

Elastic waves in inhomogeneously oriented anisotropic materials

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Abstract

Ray theory is developed for elastic waves propagating in inhomogeneously oriented anisotropic solids. These are materials of uniform density with moduli which are uniform up to a rotation of the underlying crystalline axes about a common direction, the degree of rotation varying smoothly with position. The ordinary differential equations governing the evolution of a ray have a simple form, and involve the angle of deviation between the slowness and wave velocity directions. The general theory is demonstrated for the case of SH waves in a transversely isotropic medium. The equations required for the description of SH Gaussian beams are derived, including the transport equation and the wavefront curvature equation. The theory is combined with an equivalent complex-source representation to generate an approximation to a time harmonic point source. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

One of Wickham's continued interests lay in modelling of elastic wave problems related to ultrasonic non-destructive evaluation. An example is the refraction of ultrasound in austenitic steel welds, which was the topic of two papers with Abrahams [1,2]. The problem was motivated by a clear need to quantify how ultrasound propagates in a weld region, which is a highly inhomogeneous and anisotropic medium. In the first of their papers [1], the authors presented a credible model for the weld as a material with constant density and constant crystalline moduli, but with the axes of the crystal a smoothly varying function of position. The propagation of SH waves was treated in detail, and a uniform representation for the field of a point source was derived by means of a wavenumber integral. In the second paper [2], the theory was developed to include quasi-longitudinal waves.

Here we focus on the structure and properties of the ray equations for inhomogeneous anisotropic materials of this class. The theory is applicable to materials with a smoothly varying granular structure, in the sense that the properties at each position are those of the underlying grain, modulo a rotation. The grain orientation in weld regions is caused by thermal alignment as the metal cools. Similar smoothly varying heterogeneity could arise from deformation of geological layers. We begin in Section 2 with a precise definition of the general model, followed by an analysis of the ray equations for arbitrary wave type: quasi-longitudinal or quasi-transverse. The specific case of SH wave motion is treated in greater detail in Section 3. Variational ray equations [3,4] are derived which govern the evolution of the ray tube area and the wavefront curvature matrices and provide a consistent set of equations for

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calculating the amplitude along a ray and also for modelling Gaussian beams and wave packets. Finally, we will demonstrate how Gaussian beams may be combined in a finite integral to replicate the field of a point source.

2. General theory for granular media

2.1. The governing ray equations

A granular medium is defined here as a material with constant density and elastic moduli which are constant to within a rotation of the coordinate axes. Let the unit vectors $\mathbf{e}^{(m)}$, $m = 1, 2, 3$ represent fixed Cartesian axes, and $\bar{\mathbf{e}}^{(m)}$ are the crystal (or zonal) axes of the material at the position $\mathbf{x}$. The two orthonormal triads are assumed to be related by $\bar{\mathbf{e}}^{(m)} = \mathbf{e}^{(m)T} \mathbf{Q}$, or $\bar{e}_i^{(m)} = e_j^{(m)} Q_{ji}$, where $\mathbf{Q}(\mathbf{x})$ is the transformation matrix for rotation of the grain from an arbitrary reference orientation, satisfying $\mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I}$ (summation over repeated indices is assumed). Let $\bar{C}_{ijkl}$ denote the elastic moduli in the crystal frame, then the moduli in the fixed coordinate system are

$$C_{ijkl} = Q_{mi} Q_{nj} Q_{pk} Q_{ql} \bar{C}_{mnpq}. \quad (1)$$

The granular moduli $\bar{C}_{ijkl}$, and hence the apparent moduli C_{ijkl} , are assumed to possess the standard symmetries: $C_{ijkl} = C_{klij}$ and $C_{ijkl} = C_{jikl}$. The governing elastodynamic equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \ddot{u}_i, \quad \sigma_{ij} = C_{ijkl} e_{kl}, \quad (2)$$

where u_i , $i = 1, 2, 3$ are components of the elastic displacement $\mathbf{u}(\mathbf{x}, t)$ in the fixed rectangular coordinate system. Also, $\dot{u}_i = \partial u_i / \partial t$, $e_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i) / 2$ are the elastic strains, σ_{ij} are the stresses, and ρ is the density, assumed constant.

The heterogeneity of the granular medium is smoothly varying, and hence we consider a high-frequency type of *ansatz* for the displacement

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}, t) \exp\{i\omega\phi(\mathbf{x}, t)\}, \quad (3)$$

where the phase function ϕ is smoothly varying, and $\omega \gg 1$ by assumption. Note that this *ansatz* includes, but is not limited to, the single frequency time harmonic solution, for which $\phi(\mathbf{x}, t) = \Phi(\mathbf{x}) - t$. By substituting (3) into the governing equations (2), and setting to zero the leading order term in ω (the coefficient of ω^2) we obtain

$$[C_{ijkl} \nabla_j \phi \nabla_l \phi - \rho \dot{\phi}^2 \delta_{ik}] U_k = 0. \quad (4)$$

Thus, $\mathbf{U}$ is a proper vector of the symmetric matrix $C_{ijkl} \nabla_j \phi \nabla_l \phi$. The inner product of (4) with $\mathbf{U}$ may be written as $H_+ H_- = 0$, where

$$H_{\pm}(\mathbf{x}, \nabla \phi, \dot{\phi}) = \dot{\phi} \pm [\rho^{-1} C_{ijkl} b_i b_k \nabla_j \phi \nabla_l \phi]^{1/2}, \quad (5)$$

and $\mathbf{b} = \mathbf{U} / (U_k U_k)^{1/2}$ is the unit displacement vector, also called the polarization vector. The forward and backward propagating rays for the *ansatz* (3) correspond to the Hamiltonians H_+ and H_- , respectively. Here we will focus on the characteristic $H_+ = 0$.

