (**14**) again derived the coupled isotropic thermoelastic equations and provided a solution for unidimensional structures, including a discontinuous beam. Perhaps the most thorough analysis of thermal damping in the context of the coupled equations of thermoelasticity is due to Chadwick (**15**). By considering a modal decomposition of the elastic and thermal fields, an exact relation for the complex-valued frequency of oscillation of each mode was obtained. This enabled Chadwick to derive a generalization of Zener's expression for the thermoelastic damping of an arbitrary elastic body. Chadwick subsequently derived the governing equations of thermoelasticity for thin plates and beams (**16**). The equations are in the form of coupled equations, one of which reduces to the classical equations for the structural mode, for example, flexural waves in thin plates, and the other the temperature diffusion equation, in the limit of zero coupling. The analysis in (**15**,**16**) is restricted to isotropic solids.

This paper has several objectives. The first is to demonstrate how the Zener model follows from the full equations for the coupled dynamic system by using a consistent approximation scheme. In the process we generalize Zener's approach to incorporate general elastic deformation, specifically the elastic stress and strain tensors. The main applications are to thin plate structures, for which we obtain a Zener-like result for arbitrary flexural deformation that includes the general curvature tensor. Our results will also include the possibility of thermal diffusion in the lateral direction in thin plates, which is explicitly ignored in Zener's approach, but was considered by Alblas (**11**, **12**) and Chadwick (**16**). However, it will be shown that circumstances under which lateral thermal flux becomes important coincide with the limit in which the thin plate theory is no longer applicable.

The paper is arranged as follows. Governing equations of thermoelasticity are presented in section 2 for anisotropic elastic bodies. General solutions are discussed in section 3 with no particular type of structure in mind. The theory is applied to thin plates in flexure in section 4 and a non-dimensional modal participation factor (MPF) is introduced which defines the local contribution to thermoelastic (TE) loss in terms of the plate curvature. An alternative method for deriving the TE loss of travelling flexural waves is presented in section 5 using generalized plate equations. The effects of lateral thermal diffusion are discussed in the context of travelling wave solutions in section 5.

2. Equations of thermoelasticity

2.1 *Constitutive relations of thermomechanics*

The thermomechanical variables are the bulk stress and strain, $\boldsymbol{\sigma}$ and $\mathbf{e}$, the temperature deviation, θ, from the ambient absolute temperature θ_a ($|\theta|/\theta_a \ll 1$), and the entropy deviation per unit volume, s, from the ambient entropy s_a. All quantities are defined relative to their ambient values, and would be zero in the absence of exterior motivating forces. The constitutive relations for strain and entropy in terms of the independent variables stress and temperature, $\{\boldsymbol{\sigma},\ \theta\}$, are (**17** to **19**)

$$\mathbf{e} = \mathbf{S}\boldsymbol{\sigma} + \boldsymbol{\alpha}\theta, \qquad s = C_p(\theta/\theta_a) + \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}. \tag{2.1}$$

Table 1 Thermoelastic constitutive equations. The left-most column indicates the dependent variable. Each subsequent column corresponds to a set of constitutive equations in terms of the independent variables in the first row

Dependent variable	$\boldsymbol{\sigma}, \theta$	$\mathbf{e}, \theta$	$\mathbf{e}, s$	$\boldsymbol{\sigma}, s$
$\mathbf{e} =$	$\mathbf{S}\boldsymbol{\sigma} + \boldsymbol{\alpha}\theta$			$\mathbf{S}_s\boldsymbol{\sigma} + \dfrac{\theta_a}{C_p}\boldsymbol{\alpha}\, s$
$s =$	$\dfrac{C_p}{\theta_a}\theta + \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}$	$\dfrac{C_v}{\theta_a}\theta + \boldsymbol{\beta}\cdot\mathbf{e}$		
$\boldsymbol{\sigma} =$		$\mathbf{C}\mathbf{e} - \boldsymbol{\beta}\theta$	$\mathbf{C}_s\mathbf{e} - \dfrac{\theta_a}{C_v}\boldsymbol{\beta} s$	
$\theta =$			$\dfrac{\theta_a}{C_v}(s - \boldsymbol{\beta}\cdot\mathbf{e})$	$\dfrac{\theta_a}{C_p}(s - \boldsymbol{\alpha}\cdot\boldsymbol{\sigma})$

Table 1 summarizes the alternative formulations of the same equations based on different choices of the independent variables: $\{\mathbf{e}, \theta\}$, $\{\mathbf{e}, s\}$ or $\{\boldsymbol{\sigma}, \mathrm{s}\}$.

The material constants are as follows: $\mathbf{S}$ is the fourth-order tensor of isothermal compliances, with inverse $\mathbf{C}$, and corresponding adiabatic quantities are $\mathbf{S}_s$ and $\mathbf{C}_s$. The symmetric second-order tensor of thermal expansion coefficients is $\boldsymbol{\alpha}$, and the related tensor $\boldsymbol{\beta}$ is defined as $\boldsymbol{\beta} = \mathbf{C}\boldsymbol{\alpha}$, while the quantities C_p and C_v are the heat capacities per unit volume at constant stress and strain respectively. The following relations can be verified from Table 1:

$$C_p/\theta_a = (C_v/\theta_a) + \boldsymbol{\alpha}\cdot\mathbf{C}\boldsymbol{\alpha}, \quad \mathbf{S} = \mathbf{S}_s + (\theta_a/C_p)\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}, \quad \mathbf{C} = \mathbf{C}_s - (\theta_a/C_v)\,\boldsymbol{\beta}\otimes\boldsymbol{\beta}. \tag{2.2}$$

A word about notation: $\boldsymbol{\alpha}\cdot\boldsymbol{\sigma} = \mathrm{tr}\,(\boldsymbol{\alpha}\boldsymbol{\sigma})$ is a scalar, while $\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}$ is a fourth-order tensor.

The constitutive relations in Table 1 follow from the standard thermodynamic relations

$$dU = \boldsymbol{\sigma}\cdot d\mathbf{e} + \theta_a ds, \quad dF = \boldsymbol{\sigma}\cdot d\mathbf{e} - s_a d\theta, \tag{2.3}$$

$$dG = -\mathbf{e}\cdot d\boldsymbol{\sigma} - s_a d\theta, \quad dH = -\mathbf{e}\cdot d\boldsymbol{\sigma} + \theta_a ds, \tag{2.4}$$

where U, F, G and H are, respectively, internal energy, Helmholtz free energy, Gibbs free energy and enthalpy, all per unit volume. These are related by the standard connections $U = F + TS = H + \boldsymbol{\sigma}\cdot\mathbf{e} = G + TS + \boldsymbol{\sigma}\cdot\mathbf{e}$, where here, T and S are the absolute temperature and entropy, $T = \theta_a + \theta$, $S = s_a + s$. The energy densities can be expressed, in the quadratic approximation that

is used here, as

$$2U = \mathbf{e}\cdot\mathbf{C}\mathbf{e} + (C_v/\theta_a)\,\theta^2, \quad 2F = \mathbf{e}\cdot\mathbf{C}_s\mathbf{e} - (\theta_a/C_v)\,s^2, \tag{2.5}$$

$$2G = -\boldsymbol{\sigma}\cdot\mathbf{S}_s\boldsymbol{\sigma} - (\theta_a/C_p)\,s^2, \quad 2H = -\boldsymbol{\sigma}\cdot\mathbf{S}\boldsymbol{\sigma} + (C_p/\theta_a)\,\theta^2, \tag{2.6}$$

The constitutive relations in Table 1 follow from (2.3) and (2.5) combined with the basic definitions of the thermal expansion coefficients and the heat capacity,

$$\boldsymbol{\alpha} = \left.\frac{\partial\mathbf{e}}{\partial T}\right|_{\boldsymbol{\sigma}}, \quad \Delta Q_{p,v} = C_{p,v}\Delta T. \tag{2.7}$$

2.2 *Thermoelastic relaxation governing equations*

We first present the exact governing equations and then make appropriate asymptotic approximations. The motion is assumed to be caused by external forcing with no internal applied body forces or sources of heat. The heat flow in an elastic body is governed by the energy balance

$$\theta_a\dot{s} + \operatorname{div}\mathbf{q} = 0, \tag{2.8}$$

where $\mathbf{q}$ is the heat flux. The equation for s in terms of θ and $\boldsymbol{\sigma}$ in Table 1 and (2.8) imply

$$C_p\dot{\theta} + \operatorname{div}\mathbf{q} = -\theta_a\,\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}. \tag{2.9}$$

