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Exact complex source representations of transient radiation

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Abstract

Exact expansions are presented for frequency-domain and time-domain fields generated by sources in a finite region of space. These expansions determine the field outside the minimum sphere from the values of the field on any sphere outside the source region, e.g., the far-field sphere. The basis functions are directionally localized monopole or multipole point sources located on a sphere in complex space. The frequency-domain expansions are first presented and then used along with the analytic Fourier transform to obtain the corresponding time-domain expansions.

1. Introduction

We consider the radiation from 3D scalar sources located in a finite region of space inside the minimum sphere with radius $r_{\min}$. The scalar field generated by these sources is denoted by $f(\mathbf{x}, t)$ and satisfies the wave equation with constant wave speed c. We shall derive new exact expansions for this field in which the basis functions are directionally localized, and the expansion coefficients are either determined from the values of $f(\mathbf{x}, t)$ on a sphere $r = b$, with $b \geq r_{\min}$, or from the far-field pattern of $f(\mathbf{x}, t)$. Two time-harmonic (frequency-domain) and two transient (time-domain) expansions will be presented. Each of the time-domain expansions is obtained by applying the analytic Fourier transform to the corresponding frequency-domain expansion.

The simplest frequency-domain expansion that expresses the field everywhere outside the minimum sphere in terms of its values on the minimum sphere is the Helmholtz representation [1, Section 1.29], which requires that the field as well as its normal derivative be known on the minimum sphere. A more versatile expansion that requires knowledge only of the field itself is the spherical harmonics expansion [1, Section 8.11]. This expansion can be used to compute the frequency-domain field everywhere in the region $r \geq r_{\min}$, either from the values of the field on any sphere $r = b$ with $b \geq r_{\min}$, or from the far-field pattern.

The time-domain analog of the Helmholtz representation is easily obtained and well-known [1, Section 1.17]; whereas the time-domain analogs of the spherical harmonics expansion are more complicated [2–6]. The expansion derived in [5] allows one to compute the field in the region $r \geq b$ given the field and its time integration on $r = b$. Moreover, a sampling theorem is derived in [5] that determine how many spherical modes are needed and how

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closely one has to sample in space and time to compute the expansion coefficients accurately. The expansion in [5] was derived for spherical near-field scanning in the time domain.

All the frequency and time-domain expansions discussed so far employ basis functions that are not directionally localized. In many cases where inhomogeneous media are involved, it is advantageous to have at one's disposal expansions that employ directionally localized basis functions [7–10].

The directionally localized complex point-source field of Deschamps [11] was used by Norris [7] to expand the field of a real frequency-domain point source. Specifically, in [7] the real point-source field is written as an exact superposition of point-source fields whose source points are located on the surface of a sphere in complex space. Using the analytic Fourier transform along with Norris' beam formula [7], Heyman [8] obtained an exact expansion of a real time-domain point source in terms of complex-source pulsed beams located on a sphere in complex space. By superposition, the complex point-source expansions of Norris [7] and Heyman [8] could be used to expand the field of a general source region of finite extent. However, this simple superposition would result in expansions in which the complex point sources would be located on different nonconcentric spheres. Specifically, each point in the source region would be the center of a sphere in complex space that would contain complex point sources. Such expansions would not allow one to immediately compute the fields outside the minimum sphere $r = r_{\min}$ given the far field or the field on a sphere $r = b \geq r_{\min}$.

Recently, the frequency-domain expansion of [7] was generalized to express the field of an arbitrary frequency-domain source, located in the region $r < r_{\min}$, in terms of point sources (monopoles) on the surface of a *single* sphere in complex space [12,13]. Given the frequency-domain field on any sphere $r = b \geq r_{\min}$, the expansion in [12,13] expresses the field everywhere in the region $r \geq r_{\min}$ in terms of directionally localized sources. In particular, this expansion can be used to compute the field in the region $r \geq r_{\min}$ from the far field. In the present paper, we shall derive yet another new exact expansion, which expresses an arbitrary frequency-domain field in terms of complete set of multipole point sources on the surface of a *single* sphere in complex space.

Also in this paper, the analytic Fourier transform will be used to derive the time-domain analogs of the two frequency-domain complex point-source expansions. The resulting time-domain expansions express the field outside the minimum sphere in terms of directionally localized sources and reduce to the expansion of Heyman [8] in the case of a single point source located at $r = 0$.

The paper is organized as follows. In Section 2 we present the two frequency-domain expansions and determine the low-frequency behavior of some of the expansion coefficients. A motivation for using the analytic Fourier transform to derive the time-domain results and a short presentation of the properties of analytic signals are contained in Section 3. Finally, Section 4 presents the derivation of the new time-domain formulas that express the field outside the scan sphere in terms of directionally localized sources.

2. Frequency-domain beam formulas

In this section we are concerned with solutions to the Helmholtz equation that are regular outside the sphere of radius $r_{\min}$. That is, we consider the field radiated by a source whose maximum spatial dimension is $2r_{\min}$. The time dependence $e^{-i\omega t}$ is suppressed in all the time-harmonic equations, and the frequency-domain field radiated by the sources is denoted by $f(\mathbf{x}, \omega)$ for $\omega \geq 0$.

The wave function $f(\mathbf{x}, \omega)$ satisfies the Helmholtz equation

$$\nabla^2 f(\mathbf{x}, \omega) + k^2 f(\mathbf{x}, \omega) = 0, \quad r > r_{\min}, \tag{1}$$

and the radiation condition

$$\partial f/\partial r - ikf = O(f/r), \quad r \to \infty, \tag{2}$$

with $r = |\mathbf{x}|$ and $k = \omega/c$, where c is the constant wave speed. At infinity, the field $f(\mathbf{x}, \omega)$ can be written in terms of the far-field pattern $F(\widehat{\mathbf{x}}, \omega)$ as

$$f(\mathbf{x}, \omega) \sim F(\widehat{\mathbf{x}}, \omega)\frac{e^{ikr}}{4\pi r}, \quad r \to \infty, \tag{3}$$

and the field can be expanded in terms of spherical harmonics as

$$f(\mathbf{x}, \omega) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} f_{nm}(b, \omega)\frac{h_n^{(1)}(kr)}{h_n^{(1)}(kb)} Y_{nm}(\theta, \phi), \quad r \geq r_{\min}, \tag{4}$$

where the expansion coefficients

$$f_{nm}(b, \omega) = \int_{4\pi} f(b\widehat{\mathbf{x}}, \omega) Y_{nm}^*(\theta, \phi)\, \mathrm{d}\Omega(\theta, \phi) \tag{5}$$

are determined from the field on the sphere $r = b \geq r_{\min}$. Here $h_n^{(1)}(kr)$ is the spherical Hankel function of the first kind, $Y_{nm}(\theta, \phi)$ the spherical harmonic as defined by Jackson [14], $\widehat{\mathbf{x}}$ a unit vector pointing in the direction given by the spherical angles (θ, ϕ), and $*$ denotes complex conjugation. The expansion coefficients $f_{nm}(b_1, \omega)$ determined from the field at the sphere of radius $r = b_1$ are related to the expansion coefficients $f_{nm}(b, \omega)$ through

$$f_{nm}(b_1, \omega) = \frac{h_n^{(1)}(kb_1)}{h_n^{(1)}(kb)} f_{nm}(b, \omega). \tag{6}$$

