

Acoustic diffraction by a fluid-loaded membrane corner

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The complete solution is derived for an acoustic plane wave incident on a membrane which is stretched over a line constraint, forming a wedge shaped fluid region of interior angle 2Φ . The exact two-dimensional solution is developed using Malyuzhinets functions and integral transforms. Explicit formulae are presented for the acoustic and structural diffraction coefficients for $\Phi > \pi/2$ and numerical results are given for the frequency and angular dependence of the membrane wave excitation coefficients for identical membranes on either side of the vertex.

1. Introduction

Sound diffraction by edges of elastic structures immersed in an acoustic medium is an important and challenging problem that has received considerable attention from acousticians for the last four decades, starting with the papers by Malyuzhinets (1958*a*) and Lamb (1959) on the diffraction by a thin elastic half-plane. In the case of rectangular geometries (half-planes and their junctions at right angles) the problem has been exhaustively studied by applying the Wiener–Hopf mathematical technique (relevant references can be found, for example, in Norris & Wickham (1995)). The solution of similar problems for arbitrary angle geometry requires distinctly different approaches which utilize other integral transforms such as Sommerfeld–Malyuzhinets (Malyuzhinets 1955*a, b*, 1958*b, c*, 1960; Williams 1959), Laplace (Senior 1959), or Kontorovich–Lebedev (Lebedev & Skalskaya 1962; Abrahams 1986, 1987). Malyuzhinets’s technique proved to be the most convenient since it enables treating with equal facility the diffraction by both rectangular (Malyuzhinets 1958*a*, 1971; Malyuzhinets & Tuzhilin 1970) and arbitrarily angled structures (Tuzhilin 1973).

A closed form analytical solution to the particular case of the acoustic diffraction by a fluid wedge with two plane elastic boundaries has been presented in Osipov (1993). The general solution to the related class of boundary contact problems involving the two-dimensional Helmholtz equation in a wedge shaped region supplemented with higher order boundary conditions has been given in Osipov (1994). The procedure modifies Malyuzhinets’s method so as to allow for arbitrary polynomial expressions as symbols of the boundary condition operators or, what is the same, each problem-specific distribution of the associated Brewster angles in the complex plane. A recent paper by Abrahams & Lawrie (1995) gives a theoretical treatment of membrane waves at a corner of arbitrary angle based upon another extension of

the Malyuzhinets technique. Their *ansatz* may be viewed as Osipov's general solution for the particular case of third order boundary conditions in which all complex Brewster angles have positive real parts analytically continued to the case of a real membrane in which one of the three associated Brewster angles has its real part negative. Unfortunately, this leads to unphysical behaviour of the solution as a function of the incidence angle since such a procedure adds a forbidden pole to this function, resulting therefore in an unbounded response of the system at a particular value of the incidence angle.

In this paper we study the diffraction of a time harmonic plane sound wave in a wedge shaped fluid region bounded by a pair of plane membranes on which the acoustic pressure is subject to third order boundary conditions. Based on the modified Malyuzhinets method (Osipov 1993, 1994), an accurate solution in the form of a Sommerfeld integral is deduced and investigated analytically in the far-field by applying the saddle point method, and also numerically, thereby demonstrating the physical and practical validity of the solution. A similar problem for an angular joint of thin elastic plates modelled by conventional fifth order boundary conditions is analysed in Osipov (1997) and Norris & Osipov (1997).

2. Basic equations

The equations of motion and continuity for a membrane with fluid on one side are

$$T \frac{\partial^2 w}{\partial s^2} - m \frac{\partial^2 w}{\partial t^2} = p, \quad \rho \frac{\partial^2 w}{\partial t^2} = -\frac{\partial p}{\partial n}. \quad (2.1)$$

Here, w is the displacement into the fluid, p is the acoustic pressure at the membrane surface, $\partial/\partial s$, $\partial/\partial n$ denote the tangential and normal derivatives with respect to the membrane surface, and m , T , and ρ are the membrane density, membrane tension, and the fluid density. We are interested in the time harmonic acoustic pressure field $p = p(r, \phi)e^{-i\omega t}$ satisfying (2.1) on the faces of a wedge and the Helmholtz equation outside the wedge,

$$\nabla^2 p + k^2 p = 0, \quad 0 < r < \infty, \quad -\Phi < \phi < \Phi, \quad (2.2)$$

where $k = \omega/c$ is the acoustic wave number. The incident field is supposed to be

$$p_i(r, \phi, \phi_0) = P_0 \exp(-ikr \cos(\phi - \phi_0)), \quad (2.3)$$

which represents a plane wave for $\phi_0 \in [-\Phi, \Phi]$ and a subsonic membrane surface wave travelling inwards on $\phi = \pm\Phi$ for the incident angle $\phi_0 = \pm\Phi \mp \theta_1^\pm$, where the complex Brewster angles $\theta_1^\pm$ are defined below. The incident wave and the membrane are depicted in figure 1. Following Osipov (1994) we consider the boundary value problem for $p(r, \phi)$, which satisfies the Helmholtz equation (2.2) and the higher order boundary conditions

$$\mathcal{L}_\pm \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \phi} \right) p(r, \phi) = 0, \quad 0 < r < \infty, \quad \phi = \pm\Phi, \quad (2.4)$$

where $\mathcal{L}_\pm$ are differential operators by which the material properties of the boundaries are being modelled. For the membrane, we have

$$\mathcal{L}_\pm \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \phi} \right) = \mp a_\pm \left(1 + \frac{M_\pm^2}{k^2} \frac{\partial^2}{\partial r^2} \right) \frac{1}{r} \frac{\partial}{\partial \phi} - 1, \quad (2.5)$$

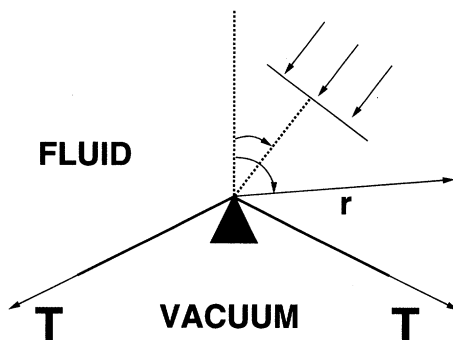


Figure 2. The integration contours.