Irreversibility is introduced by requiring the heat flux to satisfy a generalized form of Fourier's relation (**19**)

$$\mathbf{q} + \tau_r\dot{\mathbf{q}} + \mathbf{K}\nabla\theta = 0, \tag{2.10}$$

where $\mathbf{K}$ is the positive definite thermal conductivity tensor, and τ_r is the thermal relaxation time (**19**). The Cattaneo–Vernotte heat flux equation (2.10) includes the classical and more commonly used Kirchhoff law in the limit as $\tau_r \to 0$. The parameter τ_r is sometimes introduced to ensure finite speeds in the theory (**20**). Our objective is to solve the linear system of partial differential equations, (2.9) and (2.10), for θ as a function of the forcing in the right member of (2.9). The heat flux can be eliminated to give a single equation for θ,

$$C_p\left(\dot{\theta} + \tau_r\ddot{\theta}\right) - \operatorname{div}\mathbf{K}\nabla\theta = -\left(1 + \tau_r\frac{\partial}{\partial t}\right)\theta_a\,\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}. \tag{2.11}$$

A closed system of equations is obtained by applying the dynamic equilibrium condition

$$\operatorname{div}\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}} = 0, \tag{2.12}$$

where ρ is the mass per unit volume and $\mathbf{u}$ is the elastic displacement vector, related to the strain via $\mathbf{e} = \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)/2$. It is shown in Appendix A that closed-form solutions of the coupled system of equations (2.11) and (2.12) are generally feasible only under restricted conditions. These require, essentially, that the material must be elastically isotropic, which is too limiting for our purposes.

3. Solution for arbitrary structures

3.1 *Asymptotic approximation*

The key quantities are the positive definite compliance and stiffness differences $\Delta\mathbf{S} \equiv \mathbf{S} - \mathbf{S}_s$ and $\Delta\mathbf{C} \equiv \mathbf{C}_s - \mathbf{C}$. These determine the energy decrement between the final (isothermal) relaxed and initial unrelaxed (adiabatic) states, and they follow from (2.2) as $\Delta\mathbf{S} = (\theta_a/C_p)\boldsymbol{\alpha}\boldsymbol{\alpha}^T$ and $\Delta\mathbf{C} \equiv \mathbf{C}_s - \mathbf{C} = (\theta_a/C_v)\boldsymbol{\beta}\boldsymbol{\beta}^T$. Zener's approach is based on a separation between the mechanics and the thermodynamics. By assumption, the *total difference* between the relaxed and unrelaxed energies is small. Specifically $\Delta\mathcal{E}_0/\mathcal{E}_0 \ll 1$, where the mechanical energy $\mathcal{E}_0 = \frac{1}{2}\boldsymbol{\sigma}_0 \cdot \mathbf{e}_0$ is defined by $\mathbf{e}_0$ and $\boldsymbol{\sigma}_0$, which are related by the purely mechanical equation $\boldsymbol{\sigma}_0 = \mathbf{C}\mathbf{e}_0$ (ignoring temperature and entropy variations). Thus, $\mathcal{E}_0 = \frac{1}{2}\boldsymbol{\sigma}_0 \cdot \mathbf{S}\boldsymbol{\sigma}_0$ and the decrement may be defined as

$$\Delta\mathcal{E}_0 = \boldsymbol{\sigma}_0 \cdot \Delta\mathbf{S}\,\boldsymbol{\sigma}_0 = (\theta_a/C_p)\big(\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}_0\big)^2. \tag{3.1}$$

Alternatively, $\Delta\mathcal{E}_0 \approx \mathbf{e}_0 \cdot \Delta\mathbf{C}\mathbf{e}_0 = (\theta_a/C_v)\big(\boldsymbol{\beta} \cdot \mathbf{e}_0\big)^2$, where the approximation is due to the assumed purely mechanical relation between $\boldsymbol{\sigma}_0$ and $\mathbf{e}_0$. The main point is that the relative change in energy between the unrelaxed and relaxed states in either case is the same to leading order in ϵ, where the non-dimensional parameter governing TE damping is

$$\epsilon = E\theta_a\alpha^2/C_p. \tag{3.2}$$

This definition of ϵ is chosen to equal the relative change in elastic moduli $\Delta c/c_0$ for TE relaxation of a thin beam, equation (1.2). It can also be expressed as

$$\epsilon = \tfrac{1}{3}(1-2\nu)(C_p - C_v)/C_p, \tag{3.3}$$

where ν is the isothermal Poisson's ratio. Chadwick (**15**) employed a slightly different non-dimensional parameter (denoted here by ϵ_c to distinguish it from ϵ)

$$\epsilon_c = \frac{1}{3}\left(\frac{1+\nu}{1-\nu}\right)\frac{C_p - C_v}{C_v}. \tag{3.4}$$

It is clear that the non-dimensional parameters are closely related and, in particular, of the same order of magnitude.

3.2 *Solution by projections for anisotropic systems*

The coupled equations (2.11) and (2.12) are solved using a regular perturbation procedure in the asymptotic parameter $\epsilon \ll 1$. We will achieve the solution using a projection method, similar to Zener's approach. A separation of variables reduces the problem to coupled ordinary differential equations in time. Anisotropy in the elastic material does not permit a modal expansion with a common set of scalar eigenfunctions, the basis for Zener's method, and the key to a generalization of his method to the limited but important case of isotropic solids (**15**); see Appendix A. However, even in the case of the exact solution obtained by Chadwick (**15**), the interesting phenomena are obtained by the leading-order approximation to the complex-valued frequencies. It therefore makes more sense ultimately to proceed by a regular asymptotic approximation at the stage of the coupled equations (2.11) and (2.12). In this approach we view them as decoupled to leading order, whereby

the elasticity problem is solved with no thermal effects. That is, we consider the elasticity as an uncoupled but vital forcing term in the 'thermal equation' (2.11).

Thus, to leading order equation (2.11) gives an uncoupled equation for temperature with the stress entering on the right-hand side as a forcing term. It is important to note that the forcing in (2.11) is proportional to $\boldsymbol{\alpha} \cdot \dot{\boldsymbol{\sigma}}$. When the thermal expansion tensor is isotropic, the forcing depends upon the rate of hydrostatic stress, even when the material is elastically anisotropic. Thus, it is the hydrostatic stress, *not* strain, that governs the TE loss.

Further progress is made using projections onto a set of eigenfunctions. In fact, this is similar to the method first proposed by Zener (**8**), which treated a simpler decoupled thermoelasticity problem. We first discuss the generalization of Zener's method to (2.11) as it allows us to determine the final answer in a form similar to the familiar and classical result of Zener for an elastic beam in flexure. We will later compare the general solution with direct solutions for particular configurations. Assume the temperature can be represented as

$$\theta(\mathbf{x}, t) = \sum_{n=0}^{\infty} \theta_n(t)\phi_n(\mathbf{x}), \tag{3.5}$$

where $\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$ and the eigenvalues τ_n and eigenfunctions ϕ_n satisfy

$$C_p^{-1}\mathrm{div}\,\mathbf{K}\nabla\phi_n + \tau_n^{-1}\phi_n = 0, \quad n = 0,\ 1,\ 2, \ldots, \tag{3.6}$$

plus appropriate boundary conditions (for example, no flux). The amplitudes solve

$$\dot{\theta}_n + \tau_r\ddot{\theta}_n + \tau_n^{-1}\theta_n = -\left(1 + \tau_r\frac{\partial}{\partial t}\right)\frac{\theta_a}{C_p}\langle\phi_n,\ \boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}\rangle \tag{3.7}$$

where the brackets indicate the inner product $\langle f, g\rangle = \int dV f(\mathbf{x})\, g(\mathbf{x})$.