The far-field pattern can also be expanded in terms of spherical harmonics

$$F(\widehat{\mathbf{x}}, \omega) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} F_{nm}(\omega) Y_{nm}(\theta, \phi), \tag{7}$$

where the expansion coefficients for the far-field pattern

$$F_{nm}(\omega) = \int_{4\pi} F(\widehat{\mathbf{x}}, \omega) Y_{nm}^*(\theta, \phi)\, \mathrm{d}\Omega(\theta, \phi) \tag{8}$$

are related to the expansion coefficients $f_{nm}(b, \omega)$ through

$$F_{nm}(\omega) = \frac{4\pi(-\mathrm{i})^{n+1}}{kh_n^{(1)}(kb)} f_{nm}(b, \omega). \tag{9}$$

It is assumed that outside the source region $f(\mathbf{x}, \omega) \to f_1(\mathbf{x})$ as $\omega \to 0$, where $f_1(\mathbf{x})$ is a bounded frequency-independent function. This assumption assures that the frequency-domain field has no singularities (with respect to ω) near $\omega = 0$. Furthermore, it will be assumed that a positive constant α exists such that $f(\mathbf{x}, \omega) = \mathrm{O}(e^{-\alpha\omega})$ as $\omega \to \infty$. This condition is met by any known generator and assures that the time-domain field is infinitely differentiable with respect to time.

Before presenting the frequency-domain beam formulas, we determine the behavior of the frequency-dependent expansion coefficients $F_{nm}(\omega)$ for small ω. From the identity [12, (5.4)] and the fact that $j_n(x) = \mathrm{O}(x^n)$ as $x \to 0$, we find that

$$\int_{4\pi} (\widehat{\mathbf{x}}' \cdot \mathbf{x})^q Y_{nm}(\theta', \phi')\, \mathrm{d}\Omega(\theta', \phi') = 0, \quad n > q, \tag{10}$$

where $\mathbf{x}$ is an arbitrary vector. Letting the surface in the Helmholtz representation be the sphere $r = b$, the far-field pattern can be written as

$$F(\widehat{\mathbf{x}}, \omega) = -b^2 \int_{4\pi} \left\{ \mathrm{i}k\widehat{\mathbf{x}} \cdot \widehat{\mathbf{x}}' f(b\widehat{\mathbf{x}}', \omega) + \frac{\partial f}{\partial r'}(b\widehat{\mathbf{x}}', \omega) \right\} \mathrm{e}^{-\mathrm{i}kb\widehat{\mathbf{x}} \cdot \widehat{\mathbf{x}}'} \, \mathrm{d}\Omega(\theta', \phi'). \tag{11}$$

Inserting the far-field expression (11) into formula (8) for the far-field expansion coefficients and employing (10) shows that $F_{nm}(\omega) = \mathrm{O}(\omega^n)$ as $\omega \to 0$. This low-frequency result for $F_{nm}(\omega)$ is derived under the assumption that the spectrum of $f(\mathbf{x}, \omega)$ has no singularity at $\omega = 0$ and will prove useful in the subsequent time-domain derivation. Of course, if $f(\mathbf{x}, \omega) \to 0$ as $\omega \to 0$, the coefficients $F_{nm}(\omega)$ approach zero faster than ω^n.

2.1. First frequency-domain beam formula

This section presents the first of the frequency-domain beam formulas that will be translated into the time domain in Section 4. In [12,13] we showed that the frequency-domain field $f(\mathbf{x}, \omega)$ for $\omega \geq 0$ can be written in terms of complex point-source fields as follows:

$$f(\mathbf{x}, \omega) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \int_{4\pi} Y_{nm}(\theta', \phi') \frac{F_{nm}(\omega)}{4\pi \mathrm{i}^{-n} j_n(\mathrm{i}ka)} \frac{\mathrm{e}^{\mathrm{i}ks}}{4\pi s} \, \mathrm{d}\Omega(\theta', \phi'), \qquad r \geq r_{\min}, \tag{12}$$

where a is a positive constant satisfying $r_{\min} \geq a$, and $j_n(kr)$ is the spherical Bessel function. Furthermore, $s = |\mathbf{x} - \mathrm{i}a\boldsymbol{\nu}| = s_\mathrm{r} + \mathrm{i}s_\mathrm{i}$ is the complex length whose real and imaginary parts are given by s_r and s_i, respectively. The unit vector $\boldsymbol{\nu}$ defines the point on the unit sphere given by the primed spherical coordinates (θ', ϕ'), and s has a branch cut on the disk whose radius, center, and normal are given by a, $\mathbf{x} = 0$, and $\boldsymbol{\nu}$, respectively. The real part of s satisfies $s_\mathrm{r} \geq 0$, whereas the imaginary part of s satisfies $-a \leq s_\mathrm{i} \leq a$ with its minimum value $-a$ attained only when $\widehat{\mathbf{x}} = \boldsymbol{\nu}$ and its maximum value a attained only when $\widehat{\mathbf{x}} = -\boldsymbol{\nu}$. Employing (9), we can also write $f(\mathbf{x}, \omega)$ in terms of the coefficients $f_{nm}(b, \omega)$ that are determined from the field on the sphere $r = b$. Comparing the complex point-source expansion (12) to the standard spherical harmonics expansion (4) with (9) inserted, one finds that the multipole functions $h_n^{(1)}(kr) Y_{nm}(\theta, \phi)$ can be written in terms of the complex point sources as

$$h_n^{(1)}(kr) Y_{nm}(\theta, \phi) = \frac{1}{\mathrm{i}k j_n(\mathrm{i}ka)} \int_{4\pi} Y_{nm}(\theta', \phi') \frac{\mathrm{e}^{\mathrm{i}ks}}{4\pi s} \, \mathrm{d}\Omega(\theta', \phi'), \tag{13}$$

which is a generalization of Norris' result [7].

