Flexural waves on narrow plates

Andrew N. Norris[a)]
Department of Mechanical & Aerospace Engineering, Rutgers University, 98 Brett Road, Piscataway, New Jersey 08854-8058

(Received 16 July 2002; revised 24 January 2003; accepted 28 January 2003)

Flexural wave speeds on beams or plates depend upon the bending stiffnesses which differ by the well-known factor $(1-\nu^2)$. A quantitative analysis of a plate of finite lateral width displays the plate-to-beam transition, and permits asymptotic analysis that shows the leading order dependence on the width. Orthotropic plates are analyzed using both the Kirchhoff and Kirchhoff–Rayleigh theories, and isotropic plates are considered for Mindlin's theory with and without rotational inertia. A frequency-dependent Young's modulus for beams or strips of finite width is suggested, although the form of the correction to the modulus is not unique and depends on the theory used. The sign of the correction for the Kirchhoff theory is opposite to that for the Mindlin theory. These results indicate that the different plate and beam theories can produce quite distinct behavior. This divergence in predictions is further illustrated by comparison of the speeds for antisymmetric flexural, or torsional, modes on narrow plates. The four classical theories predict limiting wave speeds as the plate width vanishes, but the values are different in each case. The deviations can be understood in terms of torsional waves and how each theory succeeds, or fails, in approximating the effect of torsion. Dispersion equations are also derived, some for the first time, for the flexural edge wave in each of the four ''engineering'' theories. © *2003 Acoustical Society of America.*
[DOI: 10.1121/1.1561493]

PACS numbers: 43.40.Cw, 43.40.Dx [JJM]

I. INTRODUCTION

The wave number of a flexural wave in a beam of rectangular cross-section or in a plate is $k=(\omega^2 m/D)^{1/4}$ according to classical (Kirchhoff, Euler–Bernoulli) theory, where D is the bending stiffness, m is the mass density (per unit length or area) and ω is the circular frequency. The bending stiffness for the beam is $D=EI$, while that of the plate of the same thickness as the beam is $D=EI/(1-\nu^2)$, where E is Young's modulus, I is the moment of inertia of the cross-section, and ν is Poisson's ratio. The appearance of the factor $1/(1-\nu^2)$ can be explained in terms of the different assumptions in each theory. Both make use of the Kirchhoff kinematic assumption; the plate theory assumes plane stress, while the beam theory is based on the assumption of uniaxial stress. The factor can therefore be attributed to the different assumed forms for the stress in the structure. The uniaxial stress approximation is clearly reasonable for a beam, bar, or rod that is thin in the cross directions, both transverse and lateral. The transverse direction is defined as the direction perpendicular to the plate, and a beam of rectangular cross-section can therefore be considered as the limit of a plate where the lateral dimension is small.

The purpose of this paper is to examine how the beam and plate theories are reconciled, that is, how the transition occurs between the uniaxial and plane stress theories. We will demonstrate explicitly how the beam limit occurs, and in particular will examine the leading order correction to beam theory that includes the dependence on the lateral width. The analysis is performed in the context of several classical theories—the four engineering theories,[1] beginning with the Kirchhoff theory for orthotropic plates and we show that the beam theory prediction for the wave number falls out in the limit of zero plate width. The analysis yields surprising differences. Thus, the four theories contribute different physical aspects which lead to serious differences in the high-frequency limit, but they all agree in the low or quasi-static limit where the Euler–Bernoulli predictions remains inviolate. In this context, the results here show the surprising result, a surprise to the author anyway, that the first correction to the beam theory prediction from the four theories are all distinct. While the main results are for flexural waves symmetric about the center line of the beam or plate, we also analyze the situation where the flexural wave is totally antisymmetric about the center line, and again demonatrate that the four theories provide distinct predictions. Comparison of the symmetric and asymmetric modes offers some explanation for the variations in the prediction for the first correction to the flexural wave (symmetric case).

The methodology adopted here uses known plate theories as the starting point to examine the narrow plate limit. One could also begin with the exact theory of elasticity and derive reduced order theories appropriate to the narrow plate. This approach is outlined briefly in Appendix B for a reduced order model consistent with the Kirchoff hypothesis, generating a uni-dimensional beam theory. It is shown that the predictions of this reduced order model are entirely consistent with those found using the Kirchhoff plate theory. It is expected that similar connections could be obtained between the present results for the Mindlin plate theory and a higher order reduced parameter model for narrow plates that includes shear and rotary inertia,[2] but the analysis is overly complicated and beyond the present study.

[a)]Electronic mail: norris@jove.rutgers.edu

We begin in the next section with the classical plate equations, and derive the dispersion relation for waves propagating in a plate of finite lateral width. A mode which is symmetric about the center line is shown to exist for all frequencies, and it reduces to the beam mode in the appropriate limit. We also show that this mode has the flexural edge wave speed[3] as asymptote for large width. Subsequent sections consider the same problem in the context of the refined models, the Kirchhoff–Rayleigh and Mindlin plates theories. In each case we illustrate how the appropriate beam theory naturally drops out in the limit as the lateral width becomes small, and we derive the leading order correction to the beam theory that includes the width for asymptotically small values. Modes asymmetric about the plate center line are also examined for the four theories, and the behavior of the lowest order mode as the width vanishes is examined, and compared with the symmetric results.

II. KIRCHHOFF PLATE THEORIES

The classical Kirchhoff theory ignores shearing of cross sections and rotational inertia, effects that are included in the Mindlin plate theory, discussed in the next section. We first consider the classical theory, and then the related Kirchhoff–Rayleigh theory which includes rotational inertia. Both theories are examined in the context of orthotropic plates for greater generality.

A. Classical Kirchhoff plate theory

The plate occupies $-\infty<x<\infty$, $-b\leqslant y\leqslant b$, $-h\leqslant z\leqslant h$, with flexural motion in the z- direction. The governing equations for the displacement $w(x,y,t)$, in the absence of external loading, are[4]

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=m\frac{\partial^2 w}{\partial t^2}, \tag{1}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0, \tag{2}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0, \tag{3}$$

where $m=2\rho h$ is the mass density per unit area. The plate is assumed to be orthotropic with axes of symmetry in the x and y directions, for which the moments M_x, M_y, and M_{xy} are

$$M_x=-D_x\frac{\partial^2 w}{\partial x^2}-D_0\frac{\partial^2 w}{\partial y^2}, \tag{4}$$

$$M_y=-D_y\frac{\partial^2 w}{\partial y^2}-D_0\frac{\partial^2 w}{\partial x^2}, \tag{5}$$

$$M_{xy}=-2D_{xy}\frac{\partial^2 w}{\partial x\partial y}. \tag{6}$$

Substituting from Eqs. (2)–(6) into (1) yields the flexural wave equation

$$D_x\frac{\partial^4 w}{\partial x^4}+2(D_0+2D_{xy})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_y\frac{\partial^4 w}{\partial y^4}+m\frac{\partial^2 w}{\partial t^2}=0. \tag{7}$$

This possesses a wave solution of the form $w(x,t)=\mathrm{Re}[Ae^{i(k_x x-\omega t)}]$, where k_x is the wave number of a wave traveling in the x direction in a plate of infinite width, and for later use we also define the analog for the y direction,

$$k_x=\left(\frac{m\omega^2}{D_x}\right)^{1/4}, \tag{8a}$$

$$k_y=\left(\frac{m\omega^2}{D_y}\right)^{1/4}. \tag{8b}$$

Consider, for instance, the limiting case of an isotropic plate, for which the bending stiffnesses reduce to

$$D_x=D_y=D,\quad D_0=\nu D,\quad D_{xy}=\tfrac{1}{2}(1-\nu)D, \tag{9}$$

with

$$D=\frac{EI}{1-\nu^2},\quad I=\frac{2}{3}h^3, \tag{10}$$

where ν is Poisson's ratio and E is the Young's modulus, and, hence, $k_x=((1-\nu^2)m\omega^2/EI)^{1/4}$. By comparison, the flexural wave number in a beam, a purely 1D construct, is $k=(m\omega^2/EI)^{1/4}$. In the following we will determine how this factor of $(1-\nu^2)$ arises, simultaneously examining the analogous situation for the orthotropic plate. We note here that an orthotropic plate composed of an anisotropic material with in-plane extensional (Young's) moduli E_1 and E_2, shear modulus G_{12}, and generalized Poisson's ratios ν_{12} and ν_{21} related by $\nu_{12}E_2=\nu_{21}E_1$ has

$$D_x=\frac{IE_1}{1-\nu_{12}\nu_{21}},\quad D_y=\frac{\nu_{21}}{\nu_{12}}D_x,$$
$$D_0=\nu_{21}D_x,\quad D_{xy}=IG_{12}. \tag{11}$$

In general, the bending stiffnesses D_x, D_y, D_0, and D_{xy} satisfy the inequalities $D_x+D_y>0$, $D_xD_y-D_0^2>0$, $D_{xy}>0$, as a result of the fact that the flexural strain energy is necessarily a positive quantity.