The ray equations follow from the Hamilton–Jacobi equations for the Hamiltonian. Let τ be the parameter along the ray, conjugate to $\dot{\phi}$, so that

$$\frac{dt}{d\tau} = \frac{\partial H_+}{\partial \dot{\phi}} \equiv 1, \quad (6)$$

hence τ is identical to the travel time on the ray. Also, $d\dot{\phi}/d\tau = \partial H_+/\partial t \equiv 0$, so that we may take $\dot{\phi} = -1$ on the ray with no loss in generality. This choice makes the comparison of the present theory with time-harmonic ray theory easy ($e^{-i\omega t}$), and also means that the polarization vector satisfies

$$\rho^{-1} C_{ijkl} p_j p_l b_k = b_i, \quad (7)$$

where $\mathbf{p} = \nabla\phi$ is the slowness vector along the ray $\mathbf{x}(\tau)$. Define the wave velocity vector $\mathbf{c}$,

$$c_j = \rho^{-1} C_{ijkl} b_i b_k p_l, \quad (8)$$

then (7) implies the identity $\mathbf{p}^T \mathbf{c} = 1$, a well-known relationship between the slowness and velocity vectors, which are polar reciprocal to one another.

Finally, the ray equations for the evolution of $\mathbf{x}(\tau)$ and $\mathbf{p}(\tau)$ are

$$\frac{dx_i}{d\tau} = \frac{\partial H_+}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial H_+}{\partial x_i}. \quad (9)$$

It follows from (6) through (9) that $d\phi/d\tau = 0$ and hence the phase is constant along the ray.

2.2. Grains rotated about a common axis

We restrict subsequent attention to the class of granular materials in which the grains are all rotated about a single direction, denoted by the unit axial vector $\mathbf{a}$. Let $\theta(\mathbf{x})$ be the angle of rotation for the crystal at position $\mathbf{x}$, then we may write the transformation matrix $\mathbf{Q}(\theta)$ for rotation by angle θ about the $\mathbf{a}$ -direction as $\mathbf{Q} = e^{\theta \mathbf{R}}$, where $\mathbf{R}^T = -\mathbf{R}$ is a constant skew-symmetric matrix with components $R_{ij} = -e_{ijk} a_k$ [5]. Now $\partial H_+/\partial x_i = (dH_+/d\theta) \nabla_i \theta$, and along the ray we have

$$\frac{dH_+}{d\theta} = \rho^{-1} R_{im} C_{mjkl}(\theta) b_i b_k p_j p_l + \rho^{-1} R_{jm} C_{imkl}(\theta) b_i b_k p_j p_l = \mathbf{b}^T \mathbf{R} \mathbf{b} + \mathbf{p}^T \mathbf{R} \mathbf{c}, \quad (10)$$

where we have used (7) and (8) to simplify the expression. However, the skew-symmetry of $\mathbf{R}$ implies that $\mathbf{b}^T \mathbf{R} \mathbf{b}$ vanishes, and therefore, using (9) and (10), and the definition of $\mathbf{R}$, we deduce the explicit form of the ray equations

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{c}, \quad (11a)$$

$$\frac{d\mathbf{p}}{d\tau} = [\mathbf{a}, \mathbf{p}, \mathbf{c}] \nabla \theta(\mathbf{x}), \quad (11b)$$

where $[\mathbf{a}, \mathbf{p}, \mathbf{c}] = e_{ijk} a_i p_j c_k$ is the vector triple product.

Eq. (11a) is standard, and simply states that the ray follows the local wave velocity. The second equation for the rate of change of the slowness is new. It implies that the components of the slowness orthogonal to $\nabla \theta(\mathbf{x})$ do not change, and hence these components of $\mathbf{p}$ are constants of the motion if $\nabla \theta$ is everywhere in the same direction. This is a standard type of result for inhomogeneous media, but the present theory depends critically upon the local anisotropy. In order to understand the role of the anisotropy in the slowness equation (11b) consider the situation in which the slowness and wave velocity vectors remain in the plane orthogonal to $\mathbf{a}$ (this is true of the SH example considered in the next section). Let ψ denote the angle between the $\mathbf{p}$ and $\mathbf{c}$ vectors, see Fig. 1. Then, $\mathbf{p}^T \mathbf{c} = pc \cos \psi$, and $[\mathbf{a}, \mathbf{p}, \mathbf{c}] = pc \sin \psi$ and consequently using $\mathbf{p}^T \mathbf{c} = 1$ we have

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{c}, \quad \frac{d\mathbf{p}}{d\tau} = \tan \psi \nabla \theta(\mathbf{x}). \quad (12)$$

Hence, the rate of change of $\mathbf{p}$ is related to the local anisotropy through the deviation angle ψ .

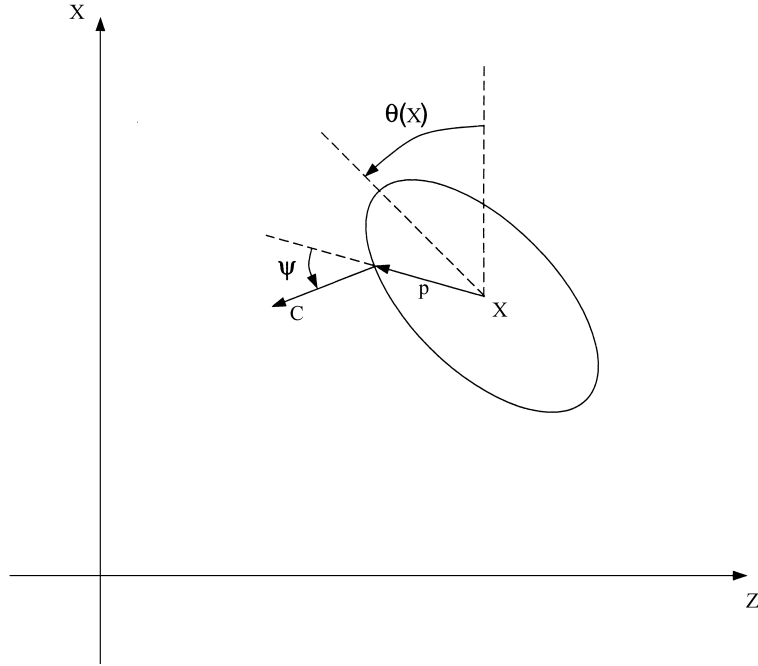


Fig. 1. A section of a slowness surface, showing the slowness and wave velocity vectors. This is the type of slowness surface considered for SH waves, where the ratio of the lengths of the major to minor axes of the ellipse is $\sqrt{1 + \gamma^{-1}}$, $\gamma > 0$.