2.2. Second frequency-domain beam formula

Another frequency-domain beam formula can be obtained from Wittmann's identity [15, Eqs. (17), (70)]

$$h_n^{(1)}(kr) Y_{nm}(\theta, \phi) = \mathrm{i}^{-n} \widetilde{P_{nm}^{\omega}} \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr}, \tag{14}$$

where the operator $\widetilde{P_{nm}^{\omega}}$ is given by

$$\widetilde{P_{nm}^{\omega}} = (-1)^m \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-m)!}{(n+m)!}} \left(\frac{1}{\mathrm{i}k} \partial_x + \mathrm{i} \frac{1}{\mathrm{i}k} \partial_y \right)^m P_n^{(m)} \left(\frac{1}{\mathrm{i}k} \partial_z \right), \qquad m \geq 0, \tag{15}$$

with $\partial_x = \partial/(\partial x)$, $\partial_y = \partial/(\partial y)$, and $\partial_z = \partial/(\partial z)$. Here $P_n^{(m)}(x)$ is simply the m'th derivative of the Legendre polynomial (it is *not* the associated Legendre function), and for $m < 0$ the operator $\widetilde{P_{nm}^{\omega}}$ can be determined from

$\widetilde{P^{\omega}_{n,-m}} = (-1)^m \widetilde{P^{\omega}_{nm}}^*$. According to Norris [7], the real point source field can be expressed in terms of complex point sources as

$$\frac{\mathrm{e}^{ikr}}{4\pi r} = \frac{1}{4\pi j_0(\mathrm{i}ka)} \int\limits_{4\pi} \frac{\mathrm{e}^{iks}}{4\pi s} \,\mathrm{d}\Omega(\theta', \phi'). \tag{16}$$

Combining Norris' beam expansion (16) with Wittmann's operator formula (14), it is found that

$$h_n^{(1)}(kr) Y_{nm}(\theta, \phi) = \frac{\mathrm{i}^{-n}}{\mathrm{i}kj_0(\mathrm{i}ka)} \int\limits_{4\pi} \widetilde{P^{\omega}_{nm}} \frac{\mathrm{e}^{iks}}{4\pi s} \,\mathrm{d}\Omega(\theta', \phi') \tag{17}$$

and thus the basis functions in the spherical expansion (4) are expressed in terms of multipoles with complex source points. Some of these complex multipoles have been discussed by Shin and Felsen [17] who showed that they reduce to Hermite–Gaussian beams in the paraxial region. Inserting expression (17) into (4) and using (9) gives us the desired general beam expansion

$$f(\mathbf{x}, \omega) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \frac{F_{nm}(\omega)}{j_0(\mathrm{i}ka)} \int\limits_{4\pi} \widetilde{P^{\omega}_{nm}} \frac{\mathrm{e}^{iks}}{16\pi^2 s} \,\mathrm{d}\Omega(\theta', \phi'), \qquad r \geq r_{\min}. \tag{18}$$

One may insert (9) into (18) to get a beam formula that expresses the field outside the minimum sphere in terms of the field on any sphere $r = b \geq r_{\min}$.

3. Analytic signals

In the following section, we shall make use of the analytic Fourier transform [16] to derive the time-domain analogs of the exact beam formulas (12) and (18). One of the reasons for using analytic Fourier transform, rather than the real standard Fourier transform, is that the beam formulas (12) and (18) hold only for $\omega \geq 0$ and contain the factors e^{-ka} and $1/j_n(\mathrm{i}ka)$.

To use the real standard Fourier transform, one has to define the frequency-domain fields for all real frequencies, and these frequency-domain fields must satisfy the relation $h(-\omega) = h^*(\omega)$. For example, the factor e^{-ka} (valid only for $\omega \geq 0$) would translate into $\mathrm{e}^{-|k|a}$ (valid for all real ω) and give rise to a convolution involving the function

$$\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-|k|a} \mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}\omega = \frac{2(a/c)}{(a/c)^2 + t^2}. \tag{19}$$

As will be demonstrated below, if instead the analytic Fourier transform is used, the factor e^{-ka} gives rise to a simple complex time shift and no convolution need to be introduced. One also finds that if the analytic Fourier transform is used, the factor $1/j_n(\mathrm{i}ka)$ can be represented in the time-domain by time integrals, time derivatives, and complex time shifts. No convolution need to be introduced. This ability to represent complicated operations in the time-domain, without introducing convolutions, is the main reason for using the analytic Fourier transform in this work.

Some of the properties and definitions of the analytic field $h^+(t)$, corresponding to the real time-domain field $h(t)$, will now be listed. The real time-domain field can be determined from its standard Fourier transform through the equations

$$h(t) = \int\limits_{-\infty}^{+\infty} h(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}\omega, \qquad h(\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} h(t) \mathrm{e}^{\mathrm{i}\omega t} \,\mathrm{d}t. \tag{20}$$

The analytic field $h^+(t)$ can be defined in terms of the Fourier transform $h(\omega)$ by

$$h^+(t) = 2\int_0^{+\infty} h(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \quad \mathrm{Im}(t) \le 0, \tag{21}$$

or directly in terms of the real field $h(t)$ by

$$h^+(t) = \begin{cases} h(t) - \mathrm{i}\mathcal{H}h(t) & \text{for real } t, \\ \dfrac{1}{\mathrm{i}\pi}\displaystyle\int_{-\infty}^{+\infty} \frac{h(t')}{t-t'}\,\mathrm{d}t' & \text{for } \mathrm{Im}(t) < 0, \end{cases} \tag{22}$$

where

$$\mathcal{H}h(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{h(t')}{t-t'}\,\mathrm{d}t' \tag{23}$$

is the Hilbert transform of $h(t)$.

From the integral (21) it is seen that $h^+(t)$ is analytic in the region $\mathrm{Im}(t) < 0$ and is defined only for $\mathrm{Im}(t) \le 0$. Also, (22) shows that the real signal $h(t)$ can be recovered from the analytic signal $h^+(t)$ through the equation

$$h(t) = \mathrm{Re}(h^+(t)) \quad \text{for real } t. \tag{24}$$

One may show that if $h(\omega) = h_1(\omega)h_2(\omega)$, then for $\mathrm{Im}(t) \le 0$

$$h^+(t) = \frac{1}{2\pi}h_1^+ \otimes h_2(t) = \frac{1}{2\pi}h_2^+ \otimes h_1(t) = \frac{1}{4\pi}h_1^+ \otimes h_2^+(t) = \frac{1}{4\pi}h_2^+ \otimes h_1^+(t), \tag{25}$$

where the convolution operator $\otimes$ is defined by

$$h_1 \otimes h_2(t) = \int_{-\infty}^{+\infty} h_1(t-t')h_2(t')\,\mathrm{d}t'. \tag{26}$$

We will now investigate the decay of the analytic signal $h^+(t)$ at infinity in the complex t plane and write $t = t_1 + \mathrm{i}t_2$, where both t_1 and t_2 are real and $t_2 \le 0$. Assume first that $h(\omega) = \mathrm{O}(\omega^n)$ as $\omega \to 0$ and that there exists an $\alpha > 0$ such that $h(\omega) = \mathrm{O}(\mathrm{e}^{-\omega\alpha})$ as $\omega \to +\infty$. From Watson's Lemma [18, p. 103], it is immediately found that $h^+(t_1 + \mathrm{i}t_2) = \mathrm{O}(t_2^{-n-1})$ as $t_2 \to -\infty$ when t_1 is fixed. Similarly, from [18, p. 232] it is found that $h^+(t_1 + \mathrm{i}t_2) = \mathrm{O}(t_1^{-n-1})$ as $|t_1| \to +\infty$ when t_2 is fixed.

