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# Stress invariance and redundant moduli in three-dimensional elasticity

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A three-dimensional framework is established for generating invariant stress configurations and associated shifts in the elastic compliance. Under these shifts the stress throughout an elastic body is unaltered, while the compatibility equations for the strain are automatically satisfied. The types of invariant stress fields and translations of the compliance identified here generalize the results of Cherkaev, Lurie & Milton (CLM) for planar elasticity. The key to the classification is the partitioning of the fourth-order compliance tensor into symmetric and antisymmetric components. The CLM theorem and its generalization are closely linked to the six-dimensional antisymmetric part of the compliance, and several examples are given of stress invariance under shifts of these elements of the compliance tensor.

**Keywords: elasticity; anisotropy; stress; compatibility; compliance; invariance**

## 1. Introduction

Stress invariance under arbitrary changes in elastic moduli is a relatively new but significant concept. Cherkaev *et al.* (1992) demonstrated that the stress in a planar elastic material is unchanged if one single combination of elastic moduli is allowed to vary. The specific result of Cherkaev *et al.* (1992), known as the CLM theorem, is that the stress and the traction are invariant to equal but opposite modifications of the planar bulk and shear moduli by a linear shift $\lambda$. Cherkaev *et al.* (1992) introduced the notion of equivalent elastic materials, which are characterized by an arbitrary value of this free parameter. For instance, by considering the equivalent class of materials with shifted compliances it is clear that even if the planar bulk and shear moduli are spatially varying, the *effective* compliances of the equivalent composite are simply shifted by $\lambda$ and $-\lambda$. This has important implications for the effective properties of planar composites. For instance, it means that the effective Young's modulus of a planar body with holes is independent of the Poisson's ratio of the material.

The CLM result was subsequently developed by many others (Thorpe & Jasiuk 1992; Dundurs & Markenscoff 1993; Jasiuk *et al.* 1994; Moran & Gosz 1994; Chen 1995*a*, *b*; Ostoja-Starzewski & Jasiuk 1995; Zheng & Hwang 1996, 1997; Dundurs & Jasiuk 1997; He 1997, 1998). For instance, Dundurs & Markenscoff (1993) generalized the CLM result to include equivalent materials with an affine (linear) shift in the same modulus. The major thrust of these studies concern the implications for planar composites and the connection of the CLM theorem to previously known results in two-dimensional elasticity. Thus, the fact that the effective properties of a planar two-phase composite material depend upon only two non-dimensional combinations of the moduli (the Dundurs constants) is closely related to the CLM theorem (Thorpe & Jasiuk 1992; Dundurs & Markenscoff 1993). The CLM result has also been extended to include other types of elastic effects, such as anisotropy (Moran & Gosz 1994; He 1997), Cosserat elasticity (Ostoja-Starzewski & Jasiuk 1995) and piezoelectricity (Chen 1995*a*, *b*).

The stress invariance effect has only been studied within the context of planar elasticity, a subset of elasticity in which the stress and strain are two-dimensional objects, the classic examples being the plane-strain and plane-stress problems. Planar elasticity is closely related to, but not quite the same as, two-dimensional elasticity, which is defined as states of stress that are everywhere independent of one coordinate. In this paper we develop a three-dimensional framework for investigating stress invariance. The idea is to find all possible variations in the elastic compliances which do not effect the elastic compatibility equations. The general problem as stated represents six second-order differential restraints on the possible 21 moduli. This difficulty has been resolved by restricting attention to smaller subspaces of moduli, using the decomposition scheme of Backus (1970). The CLM theorem turns out to be related to a single parameter shift in a six-dimensional subspace of the moduli: those which are associated with the totally antisymmetric part of the elastic modulus tensor. By varying other parameters in this subspace of compliances we deduce generalizations of the CLM result, applicable to two-dimensional elasticity. Thus, the three-dimensional theory presented here includes the CLM result as a special case.

## 2. Governing equations

The stress $\boldsymbol{\sigma}$ is a second-order symmetric tensor field, with components $\sigma_{ij}$ relative to an orthonormal basis. The stress solves, or satisfies, the following traction boundary value problem in three dimensions:†

$$\sigma_{ij,j} = f_i \quad \text{in } V, \tag{2.1}$$

$$\sigma_{ij} n_j = \tau_i \quad \text{on } S. \tag{2.2}$$

The strain $\boldsymbol{\epsilon}$ must be derivable as the symmetric part of a displacement gradient, which implies that its components satisfy the six partial differential constraints, or compatibility equations,

$$e_{ikm} e_{jln} \epsilon_{mn,kl} = 0, \tag{2.3}$$

where $e_{ikm}$ are the components of the third-order alternating tensor. The strain is related to the stress by

$$\epsilon_{ij} = C_{ijkl} \sigma_{kl}, \tag{2.4}$$

where $C_{ijkl}$ are the components of the fourth-order compliance tensor. Therefore, substituting from the stress–strain relation (2.4),

$$e_{ipr} e_{jqs} (C_{rskl} \sigma_{kl})_{,pq} = 0. \tag{2.5}$$

† Repeated lower case italic subscripts imply summation over $1, 2$ and $3$, unless indicated otherwise.

We are interested in finding other possible elastic compliances under which the same equations of equilibrium and compatibility are satisfied. Consider the different compliance tensor

$$C'_{ijkl} = C_{ijkl} + C^{(1)}_{ijkl}, \tag{2.6}$$

where the added moduli $C^{(1)}_{ijkl}$ are elasticity tensors, i.e. they satisfy the required symmetries. The same stress state holds in the presence of these shifted moduli of (2.6) if the stress satisfies

$$e_{ipr}e_{jqs}(C^{(1)}_{rskl}\sigma_{kl})_{,pq} = 0. \tag{2.7}$$

One set of permissible moduli is defined by the constraints

$$C^{(1)}_{rskl}\sigma_{kl} = 0, \tag{2.8}$$

which is essentially a set of six equations constraining the 21 moduli. As such, it is not very useful. For instance, it is not possible to deduce the CLM result for plane stress from (2.8). Thus, by changing certain moduli, as the CLM result allows, the stress solution remains unchanged, but the strain might be altered. Equation (2.8) restricts the variation of the moduli to those which leave the strain unchanged.

These considerations indicate that there are two distinct types of invariance that should be distinguished. The first is of the type established by CLM, in which an arbitrary, although restricted, state of stress remains unchanged under the shift in moduli. The possible variation in the moduli do not depend directly upon the stresses. Alternatively, for a given state of stress, we can find particular changes in the moduli for which the compatibility conditions do not change. The second type of invariance is much broader than the first, because it allows that the class of moduli changes are stress related. As an example of the second type of invariance, consider a hydrostatic stress $\boldsymbol{\sigma} = -p_0(\boldsymbol{x})\boldsymbol{I}$ in an isotropic solid, implying $\boldsymbol{\epsilon} = -(p_0\kappa/3)\boldsymbol{I}$, where $\kappa$ is the bulk compliance. The compatibility conditions do not involve the shear compliance and therefore remain unchanged for any value of this modulus. In this case the free modulus does not appear in the compatibility relations.

## 3. Symmetric and antisymmetric tensors

### (*a*) *Definitions*

An elasticity tensor, of which the compliance $C_{ijkl}$ is an example, is a fourth-order tensor with the properties

$$C_{ijkl} = C_{klij}, \quad C_{ijkl} = C_{jikl}. \tag{3.1}$$

A fourth-order elasticity tensor with components $S_{ijkl}$ satisfying (3.1) is *totally symmetric* if in addition it has the property that

$$S_{ijkl} = S_{ikjl}. \tag{3.2}$$

Similarly, an elasticity tensor $A_{ijkl}$ is called *antisymmetric* if

$$A_{ijkl} + A_{ikjl} + A_{ilkj} = 0. \tag{3.3}$$

A totally symmetric tensor is unchanged under any permutation of the indices, and hence we can create a totally symmetric tensor from any elasticity tensor by taking the appropriate average over the indicial permutations (Backus 1970). The remainder is the antisymmetric part of the elasticity tensor. Any elasticity tensor can be uniquely partitioned into the sum of symmetric and antisymmetric tensors: thus,

$$C_{ijkl} = S_{ikjl} + A_{ijkl}, \tag{3.4}$$

where

$$S_{ikjl} = \tfrac{1}{3}(C_{ijkl} + C_{ikjl} + C_{ilkj}), \quad A_{ikjl} = \tfrac{1}{3}(2C_{ijkl} - C_{ikjl} - C_{ilkj}). \tag{3.5}$$

The left member of equation (3.3) expresses the fact that the totally symmetric part of an antisymmetric tensor must be zero.