Our approach is to consider the beam as the limit of a plate of vanishing width. In order to take the proper limit we must enforce free-free boundary conditions on the edges $y=\pm b$, viz.,

$$M_y(x,\pm b,t)=0,\quad V_y(x,\pm b,t)=0,\quad -\infty<x<\infty, \tag{12}$$

where $V_y=Q_y+\partial M_{xy}/\partial x$ is the Kirchhoff shear force.[4] We consider time harmonic solutions of the form $w(x,y,t)=\mathrm{Re}[W(y)e^{i(kx-\omega t)}]$. The most general solution that is symmetric about the center line $-\infty<x<\infty$, $y=0$, is

$$W(y)=A_1\cosh\gamma_1 y+A_2\cosh\gamma_2 y, \tag{13}$$

where the transverse wave numbers $\gamma_{1,2}$ are

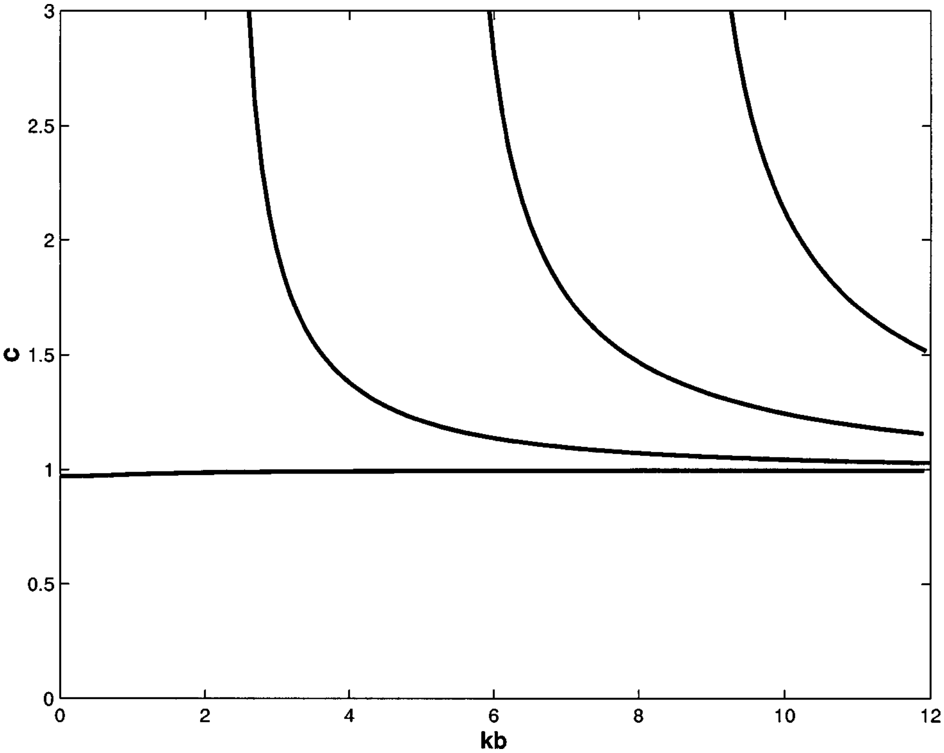

FIG. 1. Dispersion curves for symmetric flexural modes on a strip of width $2b$, from Kirchhoff theory, Eq. (16) for an isotropic plate with $\nu=\frac{1}{3}$. The quantity plotted is the phase speed $c=k_\infty/k$ relative to the speed on a plate of infinite width, $k_\infty^4=m\omega^2(1-\nu^2)/EI$.

$$\gamma_j=\left\{\left(\frac{D_0+2D_{xy}}{D_y}\right)k^2+(-1)^j\left[\left(\frac{D_0+2D_{xy}}{D_y}\right)^2k^4+\frac{D_x}{D_y}(k_x^4-k^4)\right]^{1/2}\right\}^{1/2},\quad j=1,2. \tag{14}$$

Note that $\gamma_2>0$, while γ_1 is positive for subsonic ($k>k_x$) solutions, and pure imaginary, $\gamma_1=i|\gamma_1|$, for supersonic ($k<k_x$) solutions. Applying the boundary conditions (12) gives the simultaneous equations

$$\begin{bmatrix}(D_0k^2-D_y\gamma_1^2)\cosh\gamma_1 b & (D_0k^2-D_y\gamma_2^2)\cosh\gamma_2 b\\ [(D_0+4D_{xy})k^2-D_y\gamma_1^2]\gamma_1\sinh\gamma_1 b & [(D_0+4D_{xy})k^2-D_y\gamma_2^2]\gamma_2\sinh\gamma_2 b\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \tag{15}$$

from which the dispersion relation follows as

$$(D_0k^2-D_y\gamma_1^2)^2\gamma_1^{-1}\coth\gamma_1 b$$
$$-(D_0k^2-D_y\gamma_2^2)^2\gamma_2^{-1}\coth\gamma_2 b=0. \tag{16}$$

The relation $\gamma_1^2+\gamma_2^2=(2D_0+4D_{xy})k^2/D_y$ has been used in simplifying Eq. (16).

The first few dispersion curves are plotted in Fig. 1 versus the nondimensional parameter k_xb for an isotropic plate. The nondimensional speed plotted is $c=k_x/k$, which characterizes the wave number of the wave in the x direction. Solutions with $c<1$ ($c>1$) correspond to waves that are subsonic (supersonic) relative to the reference phase speed

$$v_x\equiv\frac{\omega}{k_x}=\left(\frac{D_x\omega^2}{m}\right)^{1/4}. \tag{17}$$

Note the appearance in Fig. 1 of the modes at discrete frequencies, which in general occur when $k=0$, at discrete values of k_yb satisfying

$$\tan k_yb+\tanh k_yb=0,\quad \text{symmetric cut off,} \tag{18}$$

where k_y, defined in Eq. (8b), is the wavenumber of a wave traveling in the y direction in a plate of infinite width (see Table I).

In addition to these modes, the symmetric solution displays a mode that exists at arbitrarily low values of k_xb,

TABLE I. The first five cutoff frequencies for the Kirchhoff plate model, Eqs. (16) and (49), follow from these values of x, according to $k_yb=\pi x$.

Symmetric: $\tan(\pi x)+\tanh(\pi x)=0$.	0	$x=0.7528$	1.7500	2.7500	3.7500
Asymmetric: $\tan(\pi x)-\tanh(\pi x)=0$	$x=0$	1.2499	2.2500	3.2500	4.2500

which is the mode that reduces to the flexural wave on a beam. Before considering this limit, we note that the large-b asymptote of this mode in Fig. 1 is reached when $k_x b \gg 1$, for which (16) reduces to

$$(D_0 k^2 - D_y \gamma_1^2)^2 \gamma_2 - (D_0 k^2 - D_y \gamma_2^2)^2 \gamma_1 = 0. \tag{19}$$

This has a unique positive root[5] at $k = k_x / c_{\text{edge}}$ where $0 < c_{\text{edge}} \leq 1$ is given by

$$c_{\text{edge}}^4 = 1 - \frac{(\sqrt{D_0^2 + 4D_{xy}^2} - 2D_{xy})^2}{D_x D_y}. \tag{20}$$

The speed of a flexural wave guided by the free edge of a semi-infinite isotropic elastic thin plate was first derived by Konenkov,[3] and later by Sinha[6] and by Thurston and McKenna.[7] The flexural edge wave speed for an isotropic plate was also derived from the modes of a thin plate of finite width by taking the limit in which the width becomes infinite.[8] The edge wave decays exponentially with distance from the edge, similar to a Rayleigh wave on an elastic half-space. The existence of the edge wave on orthotropic thin plates was demonstrated by Norris,[5] who first derived the explicit expression (20). Abrahams and Norris[9] showed that it can also exist in the presence of fluid loading, a result which is perhaps surprising. However, the existence is restricted to very light fluid loading conditions: for example, thin plates of aluminum or plexiglass can support edge waves in air, although not in water.[9] The classical Kirchhoff plate theory predicts a speed for the edge wave which is in constant proportion to the flexural wave speed. The constant of proportionality is independent of the frequency and depends only on the Poisson's ratio, being slightly less than unity and equal to unity when the Poisson's ratio vanishes. As noted by Thurston and McKenna,[7] this equality reflects the fact that a flexural wave traveling parallel to the edge of a thin plate of zero Poisson's ratio gives no bending moment or shear and hence automatically satisfies the free edge conditions of the classical plate theory.