Let us now briefly describe the analytical wave fields that will be used as basis functions in the time-domain expansions. As in [8], we shall employ time-domain pulsed beams that are the time-domain analogs of the frequency-domain complex point-source fields $h(\omega)\mathrm{e}^{\mathrm{i}ks}/(4\pi s)$, where $h(\omega)$ is some frequency-domain function and s is defined in Section 2. Since $s_\mathrm{i} = \mathrm{Im}(s)$ satisfies $-a \le s_\mathrm{i} \le a$, and thus becomes negative, we cannot immediately define an analytic signal corresponding to the frequency-domain beam $h(\omega)\mathrm{e}^{\mathrm{i}ks}/(4\pi s)$ for all t with $\mathrm{Im}(t) \le 0$. The reason for this is the exponential growth of $\mathrm{e}^{\mathrm{i}ks}$ occurring when $s_\mathrm{i} < 0$. Instead we consider the beam field

$$v(\mathbf{x}, \omega) = \frac{h(\omega)\mathrm{e}^{-ka}\mathrm{e}^{\mathrm{i}ks}}{4\pi s}, \tag{27}$$

which has an analytical time signal

$$v^+(\mathbf{x}, t) = \frac{h^+(t - s_r/c - \mathrm{i}(a/c + s_i/c))}{4\pi s} \tag{28}$$

defined for all complex times with $\mathrm{Im}(t) \leq 0$. The pulsed beam (28), as well as its real counterpart $v(\mathbf{x}, t) = \mathrm{Re}(v^+(\mathbf{x}, t))$ for real t, has been described in [8, Appendix]. To illustrate the physical properties of the pulsed beams, we consider the special case where $h(\omega) = -\mathrm{e}^{-\omega t_1}$ with $t_1 > 0$ for which the analytic beam is given by

$$v^+(\mathbf{x}, t) = \frac{1}{4\pi \mathrm{i} s[t - s/c - \mathrm{i}(t_1 + a/c)]}. \tag{29}$$

Furthermore, let the normal direction of the source disk be the z direction, so that $s = \sqrt{x^2 + y^2 + (z - \mathrm{i}a)^2}$ with $\mathrm{Re}(s) \geq 0$. The source disk is then located in the region given by the equations $z = 0$ and $x^2 + y^2 < a^2$. With these choices of the time function and the source disk, one finds that the analytic far-field pattern $V^+(\widehat{\mathbf{x}}, t)$, defined such that $v^+(\mathbf{x}, t) \sim V^+(\widehat{\mathbf{x}}, t - r/c)/(4\pi r)$ as $r \to \infty$, is given by

$$V^+(\widehat{\mathbf{x}}, t) = \frac{1}{t_1 + (1 - \cos\theta)a/c - \mathrm{i}t}, \tag{30}$$

where θ is the usual spherical angle. In Fig. 1 we have plotted the real far-field pattern $V(\widehat{\mathbf{x}}, t) = \mathrm{Re}(V^+(\widehat{\mathbf{x}}, t))$ for different angles of observation. It is seen that the pulsed beam has its maximum radiation in the direction $\theta = 0$ (which is the normal to the source disk), and that the radiation decreases rapidly as one moves away from $\theta = 0$.

In the following sections, we will express the time-domain field generated by time-dependent sources located in the region $r < r_{\min}$ as a superposition of pulsed beams of the form (28).

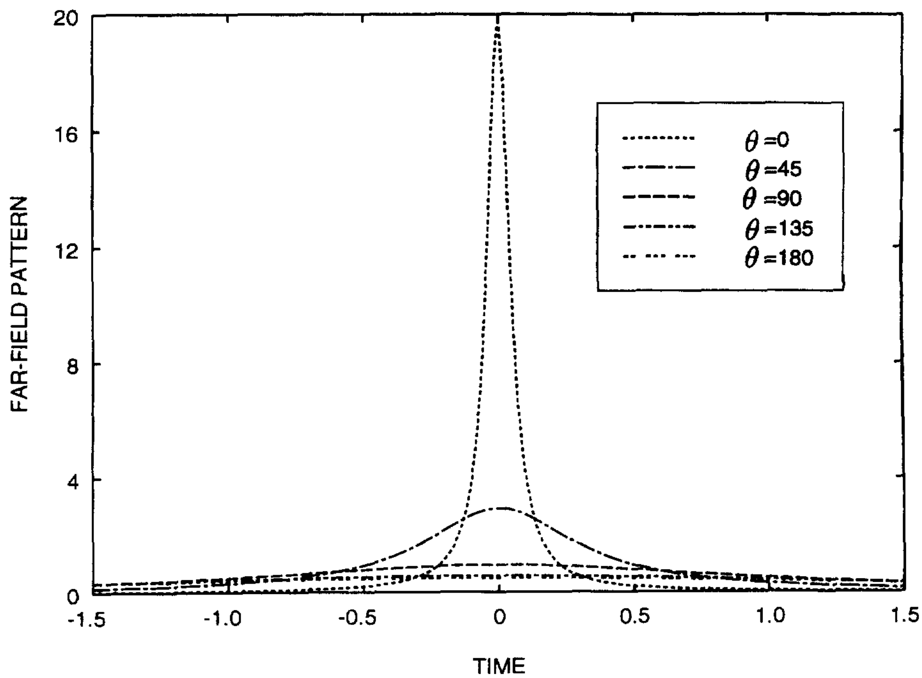

Fig. 1. The real far-field pattern of a complex pulsed beam with $t_1 = 0.05\,a/c$.

4. Time-domain beam formulas

In this section we use the analytic Fourier transform, described in Section 3, to derive the time-domain analogs of the first and second frequency-domain beam formulas given by (12) and (18), respectively.

4.1. First time-domain beam formula

Taking the analytic Fourier transform of the frequency-domain beam formula (12) shows that

$$f^+(\mathbf{x},t)=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\int_{4\pi} Y_{nm}(\theta',\phi')\,\frac{\widetilde{L_n}F_{nm}^+(t-s/c)}{16\pi^2 s}\,\mathrm{d}\Omega(\theta',\phi'),\tag{31}$$

where $\widetilde{L_n}$ is a time operator that will be determined below. This formula expresses the radiated field in terms of pulsed beams of the form (28). For each of the pulsed beams in this expansion, the direction normal to the source disk (which is also the main-beam direction) is given by the spherical integration variables (θ',ϕ'). The time function for each of these beams is $L_n F_{nm}^+(t-s/c)$, where $F_{nm}^+(t)$ is the analytic far-field expansion coefficients determined from the analytic far-field pattern by

$$F_{nm}^+(t)=\int_{4\pi} F^+(\widehat{\mathbf{x}},t)Y_{nm}^*(\theta,\phi)\,\mathrm{d}\Omega(\theta,\phi),\tag{32}$$

and the analytic far-field pattern $F^+(\widehat{\mathbf{x}},t)$ is defined such that $f^+(\mathbf{x},t)\sim F^+(\widehat{\mathbf{x}},t-r/c)/(4\pi r)$ as $r\to\infty$. In Section 4.3 it will be shown how $F_{nm}^+(t)$ can be determined from the field on any sphere $r=b$ with $b\geq r_{\min}$.