3. SH-waves in a granular transversely isotropic material

3.1. The ray equations for SH-waves

We now consider in greater detail the configuration analysed by Abrahams and Wickham [1]. The underlying grain is transversely isotropic with symmetry axis in the $\bar{z}$ -direction and isotropy in the $\bar{x}$ – $\bar{y}$ plane. The local crystal axes are related to the fixed $\{x, y, z\}$ axes by rotation about the $y = \bar{y}$ -axis through angle $\theta(x, z)$, see Fig. 1. For simplicity, we consider horizontally polarized shear (SH) waves where the displacement $v = v(x, z, t)$ is in the y -direction and the propagation vector lies in the (x, z) -plane. The only non-vanishing stress components are σ_{yz} and σ_{yx} in the fixed coordinate system, and the associated stresses in the crystal coordinates are $\sigma_{\bar{y}\bar{z}} = \bar{c}_{44}\partial v/\partial \bar{z}$ and $\sigma_{\bar{y}\bar{x}} = \bar{c}_{66}\partial v/\partial \bar{x}$. The constitutive relations between stress and strain components in the fixed coordinates are then

$$\sigma_{yz} = (\bar{c}_{44} - \bar{c}_{66}) \sin \theta \cos \theta \frac{\partial v}{\partial x} + (\bar{c}_{66} \sin^2 \theta + \bar{c}_{44} \cos^2 \theta) \frac{\partial v}{\partial z}, \quad (13)$$

$$\sigma_{yx} = (\bar{c}_{44} - \bar{c}_{66}) \sin \theta \cos \theta \frac{\partial v}{\partial z} + (\bar{c}_{66} \cos^2 \theta + \bar{c}_{44} \sin^2 \theta) \frac{\partial v}{\partial x}. \quad (14)$$

Substituting these into the single non-trivial equation of motion, $\partial \sigma_{yx}/\partial x + \partial \sigma_{yz}/\partial z = \rho \ddot{v}$, yields a partial differential equation for v

$$\begin{aligned} & \frac{\partial}{\partial x} \left[(\bar{c}_{66} \cos^2 \theta + \bar{c}_{44} \sin^2 \theta) \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial z} \left[(\bar{c}_{66} \sin^2 \theta + \bar{c}_{44} \cos^2 \theta) \frac{\partial v}{\partial z} \right] \\ & + (\bar{c}_{44} - \bar{c}_{66}) \left[\frac{\partial}{\partial z} \left(\sin \theta \cos \theta \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial x} \left(\sin \theta \cos \theta \frac{\partial v}{\partial z} \right) \right] - \rho \ddot{v} = 0. \end{aligned} \quad (15)$$

It is convenient to introduce the anisotropy parameter

$$\gamma = \frac{\bar{c}_{66}}{\bar{c}_{44} - \bar{c}_{66}} \quad (16)$$

and with no loss in generality we assume that $\bar{c}_{44} > \bar{c}_{66}$, which is tantamount to a definition of the direction $\theta = 0$. Hence $\gamma > 0$. Let $t = t'\sqrt{\gamma\rho/\bar{c}_{66}}$, then the differential equation (15) becomes, after dropping the prime on t' :

$$\nabla^T(\mathbf{N}\nabla v) - \ddot{v} = 0, \quad (17)$$

where ∇ now denotes the 2D gradient $(\partial/\partial z, \partial/\partial x)^T$, and $\mathbf{N}$ is the symmetric matrix

$$\mathbf{N}(\theta) = \begin{bmatrix} \gamma + \cos^2\theta & \sin\theta \cos\theta \\ \sin\theta \cos\theta & \gamma + \sin^2\theta \end{bmatrix}. \quad (18)$$

As before, we assume $v(x, z) = U(x, z, t)e^{i\omega\phi(x, z, t)}$. Substitution into (17) yields terms of different order in ω , and setting the term of $O(\omega^2)$ to zero yields the eiconal equation

$$(\nabla\phi)^T\mathbf{N}(\theta)\nabla\phi - \dot{\phi}^2 = 0. \quad (19)$$

In this case we have $H_+(\mathbf{x}, \nabla\phi, \dot{\phi}) = \dot{\phi} + [\nabla\phi^T\mathbf{N}(\theta)\nabla\phi]^{1/2}$, and along the ray the identity $\mathbf{p}^T\mathbf{N}(\theta)\mathbf{p} = 1$ holds. The ray equations become

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{N}\mathbf{p}, \quad (20a)$$

$$\frac{d\mathbf{p}}{d\tau} = -\frac{1}{2}(\mathbf{p}^T\mathbf{N}'\mathbf{p})\nabla\theta(x), \quad (20b)$$

where the two-vectors are $\mathbf{x} = (z, x)^T$ and $\mathbf{p} = (p_z, p_x)^T$, and $\mathbf{N}'(\theta) = d\mathbf{N}(\theta)/d\theta$. It may be checked that these equations are consistent with the general results of (11a). Thus, $\mathbf{N}\mathbf{p} = \mathbf{c}$ is the wave velocity and $-\frac{1}{2}\mathbf{p}^T\mathbf{N}'\mathbf{p} = [\mathbf{a}, \mathbf{p}, \mathbf{c}]$.