The problem is defined by equations (2.2) in the fluid, (2.3) at infinity, (2.4), (2.5) on the wedge faces, and the conditions at the tip $r = 0$, which we assume to be pinned, implying that w vanishes there, or

$$\lim_{r \rightarrow 0} \frac{1}{r} \frac{\partial p}{\partial \phi} = 0, \quad \phi = \pm \Phi. \quad (2.6)$$

3. General solution

(a) Field representation and functional equations

We seek the solution to the boundary value problem of (2.2) through (2.6) in the form of a Sommerfeld integral

$$p(r, \phi) = \frac{P_0}{2\pi i} \int_{\gamma} e^{-ikr \cos \alpha} S(\alpha + \phi) d\alpha. \quad (3.1)$$

Figure 2 depicts the integration contour $\gamma = \gamma_+ \cup \gamma_-$, where γ_+ is a loop in the upper half of the complex α -plane, beginning at $\pi/2 + i\infty$, ending at $-3\pi/2 + i\infty$, with $\text{Im } \alpha$ lying above a well defined minimum, which we determine later. The contour γ_- is the image of γ_+ under inversion about the origin $\alpha = 0$. Thus,

$$p(r, \phi) = \frac{P_0}{2\pi i} \int_{\gamma_+} e^{-ikr \cos \alpha} (S(\alpha + \phi) - S(-\alpha + \phi)) d\alpha. \quad (3.2)$$

The ‘transform’ $S(\alpha)$ which is sought should be a meromorphic function of a complex variable α , have a single pole with a unit residue at a point $\alpha = \phi_0$ in the strip $\Pi_0 = \{\alpha : |\text{Re } \alpha| \leq \Phi\}$ to reproduce the incident field (2.3), and be bounded when $\text{Im } \alpha \rightarrow \pm\infty$ satisfying the condition

$$|S(\alpha + \phi) - S(\pm\infty)| \rightarrow 0,$$

uniformly in ϕ , to ensure the boundedness and continuity of the acoustic pressure $p(r, \phi)$ at the tip $r = 0$. Note that because of (3.2) the integral (3.1) is invariant under the transformation $S(\alpha) \rightarrow S(\alpha) + \text{const}$. Choosing appropriately this constant we can set, without loss of generality,

$$S(i\infty) = -S(-i\infty), \quad (3.3)$$

which will be assumed hereinafter.

Inserting (3.1) into the boundary conditions (2.4) leads to a pair of integral identities

$$\int_{\gamma} e^{-ikr \cos \alpha} L_{\pm}(\alpha) S(\alpha \pm \Phi) d\alpha = 0, \quad 0 \leq r < \infty, \quad (3.4)$$

where the functions $L_{\pm}(\alpha) = \mathcal{L}_{\pm}(-ik \cos \alpha, -ik \sin \alpha)$ will be referred to as ‘symbols’ of the boundary condition operators. From (2.5) we have

$$L_{\pm}(\alpha) = \pm ika_{\pm} \sin \alpha (1 - M_{\pm}^2 \cos^2 \alpha) - 1. \quad (3.5)$$

For the class of functions prescribed above the integrands in (3.4) must be analytic inside the loops $\gamma_{\pm}$; as a consequence of this and Malyuzhinets’s theorem (Malyuzhinets 1958b), the integral equations (3.4) can be replaced by a set of functional equations

$$L_{\pm}(\alpha) S(\alpha \pm \Phi) - L_{\pm}(-\alpha) S(-\alpha \pm \Phi) = 2 \sin \alpha \sum_{n=1}^{N_{\pm}} C_n^{\pm} \cos^{n-1} \alpha, \quad (3.6)$$

with arbitrary constants $C_n^{\pm}$, $n = 1, 2, \dots, N_{\pm}$. The orders of the trigonometric polynomials in the right member of (3.6) are determined by the behaviour of the functions $L_{\pm}(\alpha) S(\alpha \pm \Phi)$ at infinity. In the case considered, because of the equation (3.5) and the boundedness of $S(\pm\infty)$, the terms in the left part of (3.6) may not grow faster than $O(|\sin \alpha \cos^2 \alpha|)$ as $\text{Im } \alpha \rightarrow \pm\infty$, and that means $N_{\pm} \leq 3$. Then putting $\text{Im } \alpha = \infty$ in (3.6), we arrive at the relations

$$C_3^{\pm} = \mp \frac{1}{2} ika_{\pm} M_{\pm}^2 (S(+i\infty) + S(-i\infty)),$$

which owing to (3.3) give

$$C_3^{\pm} = 0.$$

Thus, in accordance with general theory (Osipov 1994), satisfaction of the boundary conditions (2.4), (2.5) reduces the problem to a system of functional equations (3.6) with four arbitrary constants $C_n^{\pm}$, $n = 1, 2$, which are to be specified so as to meet the other conditions.

(b) The auxiliary function $\Psi(\alpha)$

In order to solve the coupled functional equations (3.6) we follow the procedure originally proposed by Malyuzhinets (1958c). It consists in transforming the initial system of functional equations (3.6) with variable coefficients $L_{\pm}(\alpha)$ to that with constant ones by substitution

$$S(\alpha) = \Psi(\alpha)\tilde{S}(\alpha), \quad (3.7)$$

where $\tilde{S}(\alpha)$ denotes a new unknown function. The function $\Psi(\alpha)$ is an auxiliary function which is supposed to be a particular solution of the homogeneous system

$$L_{\pm}(\alpha)\Psi(\alpha \pm \Phi) - L_{\pm}(-\alpha)\Psi(-\alpha \pm \Phi) = 0. \quad (3.8)$$

With this substitution, the functional equations (3.6) become

$$\tilde{S}(\alpha \pm \Phi) - \tilde{S}(-\alpha \pm \Phi) = f_{\pm}(\alpha), \quad (3.9)$$

where

$$f_{\pm}(\alpha) = \frac{2 \sin \alpha}{L_{\pm}(\alpha)\Psi(\alpha \pm \Phi)} \sum_{n=1}^2 C_n^{\pm} \cos^{n-1} \alpha. \quad (3.10)$$