We now turn to the beam limit by considering small but nonzero width, specifically $k_x b \ll 1$, by taking the leading order terms in $\coth(\gamma_{1,2} b)$. Using $\coth \xi = 1/\xi + \xi/3 + \mathrm{O}(\xi^3)$, Eq. (16) simplifies to

$$\left(D_x - \frac{D_0^2}{D_y} \right) k^4 - m\omega^2 + \frac{4D_0^2 D_{xy}}{3D_y^2} k^6 b^2 + \mathrm{O}(k^8 b^4) = 0, \tag{21}$$

which may be rewritten

$$k^4 = \frac{m\omega^2}{D^*}, \tag{22}$$

with modified stiffness D^*,

$$D^*(b) = D_x \left[1 - \frac{D_0^2}{D_x D_y} + \frac{4D_0^2 D_{xy}}{3D_x D_y} \frac{k_y^2 b^2}{\sqrt{D_x D_y - D_0^2}} + \mathrm{O}(k^4 b^4) \right]. \tag{23}$$

Substituting from (11) gives $D^* = E^* I$, with effective Youngs modulus

$$E^* = E_1 \left[1 + \frac{4G_{12}}{3E_2} \nu_{12} \nu_{21} (k_{x0} b)^2 + \mathrm{O}(k^4 b^4) \right], \tag{24}$$

where k_{x0} is the wave number of simple beam theory, $k_{x0} = (m\omega^2 / E_1 I)^{1/4}$. Thus, the beam flexural wave number that includes the leading order correction is $k = (m\omega^2 / E^* I)^{1/4} + \mathrm{O}(b^4)$.

Equation (24) suggests that the first correction to the beam limit for nonzero width $b > 0$ can be interpreted as an *increase* in uniaxial stiffness. The increase is proportional to $b^2 \sqrt{\omega}$, and the leading order approximation assumes $k_{x0} b \ll 1$. In the case of isotropy, we have

$$E^* = E \left[1 + \frac{2\nu^2 k_0^2 b^2}{3(1+\nu)} \right] + O(k_0^4 b^4), \tag{25}$$

and k_0 is the beam flexural wave number. The veracity of this leading order correction is verified by the straight line approximation in Fig. 2, which shows the nondimensional relative phase speed $c = k_\infty / k$ versus the $(k_\infty b)^2$ where k_∞ is the flexural wave number of a plate, i.e., $k_\infty = k_0 (1 - \nu^2)^{1/4}$.

B. Kirchhoff–Rayleigh plate theory

The equation of motion of an orthotropic plate is again Eq. (1), but now the effect of rotational inertia is included as follows:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = -\rho I \frac{\partial^3 w}{\partial t^2 \partial x}, \tag{26}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = -\rho I \frac{\partial^3 w}{\partial t^2 \partial y}, \tag{27}$$

The moments are as before, Eqs. (4)–(6), and the governing equation for flexural displacement becomes

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(D_0 + 2D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + m \frac{\partial^2 w}{\partial t^2} - \rho I \nabla^2 \frac{\partial^2 w}{\partial t^2} = 0. \tag{28}$$

The wave number of a wave traveling in the x direction in a plate of infinite width, formerly given by Eq. (8), is now

$$k_x = \left(\frac{\rho I \omega^2}{2D_x} + \sqrt{\left(\frac{\rho I \omega^2}{2D_x} \right)^2 + \frac{m\omega^2}{D_x}} \right)^{1/2}. \tag{29}$$

Despite the more sophisticated theory, it can be shown following the same procedures as before that the dispersion relation for the plate of finite lateral width $2b$ is again given by Eq. (16), but where $\pm \gamma_{1,2}$ are now roots of

$$D_y \gamma^4 - [2(D_0 + 2D_{xy})k^2 - \rho I \omega^2] \gamma^2 + D_x k^4 - \rho I \omega^2 k^2 - m\omega^2 = 0. \tag{30}$$

The wave number for the edge wave is defined by (19), derived from (16) in the limit $b \to \infty$, where γ_1 and γ_2 are roots of (30). The equation for the edge wave number may be expressed as a quartic in k^2. At high frequencies it becomes nondispersive, and reduces to a cubic in k^2.

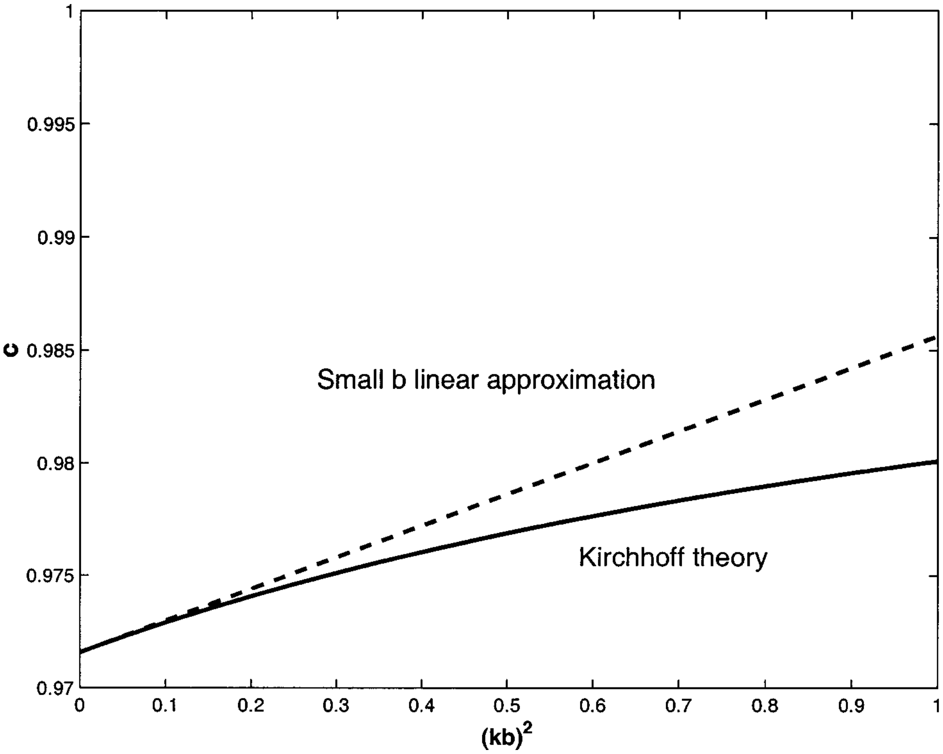

FIG. 2. The relative phase speed $c = k_x/k$ of the lowest symmetric mode versus $(k_\infty b)^2$ (frequency) for an isotropic plate ($\nu=\frac{1}{3}$) using Kirchhoff theory. The zero frequency limit is $(1-\nu^2)^{1/4}$. The dashed straight line is the small-b leading order correction based on Eq. (25).

We now examine the behavior of this plate theory as the width b vanishes. Expanding (16) to first order in b^2, subject to the constraint (30) on $\gamma_{1,2}$, yields

$$\left[1+\frac{D_0}{3}k^2b^2\right][(D_xD_y-D_0^2)k^4-D_y\rho I\omega^2k^2-D_ym\omega^2] + \frac{D_0^2}{3D_y}k^4b^2(4D_{xy}k^2-\rho I\omega^2)+\mathrm{O}(b^4)=0. \tag{31}$$

The limiting wave number as $b\to 0$ therefore solves

$$(D_xD_y-D_0^2)k^4-D_y\rho I\omega^2k^2-D_ym\omega^2=0, \tag{32}$$

that is, it is given by (29) with D_x replaced by $D^*(0)$ of Eq. (23). This is the precisely the prediction according to Kirchhoff–Rayleigh beam theory.