As an example, the isotropic compliance tensor is

$$C_{ijkl} = \frac{1+\nu}{E} I_{ijkl} - \frac{\nu}{E}\delta_{ij}\delta_{kl}, \tag{3.6}$$

where $I_{ijkl} = \frac{1}{2}12(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is the fourth-order identity tensor and $\nu$ and $E$ are the Poisson's ratio and Young's modulus, respectively. For this case,

$$S_{ikjl} = \frac{1}{3E}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad A_{ikjl} = \frac{1+3\nu}{6E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}). \tag{3.7}$$

### (*b*) *Properties of antisymmetric elasticity tensors*

An arbitrary elasticity tensor satisfying (3.1) possesses at most 21 independent elements, while the additional constraints of (3.2) imply that a symmetric elasticity tensor has at most 15 constants, and conversely, an antisymmetric tensor may be characterized by six or fewer constants. Backus (1970) demonstrated that for every antisymmetric tensor $A_{ikjl}$ there is a unique second-order symmetric tensor $S_{ij}$ such that

$$A_{ijkl} = \tfrac{1}{3}(2\delta_{ij}S_{kl} + 2\delta_{kl}S_{ij} - \delta_{ik}S_{jl} - \delta_{il}S_{jk} - \delta_{jk}S_{il} - \delta_{jl}S_{ik}). \tag{3.8}$$

The symmetric tensor $S_{ij}$ can be found in terms of $A_{ijkl}$ by contracting on the two final indices, to give

$$A_{ijkk} = \tfrac{2}{3}(S_{ij} + s\delta_{ij}), \quad \text{where } s = S_{kk}. \tag{3.9}$$

Contracting on the remaining indices gives $A_{iikk} = \frac{8}{3}s$, and hence

$$S_{ij} = \tfrac{3}{2}A_{ijkk} - \tfrac{3}{8}A_{kkll}\delta_{ij}. \tag{3.10}$$

We now demonstrate that the antisymmetric part of the tensor $C_{ijkl}$ can be represented in the alternative manner

$$A_{ijkl} = -\tfrac{1}{3}(e_{ikm}e_{jln} + e_{jkm}e_{iln})M_{mn}, \tag{3.11}$$

where $\boldsymbol{M}$ is a symmetric second-order tensor. First, we note that $A_{ijkl}$ of equation (3.11) satisfies the relation (3.3) on account of the symmetry of $M_{ij}$. Therefore, the right-hand side of (3.11) is indeed an antisymmetric tensor, and can be represented by a second-order symmetric tensor. On contracting the second pair of indices in (3.11), we obtain

$$A_{ijkk} = \tfrac{2}{3}(M_{ij} - m\delta_{ij}), \quad m = M_{kk}, \tag{3.12}$$

which can be solved for $M_{ij}$:

$$M_{ij} = \tfrac{3}{2}A_{ijkk} - \tfrac{3}{4}A_{kkll}\delta_{ij}. \tag{3.13}$$

Alternatively, we can equate the right members of equations (3.9) and (3.12), to obtain

$$\boldsymbol{M} = \boldsymbol{S} - s\boldsymbol{I} \quad \Leftrightarrow \quad \boldsymbol{S} = \boldsymbol{M} - \tfrac{1}{2}m\boldsymbol{I}. \tag{3.14}$$

This completes the proof of the representation (3.11), and gives the explicit connection between $\boldsymbol{S}$ and $\boldsymbol{M}$, either of which may be considered as basic. However, there are reasons to view $\boldsymbol{M}$ as the more fundamental of the two, because it is directly related to the inverse of the antisymmetric elasticity tensor, if it exists (see Appendix A for details). We note that the relation $(3.14)_1$ can also be expressed as

$$M_{ij} = -e_{ikm}e_{jlm}S_{kl}. \tag{3.15}$$

We may express $\boldsymbol{M}$ and $\boldsymbol{S}$ in terms of the elements of the original elasticity tensor $C_{ijkl}$, using (3.5). First, define the second-order tensors formed by contracting $C_{ijkl}$:

$$C_{ij}^{(1)} = C_{ijkk}, \quad C_{ij}^{(2)} = C_{ikjk}. \tag{3.16}$$

Then equations $(3.5)_2$ and (3.12) imply

$$\boldsymbol{C}^{(1)} - \boldsymbol{C}^{(2)} = \boldsymbol{M} - m\boldsymbol{I} \quad \Leftrightarrow \quad \boldsymbol{M} = \boldsymbol{C}^{(1)} - \boldsymbol{C}^{(2)} - \tfrac{1}{2}([\boldsymbol{C}^{(1)} - \boldsymbol{C}^{(2)}]\cdot\boldsymbol{I})\boldsymbol{I}. \tag{3.17}$$

Hence,†

$$\boldsymbol{M} = \begin{bmatrix} c_{44} - c_{23} & c_{36} - c_{45} & c_{25} - c_{46} \\ c_{36} - c_{45} & c_{55} - c_{13} & c_{14} - c_{56} \\ c_{25} - c_{56} & c_{14} - c_{56} & c_{66} - c_{12} \end{bmatrix}. \tag{3.18}$$

Let

$$a = c_{44} - c_{23}, \quad b = c_{55} - c_{13}, \quad c = c_{66} - c_{12},$$
$$d = c_{14} - c_{56}, \quad e = c_{25} - c_{46}, \quad f = c_{36} - c_{45},$$

then

$$\boldsymbol{M} = \begin{bmatrix} a & f & e \\ f & b & d \\ e & d & c \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \tfrac{1}{2}(a-b-c) & f & e \\ f & \tfrac{1}{2}(b-c-a) & d \\ e & d & \tfrac{1}{2}(c-a-b) \end{bmatrix}. \tag{3.19}$$

For example, the isotropic compliance of equation (3.6) gives

$$\boldsymbol{M} = \frac{1+3\nu}{2E}\boldsymbol{I}$$

for isotropic materials.

† The Voigt notation for fourth-order elasticity tensors is used here: $c_{IJ}$ replaces $C_{ijkl}$ where $I$ and $J$ take the values $1, 2, 3, 4, 5, 6$, corresponding to the indicial pairs $11, 22, 33, 23, 13$ and $12$, respectively.

It is useful to document the reverse relationship between the antisymmetric fourth-order tensor and its symmetric second-order tensor. Thus, for a given second-order symmetric tensor $\boldsymbol{M}$ with components given by (3.19), the antisymmetric elasticity tensor $\boldsymbol{A}$ of equation (3.11) is

$$\boldsymbol{A} = -\tfrac{1}{3}\begin{bmatrix} 0 & 2c & 2b & -2d & 0 & 0 \\ 2c & 0 & 2a & 0 & -2e & 0 \\ 2b & 2a & 0 & 0 & 0 & -2f \\ -2d & 0 & 0 & -a & f & e \\ 0 & -2e & 0 & f & -b & d \\ 0 & 0 & -2f & e & d & -c \end{bmatrix}. \tag{3.20}$$

In general, for any given symmetric second-order tensor $\boldsymbol{\pi}$ we define the antisymmetric fourth-order elasticity tensor $\boldsymbol{P}(\boldsymbol{\pi})$ by

$$P_{ijkl}(\boldsymbol{\pi}) = -\tfrac{1}{2}(e_{ikm}e_{jln} + e_{jkm}e_{iln})\pi_{mn}. \tag{3.21}$$

Thus, for instance, equation (3.11) is

$$\boldsymbol{A} = \tfrac{2}{3}\boldsymbol{P}(\boldsymbol{M}). \tag{3.22}$$

By comparing the equivalent forms (3.8) and (3.11), and using (3.14), we deduce the identity,

$$P_{ijkl}(\boldsymbol{S} - s\boldsymbol{I}) = \tfrac{1}{2}(2S_{ij}\delta_{kl} + 2S_{kl}\delta_{ij} - S_{ik}\delta_{jl} - S_{il}\delta_{jk} - S_{jk}\delta_{il} - S_{jl}\delta_{ik}), \tag{3.23}$$

from which an explicit expansion of the antisymmetric operator $\boldsymbol{P}$ follows:

$$P_{ijkl}(\boldsymbol{\pi}) = \tfrac{1}{2}(2\pi_{ij}\delta_{kl} + 2\pi_{kl}\delta_{ij} - \pi_{ik}\delta_{jl} - \pi_{il}\delta_{jk} - \pi_{jk}\delta_{il} - \pi_{jl}\delta_{ik} \\ + \pi(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})). \tag{3.24}$$

We note the following identity: for any two symmetric second-order tensors, $\boldsymbol{\pi}$ and $\boldsymbol{\chi}$,

$$P_{ijkl}(\boldsymbol{\pi})\chi_{kl} = P_{ijkl}(\boldsymbol{\chi})\pi_{kl} \quad \Leftrightarrow \quad \boldsymbol{P}(\boldsymbol{\pi})\boldsymbol{\chi} = \boldsymbol{P}(\boldsymbol{\chi})\boldsymbol{\pi}. \tag{3.25}$$

The inverse of an antisymmetric elasticity tensor $\boldsymbol{A}$ is closely related to the inverse of the fundamental matrix $\boldsymbol{M}$ according to the following.

