Flexural edge waves and Comments on "A new bending wave solution for the classical plate equation" [J. Acoust. Soc. Am. 104, 2220–2222 (1998)]

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A brief review is presented of the theory of flexural edge waves, first predicted in 1960 by Yu K. Konenkov using Kirchhoff plate theory. It is demonstrated that the flexural edge wave is also predicted by Mindlin's plate theory, and that the prediction agrees with measured data. It is noted that the edge wave was erroneously presented as a new type of bending wave solution in a recently published paper in this journal. © *2000 Acoustical Society of America.* [S0001-4966(00)01301-1]

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The existence of a flexural wave guided by the free edge of a semi-infinite isotropic elastic thin plate was first demonstrated by Konenkov[1] in 1960. The wave has properties analogous to a Rayleigh wave on an elastic half-space, in that it decays exponentially with distance from the edge. This result was apparently not very widely known in Western scientific circles, not an uncommon situation in the 1960's and 1970's, because it was rediscovered concurrently and independently by Sinha[2] and by Thurston and McKenna,[3] both published in 1974 in ''western'' journals. The first author of this Letter was likewise ignorant of Konenkov's original contribution when he proved that the edge wave also exists on anisotropic plates.[4]

In a recent article[5] in this journal, the now-classical result for the flexural edge wave was again presented as new. The author cited a paper of McKenna *et al.*[6] but apparently did not know the Thurston and McKenna paper,[3] published shortly after and in the same journal, and was also unaware of the sizeable literature on the subject.

The propagation of flexural waves guided by the tip of a wedge or the free end of a ridge on a substrate was studied in some depth by various groups in the 1970's. In 1974 Sinha,[2] using the classical plate theory, obtained an explicit expression for the speed of the flexural edge wave. At about the same time, McKenna *et al.*[6] derived the dispersion relation for a flexural plate edge wave by taking the limit of a wedge with zero internal angle. Thurston and McKenna[3] subsequently discussed in detail the existence and behavior of this wave, and obtained the same wave speed as Sinha.[2] The flexural edge wave speed was also derived from the modes of a thin plate of finite width by taking the limit in which the width becomes infinite.[7]

The classical Kirchhoff plate theory predicts a speed for the edge wave which is in constant proportion to the flexural wave speed. The constant of proportionality is independent of the frequency and depends only on the Poisson's ratio, being slightly less than unity and equal to unity when the Poisson's ratio vanishes. As noted by Thurston and McKenna,[3] this equality reflects the fact that a flexural wave traveling parallel to the edge of a thin plate of zero Poisson's ratio gives no bending moment or shear and hence automatically satisfies the free edge conditions of the classical plate theory.

The edge wave solution of the classical plate theory has been studied by many since then, for instance Refs. 8, 9, and is discussed in at least two monographs.[10,11] The speed of the flexural edge wave was measured and agrees well with finite element calculations,[9] and both the measured speed and the FEM calculations are less than the values predicted by the Kirchhoff plate theory. An explicit expression can also be obtained for the edge wave speed on an anisotropic thin plate.[4] The analog of a flexural Stoneley wave propagating along the line of contact of two joined semi-infinite plates and decaying exponentially in the perpendicular direction has been studied by Zilbergleit and Suslova.[12] Kouzov *et al.*[13] have considered the more general configuration of a starlike nodal junction of thin plates.

Two of the present authors have recently established the existence of edge supported flexural waves on fluid loaded thin plates.[14] However, submerged plates support such waves only under very light fluid loading conditions: for instance, thin plates of aluminum, brass or Plexiglas will support edge waves in air, but not in water.[14]

Previous studies of the flexural edge wave on a semi-infinite plate *in vacuo* have used either the Kirchhoff plate theory or some far more sophisticated analysis based on the full equations, e.g., FEM. We now demonstrate that the essential characteristics of the edge wave are captured by the Mindlin plate theory[15] without much additional effort beyond that required for the Kirchhoff theory. To the best of our knowledge, the edge wave has not been previously analyzed in the context of Mindlin plate theory. The Mindlin theory for flexural waves incorporates shear-deformation and rotary-inertia, two effects that are absent from the classical Kirchhoff theory. According to Mindlin's theory the displacement at (x,y,z) is $\mathbf{u}=z\boldsymbol{\psi}(\mathbf{x},t)+w(\mathbf{x},t)\mathbf{e}_z$, where $\mathbf{x}=(x,y)$ is the 2-D position on the central plane of the plate, z is the transverse coordinate, with $z=0$ the center plane of the plate and $\boldsymbol{\psi}=(\psi_x,\psi_y)$ is the in-plane vector of rotations.