A partial solution to (3.8), regular and free of zeros in the strip Π_0 , can be constructed by introducing its logarithmic derivative $Y(\alpha) = \Psi'(\alpha)/\Psi(\alpha)$. This satisfies a pair of difference equations with constant coefficients

$$Y(\alpha \pm \Phi) + Y(-\alpha \pm \Phi) = \pm \frac{R'_{\pm}(\alpha)}{R_{\pm}(\alpha)}, \quad (3.11)$$

where

$$R_{\pm}(\alpha) = -\frac{L_{\pm}(\mp\alpha)}{L_{\pm}(\pm\alpha)} = \frac{ika_{\pm}(1 - M_{\pm}^2 \cos^2 \alpha) \sin \alpha + 1}{ika_{\pm}(1 - M_{\pm}^2 \cos^2 \alpha) \sin \alpha - 1} \quad (3.12)$$

is the acoustic reflection coefficient for an infinite planar membrane with fluid on one side. The solution of the difference equations in (3.11), regular in Π_0 , is deduced via the modified Fourier transformation

$$Y(\alpha) = \int_{-i\infty}^{i\infty} e^{-it\alpha} \eta(t) dt, \quad \eta(t) = -\frac{1}{2\pi} \int_{-i\infty}^{i\infty} e^{it\alpha} Y(\alpha) d\alpha, \quad (3.13)$$

which differs from the conventional one only by a change of variables. This procedure reduces (3.11) to an algebraic problem for the Fourier transform $\eta(t)$

$$\eta(t)e^{\mp it\Phi} + \eta(-t)e^{\pm it\Phi} = \mp \frac{1}{2\pi} \int_{-i\infty}^{i\infty} e^{it\alpha} \frac{R'_{\pm}(\alpha)}{R_{\pm}(\alpha)} d\alpha, \quad (3.14)$$

whose solution, on inverse Fourier transformation, takes the form of a product of special Malyuzhinets functions $\psi_{\Phi}(\alpha)$ (Osipov 1994),

$$\left. \begin{aligned} \Psi(\alpha) &= \Psi_+(\alpha)\Psi_-(\alpha), \\ \Psi_{\pm}(\alpha) &= \prod_{n=1}^3 (\psi_{\Phi}(\alpha \pm \Phi + \tfrac{1}{2}\pi - s_n^{\pm}\theta_n^{\pm})\psi_{\Phi}(\alpha \pm \Phi - \tfrac{1}{2}\pi + s_n^{\pm}\theta_n^{\pm}))^{s_n^{\pm}}, \end{aligned} \right\} \quad (3.15)$$

where $s_n^{\pm} = \text{sgn}(\text{Re } \theta_n^{\pm})$, which will be verified later. Notice that the function $\Psi(\alpha)$ above is free both of poles and zeros in the strip Π_0 whether the real parts of $\theta_n^{\pm}$ are

positive or not, and this is in contrast to what is used by Abrahams & Lawrie (1995) where the only precaution taken is to make this function free of poles in this strip. As it will be evident from the analytical form of the solution (see equation (4.24)), any zero of $\Psi(\alpha)$ in that portion of the complex α -plane will produce a pole of the solution at a point ϕ_0 located inside the strip $|\operatorname{Re} \phi_0| \leq \Phi$, thereby introducing an unphysical singularity of the wave field for a particular value of the incidence angle.

The parameters $\theta_n^\pm$, $n = 1, 2, 3$, are complex numbers defined by

$$L_\pm(\mp\theta_n^\pm) = 0, \quad \operatorname{Re} \theta_n^\pm \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \quad (3.16)$$

and so because of (3.12) they can be interpreted as Brewster angles for the boundaries, that is, grazing angles at which an incident plane wave does not produce a reflected one.

The function $\psi_\Phi(\alpha)$ originally introduced in Malyuzhinets (1958c) is an even and meromorphic function of a complex variable α , which may be represented as (Zavadskii & Sakharova 1967; Osipov 1990a)

$$\psi_\Phi(\alpha) = \exp\left(-\frac{1}{2} \int_0^{+\infty} \frac{\cosh(t\alpha) - 1}{t \cosh(t\pi/2) \sinh(2t\Phi)} dt\right), \quad (3.17)$$

if $|\operatorname{Re} \alpha| < \pi/2 + 2\Phi$. It may be continued outside this strip by using its functional properties (Malyuzhinets 1958c):

$$\frac{\psi_\Phi(\alpha + 2\Phi)}{\psi_\Phi(\alpha - 2\Phi)} = \cot\left(\frac{\alpha}{2} + \frac{\pi}{4}\right), \quad (3.18)$$

$$\psi_\Phi(\alpha + \tfrac{1}{2}\pi)\psi_\Phi(\alpha - \tfrac{1}{2}\pi) = \psi_\Phi^2(\tfrac{1}{2}\pi) \cos\left(\frac{\pi\alpha}{4\Phi}\right), \quad (3.19)$$

$$\psi_\Phi(\alpha + \Phi)\psi_\Phi(\alpha - \Phi) = \psi_\Phi^2(\Phi)\psi_{\Phi/2}(\alpha). \quad (3.20)$$

As a function of the complex variable α , at a set of points

$$\alpha = \pm\alpha_{nm}, \quad \alpha_{nm} = \tfrac{1}{2}\pi(2m - 1) + 2\Phi(2n - 1), \quad (3.21)$$

$n = 1, 2, 3, \dots$, $\psi_\Phi(\alpha)$ has its zeros for $m = 1, 3, 5, \dots$ and its poles for $m = 2, 4, 6, \dots$. If $\Phi = n\pi/(4m)$ with m and n integer, $\psi_\Phi(\alpha)$ can be expressed through trigonometric functions

$$\psi_{n\pi/4m}(\alpha) = \prod_{q=1}^m \prod_{l=1}^n \left(\frac{\cos(\frac{1}{2}a(q, l))}{\cos(\alpha/2n + \frac{1}{2}a(q, l))} \right)^{(-1)^l}, \quad (3.22)$$

for odd values of n , and

$$\psi_{n\pi/4m}(\alpha) = \prod_{q=1}^m \prod_{l=1}^n \exp\left(\frac{(-1)^l}{\pi} \int_{a(q, l)}^{a(q, l) + \alpha/n} u \cot u du\right), \quad (3.23)$$

for even n , where

$$a(q, l) = \frac{1}{2}\pi \left(\frac{2l - 1}{n} - \frac{2q - 1}{m} \right).$$

The representation (3.15) for the auxiliary function $\Psi(\alpha)$ depends on the parameters $s_n^\pm$ identifying the signs of real parts of $\theta_n^\pm$, $n = 1, 2, 3$. To determine them, we consider two related functions

$$F_\pm(X) = M_\pm^2 X^3 + (1 - M_\pm^2)X - i/ka_\pm, \quad (3.24)$$

whose roots $X_n^\pm$, $n = 1, 2, 3$, are associated with the Brewster angles by the formula: $X_n^\pm = \sin \theta_n^\pm$. Solving the equation $F_\pm(X_n^\pm) = 0$ gives