3.3 *A general result for energy loss*

Before considering applications of (3.5) to (3.7) to particular structures we first derive a general result for TE dissipation. A measure of local structural damping may be defined in terms of the local relative loss in energy per cycle. The rate of change of local mechanical energy per unit volume is $\dot{\mathcal{E}} = \boldsymbol{\sigma} \cdot \dot{\mathbf{e}}$. We assume periodic oscillation for $\boldsymbol{\sigma}$ and $\mathbf{e}$ and determine the loss in the mechanical energy through the coupling to irreversible thermal process, θ, also periodic. Using the relation for $\mathbf{e}$ in terms of $\boldsymbol{\sigma}$ and θ in Table 1, gives

$$\dot{\mathcal{E}} = \boldsymbol{\sigma} \cdot \mathbf{S}\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \cdot \boldsymbol{\alpha}\,\dot{\theta}. \tag{3.8}$$

Taking the average over a cycle, and using $\overline{f\,\dot{f}} = 0$, where the overbar indicates a time average, implies that the local irreversible energy loss rate per unit volume is

$$\overline{\dot{\mathcal{E}}(\mathbf{x})} = -\,\overline{\theta(\mathbf{x}, t)\,\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}(\mathbf{x}, t)} = \sum_{n=0}^{\infty}\phi_n(\mathbf{x})\,\overline{\theta_n(t)\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}(\mathbf{x}, t)}. \tag{3.9}$$

The total power dissipated is thus

$$\Pi_d = -\int dV \overline{\dot{\mathcal{E}}(\mathbf{x})} = -\sum_{n=0}^{\infty} \overline{\theta_n(t)\langle \phi_n,\ \boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}\rangle}$$
$$= \sum_{n=0}^{\infty} \frac{\omega^2\tau_n}{(1-\omega^2\tau_r\tau_n)^2+\omega^2\tau_n^2}\,\frac{\theta_a}{C_p}\,\overline{\langle \phi_n,\ \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\rangle^2}, \tag{3.10}$$

where we have assumed periodic motion of period $2\pi/\omega$ and used (3.7) to express the time harmonic temperature field in terms of the stress.

The loss factor is then given by $Q^{-1} \approx \tan\delta = \Delta\mathcal{E}/(2\pi\mathcal{E}_0)$, where $\Delta\mathcal{E} = \overline{\Pi_d}\,2\pi/\omega$ is the loss per cycle and $\mathcal{E}_0$ is the total energy of oscillation. We find

$$Q^{-1} = \frac{\theta_a}{C_p\mathcal{E}_0}\sum_{n=0}^{\infty} \frac{\omega\tau_n}{(1-\omega^2\tau_r\tau_n)^2+\omega^2\tau_n^2}\,\overline{\langle \phi_n,\ \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\rangle^2}. \tag{3.11}$$

This provides a general formula for the TE loss in terms of the inhomogeneous stress. The Q of a particular mode may be straightforwardly obtained by integrating (3.11) over the volume of the oscillator and dividing by the total energy. This equation may alternatively be expressed in terms of the inhomogeneous strain; however, we find the stress formulation more convenient.

Equation (3.11) is one of the main results of the paper, as it provides a means to compute TE dissipation given a solution in terms of the inhomogeneous stress.

We remark on the summation in (3.11). If the first term in the infinite sum is dominant, as is often the case (**8**), the sum can be truncated after only one term ($n = 0$). This gives a result very similar to (1.1) except that the simple Lorentzian in the latter is replaced by the generalized Lorentzian amplitude

$$A(\omega\tau) = \frac{\omega\tau}{(1-\omega^2\tau_r\tau)^2+\omega^2\tau^2}. \tag{3.12}$$

Of course, this reduces to the classical Lorentzian when $\tau_r = 0$, which has a maximum as a function of ω when $\omega\tau = 1$. It is worth describing the properties of this generalized Lorentzian, in particular how τ_r influences the maximum. For every $\tau_r \geqslant 0$, A has a single maximum at a unique value of $\omega = \omega^*$ defined by

$$\omega^*\tau = \left[(1-4r+16r^2)^{1/2} - 1 + 2r\right]^{1/2}/(r\sqrt{6}), \quad \text{where} \quad r = \tau_r/\tau. \tag{3.13}$$

Furthermore, $\omega^*\tau \geqslant 1$ for a restricted range of τ_r. Specifically, $1 \leqslant \omega^*\tau \leqslant \sqrt{4/3}$ for $0 \leqslant \tau_r \leqslant 2\tau/3$, with $\omega^*\tau$ equal to unity at the two extremes ($\tau_r = 0$ and $\tau_r = 2\tau/3$) and $\omega^*\tau = \sqrt{4/3}$ for $\tau_r = \tau/4$. Conversely, $\omega^*\tau < 1$ for $\tau_r > 2\tau/3$. In particular, for relatively large $\tau_r \gg \tau$, the maximum is at $\omega^*(\tau\tau_r)^{1/2} = 1$. The value of A at the maximum, $A_{\max}$, increases monotonically from $A_{\max} = \frac{1}{2}$ when there is zero thermal relaxation, $\tau_r = 0$, to $A_{\max} \approx (\tau_r/\tau)^{1/2}$ for $\tau_r \gg \tau$.

4. Thermoelastically damped orthotropic thin plates

4.1 *Thin plate dynamics*

We consider plates that are orthotropic with axes of symmetry coincident with the coordinate axes. Assume, with no loss in generality, that the thermal expansion tensor is diagonal, $\boldsymbol{\alpha} =$

diag $(\alpha_1,\ \alpha_2,\ \alpha_3)$. By virtue of the thin plate configuration we can ignore the normal stress σ_{zz}, and employing the Kirchhoff approximation for the deformation, we have

$$\boldsymbol{\alpha}\cdot\boldsymbol{\sigma} = \alpha_1\sigma_{xx} + \alpha_2\sigma_{yy} = \frac{z}{1-\nu_{12}\nu_{21}}\big[(1+\nu_{21})E_1\,\alpha_1\kappa_{xx} + (1+\nu_{12})E_2\,\alpha_2\kappa_{yy}\big], \tag{4.1}$$

where $\boldsymbol{\kappa}$ is the curvature tensor with components κ_{xx}, κ_{yy} and κ_{xy}. The curvature is related to the transverse deflection of the centre-plane, $w(x, y)$, by

$$\kappa_{ij} = -\frac{\partial^2 w(x,y)}{\partial x_i \partial x_j}. \tag{4.2}$$

The quantity ν_{ij} is the Poisson ratio for strain in the j-direction caused by stress in the i-direction, and the two Poisson's ratios satisfy $\nu_{21}E_1 = \nu_{12}E_2$. The instantaneous potential energy density per unit area of a plate in flexure is

$$\mathcal{E}_{PE} = \frac{I}{2}\left\{\frac{1}{1-\nu_{12}\nu_{21}}\big(E_1\kappa_{xx}^2 + E_2\kappa_{yy}^2 + 2\nu_{21}E_1\kappa_{xx}\kappa_{yy}\big) + 4\mu\kappa_{xy}^2\right\}, \tag{4.3}$$

where μ is the in-plane shear modulus and $I \equiv \langle z,\ z\rangle = h^3/12$.

4.2 *Asymptotic solution by projections for thin plates*

Our purpose here is to obtain a general solution for the TE loss based on the assumption that the transverse diffusion of heat can be ignored. This assumption is examined in section 5 where it is shown that provided the assumptions of thin plate theory and classical heat transport are satisfied, the transverse heat flow gives rise only to small corrections to the TE loss.

The governing equation for the non-equilibrium temperature field is obtained by ignoring the transverse heat flow terms in (2.11),

$$C_p\left(\dot\theta + \tau_r\ddot\theta\right) - K_3\frac{\partial^2\theta}{\partial z^2} = -\left(1+\tau_r\frac{\partial}{\partial t}\right)\theta_a\,\boldsymbol{\alpha}\cdot\dot{\boldsymbol{\sigma}}, \tag{4.4}$$

where the dependence of the local temperature on position is governed solely by the **x**-dependence of the prescribed stress field σ, where now $\mathbf{x} = (x, y) = (x_1, x_2)$. Also, K_3 is the through-thickness element of the thermal conductivity tensor which, in keeping with the general orthotropic formulation, is $\mathbf{K} = \text{diag}\,(K_1,\ K_2,\ K_3)$.