The equations of motion in the absence of external loading are

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=\frac{\rho h^3}{12}\frac{\partial^2\psi_x}{\partial t^2}, \tag{1}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=\frac{\rho h^3}{12}\frac{\partial^2\psi_y}{\partial t^2}, \tag{2}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=\rho h\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

where ρ is the density and h the thickness. The moments M_x, M_y, and M_{xy}, and the shear forces Q_x and Q_y, are

$$\begin{aligned}
&M_x=D\left(\frac{\partial\psi_x}{\partial x}+\nu\frac{\partial\psi_y}{\partial y}\right),\quad M_y=D\left(\frac{\partial\psi_y}{\partial y}+\nu\frac{\partial\psi_x}{\partial x}\right),\\
&M_{xy}=\frac{1}{2}D(1-\nu)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right),\quad Q_x=\alpha^2\mu h\left(\frac{\partial w}{\partial x}+\psi_x\right),\\
&Q_y=\alpha^2\mu h\left(\frac{\partial w}{\partial y}+\psi_y\right),
\end{aligned} \tag{4}$$

where $D=Eh^3/12(1-\nu^2)$ is the bending stiffness, ν is Poisson's ratio, μ is the shear modulus, and $E=2(1+\nu)\mu$ is the Young's modulus. The shear modulus is modified by the factor α^2 in order to better approximate the shear forces, and α may be chosen according to different criteria, but normally, $\alpha^2\leqslant 1$.[16]

Now let the plate occupy $-\infty<x<\infty$, $y\geqslant 0$ with a free edge on $y=0$. We consider the time harmonic *ansatz* of frequency ω:

$$\begin{aligned}
&w(\mathbf{x},t)=\mathrm{Re}[(V_1+V_2)e^{-i\omega t}],\\
&\boldsymbol{\psi}(\mathbf{x},t)=\mathrm{Re}[(\beta_1\nabla V_1+\beta_2\nabla V_2-\mathbf{e}_z\times\nabla V_3)e^{-i\omega t}],
\end{aligned} \tag{5}$$

where V_1, V_2 and V_3 are given by[15]

$$V_j(\mathbf{x})=A_j e^{i\xi x-\sqrt{\xi^2-k_j^2}y},\quad j=1,2,3. \tag{6}$$

The three bulk wave numbers k_1, k_2 and k_3 and the numbers β_1 and β_2 follow by direct substitution into Eqs. (1)–(4) and are given by

$$\begin{aligned}
&k_1^2+k_2^2=k_S^2+k_P^2,\quad k_1^2k_2^2=k_S^2k_P^2-k_f^4,\quad k_3^2=\alpha^2k_1^2k_2^2/k_P^2,\\
&\beta_j=-1+k_S^2/k_j^2,\quad j=1,2,
\end{aligned} \tag{7}$$

where

$$k_S=\frac{\omega}{\alpha}\sqrt{\rho/\mu},\quad k_P=\omega\sqrt{(1-\nu^2)\rho/E},\quad k_f^2=\omega\sqrt{h\rho/D}. \tag{8}$$

By applying the boundary conditions appropriate to a free edge: vanishing shear force, bending moment and twisting moment, that is, $Q_y(x,0)=0$, $M_y(x,0)=0$, and $M_{xy}(x,0)=0$, respectively, three equations are obtained for A_j, $j=1,2,3$. Setting the resulting determinant to zero, yields, after some simplification, the following dispersion relation for the wave number ξ of the edge wave:

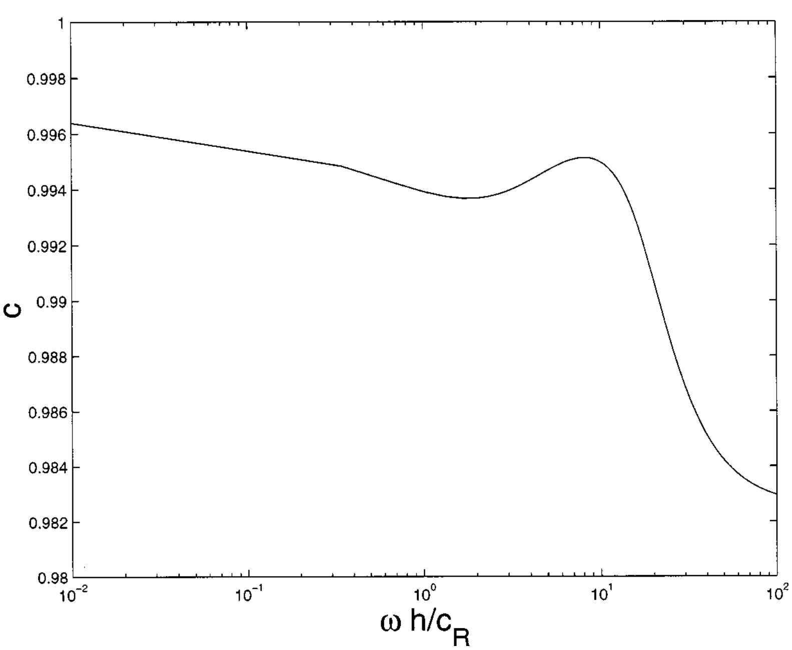

FIG. 1. The dimensionless speed $c=k_1/\xi$ according to Mindlin plate theory plotted against the dimensionless frequency $k_R h=\omega h/c_R$. The free parameters are ν=0.39, $\alpha=c_R/c_T$. The value at $k_R h=0$ is given by Eq. (10).

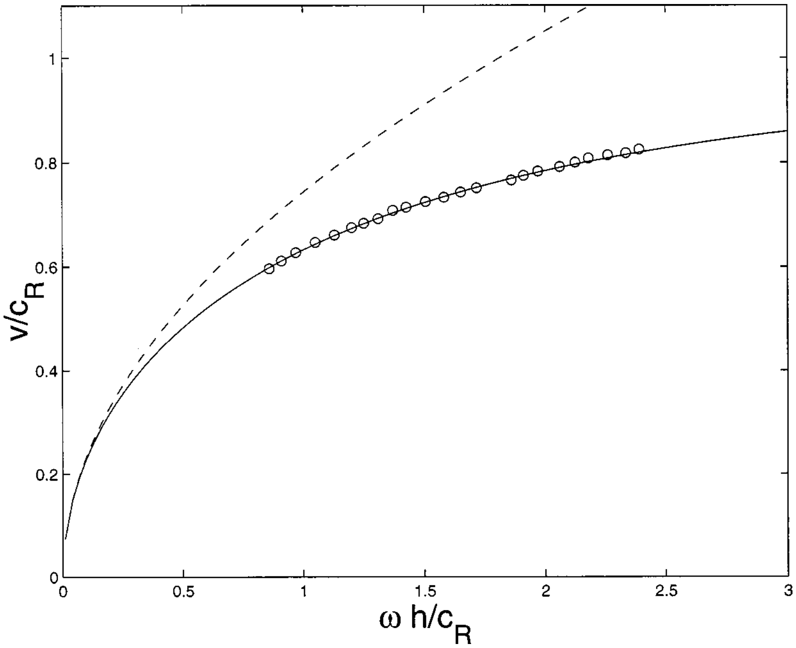

FIG. 2. The edge wave speed v relative to the Rayleigh wave speed c_R according to Mindlin theory (solid curve) and the Kirchhoff theory (dashed curve). The circles represent data from Lagasse and Oliner (Ref. 9).