$$X_n^\pm = i \frac{2}{\sqrt{3}} \frac{\sqrt{1 - M_\pm^2}}{M_\pm} \sin \beta_n^\pm, \quad (3.25)$$

where

$$\beta_n^\pm = \begin{cases} -\frac{1}{3}\pi - \beta_0^\pm, & \frac{1}{3}\pi - \beta_0^\pm, & \beta_0^\pm, & \text{if } M_\pm < 1, \\ \beta_0^\pm, & \frac{1}{3}\pi - \beta_0^\pm, & -\frac{1}{3}\pi - \beta_0^\pm, & \text{if } M_\pm > 1, \end{cases}$$

$$\sin(3\beta_0^\pm) = \frac{k_\pm}{k}, \quad k_\pm a_\pm = \frac{3\sqrt{3}M_\pm}{2(1 - M_\pm^2)^{3/2}}.$$

For real and positive values of $M_\pm$, $a_\pm$ and k , the related function $F_\pm(X)$ has either three completely imaginary roots when $0 < M_\pm \leq 1$ and $0 < k_\pm \leq k$, or one completely imaginary root $X_1^\pm$ and two complex ones $X_{2,3}^\pm$ such that $\text{Re } X_3^\pm = -\text{Re } X_2^\pm > 0$, $\text{Im } X_3^\pm = \text{Im } X_2^\pm > 0$, otherwise. The imaginary roots $\theta_1^\pm$ have $\text{Im } \theta_1^\pm < 0$ and correspond to a subsonic structural wave which propagates unattenuated along each membrane and decays exponentially into the fluid.

The analytical procedure which is used here requires removing the ambiguity concerning the values of parameters $s_n^\pm$ if $\text{Re } X_n^\pm = 0$. To do this, we introduce a small amount of dissipation in the membrane by giving the membrane tension $T_\pm$ a small negative imaginary part, thus shifting these roots from the imaginary axis in the complex X -plane to get $\text{Re } X_{1,3}^\pm > 0$, $\text{Re } X_2^\pm < 0$. In consequence of this, we have $\text{Re } \theta_{1,3}^\pm \in (0, \pi/2)$, $\text{Re } \theta_2^\pm \in (-\pi/2, 0)$ and the signs of the real parts of the $\theta_n^\pm$ are given by

$$s_n^\pm = (-1)^{n-1}, \quad n = 1, 2, 3, \quad (3.26)$$

thereby specifying the analytical form of the auxiliary function

$$\Psi(\alpha) = \frac{\Psi_1^+(\alpha) \Psi_3^+(\alpha)}{\Psi_2^+(\alpha)} \frac{\Psi_1^-(\alpha) \Psi_3^-(\alpha)}{\Psi_2^-(\alpha)}, \quad (3.27)$$

$$\Psi_n^\pm(\alpha) = \psi_\Phi(\alpha \pm \Phi + \frac{1}{2}\pi + (-1)^n \theta_n^\pm) \psi_\Phi(\alpha \pm \Phi - \frac{1}{2}\pi - (-1)^n \theta_n^\pm). \quad (3.28)$$

To check this representation, one can express the symbols $L_\pm(\alpha)$ as products

$$L_\pm(\alpha) = \pm i k a_\pm M_\pm^2 (\sin \alpha \pm \sin \theta_1^\pm) (\sin \alpha \pm \sin \theta_2^\pm) (\sin \alpha \pm \sin \theta_3^\pm), \quad (3.29)$$

and rewrite equations (3.8) for $\Psi(\alpha)$ as

$$\frac{\Psi(\alpha \pm \Phi)}{\Psi(-\alpha \pm \Phi)} = \frac{(-\sin \alpha \pm \sin \theta_1^\pm)(-\sin \alpha \pm \sin \theta_2^\pm)(-\sin \alpha \pm \sin \theta_3^\pm)}{(\sin \alpha \pm \sin \theta_1^\pm)(\sin \alpha \pm \sin \theta_2^\pm)(\sin \alpha \pm \sin \theta_3^\pm)}. \quad (3.30)$$

It follows from the functional properties of the function $\psi_\Phi(\alpha)$ (see equation (3.18)) that

$$\frac{\Psi_n^\pm(\alpha \mp \Phi)}{\Psi_n^\pm(-\alpha \mp \Phi)} = 1,$$

and

$$\frac{\Psi_n^\pm(\alpha \pm \Phi)}{\Psi_n^\pm(-\alpha \pm \Phi)} = \frac{\mp \sin \alpha - (-1)^n \sin \theta_n^\pm}{\pm \sin \alpha - (-1)^n \sin \theta_n^\pm}, \quad n = 1, 2, 3,$$

leading to the equation (3.30) if account is taken of the analytical form of $\Psi(\alpha)$ in

(3.27). The lack of zeros and poles of $\Psi(\alpha)$ in the strip Π_0 results from (3.21) and (3.26). Thus, on the assumption that the Brewster angles $\theta_n^\pm$ are distributed according to (3.26), the derived representation (3.27) completely determines the auxiliary function $\Psi(\alpha)$. In particular, when $\text{Im } \alpha \rightarrow \infty$, due to the asymptotic properties of the Malyuzhinets function (Malyuzhinets 1958c; Osipov 1990a)

$$\psi_\Phi(\alpha) = \frac{1}{\sqrt{2}} \psi_\Phi\left(\frac{1}{2}\pi\right) \exp\left(-\frac{1}{4}i\mu\alpha \operatorname{sgn}(\text{Im } \alpha)\right)(1 + o(1)), \quad \mu = \frac{\pi}{2\Phi}, \quad (3.31)$$

the auxiliary function behaves as

$$\Psi(\alpha) = \frac{1}{4}\psi_\Phi^4\left(\frac{1}{2}\pi\right) \exp(-i\mu\alpha \operatorname{sgn}(\text{Im } \alpha))(1 + o(1)). \quad (3.32)$$

Finally, the maximum value of $|\text{Im}(\theta_n^\pm)|$ must lie beneath the Sommerfeld contour γ_+ .