The analytic function $\widetilde{L_n}F_{nm}^+(t)$ is defined for $\mathrm{Im}(t)\leq a/c$ so that $\widetilde{L_n}F_{nm}^+(t-s/c)$ is defined for all t with $\mathrm{Im}(t)\leq 0$ (recall that $-a\leq \mathrm{Im}(s)\leq a$). In the frequency-domain, the operator $\widetilde{L_n}$ is represented by the real factor $[g_n(ka)]^{-1}$ with $g_n(ka)\equiv \mathrm{i}^{-n}j_n(\mathrm{i}ka)$. From [19, p. 437, Eq. (10.1.2)] it is seen that $g_n(ka)$ is a real function that increases and is positive for all $ka>0$. Moreover, $g_n(ka)\sim (ka)^n/[1\cdot 3\cdot 5\cdots(2n+1)]$ as $ka\to 0$ and $g_n(ka)\sim \mathrm{e}^{ka}/(2ka)$ as $ka\to\infty$.

Since $F_{nm}(\omega)=\mathrm{O}(\omega^n)$ as $\omega\to 0$, it follows from the results of the previous section that $F_{nm}^+(t_1+\mathrm{i}t_2)=\mathrm{O}(t_2^{-n-1})$ as $t_2\to-\infty$ for fixed t_1, and since $g_n(ka)=\mathrm{O}(\omega^n)$ as $\omega\to 0$, we find that $\widetilde{L_n}F_{nm}^+(t_1+\mathrm{i}t_2)=\mathrm{O}(t_1^{-1})$ as $|t_1|\to\infty$ for fixed t_2.

Let us begin by deriving an expression for the operator $\widetilde{L_0}$. For $ka>0$ we have

$$\frac{1}{g_0(ka)}=\frac{1}{j_0(\mathrm{i}ka)}=\frac{ka}{\sinh(ka)}=2ka\,\mathrm{e}^{-ka}\sum_{p=0}^{\infty}\mathrm{e}^{-2kap}\tag{33}$$

and to get an expansion for $1/g_0(ka)$ that is uniformly convergent for $ka\geq 0$ we note that $x^2\,\mathrm{e}^{-2xp}\leq \mathrm{e}^{-2}/p^2$ for all $x\geq 0$ and $\sum_{p=0}^{\infty}\mathrm{e}^{-2}/p^2<\infty$. Thus the series $\sum_{p=0}^{\infty}x^2\mathrm{e}^{-2xp}$ is uniformly convergent for $x\geq 0$. Writing [1]

$$\frac{1}{g_0(ka)}=\frac{2}{ka}\sum_{p=0}^{\infty}(ka)^2\mathrm{e}^{-ka(2p+1)}\tag{34}$$

[1] The series $\sum_{p=0}^{\infty}2ka\mathrm{e}^{-ka(2p+1)}$ is *not* uniformly convergent for $ka\geq 0$. To see this note that this series approaches one as $ka\to 0$, whereas all of its terms approach zero as $ka\to 0$.

and introducing the operators [2]

$$\partial_t^{-1} f^+(t) = \int_{-\infty}^{t} f^+(t')\,\mathrm{d}t', \qquad \partial_t = \frac{\partial}{\partial t} \tag{35}$$

we find that

$$\begin{aligned}
\widetilde{L}_0 F^+_{0,0}(t) &= \frac{\mathrm{i}a}{c}\partial_t^{-1} \int_0^{+\infty} (-\mathrm{i}\omega) \frac{F_{0,0}(\omega)}{g_0(ka)} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \\
&= \frac{2\mathrm{i}a}{c}\partial_t^{-1} \sum_{p=0}^{\infty} (\partial_t)^2 \int_0^{+\infty} \mathrm{e}^{-ka(2p+1)} F_{0,0}(\omega) \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \\
&= \frac{2\mathrm{i}a}{c}\partial_t^{-1} \sum_{p=0}^{\infty} (\partial_t)^2 F^+_{0,0}(t - \mathrm{i}a(1+2p)/c),
\end{aligned} \tag{36}$$

where we have used the fact that the series is uniformly convergent for $ka \geq 0$ to interchange integration and summation. Since $(\partial_t)^2 F^+_{0,0}(t_1 + \mathrm{i}t_2) = \mathrm{O}(t_1^{-3})$ we may apply the operator ∂_t^{-1} to each term in (36) to get the final expression

$$\widetilde{L}_0 F^+_{0,0}(t) = \frac{2\mathrm{i}a}{c} \sum_{p=0}^{\infty} \partial_t F^+_{0,0}(t - \mathrm{i}a(1+2p)/c). \tag{37}$$

The general asymptotic behavior $F^+_{0,0}(t_1 + \mathrm{i}t_2) = \mathrm{O}(t_2^{-1})$ as $t_2 \to -\infty$ shows that the derivative ∂_t cannot, in general, be moved outside the summation in (37).

Now consider the operator $\widetilde{L}_1$. We have

$$\frac{1}{g_1(ka)} = \frac{1}{\mathrm{i}^{-1} j_1(\mathrm{i}ka)} = \left[-\frac{\sin(\mathrm{i}ka)}{\mathrm{i}k^2a^2} + \frac{\cos(\mathrm{i}ka)}{ka}\right]^{-1} = \frac{ka/\cosh(ka)}{1 - \tanh(ka)/ka} \tag{38}$$

and since $|x^{-1}\tanh x| < 1$ for all $x > 0$ we find from (38) that

$$\frac{1}{g_1(ka)} = \frac{ka}{\cosh(ka)} \sum_{p=0}^{\infty} \left(\frac{\tanh(ka)}{ka}\right)^p \tag{39}$$

for all $ka > 0$. We want to write $1/g_1(ka)$ in terms of a series that is uniformly convergent for $ka \geq 0$ so that integration and summation can be interchanged. Begin by noting that the series $\sum_{p=0}^{\infty} x^3(x^{-1}\tanh(x))^p$ converges uniformly for $x \geq 0$ (this follows from the fact that the function $x^3(x^{-1}\tanh(x))^p$ has a maximum x_p given asymptotically by $x_p = 3/\sqrt{2p} + \mathrm{O}(1/p)$ for $p \to \infty$ and the fact that the series $\sum_{p=0}^{\infty} x_p^3(x_p^{-1}\tanh(x_p))^p$ converges). Writing

$$\frac{1}{g_1(ka)} = \frac{1}{(ka)^2\cosh(ka)} \sum_{p=0}^{\infty} (ka)^3 \left(\frac{\tanh(ka)}{ka}\right)^p, \tag{40}$$

[2] The operator ∂_t^{-1} can only be applied to functions $h^+(t)$ that satisfy $h^+(t) = \mathrm{O}(t^{-1-\epsilon})$ as $t \to -\infty$ for some $\epsilon > 0$. This condition can also be stated as $h(\omega) = \mathrm{O}(\omega^{\epsilon})$ as $\omega \to 0$.