The analysis leading to (3.10) for the power dissipated by TE effects can be repeated with the proviso that the eigenfunctions ϕ_n of the heat equation are now functions only of z, the thickness direction, and inner products should in this case be interpreted accordingly as $\langle f, g\rangle = \int dz\, f(z)\, g(z)$. Hence the analogue to (3.10) refers not to the total power lost, but instead to the rate of energy loss per unit area at position $\boldsymbol{x}$,

$$\dot{\mathcal{E}}(\boldsymbol{x}) = -\sum_{n=0}^{\infty}\frac{\omega^2\tau_n}{(1-\omega^2\tau_r\tau_n)^2+\omega^2\tau_n^2}\,\frac{\theta_a}{C_p}\,\overline{\langle\phi_n(z),\ \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}(\boldsymbol{x},z)\rangle^2}. \tag{4.5}$$

The temperature modes of interest are antisymmetric about the mid-plane (**8**) with

$$\phi_n = (2/h)^{1/2}\sin(2n+1)\frac{\pi z}{h}, \quad \tau_n = (2n+1)^{-2}\,\frac{h^2C_p}{\pi^2K_3}, \quad n = 0, 1, 2, \ldots. \tag{4.6}$$

Temperature modes that are symmetric in z have zero coupling to flexural stress components, since they are antisymmetric, and also to any membrane stresses, which are constant across the thickness. It is evident from (4.5) and (4.1) that the thermal loss in flexure depends upon the quantities (**8**)

$$f_n \equiv \langle \phi_n,\, z\rangle^2/\langle z,\, z\rangle = 96/[(2n+1)\pi]^4. \tag{4.7}$$

Combining (4.1), (4.5) and (4.7), we obtain

$$\dot{\mathcal{E}}(\boldsymbol{x}) = -\,\mathcal{E}_{\rm Diss}(\boldsymbol{\kappa}(\boldsymbol{x}))\,\frac{\bar{E}\theta_a\bar{\alpha}^2}{C_p}\sum_{n=0}^{\infty} f_n \frac{\omega^2\tau_n}{(1-\omega^2\tau_r\tau_n)^2+\omega^2\tau_n^2}, \tag{4.8}$$

where $\bar{E} = (E_1+E_2)/2$ is the average Young's modulus, $\bar{\alpha} = (\alpha_1+\alpha_2)/2$, $\boldsymbol{\kappa}$ is the curvature tensor, and the 'dissipation energy density' (per unit area) $\mathcal{E}_{\rm Diss}$ is given by

$$\mathcal{E}_{\rm Diss}(\boldsymbol{\kappa}(\boldsymbol{x})) = \frac{I}{\bar{E}\bar{\alpha}^2}\left[\frac{(1+\nu_{21})E_1\alpha_2\,\kappa_{xx} + (1+\nu_{12})E_2\alpha_2\,\kappa_{yy}}{1-\nu_{12}\nu_{21}}\right]^2. \tag{4.9}$$

Equation (4.8) is a key result pertaining to TE dissipation in structures which can be modelled as thin plates. Unlike most previous results, the predicted dissipation is inhomogeneous.

The local TE dissipation depends on position via the dependence on the local curvature tensor $\boldsymbol{\kappa}(\boldsymbol{x})$. This aspect may be explored by defining the quantity $p(\boldsymbol{\kappa}) = \mathcal{E}_{\rm Diss}(\boldsymbol{\kappa})/\mathcal{E}_0(\boldsymbol{\kappa})$, which gives a measure of the local TE energy dissipation relative to the local deformation energy. Expressing the total energy as twice the average potential energy by virtue of the virial theorem we find

$$p(\boldsymbol{\kappa}) = \frac{\left[\bar{E}\bar{\alpha}^2(1-\nu_{12}\nu_{21})\right]^{-1}\left[(1+\nu_{21})E_1\alpha_1\,\kappa_{xx} + (1+\nu_{12})E_2\alpha_2\,\kappa_{yy}\right]^2}{E_1\kappa_{xx}^2 + E_2\kappa_{yy}^2 + 2\nu_{21}E_1\kappa_{xx}\kappa_{yy} + 4\mu(1-\nu_{12}\nu_{21})\kappa_{xy}^2}. \tag{4.10}$$

The parameter p simplifies for materials of cubic symmetry, such as silicon, with $E_1 = E_2 = \bar{E} \equiv E$, $\alpha_1 = \alpha_2$ and $\nu_{12} = \nu_{21} \equiv \nu$, and hence

$$p(\boldsymbol{\kappa}) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\left(\kappa_{xx}+\kappa_{yy}\right)^2}{\kappa_{xx}^2+\kappa_{yy}^2+2\nu\kappa_{xx}\kappa_{yy}+4E^{-1}\mu(1-\nu^2)\kappa_{xy}^2}. \tag{4.11}$$

For isotropic materials, $E = 2\mu(1+\nu)$, and $p(\boldsymbol{\kappa})$ becomes

$$p(\boldsymbol{\kappa}) = \left(\frac{1+\nu}{1-\nu}\right)\frac{(\operatorname{tr}\boldsymbol{\kappa})^2}{(\operatorname{tr}\boldsymbol{\kappa})^2 - 2(1-\nu)\det\boldsymbol{\kappa}}. \tag{4.12}$$

In this case, $p(\boldsymbol{\kappa})$ depends upon the two principal invariants of the curvature: $\operatorname{tr}\boldsymbol{\kappa} = \kappa_{xx}+\kappa_{yy}$ and $\det\boldsymbol{\kappa} = \kappa_{xx}\kappa_{yy} - \kappa_{xy}^2$. Let κ_1 and κ_2 be the two principal curvatures, satisfying $\kappa_1+\kappa_2 = \operatorname{tr}\boldsymbol{\kappa}$ and $\kappa_1\kappa_2 = \det\boldsymbol{\kappa}$, then

$$p(\boldsymbol{\kappa}) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\left(\kappa_1+\kappa_2\right)^2}{\left(\kappa_1+\kappa_2\right)^2 - 2(1-\nu)\kappa_1\kappa_2}. \tag{4.13}$$

Thus,

$$0 \leqslant p(\boldsymbol{\kappa}) \leqslant 2/(1-\nu), \tag{4.14}$$

with $p(\boldsymbol{\kappa}) = 0$ at locations where the plate is locally saddle-shaped, $\kappa_1 = -\kappa_2$, and p achieves its maximum value at points where it is locally spherical, $\kappa_1 = \kappa_2$.

The bounds (4.14) also apply to materials of cubic anisotropy, and occur in the same circumstances as for the isotropic material, as can be verified from (4.11).

4.3 *TE loss factors for flexural modes and waves*

The loss factor of a particular flexural mode may be predicted via (4.8) once the displacement field $w(\boldsymbol{x})$ of the mode is known. The curvature tensor is first evaluated via the standard relation (4.2). The total energy lost from the mode per cycle is then calculated by integrating (4.8) over the volume of the plate and time averaging. In the most common situation, the displacement field will be evaluated in frequency space as a complex quantity, and the time average is obtained in the usual way as $\overline{f(t)g(t)} = \frac{1}{2}\mathrm{Re}\,[\,\tilde{f}(\omega)\tilde{g}^*(\omega)]$, where $\tilde{f}$ is the Fourier transform. Thus the TE dissipation for mode σ is given by

$$Q_\sigma^{-1} = \frac{\bar{E}\theta_a\bar{\alpha}^2}{C_p}\sum_{n=0}^{\infty} f_n \frac{\omega\tau_n}{(1-\omega^2\tau_r\tau_n)^2+\omega^2\tau_n^2}\left[\frac{\int dA\,\mathcal{E}_{\mathrm{Diss}}(\boldsymbol{\kappa})}{\mathcal{E}_{0\sigma}}\right], \tag{4.15}$$

where $\mathcal{E}_{0\sigma} = \int dA\,(\mathcal{E}_{KE}+\mathcal{E}_{PE})$ is the total energy of mode σ.

The modal energy $\mathcal{E}_{0\sigma}$ may be computed as either twice the average kinetic energy of the system or twice the average potential energy of the system, whichever is more convenient. The kinetic energy is often preferable when using experimental data because the second derivatives appearing in the curvature tensor can be ill-behaved in the presence of noise.

The expression for the TE loss factor given above is closely related to the loss factor given by Zener for a simple beam in flexure. Zener gave the result (see (1.2))

$$Q_{\mathrm{Zener}}^{-1} = \frac{E\theta_a\alpha^2}{C_p}\frac{\omega\tau_0}{1+\omega^2\tau_0^2}, \tag{4.16}$$

where we have included only the first term in the infinite sum. Our results may thus be interpreted as

$$Q_\sigma^{-1} = \frac{\int dA\,\mathcal{E}_{\mathrm{Diss}}(\boldsymbol{\kappa})}{\mathcal{E}_{0\sigma}}\,Q_{\mathrm{Zener}}^{-1} = p_\sigma\,Q_{\mathrm{Zener}}^{-1}, \tag{4.17}$$

where the quantity p_σ was called the modal participation factor in the isotropic case studied in (**4**). Clearly the MPF is closely related to the quantity $p(\boldsymbol{\kappa})$ analysed above.