(c) *General solution of the functional equations*

Consider the system (3.9). Its general solution, as with all linear non-homogeneous equations, consists of a particular solution of the non-homogeneous system plus a general solution of the corresponding homogeneous one. The particular solution of (3.9) can be deduced by a Fourier transformation (3.13) to have the form

$$\Lambda(\alpha) = \Lambda_+(\alpha) + \Lambda_-(\alpha), \quad (3.33)$$

where

$$\Lambda_\pm(\alpha) = \mp \frac{1}{8\Phi i} \int_{-\infty}^{i\infty} \tan\left(\frac{\pi}{4\Phi}(\alpha + \beta \pm \Phi)\right) f_\pm(\beta) d\beta.$$

The construction of the functions $\Lambda_\pm(\alpha)$ is essentially based on the fact that $f_\pm(\alpha)$ are odd and integrable functions of their argument when $\alpha \in i\mathcal{R}$, because of (3.5), (3.8), (3.10), and (3.32). We omit the corresponding mathematics since the correctness of the representation (3.33) can be verified immediately by substitution into the functional equations (3.9). For $\text{Im } \alpha \rightarrow \pm\infty$, the function $\Lambda(\alpha)$ vanishes in accordance with the asymptotic estimate

$$\Lambda(\alpha) = \pm \frac{i}{4\Phi} e^{\pm i\mu\alpha} \int_{-\infty}^{i\infty} e^{i\mu\beta} (f_+(\beta) + f_-(\beta)) d\beta (1 + o(1)), \quad (3.34)$$

which results from (3.33) on accounting for

$$\tan\left(\frac{\pi}{4\Phi}(\alpha + \beta \pm \Phi)\right) = \pm i(1 - 2e^{\pm i\mu(\alpha + \beta \pm \Phi)} + \dots). \quad (3.35)$$

Moreover, due to convergence of the integrals (3.33) for $\alpha \in \Pi_0$, the function $\Lambda(\alpha)$ is regular in the strip Π_0 and has no poles there.

In order to obtain the required singularity of the transform $S(\alpha)$, i.e. a simple pole with unit residue at the single point $\alpha = \phi_0$ in the strip Π_0 , it is necessary to introduce a function (Malyuzhinets 1958c)

$$\sigma(\alpha, \phi_0) = \frac{\mu \cos(\mu\phi_0)}{\sin(\mu\alpha) - \sin(\mu\phi_0)}, \quad (3.36)$$

which is a meromorphic solution of the homogeneous system

$$\sigma(\alpha \pm \Phi, \phi_0) - \sigma(-\alpha \pm \Phi, \phi_0) = 0. \quad (3.37)$$

Multiplying this by the constant $\Psi(\phi_0)^{-1}$, and adding it to $\Lambda(\alpha)$ gives the general solution of the system of functional equations (3.9) as follows,

$$\tilde{S}(\alpha) = \frac{\sigma(\alpha, \phi_0)}{\Psi(\phi_0)} + \Lambda(\alpha), \quad (3.38)$$

which is regular in Π_0 except at the point $\alpha = \phi_0$, and

$$\tilde{S}(\alpha) = O(e^{\pm i\mu\alpha}) \quad \text{as} \quad \text{Im } \alpha \rightarrow \pm\infty, \quad (3.39)$$

owing to (3.34) and (3.36).

In terms of the transform $S(\alpha)$ this yields

$$S(\alpha) = \Psi(\alpha) \left(\frac{\sigma(\alpha, \phi_0)}{\Psi(\phi_0)} + \sum_{n=1}^2 C_n^+ \Lambda_n^+(\alpha) + \sum_{n=1}^2 C_n^- \Lambda_n^-(\alpha) \right), \quad (3.40)$$

where

$$\Lambda_n^\pm(\alpha) = \mp \frac{\mu}{2\pi i} \int_{-i\infty}^{i\infty} \tan\left(\frac{\pi}{4\Phi}(\alpha + \beta \pm \Phi)\right) \frac{\sin \beta \cos^{n-1} \beta}{L_\pm(\beta) \Psi(\beta \pm \Phi)} d\beta. \quad (3.41)$$

In evaluating (3.41) we can relieve ourselves of having to use non-zero values of $\text{Re } \theta_n^\pm$ by letting the small amount of dissipation introduced in $T_\pm$ tend to zero such that $\text{Im } T_\pm \uparrow 0$, thereby obtaining the rules of bypassing the singularities of $L_\pm^{-1}(\beta)$ when integrating over the imaginary axis. The range of integration can also be reduced using (3.8), giving

$$\begin{aligned} \Lambda_n^\pm(\alpha) = & \pm \frac{\mu \sin(\mu\theta_1^\pm)}{(\sin(\mu\alpha) \mp \cos(\mu\theta_1^\pm))} \frac{\sin \theta_1^\pm \cos^{n-1} \theta_1^\pm}{L'_\pm(\mp\theta_1^\pm) \Psi(\mp\theta_1^\pm \pm \Phi)} \\ & + \frac{1}{\pi i} \mathcal{P} \int_0^{i\infty} \frac{\mu \sin(\mu\beta)}{(\sin(\mu\alpha) \mp \cos(\mu\beta))} \frac{\sin \beta \cos^{n-1} \beta}{L_\pm(\beta) \Psi(\beta \pm \Phi)} d\beta, \end{aligned} \quad (3.42)$$

where the principal value integral is over the simple pole at $\beta = -\theta_1^\pm$. A similar principal value contribution also occurs at $\beta = \theta_3^\pm$ when $0 < M_\pm \leq 1$ and $0 < k_\pm \leq k$. The integrals in (3.41) and (3.42) converge absolutely for $|\text{Im } \beta| \rightarrow \infty$ due to (3.5) and (3.32). Equations (3.40) and (3.41) define a meromorphic function $S(\alpha)$ of the complex argument α that is analytical in the strip Π_0 , apart from a point $\alpha = \phi_0$ at which it possesses a unit residue reproducing the incident field (2.3). As $\text{Im } \alpha \rightarrow \pm\infty$, the function $S(\alpha)$, because of (3.32), (3.39), remains bounded,

$$S(\pm i\infty) = \mp \frac{1}{2} i \psi_\Phi^4(\pi/2) \left(\frac{\mu \cos(\mu\phi_0)}{\Psi(\phi_0)} - \frac{1}{8\Phi} \int_{-i\infty}^{i\infty} e^{i\mu\beta} (f_+(\beta) + f_-(\beta)) d\beta \right). \quad (3.43)$$

At the same time the limit of the Sommerfeld integral (Malyuzhinets 1960) implies

$$\lim_{r \rightarrow 0} p(r, \phi) = iP_0(S(i\infty) - S(-i\infty)), \quad (3.44)$$

which means that the solution obtained provides an unambiguous limit for the acoustic pressure at $r = 0$, independent of the direction along which the observation point approaches the tip.