Alternatively, the ray equations (20a) can be written as

$$\frac{dz}{d\tau} = (\gamma + \cos^2\theta)p_z + \sin\theta \cos\theta p_x, \quad (21)$$

$$\frac{dx}{d\tau} = \sin\theta \cos\theta p_z + (\gamma + \sin^2\theta)p_x. \quad (22)$$

The component of the slowness vector perpendicular to $\nabla\theta$ is conserved along the ray if the orientation $\theta(\mathbf{x})$ only depends on one of the two coordinates. For example, suppose that θ depends only on x , then $p_z = \text{constant}$, and the value of p_x is one of the two roots of $\mathbf{p}^T\mathbf{N}\mathbf{p} = 1$, or

$$(\gamma + \sin^2\theta)p_x^2 + 2\sin\theta \cos\theta p_z p_x + (\gamma + \cos^2\theta)p_z^2 = 1. \quad (23)$$

Thus, if $p_z \equiv 0$ and the variation of θ is simply $\theta = x/h$, for some constant h , then Eqs. (21) and (22) can be solved implicitly as

$$z - z_0 = \frac{h}{2} \log \left[1 + \gamma^{-1} \sin^2 \left(\frac{x - x_0}{h} \right) \right]. \quad (24)$$

This is an example of a ray which does not turn back as a function of the “depth” x , that is, at no point is $dx/d\tau = 0$, see Fig. 2. Conversely, a ray with horizontal slowness satisfying $(1 + \gamma)^{-1/2} < |p_z| < \gamma^{-1/2}$ will turn at values of θ for which $p_x = -p_z \sin\theta \cos\theta / (\gamma + \sin^2\theta)$, or substituting into (23), where $\cos^2\theta = (1 + \gamma)(1 - \gamma p_z^2)$. The ray is confined within a waveguide of finite thickness in the x -direction, within which θ varies by $\pm\pi/2$ at most, see Fig. 2.

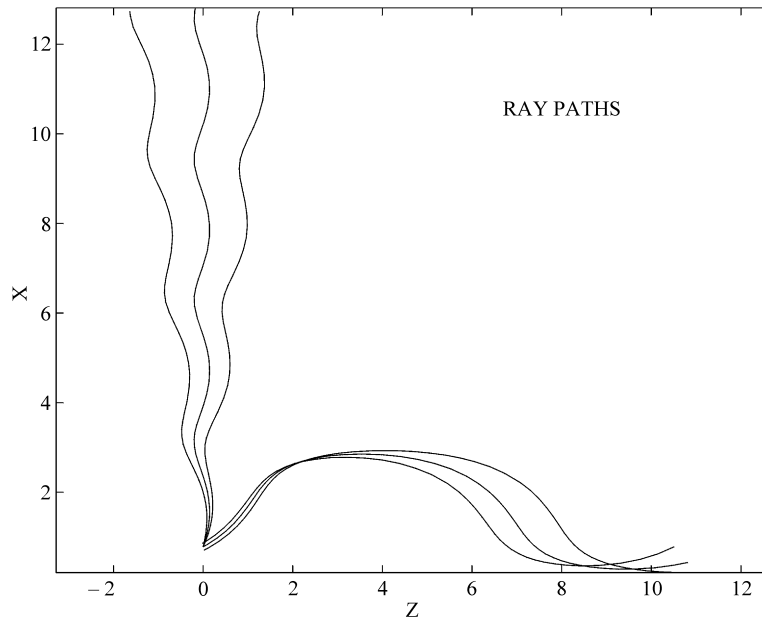


Fig. 2. Two sets of rays from the neighbourhood of the point $z_0 = 0$, $x_0 = \pi/4$ in a medium with $\theta = x$, and anisotropy $\gamma = 1$. The slowness surface at this point looks similar to Fig. 1. The three rays propagating mainly in the x -direction all originate from z_0 , x_0 , and represent a point source. The central ray has $p_z = 0$ and is an example of the periodic solution (24) (with $h = 1$). The central ray of the other set of three neighbouring rays originates from z_0 , x_0 but with a horizontal slowness within the range which guarantees horizontal channeling. The other two rays emanate from points adjacent to the first and correspond to an initially flat wavefront at that point. All rays are shown for $0 \leq \tau \leq 10$.

3.2. Variational ray equations and the transport equation

The transport of energy along a ray is governed by the transport equation, which is a consequence of the second term in the asymptotic sequence of equations in ω . We will derive the transport equation here using a general method which also produces all the quantities required to make a paraxial approximation of the wave field in the vicinity of the ray. The procedure is based on variational ray equations, which are derived from the ray equations by varying some quantity, such as the initial position of the ray. Thus, let the two-vector $\mathbf{q}$ denote a pair of independent variables, and let the 2×2 matrices $\mathbf{A}(\tau)$ and $\mathbf{B}(\tau)$ represent the partial derivatives of the ray position and its slowness with respect to $\mathbf{q}$:

$$A_{ij} = \frac{\partial x_i}{\partial q_j}, \quad B_{ij} = \frac{\partial p_i}{\partial q_j}. \quad (25)$$

Equations for the evolution of these matrices follow by a similar partial differentiation of the first-order ray equations (9). These are called the variational ray equations. Before doing so, we note the connection with paraxial ray theory, through the matrix

$$\mathbf{M}(\tau) = \mathbf{B}\mathbf{A}^{-1}, \quad (26)$$

which, based on the definitions of $\mathbf{A}$ and $\mathbf{B}$ in (25), is just the matrix of second derivatives of phase on the ray: $\mathbf{M}(\tau) = \nabla \nabla \phi$. This may also be considered as a generalized curvature matrix, because it includes the scalar curvature associated with the propagating wavefront. Recall that the analysis is in two spatial dimensions (because dependence on the third would destroy the SH nature of the solution) and hence only one parameter is required to describe the wavefront curvature. The nature of the other parameters contained within $\mathbf{M}$ will be explained below.