The MPF simplifies considerably for an isotropic medium. Using manipulations similar to those employed above, we find

$$p_\sigma = \frac{IE}{\mathcal{E}_0(1-\nu)^2}\int dA\,\overline{(\mathrm{tr}\,\boldsymbol{\kappa})^2}; \tag{4.18}$$

a result which differs from that given in (**4**) by a factor of $(1-\nu)^{-2}$. The reason for the difference is that it was assumed in (**4**) that flexural energy would be dissipated at the rate predicted by Zener for a beam. However, a plate undergoes more compressive stress than a beam and thus dissipates more energy accordingly.

It is not difficult to show, based on (4.18), that the MPF for travelling flexural waves in an isotropic plate is

$$p_\sigma = \frac{1+\nu}{1-\nu} \quad \text{for a flexural wave.} \tag{4.19}$$

This shows the difference between the TE dissipation of a cantilever beam ($p_\sigma = 1$) versus the

corresponding vibration of a plate. The identity (4.19) may be obtained by explicit substitution of a flexural waves solution, or more simply, as follows. Integrating by parts and using the governing plate equation $EI(1-\nu^2)^{-1}\nabla^4 w - \rho h\omega^2 w = 0$, gives

$$\int dA\, \overline{(\mathrm{tr}\,\boldsymbol{\kappa})^2} = \int dA\, \overline{w\nabla^4 w} = \frac{2(1-\nu^2)}{IE}\int dA\, \overline{\mathcal{E}_{KE}} = \frac{(1-\nu^2)}{IE}\,\mathcal{E}_0. \tag{4.20}$$

The last equality employs the expression for the potential energy, (4.3). Also, it is assumed that the plate boundary conditions may be ignored in the above integration, which is true for a travelling wave in a plate of 'infinite' extent.

5. Effective thin plate equations and flexural waves

The general theory is now applied to thin plates in flexure with the goal of deriving general governing equations similar to the classical Kirchhoff thin plate equations. Our objective is to provide an alternative means of calculating p_σ for flexural waves, and also to examine the range of validity of our approximation in ignoring lateral thermal diffusion.

The equation for θ, (2.11), becomes for time harmonic motion ($e^{-i\omega t}$ assumed)

$$\frac{\partial^2\theta}{\partial z^2} + k^2\theta = -k^2(\theta_a/C_p)\boldsymbol{\alpha}\cdot\boldsymbol{\sigma} - \left(\frac{K_1}{K_3}\frac{\partial^2\theta}{\partial x^2} + \frac{K_2}{K_3}\frac{\partial^2\theta}{\partial y^2}\right), \tag{5.1}$$

where $\mathbf{K} = \mathrm{diag}\,(K_1,\ K_2,\ K_3)$ and

$$k^2 = i\omega(1 - i\omega\tau_r)C_p/K_3. \tag{5.2}$$

If the stress term on the right-hand side of (5.1) is weakly dependent on x and y, then we may argue that θ inherits the same weak dependence. Provided this dependence on transverse position is uniform, the case of simple plane wave, the final term may be combined with the $k^2\theta$ term, and hence interpreted as modifying the thermal diffusion rate. Here, we ignore this and simplify (5.1) to

$$\frac{\partial^2\theta}{\partial z^2} + k^2\theta = -k^2(\theta_a/C_p)\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}. \tag{5.3}$$

The importance of the simplification (5.3) at this stage is that it allows us to derive a set of effective plate equations in which the TE damping appears directly. This approach follows on that of Alblas (**12**) and of Lifshitz and Roukes (**13**) who derived the effective equation governing the motion of a thermoelastically damped beam.

Assuming zero flux conditions at $z = \pm h/2$, and noting that the right member of (5.3) is proportional to z in flexure, $\boldsymbol{\sigma} = \langle z,\ \boldsymbol{\sigma}\rangle I^{-1} z$, we find that the solution is

$$\theta = -\frac{\theta_a}{IC_p}\langle z,\ \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\rangle\left(z - \frac{\sin kz}{k\cos(kh/2)}\right). \tag{5.4}$$

The thermally perturbed stress is therefore, using the expression for entropy in terms of temperature and strain from Table 1,

$$\boldsymbol{\sigma} = \mathbf{C}\mathbf{e} + \frac{\theta_a}{IC_p}\langle z,\ \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}_0\rangle\left(z - \frac{\sin kz}{k\cos(kh/2)}\right)\boldsymbol{\beta}. \tag{5.5}$$

The stress in the absence of thermal effects, $\boldsymbol{\sigma}_0$, is proportional to z. In order to apply (5.5) to the thin plate the standard plane stress conditions must be enforced. We consider an orthotropic plate with a symmetry plane coincident with the neutral plane ($z = 0$), for which the standard stress/strain relations for plane-stress are

$$\begin{bmatrix} \sigma_{xx}^{(0)} \\ \sigma_{yy}^{(0)} \\ \sigma_{xy}^{(0)} \end{bmatrix} = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} e_{xx}^{(0)} \\ e_{yy}^{(0)} \\ e_{xy}^{(0)} \end{bmatrix}, \tag{5.6}$$

where $\sigma_{xx}^{(0)}$, $e_{xx}^{(0)}$, etc. are the stresses and strains in the absence of the thermoelastic damping. Based on (5.5) this implies that the in-plane TE stresses are

$$\begin{aligned} \sigma_{xx} &= \sigma_{xx}^{(0)} + \frac{(\alpha_1+\alpha_2\nu_{21})\theta_a E_1}{(1-\nu_{12}\nu_{21})C_p I}\langle z,\ \alpha_1\sigma_{xx}^{(0)} + \alpha_2\sigma_{yy}^{(0)}\rangle \left(z - \frac{\sin kz}{k\cos(kh/2)}\right), \\ \sigma_{yy} &= \sigma_{yy}^{(0)} + \frac{(\alpha_2+\alpha_1\nu_{12})\theta_a E_2}{(1-\nu_{12}\nu_{21})C_p I}\langle z,\ \alpha_1\sigma_{xx}^{(0)} + \alpha_2\sigma_{yy}^{(0)}\rangle \left(z - \frac{\sin kz}{k\cos(kh/2)}\right), \\ \sigma_{xy} &= \sigma_{xy}^{(0)}. \end{aligned} \tag{5.7}$$

This provides the variation of stress through the thickness due to the temperature variation. The moments are found by taking the first moment of the stresses through the plate thickness, leading to

$$\begin{aligned} M_{xx} &= M_{xx}^{(0)} + \frac{(\alpha_1+\alpha_2\nu_{21})\theta_a E_1}{(1-\nu_{12}\nu_{21})C_p} f(kh)\left(\alpha_1 M_{xx}^{(0)} + \alpha_2 M_{yy}^{(0)}\right), \\ M_{yy} &= M_{yy}^{(0)} + \frac{(\alpha_2+\alpha_1\nu_{12})\theta_a E_2}{(1-\nu_{12}\nu_{21})C_p} f(kh)\left(\alpha_1 M_{xx}^{(0)} + \alpha_2 M_{yy}^{(0)}\right), \\ M_{xy} &= M_{xy}^{(0)}, \end{aligned} \tag{5.8}$$

where $M_{xx}^{(0)} = \langle z,\ \sigma_{xx}^{(0)}\rangle$, etc., and the function f is

$$f(\zeta) = 1 + \frac{24}{\zeta^3}\left[\frac{\zeta}{2} - \tan\frac{\zeta}{2}\right]. \tag{5.9}$$

The standard moment-curvature relations follow from (5.6) and the Kirchhoff assumption as

$$\begin{bmatrix} M_{xx}^{(0)} \\ M_{yy}^{(0)} \\ M_{xy}^{(0)} \end{bmatrix} = I \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \kappa_x^{(0)} \\ \kappa_y^{(0)} \\ \kappa_{xy}^{(0)} \end{bmatrix}. \tag{5.10}$$

The governing equation for the thin plate is

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \rho h\omega^2 w = 0. \tag{5.11}$$

Thus, we obtain the effective thin plate equation for w,

$$-\frac{I}{1-\nu_{12}\nu_{21}}\Big\{E_1 w_{xxxx}+E_2 w_{yyyy}+2\big[2\mu(1-\nu_{12}\nu_{21})+\nu_{21}E_1\big]w_{xxyy}\Big\}$$
$$-\frac{f(kh)\theta_a I}{(1-\nu_{12}\nu_{21})^2 C_p}\Big\{(\alpha_1+\alpha_2\nu_{21})^2E_1^2 w_{xxxx}+2(\alpha_1+\alpha_2\nu_{21})(\alpha_2+\alpha_1\nu_{12})E_1E_2w_{xxyy}$$
$$+(\alpha_2+\alpha_1\nu_{12})^2E_2^2w_{yyyy}\Big\}+\rho h\omega^2 w=0. \tag{5.12}$$