We refer to $S(\alpha)$ given by (3.40) as the general solution of the functional equations, since it contains four arbitrary constants $C_n^\pm$, $n = 1, 2$, satisfies (3.6) together with the regularity condition (3.43) and demonstrates the prescribed singularity at a point

$\alpha = \phi_0$, regardless of the values of these constants. In the next section we show that the constants $C_n^\pm$ can be determined from contact conditions at the tip (2.6) plus more refined conditions at infinity implying the correct behaviour of the scattered field for $r \rightarrow \infty$.

4. Determination of constants

(a) Conditions at infinity

Consider the behaviour of the Sommerfeld integral (3.1) when $r \rightarrow \infty$. To this end, we deform the contour γ into a pair of contours $\Gamma_\varepsilon(\pm\pi)$, one of which goes from $\pi - \varepsilon + i\infty$ through $\alpha = \pi$ to $\pi + \varepsilon - i\infty$, whereas the other is symmetric to it about the origin $\alpha = 0$ (figure 2). Assuming that ε is positive and arbitrarily small, one finds that these contours belong totally to those portions of the complex α -plane (hatched in figure 2) in which $\text{Im} \cos \alpha < 0$ (hereinafter labelled Π^-). For this reason, any integral with Sommerfeld's kernel $\exp(-ikr \cos \alpha)$ taken over the contours $\Gamma_\varepsilon(\pm\pi)$ vanishes as $r \rightarrow \infty$.

In the course of deforming γ , certain poles of the integrand $S(\alpha + \phi)$ may be captured which fall within the area enclosed by contours $\gamma_\pm$ at the top and bottom, and by $\Gamma_\varepsilon(\pm\pi)$ at the left and right (this area, essential to the following, is designated as Π^{ess}). If among the captured poles were those located inside the regions where $\text{Im} \cos \alpha > 0$ (we denote these parts of the complex α -plane as Π^+ and have left them blank in figure 2), then the residues at such poles would grow exponentially with $r \rightarrow \infty$, giving rise to unphysical behaviour of the scattered field at infinity. Thus, poles of such a type (named in Osipov (1994) as 'forbidden' poles) violate conditions at $r = +\infty$ and must be removed from the solution.

Let us discuss singularities of the general solution $S(\alpha + \phi)$ given by (3.40). For the function $\sigma(\alpha + \phi, \phi_0)$, they are

$$\alpha_j = -\phi + \begin{cases} \phi_0 + 4j\Phi, \\ 2\Phi - \phi_0 + 4j\Phi, \end{cases} \quad (4.1)$$

$j = 0, \pm 1, \pm 2, \dots$, among which the pole $\alpha = -\phi + \phi_0$ corresponds to the incident field from the construction of the general solution. The residues at these poles

$$p_j(r, \phi) = P_0 \frac{\Psi(\alpha_j + \phi) \cos(\mu\phi_0)}{\Psi(\phi_0) \cos(\mu(\alpha_j + \phi))} e^{-ikr \cos \alpha_j}, \quad (4.2)$$

do not include the constants $C_n^\pm$ and coincide both in phase and in amplitude with geometrical acoustics waves (Osipov 1990b). Hence, all the poles of $\sigma(\alpha, \phi_0)$ are allowable from the physical point of view.

Singularities of the auxiliary function $\Psi(\alpha + \phi)$ involving zeros of $\Psi_2^\pm(\alpha + \phi)$ and poles of $\Psi_{1,3}^\pm(\alpha + \phi)$ can be found from (3.21), (3.28) as follows:

$$\alpha = -\phi + \begin{cases} -\Phi - \pi/2 + \theta_q^+ \pm \alpha_{nm}, \\ -\Phi + \pi/2 - \theta_q^+ \pm \alpha_{nm}, \\ \Phi - \pi/2 + \theta_q^- \pm \alpha_{nm}, \\ \Phi + \pi/2 - \theta_q^- \pm \alpha_{nm}, \end{cases} \quad (4.3)$$

$q = 1, 3, m = 2, 4, 6, \dots$, and

$$\alpha = -\phi + \begin{cases} -\Phi - \pi/2 - \theta_2^+ \pm \alpha_{nm}, \\ -\Phi + \pi/2 + \theta_2^+ \pm \alpha_{nm}, \\ \Phi - \pi/2 - \theta_2^- \pm \alpha_{nm}, \\ \Phi + \pi/2 + \theta_2^- \pm \alpha_{nm}, \end{cases} \quad (4.4)$$

$m = 1, 3, 5, \dots, n = 1, 2, 3, \dots$. Simple estimates show that all the poles associated with $m \geq 3$ in (4.3), (4.4) lie outside the strip Π^{ess} , so we may restrict ourselves to $m = 1, 2$. Furthermore, one can verify that none of the poles (4.3) falls within Π^{ess} because of $\text{Re } \theta_q^\pm > 0, q = 1, 3$. As a consequence of this, we arrive at the conclusion: no poles from families (4.3), (4.4) other than those with $m = 1$ can be captured in the aforementioned contour deformation. Assuming $\text{Im } \theta_2^\pm > 0$ implies that among them the forbidden poles which are shared by Π^{ess} and Π^+ are as follows: $\alpha = -\phi + \alpha_{pn}^f$ where

$$\alpha_{pn}^f = \begin{cases} -\theta_2^+ + \Phi + 4(n-1)\Phi, & p = 1, \\ \theta_2^+ - 3\Phi - 4(n-1)\Phi, & p = 2, \\ \theta_2^- - \Phi - 4(n-1)\Phi, & p = 3, \\ -\theta_2^- + 3\Phi + 4(n-1)\Phi, & p = 4, \end{cases} \quad (4.5)$$

$n = 1, 2, \dots, N_p$, with N_p depending on the values of $\text{Re } \theta_2^\pm, \phi$ and Φ . Other poles of (4.3), (4.4) belonging to $\Pi^{\text{ess}} \cap \Pi^-$ produce the residues of the integral (3.1) that vanish when $r \rightarrow \infty$, thus being consistent with the conditions at infinity.