$$\begin{vmatrix} k_1^2 & k_2^2 & k_3^2 \\ k_S^2 & k_S^2 & 2\xi^2 \\ \dfrac{[(1-\nu)\xi^2-k_1^2](k_S^2-k_1^2)}{\sqrt{\xi^2-k_1^2}} & \dfrac{[(1-\nu)\xi^2-k_2^2](k_S^2-k_2^2)}{\sqrt{\xi^2-k_2^2}} & 2(1-\nu)\xi^2\sqrt{\xi^2-k_3^2} \end{vmatrix} = 0. \tag{9}$$

Let $k_1^2 > k_2^2$, then k_1 is the analog of the classical flexural wave number k_f in the Mindlin theory. The speed of the flexural edge wave relative to the Mindlin flexural wave is $c = k_1/\xi$. Numerical experimentation shows that there is always a root of the dispersion relation for $c \leqslant 1$, with equality again when $\nu=0$. The fact that $c=1$ for $\nu=0$ is a consequence of the fact that the flexural wave automatically satisfies the free edge conditions with no Poisson contraction effect. For nonzero ν the Poisson effect comes into play and the edge wave solution has a small but nonzero component from the other two Mindlin plate waves, and its speed is subsonic relative to the propagating flexural wave. For instance, Fig. 1 shows a plot of c versus frequency. It may be verified by direct calculation that the root of Eq. (9) reduces to the value predicted by Thurston and McKenna[3] in the limit as the dimensionless frequency $k_S h$ approaches zero, that is,

$$c = [-(1-\nu)(1-3\nu) + 2(1-\nu) \times (1-2\nu+2\nu^2)^{1/2}]^{1/4} \quad \text{as } k_S h \to 0. \tag{10}$$

The high frequency limit of the edge wave is of interest. It may be shown by taking the correct limits that at high frequencies ξ is the unique positive solution of

$$(2\xi^2-k_T^2)^2 - 4\xi^2(\xi^2-k_T^2)^{1/2}(\xi^2-k_P^2)^{1/2} = 0 \quad \text{as } k_S h \to \infty, \tag{11}$$

where $k_T = k_S/\alpha$ is the wave number of bulk transverse waves, i.e., $k_T = \omega/c_T$ where $c_T = \sqrt{\mu/\rho}$. The form of Eq. (11) is precisely the same as the equation for the Rayleigh wave on the surface of a half-space, except that the longitudinal wave number k_P in Eq. (11) is the wave number according to membrane theory; see Eq. (8). This has speed $c_P = \sqrt{E/[(1-\nu^2)\rho]}$ as compared with the larger speed of a bulk longitudinal wave, $c_L = c_P(1-\nu)/\sqrt{1-2\nu}$. The membrane Rayleigh wave speed c_R^* is the same as the true Rayleigh wave speed for a material with ν replaced by the value $\nu/(1+\nu)$, leading to a slower speed. Hence, the high frequency limit of the edge wave according to Mindlin's theory is the Rayleigh ''surface'' wave traveling along the narrow surface of a thin plate. The meaning and interpretation of this is unclear since one should be cautious of using the high frequency limit of Mindlin's theory for purposes other than intended. The original goal of Mindlin was to obtain the high frequency asymptote of the fundamental antisymmetric mode of a plate, which is itself the true Rayleigh wave speed c_R if α is chosen as $\alpha = c_R/c_T$. The occurrence of the ''membrane'' Rayleigh wave as the asymptote is therefore all the more interesting since it drops out in the limit independent of the value of α.

Finally, Fig. 2 compares the Mindlin edge wave speed with the Kirchhoff edge wave speed, and also with measured data of Lagasse and Oliner.[9] The speeds are normalized with respect to the Rayleigh wave speed. The Poisson's ratio was chosen as $\nu=0.39$ so that the Kirchhoff curve agrees with a similar curve in Ref. 9, and $\alpha=c_R/c_T$. We note that a finite element computation by Lagasse and Oliner[9] produced a curve essentially overlying the Mindlin curve of Fig. 2, within the accuracy of the data read from Ref. 9.

Note added in proof. A referee pointed out that Ambartsumyan and Belubekyan[17] came to a conclusion opposite to ours: that the more exact plate theories of Timoshenko–Mindlin type predict that localized edge wave do not exist. After examining their paper, we conclude that the analysis of Ambartsumyan and Belubekyan is in error, and that the edge wave is predicted by Mindlin plate theory.

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