For a plate made of material with cubic symmetry, this reduces to

$$\left[1+\left(\frac{1+\nu}{1-\nu}\right)\frac{E\theta_a\alpha^2}{C_p}f(kh)\right]\nabla^4 w+4(1-\nu)\left[(1+\nu)\frac{\mu}{E}-\frac{1}{2}\right]w_{xxyy}-\frac{1-\nu^2}{EI}\rho h\omega^2 w=0. \tag{5.13}$$

The second term vanishes for isotropic materials, in which case we may combine the damping with the plate stiffness to get an equation in the standard form,

$$D\nabla^4 w-\rho h\omega^2 w=0, \tag{5.14}$$

where the TE loss is now contained in the complex-valued flexural stiffness

$$D=\frac{EI}{1-\nu^2}\left\{1+\left(\frac{1+\nu}{1-\nu}\right)\frac{E\theta_a\alpha^2}{C_p}f(kh)\right\}. \tag{5.15}$$

In the more general orthotropic situation, the TE damping is inhomogeneous, as in (5.12), and cannot be interpreted in terms of a single frequency-dependent complex modulus of elasticity.

Our equation (5.14), a generalization of Zener's results for a beam (**9**), gives a homogeneous damping as a result of assuming a uniform plane wave vibration. This provides some guidance as to the loss factors of vibrating thin plate structures, but finite structures will support resonant modes that consist of a variety of wavenumbers which can interfere with one another, and thus the actual loss factor for a particular mode may differ significantly from this value. A particularly salient example is the case of a twisting mode for which the source field for temperature fluctuations, $\mathrm{tr}\,\boldsymbol{\sigma}$ (for isotropic $\boldsymbol{\sigma}$), is very small, and the resulting values of Q^{-1} may be orders of magnitude smaller than the result given in (5.14).

Corresponding results for flexural waves in circular rods are in Appendix C. In particular we note that the function f takes on a different form from that for a plate.

5.1 *Dispersion relation including in-plane variation*

It is now relatively straightforward to revise the analysis of section 4 to include in-plane variation in both the stress and the temperature. We begin by assuming that all field variables possess in-plane dependence $e^{i\zeta x}$ with wavenumber ζ. Then (5.1) becomes

$$\frac{\partial^2\theta}{\partial z^2}+\gamma^2\theta=-k^2(\theta_a/C_p)\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}, \tag{5.16}$$

where $\gamma = (k^2 - \zeta^2 K_1/K_3)^{1/2}$ and k is defined in (5.2). Solving as before, we find that the temperature is

$$\theta = -\langle z,\, \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\rangle \frac{\theta_a}{IC_p}\frac{k^2}{\gamma^2}\left(z - \frac{\sin\gamma z}{\gamma\cos(\gamma h/2)}\right), \tag{5.17}$$

and similar generalizations can be obtained for the stresses and moments, (5.5) to (5.10).

We cannot derive a governing equation, similar to (5.14), for example, since now the in-plane dependence of the stress and hence w has been assumed *a priori*. However, the in-plane wavenumber can be obtained in a self-consistent manner from the latter equation by assuming $w = w_0 e^{i\zeta x}$, where w_0 is constant. This yields an equation for ζ,

$$\zeta^4 - \omega^2(1-\nu^2)\frac{\rho h}{EI}\left\{1 + \left(\frac{1+\nu}{1-\nu}\right)\frac{E\theta_a\alpha^2}{C_p}\frac{k^2}{\gamma^2}f(\gamma h)\right\}^{-1} = 0, \tag{5.18}$$

the solution of which we discuss next.

5.2 *Effects of transverse TE dissipation*

The wavenumber of a flexural wave in an undamped thin plate is k_f, where

$$k_f^4 = \omega^2(1-\nu^2)\rho h/(EI). \tag{5.19}$$

Treating ζ as an asymptotic series in the small parameter ϵ, it is clear that the leading-order solution to the general dispersion relation of (5.18) is $\zeta = k_f + O(\epsilon)$. The next term is given by

$$\zeta = k_f\left[1 - \frac{\epsilon}{4}\left(\frac{1+\nu}{1-\nu}\right)\frac{k^2}{\gamma_f^2}f(\gamma_f h) + O(\epsilon^2)\right], \tag{5.20}$$

where $\gamma_f = (k^2 - k_f^2)^{1/2}$. Direct substitution gives

$$\gamma_f = k_0\left(1 + ial_{\mathrm{mfp}}/h - i\omega\tau_r\right)^{1/2}, \tag{5.21}$$

where $k_0 = (-i\omega C_p/K_3)^{1/2}$ and l_{mfp}, the mean free path for phonons at temperature T, is

$$l_{\mathrm{mfp}}(T) = \frac{3K_3(T)}{\bar{c}\,C_p(T)}. \tag{5.22}$$

The quantity $\bar{c}$ is the average elastic wave speed, and the order-one constant a is given by

$$a = [2(1-\nu)\rho/(3\mu)]^{1/2}\ (K_1/K_3)\,\bar{c}. \tag{5.23}$$

Values of the mean free path at room temperature are typically on the order of tens of nanometres using the value $\bar{c} = (c_L + 2c_T)/3$, where $c_T = \sqrt{\mu/\rho}$ is the transverse wave speed and $c_L = [2(1-\nu)/(1-2\nu)]^{1/2}\,c_T$ is the longitudinal speed. For isotropic conductivity the parameter a is a function of Poisson's ratio,

$$a = \left[\sqrt{8(1-\nu)} + \frac{2(1-\nu)}{\sqrt{1-2\nu}}\right]/\sqrt{27}, \tag{5.24}$$

which is of order unity.

Equation (5.21) provides an avenue to compute corrections to classical TE dissipation arising both from transverse diffusion and thermal relaxation. The asymptotic solution (5.20) indicates that a propagating flexural wave has attenuation equal to $\mathrm{Im}\,\zeta$. Using the relation between attenuation and Q, and noting that the undamped flexural wave has group velocity $2\omega/k_f$, we find $1/Q = \mathrm{Im}\,4\zeta/k_f$, or

$$Q^{-1} = -\epsilon\left(\frac{1+\nu}{1-\nu}\right)\mathrm{Im}\left\{\frac{1-i\omega\tau_r}{1-i\omega\tau_r+ial_{\mathrm{mfp}}/h}\; f\left(k_0(1-i\omega\tau_r+ial_{\mathrm{mfp}}/h)^{1/2}\right)\right\}. \tag{5.25}$$

In order to proceed further in understanding this we replace the function f by its series expansion. Equations (B.1), (B.2) and (B.3) imply

$$f\left(k_0(1-i\omega\tau_r+ial_{\mathrm{mfp}}/h)^{1/2}\right) = \sum_{n=0}^{\infty} f_n\left[1+\frac{1}{-1+i\omega\tau_n(1-i\omega\tau_r+ial_{\mathrm{mfp}}/h)}\right], \tag{5.26}$$

where τ_n, $n = 0, 1, 2, \ldots$ are defined in $(4{\cdot}6)_2$. Thus,

$$Q^{-1} = \epsilon\left(\frac{1+\nu}{1-\nu}\right)\sum_{n=0}^{\infty} f_n\,\frac{\omega\tau_n\,\left(1+\omega\tau_n(al_{\mathrm{mfp}}/h)\right)}{\left(1+\omega\tau_n(al_{\mathrm{mfp}}/h)-\omega^2\tau_n\tau_r\right)^2+\omega^2\tau_n^2}. \tag{5.27}$$

The classical result for the TE loss in a vibrating beam is (**8**)

$$Q^{-1} = \epsilon\,\frac{\omega\tau_0}{1+\omega^2\tau_0^2}. \tag{5.28}$$

This ignores the effect of τ_r, and for the purpose of comparison with (5.27) we consider the case $\tau_r = 0$ in the latter. We truncate (5.27) at the first term (which is reasonable considering $f_0/f_1 = 81$), and for further simplicity, the factor $f_0 = 96/\pi^4 = 0{\cdot}9855$ is replaced by unity, giving

$$Q^{-1} = \epsilon\left(\frac{1+\nu}{1-\nu}\right)\frac{\omega\tau_0\left(1+\omega\tau_0(al_{\mathrm{mfp}}/h)\right)}{\left(1+\omega\tau_0(al_{\mathrm{mfp}}/h)\right)^2+\omega^2\tau_0^2}. \tag{5.29}$$

Comparing (5.28) and (5.29) the first difference is the factor $(1+\nu)/(1-\nu)$ which can be attributed to the fact that it is a plate rather than a beam as discussed in the previous subsection; see (4.17) and (4.19). The major additional distinction between the present theory and the classical result is the presence of the terms involving l_{mfp} resulting from transverse diffusion, which are absent in Zener's theory and previous analyses. These corrections are always small, since the domain of validity of the thermodynamic analysis employed here is restricted to distances far greater than the mean free path, the domain $h \gg l_{\mathrm{mfp}}$.