recalling that $F_{1,m}(\omega) = \mathrm{O}(\omega)$ as $\omega \to 0$, and noting that $\partial_t \widetilde{L_1} F^+_{1,m}(t_1 + \mathrm{i}t_2) = \mathrm{O}(t_1^{-2})$ as $t_1 \to -\infty$ we find

$$\begin{aligned}\widetilde{L_1} F^+_{1,m}(t) &= \partial_t^{-1} \int_0^{+\infty} (-\mathrm{i}\omega) \frac{F_{1,m}(\omega)}{g_1(ka)} \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \\ &= \frac{c}{\mathrm{i}a} \partial_t^{-1} \sum_{p=0}^{\infty} \int_0^{+\infty} \frac{(ka)^2}{\cosh(ka)} \left(\frac{\tanh(ka)}{ka}\right)^p F_{1,m}(\omega) \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \\ &= \frac{c}{\mathrm{i}a} \sum_{p=0}^{\infty} \partial_t^{-1} \int_0^{+\infty} \frac{(ka)^2}{\cosh(ka)} \left(\frac{\tanh(ka)}{ka}\right)^p F_{1,m}(\omega) \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega, \end{aligned} \tag{41}$$

where the fact that the series in (40) is uniformly convergent for $ka \geq 0$ has been used to interchange the integration and summation. Because the Fourier integral in (41) is of order t_1^{-4} as $t_1 \to -\infty$, one can interchange the integral operator ∂_t^{-1} with the summation.

To compute the integral in (41) note that

$$\begin{aligned}\frac{(ka)^2}{\cosh(ka)} \left(\frac{\tanh(ka)}{ka}\right)^p &= \left(\frac{2\sinh(ka)}{(ka)^2 \mathrm{e}^{ka}}\right)^p \left[\frac{2(ka)^{p+2}\, \mathrm{e}^{-ka}\, \mathrm{e}^{(p+1)ka}}{(2\cosh(ka))^{p+1}}\right] \\ &= \left(\frac{1 - \mathrm{e}^{-2ka}}{(ka)^2}\right)^p \left[\frac{2(ka)^{p+2}\, \mathrm{e}^{-ka}}{(1 + \mathrm{e}^{-2ka})^{p+1}}\right], \end{aligned} \tag{42}$$

where we have arranged the factors to make the series expression for the square bracket uniformly convergent for $ka \geq 0$. For $\alpha > 1$ the series

$$\frac{x^\alpha}{1 + \mathrm{e}^{-2x}} = \sum_{q=0}^{\infty} (-1)^q x^\alpha \mathrm{e}^{-2xq} \tag{43}$$

is absolutely and uniformly convergent for $x \geq 0$. This is shown by noting that $x^\alpha \mathrm{e}^{-2xq} \leq (\alpha/(2q))^\alpha\, \mathrm{e}^{-\alpha}$ for all $x \geq 0$ and that $\sum_{q=0}^{\infty} (\alpha/(2q))^\alpha\, \mathrm{e}^{-\alpha}$ converges for $\alpha > 1$. The properties of the series (43) with $\alpha = (p+2)/(p+1)$ and the Binomial theorem show that

$$\frac{2(ka)^{p+2}\, \mathrm{e}^{-ka}}{(1 + \mathrm{e}^{-2ka})^{p+1}} = 2\sum_{q=0}^{\infty} (-1)^q (ka)^{p+2} \binom{p+q+1}{q} \mathrm{e}^{-ka(2q+1)}, \tag{44}$$

where $\binom{p}{q} = p!/q!(p-q)!$. Since series in (43) is absolutely and uniformly convergent (and bounded) for $x \geq 0$, it follows that the series in (44) (which is simply the series (43) raised to the power $p+1$) converges uniformly and absolutely for $ka \geq 0$. The operator $\widetilde{L_{1,1,p}}$ corresponding to the factor (44) can now be determined by integrating term by term to get

$$\widetilde{L_{1,1,p}} F^+_{1,m}(t) = 2\left(\frac{\mathrm{i}a}{c}\right)^{p+2} \sum_{q=0}^{\infty} (-1)^q \binom{p+q+1}{q} (\partial_t)^{p+2} F^+_{1,m}(t - \mathrm{i}a(2q+1)/c). \tag{45}$$

The second factor in the right-hand side of (42) is given by

$$\left(\frac{1 - \mathrm{e}^{-2ka}}{(ka)^2}\right)^p = (ka)^{-2p} \sum_{q=0}^{p} (-1)^q \binom{p}{q} \mathrm{e}^{-2kaq} \tag{46}$$

and its operator is

$$\widetilde{L_{1,2,p}}(\widetilde{L_{1,1,p}}F^+_{1,m})(t) = \left(\frac{c}{\mathrm{i}a}\right)^{2p} (\partial_t^{-1})^{2p} \sum_{q=0}^{p} (-1)^q \binom{p}{q} (\widetilde{L_{1,1,p}}F^+_{1,m})(t - 2\mathrm{i}aq/c). \tag{47}$$

With these definitions, the operator $\widetilde{L_1}$ in (41) is given by

$$\widetilde{L_1}F^+_{1,m}(t) = \frac{c}{\mathrm{i}a}\sum_{p=0}^{\infty} \partial_t^{-1}\widetilde{L_{1,2,p}}\widetilde{L_{1,1,p}}F^+_{1,m}(t). \tag{48}$$

It should be noted that one cannot in general interchange the operators $\widetilde{L_{1,1,p}}$ and $\widetilde{L_{1,2,p}}$ because $\widetilde{L_{1,2,p}}F^+_{1,m}(t)$ may not exist. The reason for this is that $(\partial_t^{-1})^q F^+_{1,m}(t)$ does not, in general, exist for $q > 1$.

Having determined $\widetilde{L_0}$ and $\widetilde{L_1}$ we will now derive a recursion relation that expresses $\widetilde{L_{n+1}}$ in terms of $\widetilde{L_{n-1}}$ and a simple operator $\widetilde{G_n}$. First note that the recursion relation [19, p. 439, Eq. (10.1.19)] for the spherical Bessel functions shows that

$$g_{n+1}(ka) = g_{n-1}(ka) - \frac{2n+1}{ka} g_n(ka). \tag{49}$$

Since $g_{n+1}(ka)$ is positive for $ka > 0$, Eq. (49) gives us

$$g_{n-1}(ka) > \frac{2n+1}{ka} g_n(ka) > 0 \tag{50}$$

for all $ka > 0$, and thus (49) shows that for all $n \geq 1$ and all $ka > 0$

$$\frac{1}{g_{n+1}(ka)} = \frac{1}{g_{n-1}(ka)}\left[1 - \frac{2n+1}{ka}\frac{g_n(ka)}{g_{n-1}(ka)}\right]^{-1} = \frac{1}{g_{n-1}(ka)}\sum_{p=0}^{\infty}\left(\frac{2n+1}{ka}\frac{g_n(ka)}{g_{n-1}(ka)}\right)^p, \tag{51}$$

where we have used that fact that

$$\left|\frac{2n+1}{ka}\frac{g_n(ka)}{g_{n-1}(ka)}\right| < 1 \quad \text{for } ka > 0.$$