Based upon the leading order equation (32), we may simplify (31) so that the first-order correction satisfies the following,

$$[(D_xD_y-D_0^2)k^4-D_y\rho I\omega^2k^2-D_ym\omega^2] + \frac{D_0^2}{3D_y}k^4b^2(4D_{xy}k^2-\rho I\omega^2)+\mathrm{O}(b^4)=0. \tag{33}$$

Unlike the previous case of the simple Kirchhoff theory, it does not seem possible to interpret this result in terms of an effective stiffness alone. No simplification is apparent even if we consider the isotropic version of (33),

$$[EIk^4-\rho I\omega^2k^2-m\omega^2]+\frac{\nu^2}{3}k^4b^2\left(\frac{2EIk^2}{1+\nu}-\rho I\omega^2\right) +\mathrm{O}(b^4)=0. \tag{34}$$

In particular, this equation cannot be expressed using the modified stiffness of (25) alone.

III. MINDLIN AND SHEAR PLATE THEORIES

Mindlin's theory contains the rotational inertia of the Kirchhoff–Rayleigh theory plus a shear correction. Based on the previous analysis for the Kirchhoff–Rayleigh theory it would seem likely that the Mindlin theory will not yield a simpler result, and probably more complicated than (34). Despite the extra refinements in Mindlin's theory we will see that the first correction to the beam theory, or Timoshenko's theory, only involves an effective stiffness. In fact, it will emerge that the leading order correction for the shear theory is identically zero.

A. Mindlin plate theory

The wave numbers for straight crested waves in an isotropic Mindlin plate and in a Timoshenko beam of the same thickness $2h$ are solutions of quadratic equations for k^2:

$$k^4-(k_S^2+k_P^2)k^2-k_S^2k_P^2+k_F^4=0, \tag{35}$$

$$k^4-(k_S^2+k_{P0}^2)k^2-k_S^2k_{P0}^2+k_{F0}^4=0, \tag{36}$$

respectively. Here,

$$k_P=\omega\sqrt{\frac{\rho(1-\nu^2)}{E}},\quad k_F=\left(\frac{m\omega^2(1-\nu^2)}{EI}\right)^{1/4}, \tag{37}$$

$$k_{P0}=\omega\sqrt{\frac{\rho}{E}},\quad k_{F0}=\left(\frac{m\omega^2}{EI}\right)^{1/4}\quad \text{and}\quad k_S=\frac{\omega}{\alpha}\sqrt{\frac{\rho}{\mu}},$$

where μ is the shear modulus and α is a nondimensional factor, with $0<\alpha\leq 1$.[4] The details of the Mindlin plate theory are in Appendix A, where the following dispersion relation for a plate of width $2b$ is derived,

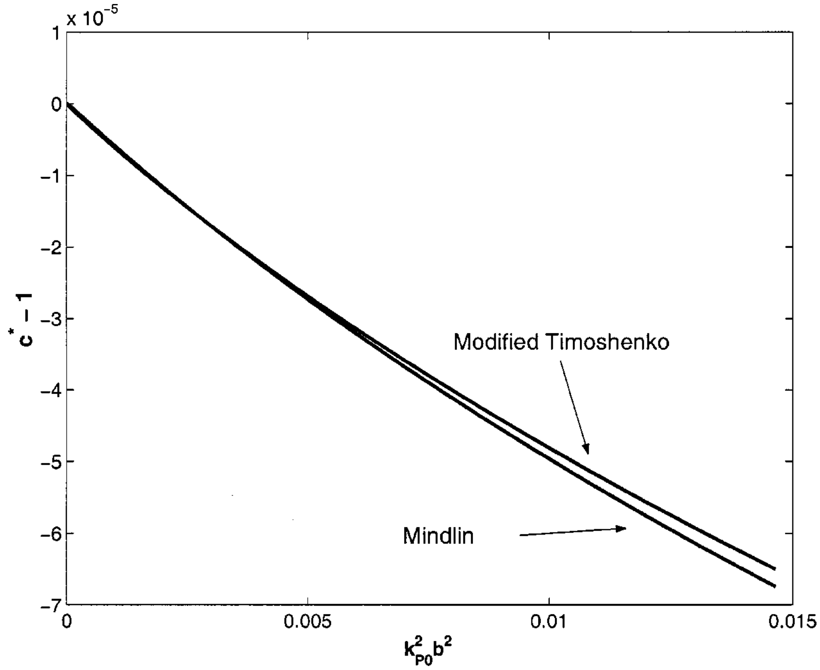

FIG. 3. Comparison of the lowest order symmetric mode for the Mindlin plate theory, and the prediction using the Timoshenko beam equation (43) and the modified Young's modulus of (45). The ratio $b/h=0.2$. The speed $c^*=k_{\rm Tim}/k$, where $k_{\rm Tim}$ is Timoshenko wave number and k is either the Mindlin wave number or the wave number predicted by the modified Timoshenko dispersion equation.

$$\left(k^2-\frac{k_1^2}{1-\nu}\right)^2(k_S^2-k_1^2)k_2^2\gamma_1^{-1}\coth(\gamma_1 b)$$
$$-\left(k^2-\frac{k_2^2}{1-\nu}\right)^2(k_S^2-k_2^2)k_1^2\gamma_2^{-1}\coth(\gamma_2 b)$$
$$+k^2k_S^2(k_1^2-k_2^2)\gamma_3\coth(\gamma_3 b)=0. \tag{38}$$

The three wavenumbers k_j, $j=1,2,3$, in this equation are defined in Eq. (A9), and $\gamma_j=\sqrt{k^2-k_j^2}$, $j=1,2,3$. Note that k_1 and k_2 are roots of the Mindlin equation (35).

It is interesting to note that the dispersion relation for the Kirchhoff–Rayleigh theory falls out of Eq. (38) by setting k_S to zero or, equivalently, allowing $\alpha\to\infty$. In the limit of infinite width, (38) reduces to

$$\left(k^2-\frac{k_1^2}{1-\nu}\right)^2(k_S^2-k_1^2)k_2^2\gamma_1^{-1}-\left(k^2-\frac{k_2^2}{1-\nu}\right)^2$$
$$\times(k_S^2-k_2^2)k_1^2\gamma_2^{-1}+k^2k_S^2(k_1^2-k_2^2)\gamma_3=0. \tag{39}$$

This equation has been examined by Norris *et al.*[10] who showed that it possesses a root at all frequencies. In the high-frequency limit $k_{S,P}h\gg 1$, the edge wave speed becomes nondispersive, and equal to the Rayleigh wave speed in plane stress,[10] given by

$$(2k^2-k_T^2)^2-4k^2(k^2-k_T^2)^{1/2}(k^2-k_P^2)^{1/2}=0, \tag{40}$$

where $k_T^2=\omega^2\rho/\mu$. Also, the cutoff frequencies of Eq. (38) for the plate of finite width are given by

$$(k_S^2-k_1^2)k_1\cot k_1 b-(k_S^2-k_2^2)k_2\cot k_2 b=0. \tag{41}$$

These define the modal cut-on frequencies, which are infinite in number but include $\omega=0$. We now consider the mode that exists down to zero frequency: the flexural mode.

Unlike the previous expansions for the Kirchhoff plate models, greater care is necessary with the Mindlin theory. Thus, in addition to the assumption $kb\ll 1$, we now need to further specify that $b\ll h$. The latter is required because when we allow the frequency to tend to zero we have $k_3 b=\mathrm{O}(b/h)$, and therefore the consistent small-b limit is reached by allowing $k_j b\ll 1$, $j=1$, 2, 3. We shall return to this point later.