Alternatively, we note that (5.29) contains two characteristic times: the 'Zener' relaxation time $\tau_0 = h^2C_P/\pi^2K_3$, and a new characteristic time defined by $\tau^* = \tau_0(al_{\mathrm{mfp}}/h)$. It is easily checked that $\omega\tau^* = (k_f h/\pi)^2$, which must be a small quantity in order for the Kirchhoff assumption to remain valid, that is, a necessary prerequisite for the validity of the plate theory is that the flexural wavelength is much longer than the thickness. Also,

$$\tau^* = h\pi^{-2}\sqrt{12(1-\nu^2)\rho/E}, \tag{5.30}$$

which suggests interpreting τ^* as the travel time of an elastic wave across the thickness. But this

time must be much less than $1/\omega$, otherwise thickness resonances can occur, again violating the conditions of the plate theory.

Thus, the restrictions on the use of both the thermal conduction and the thin plate theories requires that $\omega\tau_0(al_{\mathrm{mfp}}/h)$ is small. Nevertheless, small corrections arising from transverse diffusion can be estimated in this fashion.

The effect of non-zero τ_r can be considered by using an estimate for this relaxation time. Rudgers (**21**) provides the estimates $\tau_r = \tau_{\mathrm{mfp}}/3$, where $\tau_{\mathrm{mfp}} = l_{\mathrm{mfp}}/\bar{c}$ (Rudgers actually calculates two approximations for τ_r but they are of the same order of magnitude). The term $\omega\tau_r$ must remain small in the context of thin plate theory, otherwise the same assumptions as before are violated, for example, the wavelength is far less than the thickness. However, if the phonon mean free path is comparable with the plate thickness, then both the transverse diffusion and the Cattaneo–Vernotte terms can become important.

6. Conclusion

Equation (3.11) is one of the basic results of the paper, as it provides a means to compute TE dissipation given a solution in terms of the inhomogeneous stress. This computation is in general a complicated undertaking because it requires the determination of the eigenfunctions and eigenvalues of the heat equation in the particular geometry of interest. Nevertheless, a clear recipe for an approximate calculation of the TE dissipation in an arbitrary, anisotropic, elastic medium is given.

The theoretical scope is then narrowed to the case of thin plate structures for which the thermal heat flow simplifies dramatically. In this case, we are able to obtain an explicit formula for the TE loss in an arbitrary, anisotropic elastic system, (4.9). This equation, which shows that TE loss is in general inhomogeneous, is the other principal result obtained. It should be emphasized that the $\Delta\mathcal{E}$ derived in (4.9) has the property that it is not a global measure of damping but is local, and can be used to define TE loss at a point in a structure, and then integrated to determine the Q of an arbitrary vibrational mode. This distinction is particularly important for geometries of interest in MEMS/NEMS applications, for the TE loss rate of low-order modes in typical geometries may vary significantly with position. The local result for TE loss predicts that the attenuation of a flexural wave in a large plate is $(1 + \nu)/(1 - \nu)$ times the attenuation of a wave in a beam at the same frequency.

Finally, we have investigated the effects of transverse diffusion. Transverse diffusion effects are most easily analysed for plane wave behaviour. For this case we have derived an effective plate equation including the effects of thermoelasticity to leading order in the TE coupling. The TE dissipation can be examined fairly simply in the plane wave case because TE losses are homogeneous, and are therefore contained in a single loss factor. Corrections to TE loss resulting from transverse diffusion are found to be generally small within the domain of validity of our theoretical models.

Acknowledgement

The second author gratefully acknowledges the support of the Office of Naval Research.

References

1. D. W. Carr, S. Evoy, L. Sekaric, H. G. Craighead and J. M. Parpia, Parametric amplification in a torsional microresonator, *Appl. Phys. Lett.* **77** (2000) 1545–1547.

2. P. Mohanty, D. A. Harrington, K. L. Ekinci, Y. T. Yang, M. J. Murphy and M. L. Roukes, Intrinsic dissipation in high-frequency micromechanical resonators, *Phys. Rev.* B **66** (2002) 085416.
3. J. L. Yang, T. Ono and M. Esashi, Energy dissipation in submicrometer thick single-crystal silicon cantilevers, *J. Microelectromech. Sys.* **11** (2002) 775–783.
4. D. M. Photiadis, B. H. Houston, X. Liu, J. A. Bucaro and M. H. Marcus, Thermoelastic loss observed in a high Q mechanical oscillator, *Physica* B **316** (2002) 408–410.
5. X. Liu, J. F. Vignola, D. M. Photiadis, A. Sarkissian, B. H. Houston, R. D. Merithew and R. O. Pohl, Low temperature study of loss mechanisms of mechanical oscillators, *ibid.* **316** (2002) 393–396.
6. ——, S. F. Morse, J. F. Vignola, D. M. Photiadis, A. Sarkissian, M. H. Marcus and B. H. Houston, On the modes and loss mechanisms of a high Q mechanical resonator, *Appl. Phys. Lett.* **78** (2001) 1346–1348.
7. B. H. Houston, D. M. Photiadis, M. H. Marcus, J. A. Bucaro, X. Liu and J. F. Vignola, Thermoelastic loss in microscale resonators, *ibid.* **80** (2002) 1300–1302.
8. C. Zener, *Elasticity and Anelasticity of Metals* (University of Chicago Press, Chicago 1948).
9. ——, Theory of internal friction in reeds, *Phys. Rev.* **52** (1937) 230–235.
10. A. S. Nowick and B. S. Berry, *Anelastic Relaxation in Crystalline Solids* (Academic Press, New York 1972).
11. J. B. Alblas, On the general theory of thermo-elastic friction, *Appl. Sci. Res.* A **10** (1961) 349–362.
12. ——, A note on the theory of thermoelastic damping, *J. Thermal Stresses* **4** (1981) 333–355.
13. R. Lifshitz and M. L. Roukes, Thermoelastic damping in micro- and nanomechanical systems, *Phys. Rev.* B **61** (2000) 5600–5609.
14. V. K. Kinra and K. B. Milligan, A 2nd law analysis of thermoelastic damping, *J. Appl. Mech.* **61** (1994) 71–76.
15. P. Chadwick, Thermal damping of a vibrating elastic body, *Mathematika* **9** (1962) 38–48.
16. ——, On the propagation of thermoelastic disturbances in thin plates and rods, *J. Mech. Phys. Solids* **10** (1962) 99–109.
17. R. M. Christensen, *Mechanics of Composite Materials* (Wiley, New York 1979).
18. W. Nowacki, *Dynamic Problems in Thermoelasticity* (Noordhoff, Leyden 1989).
19. D. S. Chandrasekharaiah, Thermoelasticity with second sound: a review, *Appl. Mech. Rev.* **39** (1986) 355–376.
20. H. W. Lord and Y. Shulman, A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids* **15** (1967) 299–309.
21. A. J. Rudgers, Analysis of thermoacoustic wave propagation in elastic media, *J. Acoust. Soc. Amer.* **88** (1990) 1078–1084.
22. C. Zener, General theory of thermoelastic internal friction, *Phys. Rev.* **53** (1938) 90–99.