**Result 3.1.** The inverse tensor $\boldsymbol{P}^{-1}(\boldsymbol{\pi})$ exists iff $\boldsymbol{\pi}^{-1}$ exists, and has the explicit form

$$P^{-1}_{ijkl}(\boldsymbol{\pi}) = (\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk} - \pi_{ij}\pi_{kl})/(2\det\boldsymbol{\pi}). \tag{3.26}$$

The proof of this identity is given in Appendix A.

Finally, we note that the elements of $C_{ijkl}$ occurring in the matrix $\boldsymbol{M}$ of (3.18) also occur in six elements of the totally symmetric fourth-order tensor $S_{ijkl}$, specifically

$$\begin{bmatrix} s_{23} & s_{36} & s_{25} \\ s_{36} & s_{13} & s_{14} \\ s_{25} & s_{14} & s_{12} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} c_{23} + 2c_{44} & c_{36} + 2c_{45} & c_{25} + 2_{46} \\ c_{36} + 2c_{45} & c_{13} + 2c_{55} & c_{14} + 2c_{56} \\ c_{25} + 2c_{56} & c_{14} + 2c_{56} & c_{12} + 2c_{66} \end{bmatrix}. \tag{3.27}$$

It is important to bear this in mind when we consider shifts of the elements of $\boldsymbol{M}$, subject to the constraint that the remaining components of the compliance tensor are fixed.

### (*c*) *Properties of totally symmetric elasticity tensors*

Backus (1970) has shown that a totally symmetric elasticity tensor can be further partitioned as

$$S_{ijkl} = H_{ijkl} + (\delta_{ij}H_{kl} + \delta_{ik}H_{jl} + \delta_{il}H_{jk} + \delta_{jk}H_{il} + \delta_{jl}H_{ik} + \delta_{kl}H_{ij}) + H(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (3.28)$$

The symmetric second- and fourth-order tensors $H_{ij}$ and $H_{ijkl}$ are *harmonic*, that is, contraction over any pair of indices yields zero: $H_{kk} = H_{ijkk} = 0$. It can be shown by using this property that

$$H = \tfrac{7}{15}Q_{kk}, \quad H_{ij} = Q_{ij} - \tfrac{1}{3}\delta_{ij}Q_{kk}, \quad (3.29)$$

where

$$Q_{ij} = \tfrac{1}{7}S_{ijkk} = \tfrac{1}{21}(C_{ijkk} + 2C_{ikjk}), \quad (3.30)$$

and $H_{ijkl}$ follows from equation (3.28). Note that $H_{ij}$ and $H_{ijkl}$ can have at most five and nine independent elements, respectively, which, combined with the single scalar $H$, implies that totally symmetric elasticity tensors have no more than 15 independent elements.

### (*d*) *The compatibility equations revisited*

The compatibility conditions (2.3) may be expressed concisely, using the notation of antisymmetric fourth-order tensors, as

$$\mathrm{Rot}\,\boldsymbol{\epsilon} = 0. \quad (3.31)$$

The operator Rot is a second-order differential expression acting on symmetric second-order tensors:

$$\mathrm{Rot}\,\epsilon_{ij} \equiv -e_{ikm}e_{jln}\epsilon_{mn,kl}. \quad (3.32)$$

Referring to equations (3.21) and (3.25) we have

$$\mathrm{Rot}\,\epsilon_{ij} = P_{ijkl}(\nabla \otimes \nabla)\epsilon_{kl} = P_{ijkl,kl}(\boldsymbol{\epsilon}), \quad (3.33)$$

or

$$\mathrm{Rot}\,\boldsymbol{\epsilon} = \nabla\nabla : \boldsymbol{P}(\boldsymbol{\epsilon}). \quad (3.34)$$

This form combined with equation (3.24) yields the explicit expansion of the compatibility relations:

$$\begin{aligned}\mathrm{Rot}\,\epsilon_{ij} &= \delta_{ij}(\epsilon_{kl,kl} - \nabla^2\epsilon) + \nabla^2\epsilon_{ij} + \epsilon_{,ij} - \epsilon_{ik,kj} - \epsilon_{jk,ki} \\ &= 0, \end{aligned} \quad (3.35)$$

where $\epsilon = \epsilon_{kk}$. The compatibility conditions themselves are therefore closely related to antisymmetric fourth-order elasticity tensors.

We are now ready to examine the possible shifts in compliances which leave the compatibility equations unchanged.

## 4. Compatible antisymmetric compliance tensors

### (a) *The compatibility equations*

We first restrict our consideration to antisymmetric shifts in the compliance $C^{(1)}_{ijkl}$ as defined in equation (2.7). Thus, let

$$C^{(1)}_{ijkl} = A_{ijkl}, \tag{4.1}$$

where $A_{ijkl}$ can be expressed in terms of either of the two second-order symmetric tensors $\boldsymbol{M}$ or $\boldsymbol{S}$, using equations (3.11) or (3.8), respectively. After some preliminary investigation one finds that the representation using $\boldsymbol{M}$ is more advantageous. Thus, substituting from (3.11), equation (2.7) becomes

$$-\tfrac{2}{3}(e_{ipr}e_{jqs})(e_{rkm}e_{sln})(\sigma_{kl}M_{mn})_{,pq} = 0. \tag{4.2}$$

This can be rewritten, using (3.21), as

$$-\tfrac{2}{3}\,\mathrm{Rot}\,\boldsymbol{N} = 0, \tag{4.3}$$

where $\boldsymbol{N}$ is a symmetric second-order tensor (see (3.25)):

$$\boldsymbol{N} = \boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{M} = \boldsymbol{P}(\boldsymbol{M})\boldsymbol{\sigma}. \tag{4.4}$$

More explicit forms for $\boldsymbol{N}$ can be deduced: for instance, using (3.24),

$$\boldsymbol{N} = m\boldsymbol{\sigma} + \sigma\boldsymbol{M} - \boldsymbol{\sigma}\cdot\boldsymbol{M} - \boldsymbol{M}\cdot\boldsymbol{\sigma} + (\boldsymbol{\sigma} : \boldsymbol{M} - \sigma m)\boldsymbol{I} \tag{4.5}$$

or

$$\boldsymbol{N} = m\boldsymbol{\Sigma} + (\boldsymbol{\Sigma} : \boldsymbol{M})\boldsymbol{I} - \boldsymbol{\Sigma}\cdot\boldsymbol{M} - \boldsymbol{M}\cdot\boldsymbol{\Sigma}, \tag{4.6}$$

where

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma} - \tfrac{1}{2}\sigma\boldsymbol{I} \quad \Leftrightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\Sigma} - \Sigma\boldsymbol{I}, \tag{4.7}$$

with $\sigma = \sigma_{kk}$ and $\Sigma = \Sigma_{kk}$. The complete expansion of $\boldsymbol{N}$ follows from (4.5) as

$$\begin{aligned}\boldsymbol{N} = M_{11}&\begin{bmatrix}0 & 0 & 0\\ 0 & -\sigma_{33} & \sigma_{23}\\ 0 & \sigma_{23} & -\sigma_{22}\end{bmatrix} + M_{22}\begin{bmatrix}-\sigma_{33} & 0 & \sigma_{13}\\ 0 & 0 & 0\\ \sigma_{13} & 0 & -\sigma_{11}\end{bmatrix}\\ &+ M_{33}\begin{bmatrix}-\sigma_{22} & \sigma_{12} & 0\\ \sigma_{12} & -\sigma_{11} & 0\\ 0 & 0 & 0\end{bmatrix} + M_{23}\begin{bmatrix}2\sigma_{23} & -\sigma_{13} & -\sigma_{12}\\ -\sigma_{13} & 0 & \sigma_{11}\\ -\sigma_{12} & \sigma_{11} & 0\end{bmatrix}\\ &+ M_{13}\begin{bmatrix}0 & -\sigma_{23} & \sigma_{22}\\ -\sigma_{23} & 2\sigma_{13} & -\sigma_{12}\\ \sigma_{22} & -\sigma_{12} & 0\end{bmatrix} + M_{12}\begin{bmatrix}0 & \sigma_{33} & -\sigma_{23}\\ \sigma_{33} & 0 & -\sigma_{13}\\ -\sigma_{23} & -\sigma_{13} & 2\sigma_{12}\end{bmatrix}.\end{aligned} \tag{4.8}$$