Equations (4.1), (4.3), (4.4) cover all the singularities of $S(\alpha + \phi)$ given by (3.40). To prove this, we have to show that the function $\Lambda(\alpha)$ does not add any new poles to $S(\alpha)$ differing from those of $\Psi(\alpha)$ and $\sigma(\alpha, \phi_0)$. Let us compare the analytical properties of $\Psi(\alpha)$ and $\Omega(\alpha) = \Psi(\alpha)\Lambda(\alpha)$. By their construction, both are regular solutions in Π_0 to the functional equations (3.8) and

$$L_\pm(\alpha)\Omega(\alpha \pm \Phi) - L_\pm(-\alpha)\Omega(-\alpha \pm \Phi) = 2 \sin \alpha \sum_{n=1}^2 C_n^\pm \cos^{n-1} \alpha, \quad (4.6)$$

respectively. By virtue of these equations being regarded as functional relations they can be continued outside Π_0 . For example, these functions in the strips $\Pi_{\pm 1} = \{\alpha : \Phi < \text{Re}(\pm\alpha) \leq 3\Phi\}$ are related to themselves in Π_0 by the formulas

$$\Psi(\alpha) = \frac{L_\pm(-\alpha \pm \Phi)}{L_\pm(\alpha \mp \Phi)} \Psi(-\alpha \pm 2\Phi), \quad \alpha \in \Pi_{\pm 1}, \quad (4.7)$$

$$\begin{aligned} \Omega(\alpha) &= \frac{L_\pm(-\alpha \pm \Phi)}{L_\pm(\alpha \mp \Phi)} \Omega(-\alpha \pm 2\Phi) \\ &+ \frac{2 \sin(\alpha \mp \Phi)}{L_\pm(\alpha \mp \Phi)} \sum_{n=1}^2 C_n^\pm \cos^{n-1}(\alpha \mp \Phi), \quad \alpha \in \Pi_{\pm 1}. \end{aligned} \quad (4.8)$$

Since the functions $\Psi(-\alpha \pm 2\Phi), \Omega(-\alpha \pm 2\Phi)$ are regular if $\alpha \in \Pi_{\pm 1}$, singularities of $\Psi(\alpha), \Omega(\alpha)$ in the strips $\Pi_{\pm 1}$ must be at the points coinciding with zeros of $L_\pm(\alpha \mp \Phi)$. Similarly, one can compare singularities of $\Psi(\alpha)$ and $\Omega(\alpha)$ in any portion of the complex α -plane, and check that their poles are located at the same points

given by equations (4.3), (4.4) (though the residues may be different). This means that despite its being a meromorphic function (outside Π_0), $\Lambda(\alpha)$ has the poles which are necessarily compensated by zeros of $\Psi(\alpha)$ when multiplied by it, and therefore does not contribute to singularities of the transform $S(\alpha)$.

Thus, to guarantee the correct behaviour of the integral (3.1) as $r \rightarrow +\infty$ we have to cancel the residues of its transform $S(\alpha)$ at the forbidden poles (4.5) by equating

$$\frac{\sigma(\alpha_{pn}^f, \phi_0)}{\Psi(\phi_0)} + \Lambda(\alpha_{pn}^f) = 0, \quad n = 1, 2, \dots, N_p, \quad p = 1, 2, 3, 4. \quad (4.9)$$

For $\pi \geq \Phi \geq \pi/2$, it follows from (4.5) that $N_1 = 1$, $N_2 = 0$, $N_3 = 1$, $N_4 = 0$ and only two poles $\alpha = \alpha_{11}^f - \phi$ and $\alpha = \alpha_{31}^f - \phi$ of $S(\alpha + \phi)$ can lay in the forbidden area $\Pi^f \equiv \Pi^{\text{ess}} \cap \Pi^+$; correspondingly, the system (4.9) consists of two algebraic equations for the constants $C_m^\pm$, $m = 1, 2$. If $\Phi < \pi/2$, then further poles from (4.5) can enter Π^f . However, it can be shown (Osipov 1997) that no additional equations arise if the algebraic system (4.9) is satisfied for α_{11}^f and α_{31}^f . To put this another way, once the constants $C_m^\pm$ are obtained from the two equations (4.9) in the case $\Phi \in [\pi/2, \pi]$ for $p = 1, n = 1$ and $p = 3, n = 1$ they ensure that the residues of $S(\alpha)$ at other poles with different values of p and n are nullified, thus providing the correct behaviour of the solution at infinity for arbitrary values of Φ , including values from the interval $(0, \pi/2]$. From the mathematical viewpoint this is a consequence of the analyticity of the solution of the general boundary value problem (2.2)–(2.6) as a function of a parameter Φ , i.e. a solution which is true for a non-zero interval of Φ can be uniquely extended by analytical continuation from the initial interval to all admissible values of the parameter.

Therefore, the conditions at infinity (2.3) are satisfied by the Sommerfeld integral (3.1) with its integrand given by (3.40) if the four constants $C_n^\pm$, $n = 1, 2$, are chosen so as to solve two linear algebraic equations resulting from (4.9) for $n = 1$ and $p = 1, 3$. We reserve two upper rows for these equations in a complete system of equations which is to be formulated to determine the constants:

$$\begin{pmatrix} E_{11}^+ & E_{12}^+ & E_{11}^- & E_{12}^- \\ E_{21}^+ & E_{22}^+ & E_{21}^- & E_{22}^- \\ E_{31}^+ & E_{32}^+ & E_{31}^- & E_{32}^- \\ E_{41}^+ & E_{42}^+ & E_{41}^- & E_{42}^- \end{pmatrix} \begin{pmatrix} C_1^+ \\ C_2^+ \\ C_1^- \\ C_2^- \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}. \quad (4.10)$$

From (4.9) we have

$$g_1 = -\frac{\mu \cos(\mu\phi_0)}{\Psi(\phi_0)(\cos(\mu\theta_2^+) - \sin(\mu\phi_0))}, \quad g_2 = \frac{\mu \cos(\mu\phi_0)}{\Psi(\phi_0)(\cos(\mu\theta_2^-) + \sin(\mu\phi_0))}. \quad (4.11)$$