In deriving the variational ray equations for $\mathbf{A}$ and $\mathbf{B}$ one should be careful to note that the ray equations (20a) and (20b) are obtained from (9) by evaluating the right hand members on the rays. The variational equations involve derivatives of terms not indicated explicitly in (20a) and (20b), and are obtained directly from (9) as

$$\frac{d\mathbf{A}}{d\tau} = \mathbf{N}\mathbf{P}\mathbf{B} + \mathbf{N}'\mathbf{p}(\nabla\theta)^T\mathbf{A} - \frac{1}{2}(\mathbf{p}^T\mathbf{N}'\mathbf{p})\mathbf{N}\mathbf{p}(\nabla\theta)^T\mathbf{A}, \quad (27)$$

$$\frac{d\mathbf{B}}{d\tau} = \frac{1}{2}\mathbf{p}^T\mathbf{N}'\mathbf{p} \left[(\nabla\theta)\mathbf{p}^T\mathbf{N}\mathbf{B} + \frac{1}{2}\mathbf{p}^T\mathbf{N}'\mathbf{p}(\nabla\theta)(\nabla\theta)^T\mathbf{A} - (\nabla\nabla\theta)\mathbf{A} \right] - \frac{1}{2}\mathbf{p}^T\mathbf{N}''\mathbf{p}(\nabla\theta)(\nabla\theta)^T\mathbf{A} - (\nabla\theta)\mathbf{p}^T\mathbf{N}'\mathbf{B}, \quad (28)$$

where, $\mathbf{P} \equiv \mathbf{I} - \mathbf{p}\mathbf{p}^T\mathbf{N}$ projects onto the direction perpendicular to the slowness, i.e., $\mathbf{P}\mathbf{p} = 0$. The evolution equation for $\mathbf{M}$ follows from (26) through (28) as

$$\begin{aligned} \frac{d\mathbf{M}}{d\tau} = & -\mathbf{M}\mathbf{N}\mathbf{M} - \mathbf{M}\mathbf{N}'\mathbf{p}(\nabla\theta)^T - (\nabla\theta)\mathbf{p}^T\mathbf{N}'\mathbf{M} \\ & + \left[\mathbf{M}\mathbf{N}\mathbf{p} + \frac{1}{2}(\mathbf{p}^T\mathbf{N}'\mathbf{p})\nabla\theta \right] \left[\mathbf{M}\mathbf{N}\mathbf{p} + \frac{1}{2}(\mathbf{p}^T\mathbf{N}'\mathbf{p})\nabla\theta \right]^T - \frac{1}{2}[\mathbf{p}^T\mathbf{N}'\mathbf{p}(\nabla\nabla\theta) + \mathbf{p}^T\mathbf{N}''\mathbf{p}(\nabla\theta)(\nabla\theta)^T]. \end{aligned} \quad (29)$$

Note that this is a non-linear Riccati type of equation, whereas the equations for the matrices $\mathbf{A}$ and $\mathbf{B}$ are linear but coupled.

We now turn to the evolution equation for the scalar amplitude U on a ray, i.e. the transport equation. The terms of order ω in the expansion of the equation of motion (17) imply that U satisfies

$$2(\nabla\phi)^T\mathbf{N}\nabla U - 2\dot{\phi}\dot{U} + \text{tr}[\mathbf{N}\nabla\nabla\phi + \mathbf{N}'\nabla\phi(\nabla\theta)^T]U - \ddot{\phi}U = 0. \quad (30)$$

Along the ray we have $(\nabla\phi)^T\mathbf{N}\nabla U - \dot{\phi}\dot{U} = dU/d\tau$, so that (30) becomes

$$2\frac{dU}{d\tau} + \{\text{tr}[\mathbf{N}\mathbf{M} + \mathbf{N}'\mathbf{p}(\nabla\theta)^T] - \ddot{\phi}\}U = 0. \quad (31)$$

This is clearly a transport equation for U , but it remains to simplify the term in parentheses. In order to do this we differentiate the eiconal equation with respect to t and $\mathbf{x}$, and use the identity $\mathbf{p}^T\mathbf{N}\mathbf{p} = 1$, to obtain respectively,

$$\ddot{\phi} + \mathbf{p}^T\mathbf{N}\nabla\dot{\phi} = 0, \quad (32)$$

$$\nabla\dot{\phi} + \mathbf{p}^T\mathbf{N}\nabla\nabla\phi + \frac{1}{2}(\mathbf{p}^T\mathbf{N}'\mathbf{p})\nabla\theta = 0. \quad (33)$$

Combining these two equations with (27), and using the fact that $\nabla\nabla\phi = \mathbf{M}$ along a ray, we deduce that

$$\text{tr}[\mathbf{N}\mathbf{M} + \mathbf{N}'\mathbf{p}(\nabla\theta)^T] - \ddot{\phi} = \text{tr} \left(\frac{d\mathbf{A}}{d\tau} \mathbf{A}^{-1} \right). \quad (34)$$

Eq. (31) may now be integrated along a ray to give simply

$$U(\tau) = U(0) \left[\frac{\det \mathbf{A}(0)}{\det \mathbf{A}(\tau)} \right]^{1/2}. \quad (35)$$

Hence, $\det \mathbf{A}$ can be interpreted as the anisotropic equivalent of the ray tube area, see Fig. 3.