A second form of the compatibility equation for antisymmetric shifts in the compliance is obtained by using the representation (3.34) for Rot. This implies that (4.2) is equivalent to

$$D_{ijkl,kl} = 0, \tag{4.9}$$

where

$$\boldsymbol{D} = \tfrac{2}{3}\boldsymbol{P}(\boldsymbol{N}). \tag{4.10}$$

The fourth-order tensor $D_{ijkl}$ may be interpreted as the antisymmetric part of the fourth-order elasticity tensor $(\boldsymbol{M} \otimes \boldsymbol{\sigma} + \boldsymbol{\sigma} \otimes \boldsymbol{M})$, that is

$$D_{ijkl} = \tfrac{1}{3}(2\sigma_{ij}M_{kl} + 2\sigma_{kl}M_{ij} - \sigma_{ik}M_{jl} - \sigma_{il}M_{jk} - \sigma_{jk}M_{il} - \sigma_{jl}M_{ik}). \tag{4.11}$$

Alternatively, as an antisymmetric elasticity tensor, $D_{ijkl}$ can be represented by a symmetric second-order tensor, $\boldsymbol{R}$:

$$D_{ijkl} = \tfrac{1}{3}(2\delta_{ij}R_{kl} + 2\delta_{kl}R_{ij} - \delta_{ik}R_{jl} - \delta_{il}R_{jk} - \delta_{jk}R_{il} - \delta_{jl}R_{ik}), \tag{4.12}$$

where the second-order tensor can be found in the same manner as before, yielding

$$\boldsymbol{N} = \boldsymbol{R} - (\boldsymbol{R} \cdot \boldsymbol{I})\boldsymbol{I} \quad \Leftrightarrow \quad \boldsymbol{R} = \boldsymbol{N} - \tfrac{1}{2}(\boldsymbol{N} \cdot \boldsymbol{I})\boldsymbol{I}. \tag{4.13}$$

A third form of the compatibility equation is obtained by rewriting equation (4.2) as

$$-\tfrac{2}{3}(e_{ipr}e_{kmr})(e_{jqs}e_{lns})(\sigma_{kl}M_{mn})_{,pq} = 0. \tag{4.14}$$

Then using the identity $e_{ipr}e_{kmr} = \delta_{ik}\delta_{pm} - \delta_{im}\delta_{pk}$, we arrive at the compact form

$$(\sigma_{ij}M_{kl} + \sigma_{kl}M_{ij} - \sigma_{ik}M_{jl} - \sigma_{jk}M_{il})_{,kl} = 0. \tag{4.15}$$

This can also be deduced by substituting from equation (4.11) into equation (4.9).

Equation (4.15) is the most convenient form of the compatibility equation for antisymmetric shifts in the compliance, and we now examine it in detail. We first note that (4.15) may be expanded by direct differentiation of the final three terms, and combined with the equilibrium relations (2.1), to yield

$$\begin{aligned} 0 = (\sigma_{ij}M_{kl})_{,kl} + \sigma_{kl}M_{ij,kl} - (\sigma_{ik}M_{jl,k} + \sigma_{jk}M_{il,k})_{,l} \\ + 2f_kM_{ij,k} - f_iM_{jk,k} - f_jM_{ik,k} + f_{k,k}M_{ij} - f_{i,k}M_{jk} - f_{j,k}M_{ik}. \end{aligned} \tag{4.16}$$

It will be assumed henceforth that there are no body forces acting, $\boldsymbol{f} = \boldsymbol{0}$, so that (4.16) reduces to

$$(\sigma_{ij}M_{kl})_{,kl} + \sigma_{kl}M_{ij,kl} - (\sigma_{ik}M_{jl,k} + \sigma_{jk}M_{il,k})_{,l} = 0. \tag{4.17}$$

We now examine possible solutions for this set of equations, considered as equations for the compliance shifts $M_{ij}$. We denote these classes of solutions as compatible moduli.

### (*b*) *Example 1*

As a first example, consider a single non-zero element in $\boldsymbol{M}$:

$$M_{ij} = M_{33}\delta_{i3}\delta_{j3}, \tag{4.18}$$

for which the second-order symmetric tensor $\boldsymbol{N}$ is, from equation (4.8),

$$\boldsymbol{N} = M_{33}\begin{bmatrix} -\sigma_{22} & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.19}$$

Therefore, only the in-plane stresses† $\sigma_{\alpha\beta}$ will enter into subsequent equations for this example.

The compatibility relations follow from equations (4.3) and (4.19) as

$$\delta_{i\alpha}\delta_{j\beta}(\sigma_{\alpha\beta}M_{33})_{,33} + \delta_{i3}\delta_{j3}(\sigma_{\alpha\beta}M_{33})_{,\alpha\beta} - (\delta_{i\alpha}\delta_{j3} + \delta_{j\alpha}\delta_{i3})(\sigma_{\alpha\beta}M_{33})_{,\beta 3} = 0. \quad (4.20)$$

These represent six constraints, which may be separated into three sets of three, two and one, respectively:

$$(\sigma_{\alpha\beta}M_{33})_{,33} = 0, \quad (\sigma_{\alpha\beta}M_{33})_{,\beta 3} = 0, \quad (\sigma_{\alpha\beta}M_{33})_{,\alpha\beta} = 0. \quad (4.21)$$

The six equations in (4.21) are too general to make any definite statements about the class of admissible functions for $M_{33}$ and for $\boldsymbol{\sigma}$. In order to make some headway, we therefore assume that the in-plane stresses $\sigma_{\alpha\beta}$ are self-equilibrated, i.e.

$$\sigma_{11,1} + \sigma_{12,2} = 0, \quad \sigma_{21,1} + \sigma_{22,2} = 0. \quad (4.22)$$

This is broader than an assumption of plane stress or plane strain. The conditions (4.22) are satisfied if the two stresses $\sigma_{\alpha 3}$ are independent of $x_3$, so that their contributions to the in-plane equilibrium equations, $\sigma_{\alpha 3,3}$, are identically zero. Hence, we are assuming only that

$$\text{assumption 1:} \qquad \sigma_{\alpha 3} = q_\alpha(x_1, x_2), \quad \alpha = 1, 2. \quad (4.23)$$

Under these circumstances equations (4.21) simplify to

$$(\sigma_{\alpha\beta}M_{33})_{,33} = 0, \quad (\sigma_{\alpha\beta}M_{33,\beta})_{,3} = 0, \quad \sigma_{\alpha\beta}M_{33,\alpha\beta} = 0. \quad (4.24)$$

These imply, respectively, that

$$\sigma_{\alpha\beta}M_{33} = g_{\alpha\beta}(x_1, x_2) + x_3 h_{\alpha\beta}(x_1, x_2), \quad (4.25)$$
$$\sigma_{\alpha\beta}M_{33,\beta} = l_\alpha(x_1, x_2), \quad (4.26)$$
$$M_{33} = a_0(x_3) + a_1(x_3)x_1 + a_2(x_3)x_2. \quad (4.27)$$

Eliminating between these three equations, we deduce that

$$l_\alpha(x_1, x_2) = \frac{[g_{\alpha\beta}(x_1, x_2) + x_3 h_{\alpha\beta}(x_1, x_2)]a_\beta(x_3)}{a_0(x_3) + a_1(x_3)x_1 + a_2(x_3)x_2}. \quad (4.28)$$

The right-hand side of this must be independent of $x_3$ for arbitrarily valued in-plane stresses $\sigma_{\alpha\beta}$. This can be satisfied in either of two ways. First, if the parameters satisfy

$$(a) \qquad a_0 = b_0 f(x_3), \quad a_1 = b_1 f(x_3), \quad a_2 = b_2 f(x_3), \quad h_{\alpha\beta} = 0, \quad (4.29)$$

where $b_0$, $b_1$ and $b_2$ are constants, or secondly, if

$$(b) \qquad a_1 = 0, \quad a_2 = 0. \quad (4.30)$$

We now discuss cases $(a)$ and $(b)$ individually.