We will prove that the series in the expression

$$\frac{1}{g_{n+1}(ka)} = \frac{1}{(ka)^3 g_{n-1}(ka)}\sum_{p=0}^{\infty}(ka)^3\left(\frac{2n+1}{ka}\frac{g_n(ka)}{g_{n-1}(ka)}\right)^p \tag{52}$$

converges uniformly for $ka \geq 0$. To do this note that

$$\frac{2n+1}{x}\frac{g_n(x)}{g_{n-1}(x)} = 1 - \frac{g_{n+1}(x)}{g_{n-1}(x)} = 1 - d_{n+1}(x), \tag{53}$$

where $d_{n+1}(x)$ has the following properties: $d_{n+1}(0) = 0$, $0 < d_{n+1}(x) < 1$ for all $x > 0$, $d_{n+1}(x) \sim x^2/[(2n+1)(2n+3)]$ as $x \to 0$, and $d'_{n+1}(x) > 0$ for all $x > 0$. Uniform convergence of the series in (52) can now be investigated by considering the series $\sum_{p=0}^{\infty} x^3(1 - d_{n+1}(x))^p$. The function $x^3(1 - d_{n+1}(x))^p$ has one and only one maximum (denoted by x_p) which satisfies $3(1 - d_{n+1}(x_p)) - p x_p d'_{n+1}(x_p) = 0$. This equation for x_p and the properties of $d_{n+1}(x)$ imply that $x_p \to 0$ as $p \to \infty$. Inserting a power series expansion for $d_{n+1}(x)$ into the equation for x_p shows that $x_p \sim \sqrt{3(2n+1)(2n+3)/(2p)} + \mathrm{O}(1/p)$ as $p \to \infty$. The series $\sum_{p=0}^{\infty} x_p^3(1 - d_{n+1}(x_p))^p$

converges and consequently the series in (52) is uniformly convergent for $ka \geq 0$. Since $F_{n+1,m}(\omega) = \mathrm{O}(\omega^{n+1})$ as $\omega \to 0$ we find that

$$\widetilde{L_{n+1}} F^{+}_{n+1,m}(t) = \int_0^{+\infty} \frac{F_{n+1,m}(\omega)}{g_{n+1}(ka)} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$$
$$= \left(\frac{c}{\mathrm{i}a}\right)^{n_0-n} (\partial_t^{-1})^{n_0-n} \widetilde{L_{n-1}} \sum_{p=0}^{\infty} \int_0^{+\infty} (ka)^{n_0-n}$$
$$\times \left(\frac{2n+1}{ka}\frac{g_n(ka)}{g_{n-1}(ka)}\right)^p F_{n+1,m}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{54}$$

where $n_0 \geq 2$ can be chosen arbitrarily. The restriction that $n_0 \geq 2$ ensures that $(ka)^{n_0-n} F_{n+1,m}(\omega) = \mathrm{O}(\omega^3)$ as $\omega \to 0$.

To obtain the final operator expression for $1/g_{n+1}(ka)$ we need the operator expression for $g_n(ka)$. Note that

$$g_n(ka) = \frac{\mathrm{i}^{-n}}{2}[h_n^{(1)}(\mathrm{i}ka) + h_n^{(2)}(\mathrm{i}ka)]$$
$$= (-\mathrm{i}\omega)^{-n-1} \sum_{p=0}^{n} \frac{(n+1/2,p)}{(2\mathrm{i}a/c)^{p+1}}[(-1)^p \mathrm{e}^{ka} + (-1)^{n+1}\mathrm{e}^{-ka}](-\mathrm{i}\omega)^{n-p} \tag{55}$$

where $(n+1/2,p) = (n+p)!/[p!(n-p)!]$. Since $g_n(ka) = \mathrm{O}(\omega^n)$ as $\omega \to 0$, the summation in (55) is of order ω^{2n+1} as $\omega \to 0$. This shows that the operator for $g_n(ka)$ is given by

$$\widetilde{G_n} F^{+}_{n+1,m}(t) = (\partial_t^{-1})^{n+1} \sum_{p=0}^{n} \frac{(n+1/2,p)}{(2\mathrm{i}a/c)^{p+1}} (\partial_t)^{n-p} [(-1)^p F^{+}_{n+1,m}(t+\mathrm{i}a/c)$$
$$+ (-1)^{n+1} F^{+}_{n+1,m}(t-\mathrm{i}a/c)]. \tag{56}$$

Even though $\widetilde{G_n} f^{+}(t)$ is defined only for $\mathrm{Im}(t) \leq -a/c$, the function $\widetilde{L_{n+1}}\widetilde{G_n} f^{+}(t)$ is defined for all t with $\mathrm{Im}(t) \leq 0$ because $g_{n-1}(ka) \sim \mathrm{e}^{ka}/(2ka)$. From (54) one can now obtain the final expression

$$\widetilde{L_{n+1}} = (\partial_t^{-1})^{n_0-n} \widetilde{L_{n-1}} \sum_{p=0}^{\infty} (\partial_t)^{n_0-n} \left(\frac{2n+1}{\mathrm{i}a/c} \partial_t^{-1} \widetilde{L_{n+1}} \widetilde{G_n}\right)^p, \tag{57}$$

where $n \geq 1$ and n_0 is an arbitrary integer satisfying $n_0 \geq 2$. For $n > 1$ one may choose $n_0 = n$ and thereby simplify the expression (57).

In summary, the operators $\widetilde{L_0}$ and $\widetilde{L_1}$ are given by (37) and (45), respectively; and the operator $\widetilde{L_{n+1}}$ with $n \geq 1$ is given recursively by (57). Now that these operators are determined, the field radiated by the source region can be computed from (31) in terms of pulsed beams.

In this section it has been assumed that the field outside the source region satisfies $f(\mathbf{x},\omega) \to f_1(\mathbf{x})$ as $\omega \to 0$, where $f_1(\mathbf{x})$ is a frequency-independent function. This rather general assumption made it necessary to be careful when interchanging operators, summations, and integrations. If instead the field satisfies the more restrictive condition that $f(\mathbf{x},\omega) = 0$ for $|\omega| \leq \omega_0$ and $f(\mathbf{x},\omega) = \mathrm{O}(\mathrm{e}^{-\omega\alpha})$ as $\omega \to +\infty$ with $\alpha > 0$, the derivations would be significantly simpler. In this case, all the frequency-domain summations would be uniformly convergent for $ka \geq 0$ because the problematic point $\omega = 0$ would be avoided. The time-domain field would satisfy $f^{+}(\mathbf{x}, t_1 + \mathrm{i}t_2) = \mathrm{O}(\mathrm{e}^{-\omega_0|t_2|})$ as $t_2 \to -\infty$ and the decay of the analytic expansion coefficients $F^{+}_{n,m}(t_1 + \mathrm{i}t_2)$

would be exponential for $t_2 \to -\infty$. Thus, operators and summations could be interchanged in the expressions (37), (48), and (57). For example, when the field satisfies this more restrictive frequency-dependence one can move the operator ∂_t in (37) outside the summation.