For reference purposes we note that when the material is homogeneous, i.e. θ is constant, Eqs. (27)–(29) are easily integrated to give $\mathbf{A}(\tau) = \mathbf{A}(0) + \tau\mathbf{N}\mathbf{P}\mathbf{B}(0)$, $\mathbf{B}(\tau) = \mathbf{B}(0)$, and hence $\mathbf{M}(\tau) = \mathbf{M}(0)[\mathbf{I} + \tau\mathbf{N}\mathbf{P}\mathbf{M}(0)]^{-1}$. The amplitude then becomes, from (35), $U(\tau) = U(0)[\det(\mathbf{I} + \tau\mathbf{N}\mathbf{P}\mathbf{M}(0))]^{-1/2}$. It is common for $\mathbf{M}(0)$ to be rank deficient, for example, a propagating wavefront is defined by a single scalar, the wavefront curvature. A Gaussian beam can be defined as a propagating wavefront with complex valued curvature. In either case, $\mathbf{M}(0)$ is defined by

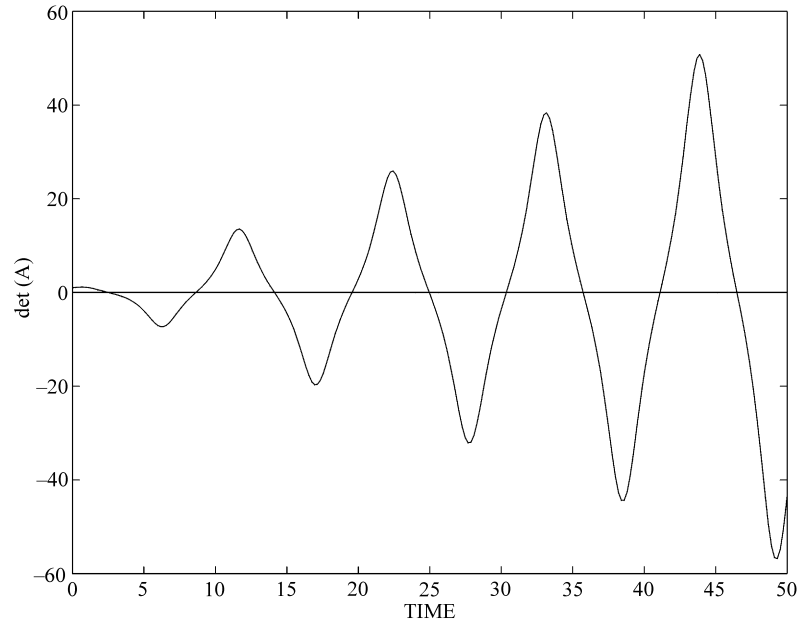


Fig. 3. The ray tube area parameter $\det \mathbf{A}(\tau)$ for the central ray of the horizontally propagating group in Fig. 2, for $0 \leq \tau \leq 50$. Zeros of $\det \mathbf{A}$ indicate intersections of the ray with caustic surfaces. Simple ray theory would predict infinite amplitudes at these points.

a single scalar and it projects onto the direction transverse to the propagation direction. In that case we have simply $\mathbf{M}(\tau) = \mathbf{M}(0)\{1 + \tau \operatorname{tr}[\mathbf{NPM}(0)]\}^{-1}$, and the amplitude is

$$U(\tau) = U(0)\{1 + \tau \operatorname{tr}[\mathbf{NPM}(0)]\}^{-1/2}, \quad \text{uniform medium.} \quad (36)$$

We include this special but simple case to show how the general theory reproduces the $\tau^{-1/2}$ decay expected for a uniform medium when $\tau \rightarrow \infty$.

A ray is specified by initial data for the ray position $\mathbf{x}(0)$ and the phase vector $\mathbf{p}(0)$. In addition, the initial values of the matrices $\mathbf{A}$ and $\mathbf{B}$ must be given, or equally well, $\mathbf{M}$ and one of either $\mathbf{A}$ or $\mathbf{B}$. For example, $\mathbf{A}(0) = \mathbf{I}$ and $\mathbf{B}(0) = 0$ corresponds to an initially plane wavefront. Conversely, at the other extreme, a point source at $\mathbf{x}(0)$ is defined by $\mathbf{A}(0) = 0$ and $\mathbf{B}(0) = \mathbf{I}$. The solutions for the point source and the plane wave yield two linearly independent pairs of $\mathbf{A}$ and $\mathbf{B}$ which can be used to represent an arbitrary initial wavefront configuration. Note that the term $\det \mathbf{A}(0)$ in the amplitude, see (35), should be set to some arbitrary non-zero value for the point source, since the initial amplitude is infinite according to this formula. Ray theory is known to produce singularities at focal points and caustics where the ray tube area shrinks to zero. These geometrical singularities are not physically realistic and can be corrected using boundary layer corrections, but at the cost of great complexity. One way to obviate the need for boundary layer theory is to represent the source as a sum, or integral, of Gaussian beams, defined next.

All of the results derived in this section can now be combined to give a paraxial approximation to the wavefield near the point $\mathbf{x}(\tau)$ on a ray. At this point the phase ϕ is equal to its initial value along the ray. We wish to find the wave field at some nearby position and time, $\mathbf{x}(\tau) + \Delta \mathbf{x}$ and $\tau + \Delta t$, which may or may not lie on the ray. At this point the amplitude may be approximated, to leading order, by the amplitude on the ray, $U(\tau)$, which is given by (35). However, the phase must be modified from its value on the ray because of the large prefactor, ω , in the exponential. Thus

$$v = U(0) \left[\frac{\det \mathbf{A}(0)}{\det \mathbf{A}(\tau)} \right]^{1/2} \exp\{i\omega(\phi(0) + \Delta\phi)\}, \quad (37)$$

where the phase can be approximated by a second order Taylor series using the derived expressions for $\nabla\phi = \mathbf{p}$, $\dot{\phi} = -1$, $\nabla\nabla\phi = \mathbf{M}$, and $\nabla\dot{\phi}$ and $\dot{\phi}$ can be found from (32) and (33). The paraxial approximation of the phase is therefore

$$\Delta\phi = \mathbf{p}^T(\tau)\Delta\mathbf{x} - \Delta t - \frac{1}{2}(\mathbf{p}^T\mathbf{N}'\mathbf{p})(\nabla\theta)^T(\Delta\mathbf{x} - \mathbf{c}\Delta t)\Delta t + \frac{1}{2}(\Delta\mathbf{x} - \mathbf{c}\Delta t)\mathbf{M}(\tau)(\Delta\mathbf{x} - \mathbf{c}\Delta t). \quad (38)$$

Eqs. (37) and (38) together define the paraxial approximation about the ray. Note that it involves both $\mathbf{M}$ and $\mathbf{A}$, we cannot use $\mathbf{M}$ alone.