† Lower case Greek subscripts indicate the restricted values 1 and 2 only.

(i) *Example 1(a): An extended CLM result*

Redefining $g_{\alpha\beta} = (b_0 + b_1 x_1 + b_2 x_2)G_{\alpha\beta}$ and $f(x_3) = 1/F(x_3)$, the first solution, (4.29), implies that the in-plane stresses and $M_{33}$ are given by

$$(a) \qquad \sigma_{\alpha\beta} = G_{\alpha\beta}(x_1, x_2)F(x_3), \quad M_{33} = (b_0 + b_1 x_1 + b_2 x_2)/F(x_3). \tag{4.31}$$

The constants $b_0$, $b_1$ and $b_2$ and the function $F$ may be arbitrary, the out-of-plane stresses $\sigma_{13}, \sigma_{23}$ satisfy (4.23) and $\sigma_{33}$ is unrestricted.

This example includes as a special case the CLM result as generalized by Dundurs & Markenscoff (1993). Thus, when $F \equiv$ const. and the only stresses of concern are $\sigma_{\alpha\beta}$, we recover the result that planar bulk and shear compliances may be shifted in an affine manner without affecting the stresses and without violating the strain compatibility conditions. In order to prove this assertion, we must show that it yields the Dundurs & Markenscoff (1993) result for isotropic planar situations: plane stress and plane strain.

An arbitrary shift of the single compliance $M_{33}$ may be effected by letting the two compliances $c_{12}$ and $c_{66}$ vary as follows: $(c_{12}, c_{66}) \to (c'_{12}, c'_{66})$, where

$$c'_{12} = c_{12} + 2\lambda, \quad c'_{66} = c_{66} - \lambda. \tag{4.32}$$

This implies that

$$M'_{33} = M_{33} + 3\lambda, \tag{4.33}$$

while the element $s_{12} = S_{1122}$ of the totally symmetric tensor remains unchanged at $s_{12} = \frac{1}{3}(c_{12} + 2c_{66})$ (see equation (3.27)). We first consider the case of isotropic plane stress, for which the in-plane stresses $\sigma_{\alpha\beta}$ are the only ones present. Under these circumstances the shifts in the plane stress bulk and shear compliances follow from equations (4.32), (B 9) and (B 10) as

$$\frac{1}{K^{(\sigma)\prime}} = \frac{1}{K^{(\sigma)}} + 4\lambda, \quad \frac{1}{\mu'} = \frac{1}{\mu} - 4\lambda. \tag{4.34}$$

Under plane strain conditions, it follows from Appendix B and equation (B 7) specifically that the isotropic plane strain moduli vary in the same manner, i.e.

$$\frac{1}{K^{(\epsilon)\prime}} = \frac{1}{K^{(\epsilon)}} + 4\lambda, \quad \frac{1}{\mu'} = \frac{1}{\mu} - 4\lambda. \tag{4.35}$$

Thus, the shift in $M_{33}$ corresponds to equal and opposite shifts in the planar bulk and shear compliances. The shift parameter $\lambda$ may, according to equation (4.31), be an affine function of the planar coordinates:

$$\lambda = b_0 + b_1 x_1 + b_2 x_2. \tag{4.36}$$

This is precisely the result of Dundurs & Markenscoff (1993), which generalized the CLM theorem: that planar stress is invariant to this type of arbitrary affine shift in compliance.

The consequences of this have been studied repeatedly and in detail, and there is no need to reproduce the results here. One can appreciate the significance of the CLM theorem by noting that, for instance, in isotropic plane stress, the Young's modulus of concern is $E$, where $1/E = 3s_{12}$, which is a constant (see equation (3.27)). Hence, the

Young's modulus $E$ is unchanged under the $\lambda$ shift. The in-plane Poisson's ratio does however, change, according to $\nu' = -c'_{12}/(3s_{12})$, or $\nu' = \nu - 2\lambda E$. This implies, for example (Cherkaev *et al.* 1992), that the effective Young's modulus of a thin plate with holes and cracks is independent of the Poisson's ratio of the plate material, because no matter how we vary the latter, the ratio between the stress and $E$ are fixed, and hence the uniaxial strain averaged over the plate is also fixed.

The present three-dimensional analysis shows that there is an extended form of the CLM theorem applicable to states of planar stress of the form (4.31) subject only to the additional constraint (4.23) on two of the remaining three stress components.

(ii) *Example 1(b)*

In this case, from equation (4.30), the stress and compliances are such that

$$(b) \qquad \sigma_{\alpha\beta} = (G_{\alpha\beta}(x_1,x_2) + x_3 H_{\alpha\beta}(x_1,x_2))F(x_3), \quad M_{33} = b_0/F(x_3). \tag{4.37}$$

This indicates that the CLM theorem extends to non-affine shifts of $M_{33}$ for states of planar stress that are affine in the third coordinate. Again, the two out-of-plane stresses $\sigma_{13}$ and $\sigma_{23}$ must satisfy the constraint (4.23).

## (c) *Example 2*

Let us first consider other possible shifts in the components of $\boldsymbol{M}$, of affine form

$$M_{ij} = M_{ij}^{(0)} + M_{ijk}^{(1)} x_k, \tag{4.38}$$

where the second- and third-order tensors $\boldsymbol{M}^{(0)}$ and $\boldsymbol{M}^{(1)}$ are constants. The symmetry of $\boldsymbol{M}$ implies that $\boldsymbol{M}^{(0)}$ is symmetric and $M_{ijk}^{(1)} = M_{jik}^{(1)}$. Equation (4.17) becomes, for this type of shift,

$$\sigma_{ij,kl}M_{kl}^{(0)} + \sigma_{ij,k}(M_{kll}^{(1)} + M_{lkl}^{(1)}) - \sigma_{ik,l}M_{jlk}^{(1)} - \sigma_{jk,l}M_{ilk}^{(1)} = 0. \tag{4.39}$$

In order to make progress we assume that the stress is two dimensional: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1,x_2)$, and only consider out-of-plane elements of $\boldsymbol{M}$, that is $M_{\alpha\beta} = 0$, or

$$M_{ij} = M_{i3}\delta_{j3} + M_{3j}\delta_{i3}, \quad M_{ij} = M_{ji}. \tag{4.40}$$

Therefore, equation (4.39) becomes

$$\sigma_{ij,\alpha}(M_{\alpha 33}^{(1)} + M_{3\alpha 3}^{(1)}) - \delta_{3j}\sigma_{ik,\alpha}M_{3\alpha k}^{(1)} - \delta_{3i}\sigma_{jk,\alpha}M_{3\alpha k}^{(1)} = 0. \tag{4.41}$$

This holds generally only if

$$M_{3\alpha k}^{(1)} = M_{\alpha 3k}^{(1)} = 0. \tag{4.42}$$

In summary, the affine shifts are limited to the diagonal elements of $\boldsymbol{M}$, i.e. $M_{33}$, but $M_{\alpha 3}$ must be constant.

Based on these findings, we therefore examine in greater detail the constant off-diagonal shift

$$M_{ij} = M_{23}(\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}), \tag{4.43}$$

for which

$$\boldsymbol{N} = M_{23} \begin{bmatrix} 2\sigma_{23} & -\sigma_{13} & -\sigma_{12} \\ -\sigma_{13} & 0 & \sigma_{11} \\ -\sigma_{12} & \sigma_{11} & 0 \end{bmatrix}. \tag{4.44}$$

In this case we make the assumptions (i) that all the stresses appearing in (4.44) are independent of $x_3$; and (ii) that $M_{23}$ is constant. We then find that the components $(\mathrm{Rot}\,\boldsymbol{N})_{\alpha\beta}$ vanish, while the remaining components are

$$(\mathrm{Rot}\,\boldsymbol{N})_{13} = -M_{23}(\sigma_{11,1} + \sigma_{12,2})_{,2}, \tag{4.45}$$

$$(\mathrm{Rot}\,\boldsymbol{N})_{23} = M_{23}(\sigma_{11,1} + \sigma_{12,2})_{,1}, \tag{4.46}$$

$$(\mathrm{Rot}\,\boldsymbol{N})_{33} = -2M_{23}\,(\sigma_{31,1} + \sigma_{32,2})_{,2}. \tag{4.47}$$

The first two of these vanish by virtue of the equilibrium equations. However, we must make the additional assumption that $\sigma_{33}$ is also independent of $x_3$ if $(\mathrm{Rot}\,\boldsymbol{N})_{33}$ is to vanish.