By considering $kb\ll 1$ and $b\ll h$, we find that $\gamma_j b\ll 1$, $j=1$, 2, and by using the leading order approximation $\coth\xi=1/\xi+\mathrm{O}(\xi)$, combined with (A9), it can be shown that Eq. (38) reduces to Eq. (36), which is precisely the dispersion relation for a flexural wave on a Timoshenko beam. Using the two-term approximation of $\coth\xi$ plus the leading order approximation given by (36), we find that the first correction to the wave number satisfies

$$k^4-(k_S^2+k_{P0}^2)k^2-k_S^2k_{P0}^2+k_{F0}^4-\frac{\nu^2}{3}(1-\nu^2)$$
$$\times b^2k_{P0}^2(k_{P0}^2k^2+k_S^2k_{P0}^2-k_{F0}^4)+\mathrm{O}(k^8b^4)=0. \tag{42}$$

The derivation of Eq. (42) was performed using the symbolic algebra program Maple. It may be written as

$$k^4-(k_S^2+k_{P*}^2)k^2-k_S^2k_{P*}^2+k_{F*}^4+\mathrm{O}(k^8b^4)=0, \tag{43}$$

where

$$k_{P*}=\omega\sqrt{\frac{\rho}{E^*}},\quad k_{F*}=\left(\frac{m\omega^2}{E^*I}\right)^{1/4}, \tag{44}$$

and

$$E^*=E\left[1-\frac{\nu^2}{3}(1-\nu^2)k_{P0}^2b^2+\mathrm{O}(b^4)\right]. \tag{45}$$

The approximation to the flexural wave number based on (43)–(45) is compared with the exact prediction from Mindlin theory in Fig. 3. We note the agreement between the lowest order mode and the asymptotic approximation. We also note that the effective Young's modulus of Eq. (45) is

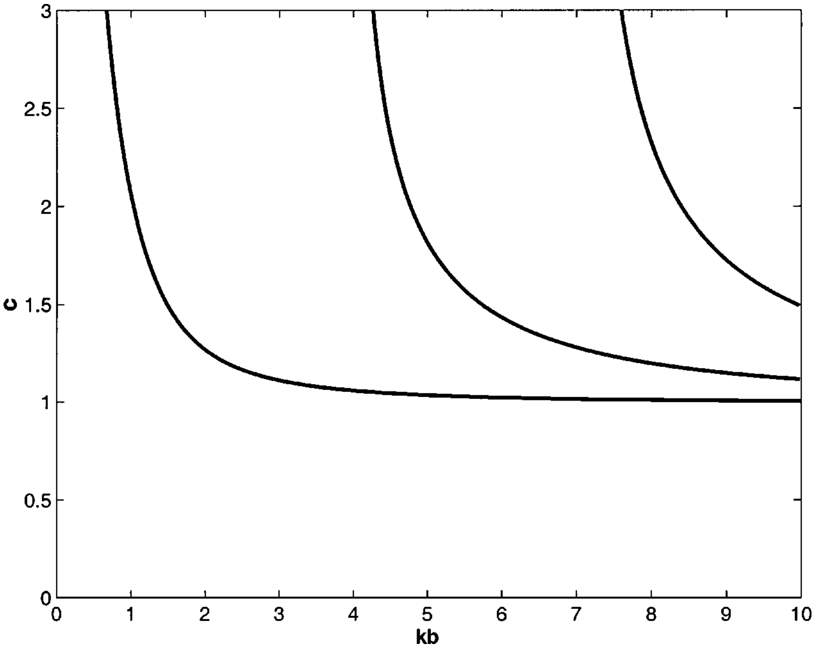

FIG. 4. Dispersion curves for the phase speed $c=k_\infty/k$ for the asymmetric modes on a plate of width $2b$ according to Kirchhoff theory, Eq. (49). Note the existence of the mode down to zero frequency, see Eq. (54).

different in form from that for the Kirchhoff theory, Eq. (25), but defer discussion until later.

B. Shear plate theory

Shear plate theory is a simplified version of Mindlin's theory, as it only considers the shear correction to the Kirchhoff model without the rotational inertia effects. Thus, the model is described by Eqs. (1)–(3) and (A3); the analysis is similar to that for the Mindlin theory, and requires a three-wave solution which is identical with the Mindlin solution if the replacement $k_P \to 0$ is made in (A9). Thus, instead of Eq. (A9), we have

$$k_j^2 = \frac{1}{2} k_S^2 - (-1)^j \sqrt{\frac{1}{4} k_S^4 + k_F^4}, \quad j=1,2, \quad k_3^2 = 3\frac{\alpha^2}{h^2}. \tag{46}$$

The dispersion relation so obtained is formally equivalent to (38), and therefore the asymptotic approximation that results is the same as for the Mindlin plate. That is, the limiting wave number reduces to the pure shear beam theory prediction as $b\to 0$. The edge wave solutions ($b\to\infty$) are also given by the Mindlin equation (39), and in the limit of high frequency the edge wave becomes nondispersive, with $k \to k_S$.

However, because $k_P=0$, we find that the first correction to the leading order equation is identically zero. That is, the narrow plate limit is given by (43) and (44) with

$$E^* = E[1+\mathrm{O}(b^4)]. \tag{47}$$

IV. ASYMMETRIC MODES

We have seen how the beam theory prediction follows from the limit of plate solutions that are symmetric about the centerline $y=0$. We now consider the analogous situation for asymmetric modes or, more correctly, pure antisymmetric modes satisfying $w(x,-y,t) = -w(x,y,t)$. These are plate modes which possess no limit in the beam theories. However, as we will see, these modes all reduce to dispersionless waves, which can be understood in terms of *torsion* rather than flexure. The question arises then whether the plate theories reduce to the correct torsional limits. As before, we consider the four engineering theories in sequence.

A. Kirchhoff plate theory

The general solution that is asymmetric about the center line $-\infty<x<\infty$, $y=0$, is

$$W(y) = A_1 \sinh \gamma_1 y + A_2 \sinh \gamma_2 y. \tag{48}$$

Applying the boundary conditions (12) on the free edges gives the dispersion relation (see Fig. 4).

$$(D_0 k^2 - D_y \gamma_1^2)^2 \gamma_1^{-1} \tanh \gamma_1 b - (D_0 k^2 - D_y \gamma_2^2)^2 \gamma_2^{-1} \tanh \gamma_2 b = 0. \tag{49}$$

Setting $k\equiv 0$ implies that the cutoff frequencies satisfy

$$\tan k_y b - \tanh k_y b = 0, \quad \text{asymmetric cut off.} \tag{50}$$

These are enumerated in Table I, and include zero, indicating a mode exists for all nonzero values of b.

Expanding the asymmetric dispersion relation (49) to leading order in $k_x b \ll 1$ gives

$$4D_{xy}k^2 - \frac{b^2}{3D_y}\{[D_0+4D_{xy}]k^4 + D_x D_y (k_x^4 - k^4)\} + \mathrm{O}(b^4) = 0. \tag{51}$$

The only consistent solution is $kb = \mathrm{O}(k_x b)^2$, with

$$k^2 = \frac{m\omega^2 b^2}{12 D_{xy}} + \mathrm{O}(b^4). \tag{52}$$

Note that the limiting wave is nondispersive, since $v=\omega/k$ is independent of frequency. Thus, using $D_{xy} = IG_{12}$ gives

$$v^2 = \frac{4G_{12}h^2}{\rho b^2} + \mathrm{O}(b^0). \tag{53}$$

In the isotropic case $(G_{12}=\mu)$, we find the limiting wave speed $v\rightarrow v_0$ as $\omega\rightarrow 0$,

$$v_0=\frac{2h}{b}c_T, \tag{54}$$

where $c_T=\sqrt{\mu/\rho}$ is the bulk transverse wave speed. We will discuss this limit later in terms of torsion theory.

B. Kirchhoff–Rayleigh plate theory

The dispersion relation for asymmetric modes is again (49), but where now we have the identities, from (30),

$$\begin{aligned} &D_y(\gamma_1^2+\gamma_2^2)=(2D_0+4D_{xy})k^2,\\ &D_y\gamma_1^2\gamma_2^2=D_xk^4-\rho I\omega^2k^2-m\omega^2. \end{aligned} \tag{55}$$

The equation for cutoff frequencies $(k=0)$ is also (50), implying the existence of a mode with a cutoff at zero frequency.