APPENDIX A

General solutions of the equations of thermoelasticity

The governing equations (2.11) and (2.12) are more commonly expressed as coupled equations of temperature and displacement. This can be achieved by first eliminating $\boldsymbol{\sigma}$ explicitly:

$$\mathrm{div}\mathbf{C}\mathbf{e} - \rho\ddot{\mathbf{u}} - \boldsymbol{\beta}\nabla\theta = 0, \tag{A.1}$$

$$\dot{\theta} + \tau_r\ddot{\theta} - C_v^{-1}\mathrm{div}\,\mathbf{K}\nabla\theta + \Big(1 + \tau_r\frac{\partial}{\partial t}\Big)\frac{\theta_a}{C_v}\boldsymbol{\beta}\cdot\dot{\mathbf{e}} = 0. \tag{A.2}$$

These may be expressed more succinctly as follows for time harmonic motion ($e^{-i\omega t}$ is assumed):

$$\mathcal{L}(\nabla,\,\omega)\mathbf{U}+\omega^2\mathbf{U}=0, \tag{A.3}$$

where

$$\mathbf{U}=\left\{\begin{array}{c}\rho^{1/2}\,\mathbf{u}\\(-i\omega)^{-1}\,(C_v/\theta_a)^{1/2}\,\theta\end{array}\right\}, \tag{A.4}$$

$$\mathcal{L}=\left[\begin{array}{cc}\rho^{-1}\,\mathbf{Q}(\nabla) & i\omega(\rho C_v/\theta_a)^{-1/2}\,\mathbf{b}(\nabla)\\ i\omega(\rho C_v/\theta_a)^{-1/2}\,\mathbf{b}^T(\nabla) & (\theta_a/C_v)\,\tilde{\kappa}(\nabla,\,\omega)\end{array}\right], \tag{A.5}$$

$Q_{ij}(\mathbf{v})=C_{ikjl}\,v_kv_l$, $\mathbf{b}(\mathbf{v})=\boldsymbol{\beta}\mathbf{v}$ and $\tilde{\kappa}(\mathbf{v},\omega)=[\theta_a(\tau_r+(-i\omega)^{-1})]^{-1}\,\mathbf{v}\cdot\mathbf{K}\mathbf{v}$. Equation (A.3) represents a generalized eigenvalue problem for the complex-valued modal frequencies.

Some simplification is possible for isotropic bodies, for which $\boldsymbol{\alpha}=\alpha\mathbf{I}$ and $\boldsymbol{\beta}=3\kappa_T\alpha\mathbf{I}$, where $\kappa_T=E/(1-2\nu)$ is the isothermal bulk modulus. Assuming the *ansatz* (**15**)

$$\mathbf{U}=\left\{\begin{array}{c}\nabla\psi\\ \lambda\psi\end{array}\right\},\qquad\text{where}\quad\nabla^2\psi+\Lambda^2\psi=0, \tag{A.6}$$

the eigenvalue problem (A.3) becomes

$$\omega^2-c_L^2\Lambda^2+\frac{i\omega\beta\lambda}{(\rho C_v/\theta_a)^{1/2}}=0, \tag{A.7}$$

$$-\frac{i\omega\beta\Lambda^2}{(\rho C_v/\theta_a)^{1/2}}+\left(\omega^2-\frac{K\Lambda^2}{C_v(\tau_r+(-i\omega)^{-1})}\right)\lambda=0, \tag{A.8}$$

where c_L is the longitudinal wave speed. Eliminating the constant λ gives an equation for ω in terms of the wavenumber Λ:

$$\left(\omega^2-c_L^2\Lambda^2\right)\left(\frac{K\Lambda^2}{C_v(1-i\omega\tau_r)}-i\omega\right)+i\omega\Lambda^2\frac{\theta_a\beta^2}{\rho C_v}=0. \tag{A.9}$$

Thus, the problem of determining the eigenfrequencies is reduced to finding solutions to the Helmholtz equation $(A.6)_2$ in the domain of interest. Chadwick (**15**) discusses this further and provides formulae for the roots of (A.9) for $\tau_r=0$, in which case it reduces to a cubic in ω.

The separation of variables approach does not work for generally anisotropic bodies. However, it is worth noting that solutions of the form (A.6) are valid as long as the thermal properties are isotropic and the elasticity tensor $\mathbf{C}$ satisfies $C_{ijkl}n_jn_kn_l=c_0n^2n_i$ for all $\mathbf{n}$, and some constant c_0. The most general form of anisotropy with this property has stiffness (using Voigt notation)

$$[\mathbf{C}]=\left[\begin{array}{cccccc}c_0 & c_0-2c_{66} & c_0-2c_{55} & -2c_{56} & 0 & 0\\ c_0-2c_{66} & c_0 & c_0-2c_{44} & 0 & -2c_{46} & 0\\ c_0-2c_{55} & c_0-2c_{44} & c_0 & 0 & 0 & -2c_{45}\\ -2c_{56} & 0 & 0 & c_{44} & c_{45} & c_{46}\\ 0 & -2c_{46} & 0 & c_{45} & c_{55} & c_{56}\\ 0 & 0 & -2c_{45} & c_{46} & c_{56} & c_{66}\end{array}\right]. \tag{A.10}$$

APPENDIX B

Exact results for the function f

Some exact results are presented for the function f of (5.9) occurring in the TE theory for beams and plates in flexure. We begin with the representation

$$f(\xi) = 1 + \sum_{j=0}^{\infty} \frac{f_j}{\frac{1}{2}\Omega_j - 1}, \tag{B.1}$$

where f_j, $j = 0, 1, 2, \ldots$, are defined in (4.7) and

$$\Omega_j = 2\xi^2/[(2j+1)\pi]^2. \tag{B.2}$$

The infinite series in (B.1) is a consequence of the fact that the left member is a meromorphic function of ξ with simple poles at $\xi = \pm(2j+1)\pi$, and residues that are readily determined. Note that

$$\sum_{j=0}^{\infty} f_j = 1. \tag{B.3}$$

It follows from (B.1) that for real-valued ξ,

$$\text{Im } f\left((1+i)(1-ir)^{1/2}\xi\right) = -\sum_{j=0}^{\infty} f_j \frac{\Omega_j}{(1-r\Omega_j)^2 + \Omega_j^2}. \tag{B.4}$$

We also note the identity

$$f((1+i)\xi) = 1 - \frac{6}{\xi^2}\left\{i + \frac{(1-i)}{\xi}\left(\frac{\sinh\xi - i\sin\xi}{\cosh\xi + \cos\xi}\right)\right\}, \tag{B.5}$$

which is useful in the case of $\tau_r = 0\,(r = 0)$, as it provides a closed form expression for the TE damping in that case via (**13**)

$$\text{Im } f((1+i)\xi) = -\frac{6}{\xi^2}\left\{1 - \frac{1}{\xi}\left(\frac{\sinh\xi + \sin\xi}{\cosh\xi + \cos\xi}\right)\right\}. \tag{B.6}$$

The exact result for a beam in flexure was derived by Lifshitz and Roukes (**13**), and corresponds to the use of the function f of (5.9). The equivalence of their derivation and Zener's prediction (**8**) is confirmed by (B.4) (both Zener, and Lifshitz and Roukes considered the case $r = 0$). Thus, while the closed form expression of Lifshitz and Roukes is novel, it is simply a more concise expression of the infinite series of Zener. Both are based on the same implicit or explicit first-order approximation arising from the small difference between the adiabatic and isothermal systems.

In fact, the analysis of Lifshitz and Roukes (**13**) is a special case of the more general problem solved by Alblas (**12**), in which the lateral (third) dimension of the beam is taken into account and the boundary-value problem for the temperature is more general. However, Alblas considered the situation in which the boundary condition for the temperature variation is zero, and as a result he obtained a different functional form of the TE damping.

APPENDIX C

Thermoelasticity theory for a circular rod

In the case of a circular rod of radius a we find that the solution to (5.3) is

$$\theta = -(I_c c_p)^{-1}\langle z,\, \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\rangle \left(r - \frac{J_1(kr)}{kJ_1'(ka)}\right)\cos\psi, \tag{C.1}$$

where $I_c = \langle z,\ z\rangle\ = \pi a^4/4$, $z = r\cos\psi$ and k is defined in (5.2). The function associated with the moment of the temperature is $f(ka)$, where now

$$f(\zeta) = 1 + \frac{4}{\zeta^3}\left[\zeta - \frac{J_1(\zeta)}{J_1'(\zeta)}\right]. \tag{C.2}$$

Zener (**22**) analysed the circular rod using the projection method involving a series rather than a closed form expression. By comparing the results here with those of Zener, we conclude that (**22**)

$$f_n = \frac{8}{q_n^2(q_n^2 - 1)}, \quad \tau_n = \frac{a^2 C_p}{q_n^2 K}, \text{ where } \quad J_1'(q_n) = 0,\ n = 0, 1, 2, \ldots. \tag{C.3}$$