Note that in order to effect the shift (4.43), all of the stresses except $\sigma_{22}$ must be independent of the third direction. This is slightly different from the situation for example 1, where constant and affine shifts of the diagonal element $M_{33}$ can be achieved when all stresses except $\sigma_{33}$ have two-dimensional dependence.

In summary, by combining example 1 for $M_{33}$ with example 2 for $M_{23}$ and the similar shift $M_{13}$, it is clear that the CLM result may be generalized as follows.

**Result 4.1.** Any shift of the form $\boldsymbol{M} \to \boldsymbol{M}'$, where

$$\boldsymbol{M}' = \boldsymbol{M} + \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha & \beta & \lambda \end{bmatrix}, \tag{4.48}$$

and the remaining components of the compliance are fixed, will leave two-dimensional stress fields, $\boldsymbol{\sigma}(x_1, x_2)$, unchanged for arbitrary constants $\alpha$, $\beta$ and $\lambda$.

### (*d*) *Plane strain*

It is useful to make the connection between the general three-dimensional shift (4.48) and the CLM theorem for planar elasticity. In particular, we examine the relevance of the former to plane strain. Thus, we consider translation of the compliances from those of an orthotropic material to equivalent triclinic materials by shifting $M_{i3} = M_{3i}$, $i = 1, 2, 3$. This generates a material with

$$c_{15} = c_{16} = c_{24} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = 0, \tag{4.49}$$

and the in-plane stress–strain relation follows from equation (B 6):

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} c_{11} - \dfrac{c_{13}^2}{c_{33}} - \dfrac{c_{14}^2}{c_{44}} & c_{12} - \dfrac{c_{13}c_{23}}{c_{33}} & -\dfrac{c_{14}c_{46}}{c_{44}} \\ c_{12} - \dfrac{c_{13}c_{23}}{c_{33}} & c_{22} - \dfrac{c_{23}^2}{c_{33}} - \dfrac{c_{25}^2}{c_{55}} & -\dfrac{c_{25}c_{56}}{c_{55}} \\ -\dfrac{c_{14}c_{46}}{c_{44}} & -\dfrac{c_{25}c_{56}}{c_{55}} & c_{66} - \dfrac{c_{46}^2}{c_{44}} - \dfrac{c_{56}^2}{c_{55}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix}. \tag{4.50}$$

Now let $c_{IJ}$ represent the base compliances and $c'_{IJ}$ the shifted compliances, so that

$$c'_{14} = 2a, \quad c'_{56} = -a, \tag{4.51}$$
$$c'_{25} = 2b, \quad c'_{46} = -b, \tag{4.52}$$
$$c'_{66} = c_{66} + c, \quad c'_{12} = c_{12} - 2c, \tag{4.53}$$
$$c'_{13} = c_{13}, \quad c'_{23} = c_{23}, \tag{4.54}$$
$$c'_{II} = c_{II}, \quad I = 1, 2, \ldots, 5 \text{ (no sum)}, \tag{4.55}$$

with the remaining elements zero. The shifts in the compliances are chosen so that the elements of the totally symmetric tensor $S_{ijkl}$ are unchanged (see equation (3.27)). Thus,

$$M_{23} = 3a, \quad M_{13} = 3b, \quad M_{33} = c_{66} - c_{12} + 3c. \tag{4.56}$$

Substituting $c_{IJ} \to c'_{IJ}$ in equation (4.50) and evaluating them according to the above prescription, gives the effective in-plane compliance matrix

$$[C^{(\mathrm{eff})}] = [C^{(0)}] + [C^{(\mathrm{shift})}], \tag{4.57}$$

where

$$[C^{(0)}] = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{bmatrix}, \tag{4.58}$$

and the shift of the compliance is

$$[C^{(\mathrm{shift})}] = \begin{bmatrix} -4a^2/c_{44} & -2c & 2ab/c_{44} \\ -2c & -4b^2/c_{55} & 2ab/c_{55} \\ 2ab/c_{44} & 2ab/c_{55} & c - a^2/c_{55} - b^2/c_{44} \end{bmatrix}. \tag{4.59}$$

The single compatibility equation for plane strain, (B 2), implies, using (4.59),

$$\begin{aligned} a^2\left(\frac{\sigma_{11}}{c_{44}}\right)_{,22} &+ b^2\left(\frac{\sigma_{22}}{c_{55}}\right)_{,11} - \left[\left(\frac{a^2}{c_{55}} + \frac{b^2}{c_{44}}\right)\sigma_{12}\right]_{,12} \\ &+ ab\left[\left(\frac{\sigma_{11}}{c_{44}}\right)_{,12} - \left(\frac{\sigma_{12}}{c_{44}}\right)_{,22} + \left(\frac{\sigma_{22}}{c_{55}}\right)_{,12} - \left(\frac{\sigma_{12}}{c_{55}}\right)_{,11}\right] \\ &+ \tfrac{1}{2}c(\sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12}) = 0. \end{aligned} \tag{4.60}$$

The term involving $c$ is identically zero on account of the equilibrium equations. The remainder can be analysed by introducing the Airy stress function

$$\sigma_{11} = \phi_{,22}, \quad \sigma_{22} = \phi_{,11}, \quad \sigma_{12} = -\phi_{,12}. \tag{4.61}$$

Assume, for the sake of simplicity, that $c_{44}$ and $c_{55}$ are constants, then (4.60) becomes

$$\frac{b^2}{c_{55}}\phi_{,1111} + \frac{a^2}{c_{44}}\phi_{,2222} + \left(\frac{a^2}{c_{55}} + \frac{b^2}{c_{44}}\right)\phi_{,1122} + \frac{2ab}{c_{55}}\phi_{,1112} + \frac{2ab}{c_{44}}\phi_{,1222} = 0. \tag{4.62}$$

This cannot be satisfied for arbitrary constants $a$, $b$, $c_{44}$ and $c_{55}$, and we therefore conclude the following.

**Result 4.2.** The stress invariant shifts associated with $M_{13}$ and $M_{23}$ are not compatible with plane strain in general. In other words, the $M_{13}$ and $M_{23}$ shifts leave the two-dimensional state of stress invariant, but they do not maintain a state of plane strain.

This is not surprising when one considers that the invariance was based upon the condition that the stress is everywhere maintained at a fixed value. The strain, on the other hand, is not fixed and will generally vary, although it still satisfies the compatibility conditions. In this case the variation in strain destroys the plane-strain configuration. The deviation from plane strain can be seen from (B 3), which gives for the out-of-plane strains

$$\epsilon_{13} = 2b\sigma_{22} - 2a\sigma_{12}, \quad \epsilon_{23} = 2a\sigma_{11} - 2b\sigma_{12}, \quad \epsilon_{33} = 0. \tag{4.63}$$

### (*e*) *Example 3*

Consider a constant shift of $\boldsymbol{M}$, for which the basic equation (4.17) reduces to

$$M_{kl}\sigma_{ij,kl} = 0 \quad \Leftrightarrow \quad \boldsymbol{M} : \nabla\nabla\boldsymbol{\sigma} = 0. \tag{4.64}$$

We have already investigated solution pairs for $(\boldsymbol{M}, \boldsymbol{\sigma})$ in rectangular coordinates, now let us consider a cylindrical configuration. The operator $\boldsymbol{M} : \nabla\nabla$ takes the following form in cylindrical coordinates, $(r, \theta, z) = (\sqrt{x_1^2 + x_2^2}, \arctan(x_2/x_1), x_3)$:

$$M_{kl}g_{,kl} = M_{rr}g_{,rr} + M_{\theta\theta}\left(\frac{g_{,\theta\theta}}{r^2} + \frac{g_{,r}}{r}\right) + M_{zz}g_{,zz} + 2M_{r\theta}\left(\frac{g_{,r\theta}}{r} - \frac{g_{,\theta}}{r^2}\right) + 2M_{rz}g_{,rz} + 2M_{\theta z}\frac{g_{,\theta z}}{r}. \tag{4.65}$$

The previously obtained results for planar elasticity are immediately apparent from this expression: let $\boldsymbol{\sigma} = \boldsymbol{\sigma}(r, \theta)$, then the stress is invariant to arbitrary changes of $M_{zz}$, $M_{rz}$ and $M_{\theta z}$.