The mode with zero cutoff may be examined using Eq. (55), yielding

$$\begin{aligned} &4D_{xy}k^2-\rho I\omega^2-\frac{b^2}{3D_y}\{[(D_0+4D_{xy})k^2-\rho I\omega^2]^2\\ &\quad+D_xD_y(k_x^4-k^4)-D_y\rho I\omega^2k^2\}+\mathrm{O}(b^4)=0. \end{aligned} \tag{56}$$

Hence,

$$k^2=\frac{\rho I\omega^2}{4D_{xy}}\left(1+\frac{b^2}{h^2}\right)+\mathrm{O}(k^4b^2). \tag{57}$$

This is again nondispersive, with limiting phase (wave) speed for the isotropic plate

$$v_0=\frac{2c_T}{\sqrt{1+b^2/h^2}}. \tag{58}$$

Again, we defer discussion of this result until later.

C. Mindlin plate theory

It may be shown using the same procedure as for the symmetric modes that the dispersion relation for asymmetric modes according to the Mindlin theory is

$$\begin{aligned} &\left(k^2-\frac{k_1^2}{1-\nu}\right)^2(k_S^2-k_1^2)k_2^2\gamma_1^{-1}\tanh(\gamma_1b)\\ &\quad-\left(k^2-\frac{k_2^2}{1-\nu}\right)^2(k_S^2-k_2^2)k_1^2\gamma_2^{-1}\tanh(\gamma_2b)\\ &\quad+k^2k_S^2(k_1^2-k_2^2)\gamma_3\tanh(\gamma_3b)=0. \end{aligned} \tag{59}$$

Setting $k=0$ implies that the cutoff frequencies satisfy

$$(k_S^2-k_1^2)k_1\tan k_1b-(k_S^2-k_2^2)k_2\tan k_2b=0, \tag{60}$$

which includes zero as a cut-on frequency.

Retaining the first two terms in the expansion of (59) in terms of b yields

$$\begin{aligned} &k_p^2\left(1-\frac{k^2b^2}{3}\right)+\frac{b^2}{3}\left\{-(1-\nu^2)k^4+2k^2\left[(1+\nu)k_p^2-2\frac{k_F^4}{k_S^2}\right]\right.\\ &\quad\left.+k_p^4+k_F^4\right\}+\mathrm{O}(b^4)=0. \end{aligned} \tag{61}$$

The dominant terms in (61) for both small b and low frequency are

$$k_p^2+\frac{b^2}{3}k_F^4\left[1-4\frac{k^2}{k_S^2}\right]\sim 0. \tag{62}$$

Hence the limiting value of the phase speed $v=\omega/k\rightarrow v_0$ at zero frequency is

$$v_0=\frac{2\alpha c_T}{\sqrt{1+h^2/b^2}}, \tag{63}$$

which is discussed in the next section.

D. Mindlin shear theory

If the rotational inertia is ignored and the pure shear theory is employed, then formally the result can be obtained from the analysis for the Mindlin theory by taking the limit of $k_P\rightarrow 0$. The outcome is that expression (62) simplifies to

$$\frac{b^2}{3}k_F^4\left[1-4\frac{k^2}{k_S^2}\right]\sim 0, \tag{64}$$

or $k=k_S/2$. Thus, the zero frequency limit for the phase speed becomes in this case

$$v_0=2\alpha c_T. \tag{65}$$

Note that the resulting wave number is independent of both the half-width b and the semithickness h.

V. DISCUSSION

We have derived the dispersion relations for symmetric and asymmetric modes on a plate of finite width according to the four engineering theories. In the limit of zero width, $b\rightarrow 0$, each plate theory predicts that the symmetric mode has the wave number of the corresponding beam theory. The four plate theories also predict the existence of an edge wave solution in the limit of infinite width, $b\rightarrow\infty$. The edge wave speed for the Kirchhoff theory is given by (20), and by (39) for the Mindlin and Mindlin-shear theories, respectively. In the high-frequency limit only the Kirchhoff edge wave remains dispersive; the other three theories predict constant but different values for the edge wave speed. We have also examined the behavior of the asymmetric mode in the narrow plate limit, and found in each case that the mode is nondispersive at zero frequency, but with different limiting values. We now discuss the symmetic and asymmetric cases in more detail.

A. Symmetric or flexural mode

The two main results are Eqs. (25) and (45), for the effective Young's moduli of narrow plates or beams of rectangular cross section. By writing the fundamental Kirchhoff beam wave number as $m\omega^2/EI=3k_{P0}^2/h^2$, where k_{P0} is defined in (37), the corrections to the beam theories can be expressed

$$\frac{E^*}{E}-1=\begin{cases}\dfrac{2\nu^2}{\sqrt{3}(1+\nu)}\dfrac{k_{P0}b^2}{h}+\mathrm{O}(b^4), & \text{Kirchhoff},\\ -\dfrac{\nu^2}{3}(1-\nu^2)k_{P0}^2b^2+\mathrm{O}(b^4), & \text{Mindlin},\\ 0+\mathrm{O}(b^4), & \text{Mindlin-Shear only}.\end{cases} \tag{66}$$

The Kirchhoff–Rayleigh theory is not included as it does not reduce to an effective Young's modulus E^*.

The effective stiffness differs for each theory, and the differences are significant in terms of the leading order dependence on the width. Thus, the signs for the correction in (66) are such that in Kirchhoff theory the beam is stiffened, it softens for the Mindlin theory, and in the Mindlin-shear theory the leading order correction is identically zero. In addition to this fundamental deviation between the conflicting theories, we note the distinct frequency dependence of the correction terms: linear in ω for the Kirchhoff correction, and quadratic for the Mindlin plate theory. It is interesting to note that the latter does not depend on the shear correction factor α. Yet, when we consider the Mindlin theory without the rotational inertia, that is the pure shear theory, the correction vanishes. These asymptotic corrections have been verified by numerical examples, see Figs. 2 and 3.

It is perhaps useful to compare the corrections in (66) with the well-known corrections to the various beam theories as a function of frequency. In this case the variation depends on the beam semi-thickness h, which enters into the beam theories as follows:

$$k^2h^2=\sqrt{3}k_{P0}h+\begin{cases}0, & \text{Kirchhoff},\\ \tfrac{1}{2}k_{P0}^2h^2+\mathrm{O}(k_{P0}^3h^3), & \text{Kirchhoff–Rayleigh},\\ \tfrac{1}{2}k_S^2h^2+\mathrm{O}(k_S^3h^3), & \text{Mindlin–shear only},\\ \tfrac{1}{2}(k_S^2+k_{P0}^2)h^2+\mathrm{O}(k_S^3h^3), & \text{Mindlin}.\end{cases} \tag{67}$$

Each of the refinements to the classical Kirchhoff or Euler–Bernoulli beam theory predicts an increase in the wave number, with the increase dependent on the model. Furthermore, the leading order correction in each case has the same frequency and thickness dependence, that is, each correction in (67) is quadratic in frequency and in the beam thickness, and may be characterized by another effective Young's modulus, E^{**}, where

$$\frac{E^{**}}{E}-1=-\sqrt{\tfrac{2}{3}}k_{P0}h\times\begin{cases}0, & \text{Kirchhoff},\\ 1+\mathrm{O}(k_{P0}^2h^2), & \text{Kirchhoff–Rayleigh},\\ 2(1+\nu)/\alpha+\mathrm{O}(k_{P0}^2h^2), & \text{Mindlin–shear only},\\ 1+2(1+\nu)/\alpha+\mathrm{O}(k_{P0}^2h^2), & \text{Mindlin}.\end{cases} \tag{68}$$

It is interesting to now compare Eqs. (66) and (68), which summarize the leading order corrections for plate width and beam thickness, respectively. Beam thickness has the consistent effect of reducing the effective Young's modulus for each of the refinements to the classical Kirchhoff beam theory. However, the finite width of the plate affects each model quite differently, as it leads to an initial increase in E^* for the Kirchhoff plate theory, and a decrease for the Mindlin theory. The distinction between the beam and plate models can be explained as a Poisson effect, which disappears when $\nu=0$. Thus, for very wide plates, both models reduce to the plane stress limit, for which $E^*\to E/(1-\nu^2)\geqslant E$, with equality only if $\nu=0$. Note that Eq. (66) retains this feature, as all the corrections to E^* are $\mathrm{O}(\nu^2)$, with frequency-dependent factors. The surprising feature of Eq. (66) is that Mindlin theory initially predicts a decrease in the stiffness, although it eventually does increase to the plane stress value as kb increases. In fact, numerical experiments indicate that the prediction of (66) is only valid for extremely small values of $k_{P0}b$ (for which the correction is itself even smaller).