The simple case where $F_{nm} = 0$ for $(n, m) \neq (0, 0)$ was considered by Heyman [8] and corresponds to the source region consisting of one single point source located at the origin of the coordinate system. Choosing $F_{0,0}(\omega) = \sqrt{4\pi} h(\omega)$ such that $f(\mathbf{x}, \omega) = h(\omega)\mathrm{e}^{\mathrm{i}kr}/(4\pi r)$, Eq. (31) becomes

$$f^+(\mathbf{x}, t) = \frac{h^+(t - r/c)}{4\pi r} = \frac{\mathrm{i}a}{2\pi c} \sum_{p=0}^{\infty} \partial_t \int_{4\pi} \frac{h^+(t - s/c - \mathrm{i}a(1 + 2p)/c)}{4\pi s} \, \mathrm{d}\Omega(\theta', \phi'), \tag{58}$$

which is identical to Heyman's result [8, Eqs. (11)–(13)].[3]

4.2. Second time-domain beam formula

The second time-domain beam formula, which is much simpler than the first time-domain beam formula, is obtained by taking the analytic Fourier transform of (18). Before doing this we note that the operator $\widetilde{P^{\omega}_{nm}}$ in (15) is of order ω^{-n} as $\omega \to 0$. Since $F_{nm}(\omega) = \mathrm{O}(\omega^n)$ as $\omega \to 0$, it follows that $(\partial_t^{-1})^n F^+_{nm}(t)$ exists and $\widetilde{P^t_{nm}} F^+_{nm}(t)$ is well defined (the operator ∂_t^{-1} is given by (35)). We can now take the analytic Fourier transform of (18) to get the time-domain beam formula

$$f^+(\mathbf{x}, t) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \int_{4\pi} \widetilde{L_0} \widetilde{P^t_{nm}} \frac{F^+_{nm}(t - s/c)}{16\pi^2 s} \, \mathrm{d}\Omega(\theta', \phi'), \tag{59}$$

where $\widetilde{L_0}$ is the time operator given by (37), and $\widetilde{P^t_{nm}}$ is the time-domain analog of $\widetilde{P^{\omega}_{nm}}$ given by

$$\widetilde{P^t_{nm}} = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} c^m (\partial_t^{-1}\partial_x + \mathrm{i}\partial_t^{-1}\partial_y)^m P_n^{(m)}(-c\partial_t^{-1}\partial_z), \quad m \geq 0. \tag{60}$$

For $m < 0$ the operator $\widetilde{P^t_{nm}}$ can be determined from $\widetilde{P^t_{n,-m}} = (-1)^m \widetilde{P^t_{nm}}^{\,*}$.

Formula (59) expresses the radiated field in terms of analytic time-domain multipole fields, which are obtained by taking spatial and time derivatives of the pulsed beams in (28). For each of the pulsed beams in this expansion, the direction normal to the source disk is given by the spherical integration variables (θ', ϕ'). The analytic far-field expansion coefficients $F^+_{nm}(t)$ are determined from the analytic far-field pattern by (32). Some of the multipole fields in (59) have been discussed by Heyman and Beracha [20]. For the special case where $F_{nm} = 0$ for $(n, m) \neq (0, 0)$ expression (59) reduces to (58), and is thus in agreement with Heyman [8].

4.3. Expansion coefficients in terms of the field on a sphere $r = b$

The time-domain beam formulas (31) and (59) express the field outside the minimum sphere in terms of the analytic far-field expansion coefficients $F^+_{nm}(t)$. To get beam formulas that express the field in terms of the analytic expansion coefficients $f^+_{nm}(b, t)$, which are determined from the analytic field on the sphere $r = b$, we shall express $F^+_{nm}(t)$ in terms of $f^+_{nm}(b, t)$.

[3] The length b in Heyman's paper [8] is given in terms of our a by the equation $b = \mathrm{i}a$.

Begin by noting that the frequency spectrum $h(\omega)$ of any real function $h(t)$ satisfies $h^*(-\omega) = h(\omega)$ for all real ω. This relation is satisfied by the factor $4\pi \mathrm{i}^{-n-1}/[kh_n^{(1)}(kb)]$ occurring in the frequency-domain formula (9). Consequently, the convolution rule (25) for analytic fields shows that

$$F_{nm}^{+}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_{nm}^{+}(b, t - t') P_n(b, t') \, \mathrm{d}t', \tag{61}$$

where $P_n(b, t)$ is the real (not analytic) time-domain function

$$P_n(b, t) = \int_{-\infty}^{+\infty} \frac{4\pi(-\mathrm{i})^{n+1}}{kh_n^{(1)}(kb)} \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega, \tag{62}$$

which is defined only for real t.

The integral in (62) can be calculated by the use of contour integration in the complex ω plane, and it can be expressed in terms of residues as [6, Eq. (12)]

$$P_n(b, t) = 8\pi^2 b \left[\delta(t + b/c) - (-\mathrm{i})^n \frac{c^2}{b^2} u(t + b/c) \sum_{q=1}^{n} \frac{\mathrm{e}^{-\mathrm{i}\omega_{nq} t}}{\omega_{nq} h_n^{(1)'}(\omega_{nq} b/c)} \right], \tag{63}$$

where $u(t)$ is the unit step function given by $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t \geq 0$. The constants ω_{nq} are the zeros of the function $h_n^{(1)}(b\omega_{nq}/c)$. It can be shown that $h_n^{(1)}(z)$ has n simple zeros located symmetrically around the imaginary axis in the half plane $\mathrm{Im}(z) < 0$ inside the circle $|z| = n$ [21]. An asymptotic formula that determines these zeros for large n is presented in [22]. One may use the symmetry of the zeros ω_{nq} to show directly that the right-hand side of (63) indeed is real.

Inserting expression (63) for $P_n(b, t)$ into the convolution integral (61) gives

$$F_{nm}^{+}(t) = 4\pi b f_{nm}^{+}(b, t + b/c) - 4\pi(-\mathrm{i})^n \frac{c^2}{b} \sum_{q=1}^{n} \int_{-b/c}^{+\infty} \frac{f_{nm}^{+}(b, t - t')\mathrm{e}^{-\mathrm{i}\omega_{nq} t'}}{\omega_{nq} h_n^{(1)'}(b\omega_{nq}/c)} \, \mathrm{d}t', \tag{64}$$

which may in turn be inserted into (31) and (59) to get beam formulas that express the field for $r \geq r_{\min}$ in terms of its values on the sphere $r = b$.

5. Conclusion

In this paper we have presented new exact expansions of fields generated by sources in a finite region of space. We started by presenting two frequency-domain expansions. One of these uses Deschamps' complex points-source fields as basis functions. The other employs as basis functions multipoles obtained by applying Wittmann's operator for the spherical harmonics [15] to Deschamps' complex point-source fields.

Two time-domain expansions were then obtained by applying the analytic Fourier transform to each of the frequency-domain expansions. These time-domain expansions express the field of the source region in terms of pulsed beams.

For all expansions the basis functions are directionally localized and the expansion coefficients are obtained from the values of the field on a sphere, e.g., the far-field sphere. These new expansions can be useful for analyzing time-harmonic as well as transient radiation generated by a general source in an inhomogeneous medium. Numerical implementation of the time-domain beam formulas still has to be performed to investigate their usefulness.

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