Alternatively, consider a state of stress that is axially symmetric, that is, independent of $\theta$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(r, z)$. Then equation (4.64) will be automatically satisfied by any shifts in the moduli $M_{r\theta}$ and $M_{\theta z}$. Hence, axisymmetric states of stress are invariant to changes in these components of $\boldsymbol{M}$. Note that the set of admissible compliances does not include $M_{\theta\theta}$.

### (*f*) *Comments*

Finally, we comment on other admissible solutions to the general compatibility conditions for the antisymmetric compliance. Equation (4.3) means that any additional compliances which are such that the second-order tensor $\boldsymbol{N}$ is constant or affine will be compatible. More generally, any $\boldsymbol{N}$ of the following form will suffice:

$$\boldsymbol{N} = \tfrac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\mathrm{T}}), \tag{4.66}$$

where $\boldsymbol{v}$ represents an arbitrary vector field. By compatible we mean that the additional compliance does not effect the stress solution. Thus, exactly the same stress will exist throughout the body, and the same tractions will exist on the surface of the body.

One possible set of compliances is generated by requiring that $\boldsymbol{M}$ is constrained to vary with the stress such that

$$\boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{M} = \tfrac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\mathrm{T}}). \tag{4.67}$$

Based upon equation (4.67) and the symmetry $P^{-1}_{ijkl} = P^{-1}_{ijlk}$, we have

$$\boldsymbol{M} = \boldsymbol{P}^{-1}(\boldsymbol{\sigma})\nabla\boldsymbol{v}.$$

Substituting for $\boldsymbol{P}^{-1}$ from equation (3.26) gives the explicit expression for the general form of the redundant $\boldsymbol{M}$:

$$M_{ij} = \frac{1}{2\det\boldsymbol{\sigma}}[\sigma_{ik}\sigma_{jl}(v_{k,l} + v_{l,k}) - \sigma_{ij}\sigma_{kl}v_{k,l}]. \tag{4.68}$$

Consider the case of the single shift of the element $M_{33}$ (example 1). The in-plane stresses are self-equilibrated (see equation (4.22)) in this case, and can therefore be represented by the Airy stress function $\phi$. By comparing the right-hand side of equation (4.67) with the explicit expression (4.19) for $\boldsymbol{N} = \boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{M}$, and also using equation (4.61), we see that the vector $\boldsymbol{v}$ can be identified as

$$\boldsymbol{v} = -M_{33}\nabla\phi. \tag{4.69}$$

This alternative view illustrates that the general stress-dependent shift can reproduce the constant or stress-independent shift if the vector $\boldsymbol{v}$ is chosen appropriately. A much broader class of shifts is obtained by taking other choices for $\boldsymbol{v}$. The consequences of this freedom are not immediately evident but perhaps deserve further study.

## 5. Compatibility for totally symmetric compliance tensors

We now consider totally symmetric shifts in the elasticity $C^{(1)}_{ijkl}$. We have seen that the original CLM result is concerned with the antisymmetric part of the compliance, therefore it would be particularly interesting if there are analogous properties associated with the symmetric part.

Referring to equation (3.28), it is clear that the simplest case is the scalar $H$, for which $C^{(1)}_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ and $\boldsymbol{C}^{(1)}\boldsymbol{\sigma} = 2\boldsymbol{\sigma} + \sigma\boldsymbol{I}$. Equation (2.7) therefore becomes, after some simplification,

$$2\,\mathrm{Rot}\,\boldsymbol{\sigma} + \nabla\nabla\sigma - \boldsymbol{I}\nabla^2\sigma = 0. \tag{5.1}$$

It is not obvious whether there are any general states of stress under which this set of equations can be satisfied identically, apart from trivial examples, such as $\boldsymbol{\sigma} = \text{const}$.

Next, consider the contribution from the harmonic second-order tensor $H_{ij}$, for which

$$C^{(1)}_{ijkl} = \delta_{ij}H_{kl} + \delta_{ik}H_{jl} + \delta_{il}H_{jk} + \delta_{jk}H_{il} + \delta_{jl}H_{ik} + \delta_{kl}H_{ij}. \tag{5.2}$$

In this case, equation (2.7) implies that

$$\mathrm{Rot}(\boldsymbol{I}(\boldsymbol{H} : \boldsymbol{\sigma}) + \sigma\boldsymbol{H} + 2\boldsymbol{\sigma}\cdot\boldsymbol{H} + 2\boldsymbol{H}\cdot\boldsymbol{\sigma}) = 0. \tag{5.3}$$

We then use the identity (3.35) for the tensor in equation (5.3), which gives the expanded form of (5.3):

$$
\begin{aligned}
0 = \delta_{ij}[\nabla^2(\sigma H + 5\boldsymbol{\sigma} : \boldsymbol{H}) - (\sigma H_{kl})_{,kl} - 4(\sigma_{km}H_{ml})_{,kl}] \\
- \nabla^2(\sigma H_{ij} + 2\sigma_{ik}H_{kj} + 2\sigma_{jk}H_{ki}) - (\sigma H + 5\boldsymbol{\sigma} : \boldsymbol{H})_{,ij} \\
+ (\sigma H_{ik} + 2\sigma_{im}H_{mk} + 2H_{im}\sigma_{mk})_{,kj} + (\sigma H_{jk} + 2\sigma_{jm}H_{mk} + 2H_{jm}\sigma_{mk})_{,ki}.
\end{aligned} \tag{5.4}
$$

It is not apparent from this rather complicated expression whether there are any generic shifts and associated stress states corresponding to the harmonic tensor $\boldsymbol{H}$.

Finally, the compatibility conditions for the nine-dimensional harmonic tensor $H_{ijkl}$ of equation (3.28) becomes

$$e_{ipr}e_{jqs}(H_{rskl}\sigma_{kl})_{,pq} = 0. \tag{5.5}$$

This is essentially the same as the starting point, equation (2.5), except that $H_{ijkl}$ contains fewer elements. However, based on the obvious generality of equation (5.5) we will not pursue this line of investigation further here. This completes the examination of the admissibility of the various components of the elastic compliance $C_{ijkl}$.

## 6. Conclusion

We have examined the possible invariance of the elastic stress field under changes, or shifts, in the compliance tensor. Several individual parts of the fourth-order elasticity tensor have been considered separately, following the partition scheme proposed by Backus (1970). This partition is independent of the existence, or lack thereof, of any underlying material symmetry in the material. We have found that the six-dimensional antisymmetric part of the elastic compliance is associated with two-dimensional and quasi-two-dimensional stress states. These include the previously discussed planar states of stress for which the CLM theorem was originally obtained, and subsequently generalized by Dundurs & Markenscoff (1993).

The present results illustrate clearly the position of the planar CLM theorem within the three-dimensional theory of elasticity. They also provide the means to generate more general states of stress invariance, with possible applications to estimating the effective properties of composite materials.