A Gaussian wave packet is defined by an initial value of $\mathbf{M}$ which has a non-zero imaginary part. In order to be physically meaningful in terms of the paraxial approximation (37), the imaginary part of $\mathbf{M}(0)$ must be non-negative. If it is positive definite then the initial waveform is a localized pulse of energy that decays as a Gaussian in all directions. The subsequent waveform will maintain this structure with the Gaussian envelope changing as the ray position, which defines the centre of the GWP, progresses. It is interesting to note that the terms in the phase which are quadratic in the deviations $\Delta\mathbf{x}$ and Δt depend upon the wave velocity $\mathbf{c}$. The solution as given decays in any direction away from the ray position. However, if the imaginary part of $\mathbf{M}(0)$ is rank degenerate with $\mathbf{c}^T \text{Im } \mathbf{M}(0)\mathbf{c} = 0$ then the initial disturbance decays only in the direction transverse to the initial ray direction $\mathbf{c}$, but not parallel to $\mathbf{c}$. This type of initial disturbance is a limiting case of the wave packet, and is called a Gaussian beam [6,7].

3.3. Approximation of a point source by Gaussian beams

We now apply the previous analysis to approximate the SH-wavefield due to a time harmonic point source in the transversely isotropic granular medium. The idea is to first replace the point source by an equivalent distribution of points sources, and then to move these into complex space. The propagation of each of these complex sources can then be achieved using ray theory for Gaussian beams, and their sum provides a uniform solution everywhere in space. The present analysis is an extension of a similar procedure for isotropic media discussed in [4].

We begin with the exact solution for a point source in a uniform but anisotropic medium, satisfying (see (17))

$$\nabla^T \mathbf{N}(\theta_0) \nabla v + \omega^2 v = -\delta(\mathbf{x}), \quad (39)$$

for some constant θ_0 . Introduce scaled coordinates Z and X defined by

$$Z = (1 + \gamma)^{-1/2} \mathbf{e}_{\theta_0}^T \mathbf{x}, \quad X = \gamma^{-1/2} \mathbf{e}_{\theta_0+\pi/2}^T \mathbf{x}, \quad (40)$$

where the unit vector $\mathbf{e}_{\theta_0}$ coincides with the crystal symmetry axis, $\mathbf{e}_{\bar{z}}$, and $\mathbf{e}_{\theta_0+\pi/2} = \mathbf{e}_{\bar{x}}$, that is, $\mathbf{e}_{\theta} = \cos \theta \mathbf{e}_{\bar{z}} + \sin \theta \mathbf{e}_{\bar{x}}$. Hence, Eq. (39) becomes

$$\frac{1}{R} \frac{\partial}{\partial R} R \frac{\partial v}{\partial R} + \omega^2 v = -[(1 + \gamma)\gamma]^{-1/2} \frac{\delta(R)}{2\pi R}, \quad R = (X^2 + Z^2)^{1/2}, \quad (41)$$

and the solution is

$$v = [(1 + \gamma)\gamma]^{-1/2} \frac{i}{4} H_0^{(1)}(\omega R), \quad (42)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

Define the unphysical position vector $\mathbf{X} = Z\mathbf{e}_{\theta_0} + X\mathbf{e}_{\theta_0+\pi/2}$ and let $V(\mathbf{X}) = v(\mathbf{x})$. Then by Huyghen's principle of virtual sources, we can replace the point source at $\mathbf{X} = 0$ by a uniform distribution of point sources on a circle of arbitrary radius ϵ in $\mathbf{X}$ -space and centred at the origin, see [4] for further details. Thus

$$V(\mathbf{X}) = \frac{1}{2\pi J_0(\omega\epsilon)} \int_0^{2\pi} V(\mathbf{X} - \epsilon \mathbf{e}_\beta) d\beta. \quad (43)$$

The corresponding point source solution in physical variables z and x is therefore

$$v(\mathbf{x}) = \frac{i[(1+\gamma)\gamma]^{-1/2}}{8\pi J_0(\omega\epsilon)} \int_0^{2\pi} H_0^{(1)}(\omega\Phi) d\beta, \quad (44)$$

where

$$\Phi = ([(1+\gamma)^{-1/2}(z \cos \theta_0 + x \sin \theta_0) - \epsilon \cos \beta]^2 + [\gamma^{-1/2}(x \cos \theta_0 - z \sin \theta_0) - \epsilon \sin \beta]^2)^{1/2}. \quad (45)$$

Note that the representation (43) is only valid outside the circle $R = |\epsilon|$, and hence the field defined by (44) is a valid solution to the homogeneous wave equation only outside the ellipse

$$(1+\gamma)^{-1}(z \cos \theta_0 + x \sin \theta_0)^2 + \gamma^{-1}(x \cos \theta_0 - z \sin \theta_0)^2 = \epsilon^2. \quad (46)$$

The virtual sources in (44) are located on the same ellipse in real space. In summary, the integral in (44) is an exact representation of the point source solution in a uniform but anisotropic medium for arbitrary positive ϵ .