B. Asymmetric or torsional mode

The results of Sec. III are summarized by the following equation for the phase speeds $v=\omega/k$ in the zero frequency limit, from Eqs. (54), (58), (63), and (65),

$$v_0=c_T\times\begin{cases}2\dfrac{h}{b}, & \text{Kirchhoff},\\ 2\left(1+\dfrac{b^2}{h^2}\right)^{-1/2}, & \text{Kirchhoff–Rayleigh},\\ 2\alpha, & \text{Mindlin–shear only},\\ 2\alpha\left(1+\dfrac{h^2}{b^2}\right)^{-1/2}, & \text{Mindlin}.\end{cases} \tag{69}$$

Note that these are all nondispersive, yet it is remarkable how they provide quite different results.

Each of these limiting values can be understood as an approximation to the torsional mode for a plate. A proper analysis for the torsional wave in the zero frequency or quasistatic limit requires an estimate of the torsional rigidity C, from which the wave speed is calculated as $v=\sqrt{C/\rho J}$, where J is the centroidal moment of inertia.[4] The rectangular cross-section of the plate implies $J=(b^2+h^2)A/3$, where $A=4bh$. The torsional stiffness of a rectangular rod is not available in closed form, although it does satisfy the inequality $C<\mu J$, implying that $v<c_T$. However, we may approximate C for rectangular cross sections that are far from square,[11] thus

$$C=\frac{\mu}{3}A^2\times\begin{cases}\dfrac{h}{b} & \text{for } b\gg h,\\[2mm] \dfrac{b}{h} & \text{for } b\ll h.\end{cases} \tag{70}$$

Hence, in the two extreme but interesting cases of very wide and extremely narrow plates, the torsional wave speed is

$$v=c_T\times\begin{cases}2\left(1+\dfrac{b^2}{h^2}\right)^{-1/2} & \text{for } b\gg h,\\[2mm] 2\left(1+\dfrac{h^2}{b^2}\right)^{-1/2} & \text{for } b\ll h.\end{cases} \tag{71}$$

Referring to Eq. (69) we see that the Kirchhoff and Kirchhoff–Rayleigh plate theories each predict the correct behavior for wide plates, $b\gg h$, in the quasistatic limit. However, neither is correct in the limit of interest here, when the plate is narrow, $b\ll h$. In that case the Kirchhoff theory predicts an unphysically large speed, and the Kirchhoff–Rayleigh theory gives a finite limit, $v_0\to 2c_T$, which is also incorrect (and unphysically large). Only the Mindlin theory predicts the correct behavior for $b\ll h$, if α is taken to be unity, which is not unreasonable. The shear-only model predicts a finite limiting speed, which is incorrect. Thus we are led to conclude that of the four theories only Mindlin's gives a proper limit for very narrow plates.

This result is perhaps surprising. A very narrow plate with $b\ll h$ undergoing asymmetric "flexural" motion is more aptly described as a plate oriented at 90° with the motion similarly rotated. Despite this extreme test of the model, Mindlin's plate theory is capable of predicting the correct limiting wave speed—the speed of a torsional wave. Finally, it should be noted that the Mindlin expansion of (61) is not sufficient to give the wide plate torsional wave limit. The reason for this is as follows. Let $\lambda=b/h\gg 1$, and $\epsilon=k_Tb\ll 1$, such that $\epsilon\lambda=o(1)$. Then, multiplying the expression (61) by b^2 to make it nondimenional, we find that it is $O(\epsilon^2\lambda^4)$ and this arises from the single term $-4b^4k^2k_F^4/(3k_S^2)$. In order to cancel this leading order term we need to expand (59) to at least the next order, but leave that as a separate exercise for the interested reader.

VI. CONCLUSION

We have examined the transition between the plate and beam regimes and how both the symmetric (flexural) and asymmetric (torsional) modes depend upon the width of the plate in the lateral direction. Analytical asymptotic predictions for narrow plates have been illustrated by numerical results, and suggest the use of a frequency-dependent Young's modulus for describing the flexural wave on beams or strips of finite width, although the form of the correction to the modulus is not unique and depends on the theory used. The sign of the correction for the Kirchhoff theory is opposite to that for the Mindlin theory. Analysis of the asymmetric or torsional mode also displays quite distinct behavior for the four classical engineering theories applied to narrow plates. It is argued that only the Mindlin theory provides a realistic result in this case, and by extension, the Mindlin theory is recommended for considering the symmetric (flexural) waves in beams of rectangular cross-section and finite width.

As part of the analysis we have also derived the dispersion equations for edge-guided waves in each of the four classical plate theories. This is given by Eq. (19) for orthotropic plates in the Kirchhoff theory, with explicit solution (20). This is the only case of the four for which the edge wave is nondispersive and for which the edge wave speed has an explicit expression, as in Eq. (20). Equation (19) also defines the edge wave for the Kirchhoff–Rayleigh theory, where γ_1 and γ_2 are now defined by Eq. (30). The edge wave in the Mindlin theory with and without rotational inertia is defined by Eq. (39). The wave numbers k_1, k_2, and k_3 are defined by Eq. (A9) for the Mindlin theory, and by Eq. (46) for the Mindlin–shear theory. This is the first time that the edge wave equations have been derived or discussed for the Kirchhoff–Rayleigh and the Mindlin–shear theories.

APPENDIX A: MINDLIN PLATE THEORY

Mindlin's theory for an isotropic plate is Eq. (1), with two additional variables corresponding to angles of rotation, $\psi_x(x,y,t)$ and $\psi_y(x,y,t)$,[4] with

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=\rho I\frac{\partial^2\psi_x}{\partial t^2}, \tag{A1}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=\rho I\frac{\partial^2\psi_y}{\partial t^2}, \tag{A2}$$

$$M_x=E_pI\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right),\quad M_y=E_pI\left(\frac{\partial\psi_y}{\partial y}+\nu\frac{\partial\psi_x}{\partial x}\right),$$

$$M_{xy}=\mu I\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right), \tag{A3}$$

$$Q_x=\alpha^2 2h\mu\left(\frac{\partial w}{\partial x}+\psi_x\right),\quad Q_y=\alpha^2 2h\mu\left(\frac{\partial w}{\partial y}+\psi_y\right).$$

Note that the thickness-integrated shear modulus appearing in the shear forces Q_x and Q_y is modified by the factor α^2 in order to better approximate shear forces in the plate, and α may be chosen according to different criteria, but normally $\alpha^2\leq 1$.[4] We consider solutions of the form

$$\{w(x,y,t),\psi_x(x,y,t),\psi_y(x,y,t)\}=\mathrm{Re}\{W(y),\Psi_x(y),\Psi_y(y)\}e^{i(kx-\omega t)}, \tag{A4}$$

where[10]

$$W(y)=A_1\cosh\gamma_1y+A_2\cosh\gamma_2y, \tag{A5}$$

$$\Psi_x(y)=ik\beta_1A_1\cosh\gamma_1y+ik\beta_2A_2\cosh\gamma_2y+\gamma_3A_3\cosh\gamma_3y, \tag{A6}$$

$$\Psi_y(y)=\gamma_1\beta_1A_1\sinh\gamma_1y+\gamma_2\beta_2A_2\sinh\gamma_2y-ikA_3\sinh\gamma_3y, \tag{A7}$$

and

$$\gamma_j = \sqrt{k^2 - k_j^2}, \quad j = 1,2,3, \quad \beta_j = -1 + k_S^2/k_j^2, \quad j = 1,2, \tag{A8}$$

$$k_j^2 = \tfrac{1}{2}(k_S^2 + k_P^2) - (-1)^j \sqrt{\tfrac{1}{4}(k_S^2 - k_P^2)^2 + k_F^4}, \quad j = 1,2, \tag{A9a}$$

$$k_3^2 = \frac{2k_1^2 k_2^2}{(1-\nu)k_S^2}. \tag{A9b}$$

The wave numbers k_P, k_F, and k_S are defined in Eq. (37). The wave number k_1 describes a straight crested flexural wave in a plate according to Mindlin's theory. The corresponding wave numbers for a Timoshenko beam are defined by Eq. (36), and are given by k_1, k_2 of Eq. (A9) with the replacements k_P, $k_F \to k_{P0}$, k_{F0}, that is, the stiffness for uniaxial extension is substituted for the plate stiffness.