## Appendix A. The inverse of an antisymmetric elasticity tensor

We prove that the inverse of the fourth-order antisymmetric tensor $\boldsymbol{P}(\boldsymbol{\pi})$ of equation (3.21) is

$$P^{-1}_{ijkl}(\boldsymbol{\pi}) = \frac{1}{2\det\boldsymbol{\pi}}(\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk} - \pi_{ij}\pi_{kl}). \tag{A 1}$$

We must verify that this satisfies the required relations for the inverse of an elasticity tensor:

$$P^{-1}_{ijkl}P_{klmn} = P_{ijkl}P^{-1}_{klmn} = I_{ijkl}. \tag{A 2}$$

Thus, from (3.21),

$$P_{ijpq}\frac{1}{2\det\boldsymbol{\pi}}(\pi_{pk}\pi_{ql}+\pi_{pl}\pi_{qk}-\pi_{pq}\pi_{kl})=\pi_{ij}^{-1}\pi_{kl}+\frac{1}{2\det\boldsymbol{\pi}}P_{ijpq}[\pi_{pk}\pi_{ql}+\pi_{qk}\pi_{pl}], \tag{A 3}$$

where we have used $e_{ikm}e_{jln}\pi_{kl}\pi_{mn}=2(\det\boldsymbol{\pi})\pi_{ij}^{-1}$. In general, $\boldsymbol{\pi}$ can be expressed as

$$\boldsymbol{\pi}=\sum_t \lambda_t \boldsymbol{e}^{(t)}\otimes\boldsymbol{e}^{(t)}, \tag{A 4}$$

where $\{\boldsymbol{e}^{(1)},\boldsymbol{e}^{(2)},\boldsymbol{e}^{(3)}\}$ form an orthonormal triad, such that

$$e_{ijk}\,e_j^{(t)}e_k^{(u)}=e_{stu}e_i^{(s)},\quad \det\boldsymbol{\pi}=\lambda_1\lambda_2\lambda_3. \tag{A 5}$$

Thus,

$$P_{ijpq}[\pi_{pk}\pi_{ql}+\pi_{qk}\pi_{pl}]=-\sum_{t,u,v}\lambda_t\lambda_u\lambda_v e_{stu}e_{rtv}e_i^{(s)}e_j^{(r)}(e_k^{(u)}e_l^{(v)}+e_l^{(u)}e_k^{(v)}). \tag{A 6}$$

Noting that

$$\frac{1}{\det\boldsymbol{\pi}}e_{stu}\lambda_t\lambda_u\lambda_v=e_{stu}\frac{\lambda_v}{\lambda_s}\quad\text{(no sum)} \tag{A 7}$$

and hence that $\lambda_t$ does not appear in the right member of (A 6), we may therefore use the identity

$$e_{stu}e_{rtv}=\delta_{sr}\delta_{uv}-\delta_{sv}\delta_{ur} \tag{A 8}$$

to get

$$\begin{aligned}\frac{1}{2\det\boldsymbol{\pi}}&P_{ijpq}[\pi_{pk}\pi_{ql}+\pi_{qk}\pi_{pl}]\\&=\frac{1}{2}\sum_{s,u}(e_i^{(s)}e_l^{(s)}e_k^{(u)}e_j^{(u)}+e_j^{(s)}e_l^{(s)}e_k^{(u)}e_i^{(u)})-\sum_{s,u}\frac{\lambda_u}{\lambda_s}e_i^{(s)}e_j^{(s)}e_k^{(u)}e_l^{(u)}.\end{aligned} \tag{A 9}$$

We can identify the first and second sums in the right member as $I_{ijkl}$ and $\pi_{ij}^{-1}\pi_{kl}$, respectively. Hence, equation (A 3) is exactly the identity (A 2).

Finally, it is clear that the inverse of $\boldsymbol{P}$ does not exist if $\det\boldsymbol{\pi}$ vanishes. This completes the proof of (A 1).

## Appendix B. Two-dimensional elasticity, plane strain and plane stress

Two-dimensional elastic configurations possess stress and strain that depend upon two spatial coordinates, say $x_1$ and $x_2$. The displacement $\boldsymbol{u}=(u_1,u_2,u_3)$ is also a function of these two coordinates and is independent of $x_3$. Thus, $\epsilon_{33}\equiv u_{3,3}=0$ and $\epsilon_{13}=\frac{1}{2}u_{3,1}$, $\epsilon_{23}=\frac{1}{2}u_{3,2}$. We therefore have the two constraints

$$\epsilon_{33}=0,\quad \epsilon_{13,2}-\epsilon_{23,1}=0. \tag{B 1}$$

Of the six compatibility relations in equation (2.3), only one provides an additional non-trivial condition:

$$2\epsilon_{12,12} - \epsilon_{11,22} - \epsilon_{22,11} = 0. \tag{B 2}$$

Two-dimensional elasticity therefore imposes three constraints on the strains: (B 1) and (B 2).

We note that the out-of-plane strains are given by

$$\begin{bmatrix}\epsilon_{13}\\ \epsilon_{23}\\ \epsilon_{33}\end{bmatrix} = \begin{bmatrix} c_{15} & c_{25} & c_{56}\\ c_{14} & c_{24} & c_{46}\\ c_{13} & c_{23} & c_{36}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ 2\sigma_{12}\end{bmatrix} + \begin{bmatrix} c_{55} & c_{45} & c_{35}\\ c_{45} & c_{44} & c_{34}\\ c_{35} & c_{34} & c_{33}\end{bmatrix}\begin{bmatrix}2\sigma_{13}\\ 2\sigma_{23}\\ \sigma_{33}\end{bmatrix}. \tag{B 3}$$

The constraints (B 1) and (B 2) are not sufficient to express, for example, the out-of-plane stresses $\sigma_{i3}$, $i = 1, 2, 3$, in terms of the in-plane stresses $\sigma_{\alpha\beta}$.

In *plane strain* configurations the displacement in the out-of-plane direction is zero, $u_3 = 0$, the in-plane displacements are independent of $x_3$ and consequently

$$\epsilon_{i3} = 0, \quad i = 1, 2, 3. \tag{B 4}$$

These three conditions imply that the in-plane and out-of-plane stress components are related by equations (B 3) and (B 4). The remaining stress–strain relations are

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{16}\\ c_{12} & c_{22} & c_{26}\\ c_{16} & c_{26} & c_{66}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ 2\sigma_{12}\end{bmatrix} + \begin{bmatrix} c_{15} & c_{14} & c_{13}\\ c_{25} & c_{24} & c_{23}\\ c_{56} & c_{46} & c_{36}\end{bmatrix}\begin{bmatrix}2\sigma_{13}\\ 2\sigma_{23}\\ \sigma_{33}\end{bmatrix}. \tag{B 5}$$

Eliminating the out-of-plane stresses, we obtain the in-plane stress–strain relations:

$$\begin{aligned}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{12}\end{bmatrix} &= \begin{bmatrix} c_{11} & c_{12} & c_{16}\\ c_{12} & c_{22} & c_{26}\\ c_{16} & c_{26} & c_{66}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ 2\sigma_{12}\end{bmatrix}\\ &\quad - \begin{bmatrix} c_{15} & c_{14} & c_{13}\\ c_{25} & c_{24} & c_{23}\\ c_{56} & c_{46} & c_{36}\end{bmatrix}\begin{bmatrix} c_{55} & c_{45} & c_{35}\\ c_{45} & c_{44} & c_{34}\\ c_{35} & c_{34} & c_{33}\end{bmatrix}^{-1}\begin{bmatrix} c_{15} & c_{25} & c_{56}\\ c_{14} & c_{24} & c_{46}\\ c_{13} & c_{23} & c_{36}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ 2\sigma_{12}\end{bmatrix}.\end{aligned} \tag{B 6}$$

A plane-strain bulk modulus may be defined by application of the hydrostatic stress $\sigma_{\alpha\beta} = p\delta_{\alpha\beta}$, yielding $\epsilon_{\alpha\alpha} = p/K^{(\epsilon)}$. Hence, for an orthotropic material,

$$\frac{1}{K^{(\epsilon)}} = c_{11} + c_{22} + 2c_{12} - \frac{(c_{13}+c_{23})^2}{c_{33}}. \tag{B 7}$$

In the isotropic case the only non-zero compliances of relevance are

$$c_{11} = \frac{1}{E}, \quad c_{12} = -\frac{\nu}{E}, \quad c_{66} = \frac{(1+\nu)}{2E}, \tag{B 8}$$

and $c_{33} = c_{11}$, $c_{13} = c_{23} = c_{12}$. The isotropic plane-strain bulk modulus is therefore

$$\frac{1}{K^{(\epsilon)}} = \frac{2(1+\nu)(1-2\nu)}{E}.$$

The plane-strain shear modulus is just the usual three-dimensional modulus:

$$\mu = 1/(4c_{66}). \tag{B 9}$$

In plane stress we have $\sigma_{i3} = 0$, for $i = 1, 2, 3$, and therefore only $C_{\alpha\beta\gamma\delta}$ are important. The isotropic plane-stress bulk modulus is defined by $\epsilon_{\alpha\alpha} = \sigma_{\alpha\alpha}/(2K^{(\sigma)})$, and so

$$1/K^{(\sigma)} = 2c_{11} + 2c_{12}, \tag{B 10}$$

or $K^{(\sigma)} = E/[2(1-\nu)]$. The isotropic plane-stress shear modulus is again given by equation (B 9), and is therefore the same as the usual shear modulus.

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