Now suppose that ϵ is complex valued, $\epsilon = \epsilon_1 + i\epsilon_2$ with $\epsilon_2 > 0$. Then the representation (43) is still formally valid, although its region of validity is not immediately apparent. First, we need to define the branch of the square root function for Φ . For simplicity, let $\epsilon_1 = 0$, then

$$\Phi = [R^2 - \epsilon_2^2 - 2i\epsilon_2(Z \cos \beta + X \sin \beta)]^{1/2}. \quad (47)$$

We define the branch cut as the line segment in $\mathbf{X}$ -space $\{\mathbf{X} : \mathbf{X} = s\epsilon_2\mathbf{e}_{\beta+\pi/2}, -1 < s < 1\}$. Along this line segment the phase Φ is purely imaginary and of equal magnitude but of opposite sign for two neighbouring points on either side of the cut. The branch cut is then completely specified by the choice $-i\Phi < 0$ on the side directed towards $\mathbf{e}_\beta$ and $\text{Re } \Phi > 0$ for all $\mathbf{X}$ not on the cut. In the high frequency limit ($\omega\epsilon_2 \gg 1$) the solution is largest at the centre of the side facing $\mathbf{e}_\beta$, and exponentially smaller elsewhere on the cut. We can easily generalize these results to the case of arbitrary ϵ with $\epsilon_2 > 0$. By considering the placement of the branch cuts in real space for each value of β it follows that the representation (44) is valid only in the region exterior to the ellipse

$$(1+\gamma)^{-1}(z \cos \theta_0 + x \sin \theta_0)^2 + \gamma^{-1}(x \cos \theta_0 - z \sin \theta_0)^2 = \epsilon\epsilon^*, \quad (48)$$

where $*$ denotes the complex conjugate. It follows from the definition of the cut that along the line defining the forward, or central ray, $\mathbf{x} = R\mathbf{e}_\beta$, $R > \epsilon_1$, we have $\Phi = R - \epsilon$, and so the imaginary part of the phase remains constant along this line. At points adjacent to this line, i.e. $X = R \sin(\beta + \delta)$, $z = R \cos(\beta + \delta)$, where $\delta \ll 1$, we have

$$\Phi = R - \epsilon + \frac{2\epsilon}{R - \epsilon} \sin^2 \frac{\delta}{2}. \quad (49)$$

Therefore the imaginary part of the phase increases as the point moves away from the central line. In the farfield, where $H_0^{(1)}$ can be asymptotically approximated as an exponential, this causes a Gaussian decay in amplitude in the direction orthogonal to the central ray direction $\mathbf{e}_\beta$. Thus, it becomes asymptotically equivalent to a Gaussian beam.

Each element in the representation integral (44) therefore corresponds to a Gaussian beam with its central ray directed in physical $\mathbf{x}$ -space in the direction of $(1+\gamma)^{1/2} \cos \beta \mathbf{e}_{\theta_0} + \gamma^{1/2} \sin \beta \mathbf{e}_{\theta_0+\pi/2}$. This identification is an asymptotic approximation, based on the premise that $\omega \text{Im } \epsilon \gg 1$. However, it allows us the possibility to extend a local approximation of the field of the point source to the far field. The means for achieving this are already at our disposal: the ray equations with appropriate initial conditions for each beam. Specifically, the initial starting point of each ray is the origin. Initial values of Φ , $\mathbf{p}$ and $\mathbf{M}$ for each beam then follow by evaluating Φ and its first two derivatives at the origin. Thus, $\Phi(0, \beta) \equiv \Phi(\tau, \beta)|_{\tau=0} = -\epsilon$, and the initial phase vector is

$$\mathbf{p}(0, \beta) = (1+\gamma)^{-1/2} \cos \beta \mathbf{e}_{\theta_0} + \gamma^{-1/2} \sin \beta \mathbf{e}_{\theta_0+\pi/2}. \quad (50)$$

The initial ray tube area matrix can be taken as $\mathbf{A}(0, \beta) = \mathbf{I}$, and $\mathbf{M}(0, \beta) = \mathbf{B}(0, \beta)$ follows from the second derivatives of Φ as

$$\mathbf{M}(0, \beta) = (-\epsilon)^{-1} \mathbf{p} \left(0, \beta + \frac{\pi}{2} \right) \otimes \mathbf{p} \left(0, \beta + \frac{\pi}{2} \right). \quad (51)$$

The initial amplitude of each beam is the same, and equal to

$$U(0, \beta) = [(1 + \gamma)\gamma]^{-1/2} \frac{i}{8\pi} \frac{H_0^{(1)}(-\omega\epsilon)}{J_0(\omega\epsilon)}. \quad (52)$$

The preceding algorithm enables us to generate a uniform approximation to the point source field in a smoothly varying medium. Each beam is propagated and its influence at a field point determined using the paraxial approximation, (37) and (38). The contribution from all beams is then summed in accordance with the integral (44). The summation procedure follows that described by White et al. [7] for the isotropic wave equation. This reference also provides a basis for estimating the accuracy of the Gaussian beam summation in specific configurations.

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References

- [1] I.D. Abrahams, G.R. Wickham, The propagation of elastic waves in a certain class of inhomogeneously anisotropic materials. I. The refraction of a horizontally polarized shear wave source, *Proc. R. Soc. Lond. A* 436 (1992) 449–478.
- [2] G.R. Wickham, I.D. Abrahams, The refraction of ultrasound in grainy and inhomogeneous fibre reinforced materials, *Proc. I.O.A.* 13 (2) (1991) 125–135.
- [3] A.N. Norris, A theory of pulse propagation in anisotropic elastic solids, *Wave Motion* 9 (1987) 509–532.
- [4] A.N. Norris, Complex point source representation of real point sources, *J. Opt. Soc. Am. A* 3 (1989) 2005–2010.
- [5] M.E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, New York, 1981.
- [6] A.N. Norris, B.S. White, J.R. Schrieffer, Gaussian wave packets in inhomogeneous media with curved interfaces, *Proc. R. Soc. Lond. A* 412 (1987) 93–123.
- [7] B.S. White, A.N. Norris, A. Bayliss, R. Burridge, Some remarks on the Gaussian beam summation method, *Geophys. J. R. Astr. Soc.* 89 (1987) 579–636.