The boundary conditions on $y = \pm b$ require the simultaneous vanishing of Q_y, M_y and M_{xy}, implying the dispersion relation

$$\begin{vmatrix} (1+\beta_1)\gamma_1 \sinh \gamma_1 b & (1+\beta_2)\gamma_2 \sinh \gamma_2 b & -ik \sinh \gamma_3 b \\ \beta_1(\gamma_1^2 - \nu k^2)\cosh \gamma_1 b & \beta_2(\gamma_2^2 - \nu k^2)\cosh \gamma_2 b & (1-\nu)k^2 \gamma_3 \cosh \gamma_3 b \\ 2ik\gamma_1\beta_1 \sinh \gamma_1 b & 2ik\gamma_2\beta_2 \sinh \gamma_2 b & (k^2+\gamma_3^2)\sinh \gamma_3 b \end{vmatrix} = 0. \tag{A10}$$

Using row and column manipulation, this can be rearranged as

$$\begin{vmatrix} k_1^2 & k_2^2 & k_3^2 \\ k_S^2 & k_S^2 & 2k^2 \\ \left(k^2 - \dfrac{k_1^2}{1-\nu}\right)(k_S^2 - k_1^2)\dfrac{\coth \gamma_1 b}{\gamma_1} & \left(k^2 - \dfrac{k_2^2}{1-\nu}\right)(k_S^2 - k_2^2)\dfrac{\coth \gamma_2 b}{\gamma_2} & 2k^2\gamma_3 \coth \gamma_3 b \end{vmatrix} = 0. \tag{A11}$$

This can be further reduced to Eq. (38) by using (A9b).

APPENDIX B: A REDUCED PLATE THEORY FOR NARROW PLATES

An alternative procedure for examining the dependence of the flexural wave speed on the width of a narrow plate is to derive a theory appropriate to this limit. In this Appendix we demonstrate that the first corrections to the beam theory prediction, Eqs. (22) and (23), are obtained for the Kirchhoff theory using this approach.

Starting with the Kirchhoff kinematic ansatz, $u(x,y,z,t) = -zW_x(x,y,t)$, $v(x,y,z,t) = -zW_y(x,y,t)$, $w(x,y,z,t) = W(x,y,t)$, along with the assumption of plane stress, implies the Lagrangian density per unit area

$$\mathcal{L}(W(x,y,t)) = \frac{1}{2}mW_t^2 - \frac{EI}{2(1-\nu^2)}[W_{xx}^2 + W_{yy}^2 + 2\nu W_{xx}W_{yy} + 2(1-\nu)W_{xy}^2]. \tag{B1}$$

For the narrow plate, we make the further assumption

$$W(x,y,t) = w(x,t) + yp(x,t) + \frac{y^2}{2}q(x,t). \tag{B2}$$

Substituting into (B1) and integrating over the plate width $-b < y < b$ yields a Lagrangian density per unit length

$$\mathrm{L}(w,p,q) = \mathrm{L}_0(w,q) + \mathrm{L}_1(p), \tag{B3}$$

with decoupled terms

$$\mathrm{L}_0(w,q) = \frac{m}{2}(w_t^2 + Kq_t^2 + Jw_t q_t) - \frac{EI}{2(1-\nu^2)}[w_{xx}^2 + q^2 + Kq_{xx}^2 + Jw_{xx}q_{xx} + 2\nu q w_{xx} + \nu Jq q_{xx} + 2(1-\nu)Jq_x^2], \tag{B4}$$

$$\mathrm{L}_1(p) = \frac{m}{2}Jp_t^2 - \frac{EIJ}{2(1-\nu^2)}p_{xx}^2 - \frac{EI}{1+\nu}p_x^2, \tag{B5}$$

where $J = b^2/3$, $K = b^4/20$. These in turn imply the Euler–Lagrange equations

$$\frac{EI}{1-\nu^2}\left(w_{xxxx} + \frac{J}{2}q_{xxxx} + \nu q_{xx}\right) + mw_{tt} + \frac{J}{2}q_{tt} = 0, \tag{B6}$$

$$\frac{EI}{1-\nu^2}\left[q + Kq_{xxxx} + \frac{J}{2}w_{xxxx} + \nu w_{xx} - (2-3\nu)Jq_{xx}\right] + mKq_{tt} + \frac{J}{2}mw_{tt} = 0, \tag{B7}$$

$$\frac{EI}{1-\nu^2}p_{xxxx} - \frac{2EI}{(1-\nu^2)J}p_{xx} + mp_{tt} = 0. \tag{B8}$$

Note that the torsional (asymmetric) mode, p, decouples from the symmetric mode, (w,q). Dispersions relations can be easily determined for each mode from Eqs. (B6)–(B8),

$$(k^4 - k_\infty^4)[1 - \tfrac{2}{3}(1-\nu)k^2b^2 + \tfrac{1}{45}(k^4 - k_\infty^4)b^4] - \nu k^4 = 0, \tag{B9}$$

$$\frac{k^4}{k_\infty^4} + 4c_T^2\frac{h^2k^2}{b^2\omega^2} = 1, \tag{B10}$$

where $k_\infty = (m\omega^2(1-\nu^2)/EI)^{1/4}$ is the wide plate flexural wave number and $c_T = \sqrt{\mu/\rho}$. Both dispersion relations yield $k \to k_\infty$ as $b \to \infty$. A straightforward expansion of the dispersion relations in the narrow plate limit $kb \ll 1$ shows that Eq. (B9) reproduces the leading order correction of Eq. (25), while Eq. (B10) gives the constant wave speed for the Kirchhoff theory in Eq. (69). Thus, this particular reduced order plate theory for the narrow plate is entirely consistent with the Kirchoff plate theory, which is not surprising since the ansatz (B2) is a special case of the general displacement included in the Kirchhoff theory.

A similar but necessarily more complicated reduced plate theory for narrow plates is discussed by Russell and White[2] within the context of a Timoshenko-type beam model.

[1] S. M. Han, H. Benaroya, and T. Wei, ''Dynamics of flexurally vibrating beams using four engineering theories,'' J. Sound Vib. **225**(5), 935–988 (1999).

[2] D. L. Russell and L. W. White, ''The bowed narrow plate model,'' Elec. J. Diff. Eqs. **2000**(**27**), 1–19 (2000).

[3] Y. K. Konenkov, ''A Rayleigh-type flexural wave,'' Sov. Phys. Acoust. **6**, 122–123 (1960).

[4] K. F. Graff, *Wave Motion in Elastic Solids* (Dover, New York, 1991).

[5] A. N. Norris, ''Flexural edge waves,'' J. Sound Vib. **171**, 571–573 (1994).

[6] B. K. Sinha, ''Some remarks on propagation characteristics of ridge guides for acoustic waves at low frequencies,'' J. Acoust. Soc. Am. **56**, 16–18 (1974).

[7] R. N. Thurston and J. McKenna, ''Flexural acoustic waves along the edge of a plate,'' IEEE Trans. Sonics Ultrason. **21**, 296–297 (1974).

[8] V. A. Veshev, I. I. Klyukin, D. P. Kouzov, and V. D. Lukyanov, ''On oscillating energy propagation in a thin elastic plate of a finite width,'' Sov. Phys. Acoust. **23**, 129–131 (1977).

[9] I. D. Abrahams and A. N. Norris, ''On the existence of flexural edgewaves on submerged elastic plates,'' Proc. R. Soc. London, Ser. A **456**, 1559–1582 (2000).

[10] A. N. Norris, V. V. Krylov, and I. D. Abrahams, ''Flexural edge waves and Comments on 'A new bending wave solution for the classical plate equation','' J. Acoust. Soc. Am. **107**, 1781–1784 (2000).

[11] S. Timoshenko, *Theory of Elasticity*, 3rd ed. (McGraw-Hill, New York, 1970).