To write down the elements of matrix $\hat{\mathbf{E}}$, it is necessary to continue the function $\Lambda(\alpha)$ outside the strip Π_0 because the points $\alpha = \alpha_{11}^f - \phi$, $\alpha = \alpha_{31}^f - \phi$ do not reside in Π_0 . For instance, $\alpha_{11}^f \in \Pi_1$, $\alpha_{31}^f \in \Pi_{-1}$ if $\Phi > \pi/4$ and, consequently, based on (4.7), (4.8), $\Lambda(\alpha)$ can be found from the relations

$$\Lambda(\alpha) = \Lambda(-\alpha \pm 2\Phi) + \frac{2 \sin(\alpha \mp \Phi)}{L_\pm(-\alpha \pm \Phi) \Psi(-\alpha \pm 2\Phi)} (C_1^\pm + C_2^\pm \cos(\alpha \mp \Phi)), \quad \alpha \in \Pi_{\pm 1}, \quad (4.12)$$

which result from the functional equations (3.9) satisfied by $\Lambda(\alpha)$, and give

$$\left. \begin{aligned} E_{1n}^+ &= \Lambda_n^+(\Phi + \theta_2^+) - \frac{2 \sin \theta_2^+ \cos^{n-1} \theta_2^+}{L_+(\theta_2^+) \Psi(\Phi + \theta_2^+)}, \\ E_{1n}^- &= \Lambda_n^-(\Phi + \theta_2^+), \\ E_{2n}^+ &= \Lambda_n^+(-\Phi - \theta_2^-), \\ E_{2n}^- &= \Lambda_n^-(-\Phi - \theta_2^-) + \frac{2 \sin \theta_2^- \cos^{n-1} \theta_2^-}{L_-(-\theta_2^-) \Psi(-\Phi - \theta_2^-)}, \quad n = 1, 2. \end{aligned} \right\} \quad (4.13)$$

The functions $\Lambda_n^\pm(\alpha)$ and $\Psi(\alpha)$ have arguments only in Π_0 and hence they can be evaluated directly from equations (3.27) and (3.42). If $0 < M_\pm \leq 1$ and $0 < k_\pm \leq k$ additional principal value contributions occur at $\beta = \theta_2^\pm$ for the term $\Lambda_n^+(\Phi + \theta_2^+)$ in E_{1n}^+ and $\Lambda_n^-(-\Phi - \theta_2^-)$ in E_{2n}^- , respectively (the associated residues cancel one half of the final terms in (4.13)). In the case of $\Phi \leq \pi/4$ (a narrow angular region filled with fluid) the matrix elements $E_{mn}^\pm$, $m, n = 1, 2$, can be deduced in a similar manner by successive use of the continuity relations (4.12).

(b) Conditions at the tip

To complete the problem of determining the constants $C_n^\pm$, $n = 1, 2$, two linearly independent constraints on the solution are necessary in addition to the regularity conditions (4.9). Note that the number K of the required constraints is in complete agreement with the general relation (Osipov 1994)

$$K = \text{entire } \frac{1}{2}(N_+ + N_- - 1), \quad (4.14)$$

since the order of the boundary conditions operators $\mathcal{L}_\pm$ are $N_+ = N_- = 3$. For these conditions we employ the conditions at the tip (the so called contact conditions) which are given by (2.6) for a pinned membrane.

From the representation (3.2) we have

$$\frac{1}{r} \frac{\partial p}{\partial \phi}(r, \pm \Phi) = -\frac{kP_0}{2\pi} \int_{\gamma_\pm} e^{-ikr \cos \alpha} \sin \alpha (S(\alpha \pm \Phi) + S(-\alpha \pm \Phi)) d\alpha. \quad (4.15)$$

It is evident from the Malyuzhinets formula for the Sommerfeld integral (3.44) that its limit when $r \rightarrow 0$ is conditioned by the behaviour of its integrand as $\text{Im } \alpha \rightarrow \infty$. In order to estimate this behaviour, rewrite the functional equations (3.6) as follows:

$$S(\alpha \pm \Phi) + S(-\alpha \pm \Phi) = \left(1 + \frac{L_\pm(-\alpha)}{L_\pm(\alpha)}\right) S(-\alpha \pm \Phi) + \frac{2 \sin \alpha}{L_\pm(\alpha)} (C_1^\pm + C_2^\pm \cos \alpha). \quad (4.16)$$

When $\text{Im } \alpha \rightarrow \infty$,

$$1 + \frac{L_\pm(-\alpha)}{L_\pm(\alpha)} = \frac{\mp 2i}{ka_\pm M_\pm^2 \sin \alpha \cos^2 \alpha} + O\left(\frac{1}{\cos^4 \alpha}\right),$$

and equation (4.16) becomes

$$S(\alpha \pm \Phi) + S(-\alpha \pm \Phi) = \frac{2 \sin \alpha}{L_\pm(\alpha)} (C_1^\pm + C_2^\pm \cos \alpha) + O\left(\frac{1}{\sin \alpha \cos^2 \alpha}\right),$$

because $S(\alpha)$ is bounded at $\alpha = \pm i\infty$. Expanding $L_\pm(\alpha)$ for large $\text{Im } \alpha$ yields

$$\sin \alpha (S(\alpha \pm \Phi) + S(-\alpha \pm \Phi)) = \frac{\pm 2i \sin \alpha}{ka_\pm M_\pm^2 \cos \alpha} \left(C_2^\pm + \frac{C_1^\pm}{\cos \alpha} + O\left(\frac{1}{\cos^2 \alpha}\right) \right). \quad (4.17)$$

Note that only the first term in the parenthesis contributes to the limiting value of the integral (4.15) since the others tend to zero as $\text{Im } \alpha \rightarrow \infty$. Therefore, one has

$$\lim_{r \rightarrow 0} \frac{1}{r} \frac{\partial p}{\partial \phi}(r, \pm \Phi) = \mp \frac{i P_0 C_2^\pm}{\pi a_\pm M_\pm^2} \lim_{r \rightarrow 0} \int_{\gamma_+} e^{-ikr \cos \alpha} \tan \alpha \, d\alpha, \quad (4.18)$$

and calculating the expression

$$\lim_{r \rightarrow 0} \int_{\gamma_+} e^{-ikr \cos \alpha} \tan \alpha \, d\alpha = -2\pi i, \quad (4.19)$$

one arrives at the formula

$$\lim_{r \rightarrow 0} \frac{1}{r} \frac{\partial p}{\partial \phi}(r, \pm \Phi) = \mp \frac{2P_0}{a_\pm M_\pm^2} C_2^\pm. \quad (4.20)$$

Thus, for a membrane pinned at the tip,

$$C_2^\pm = 0, \quad (4.21)$$

and the system (4.10) reduces to

$$\begin{pmatrix} E_{11}^+ & E_{12}^+ & E_{11}^- & E_{12}^- \\ E_{21}^+ & E_{22}^+ & E_{21}^- & E_{22}^- \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1^+ \\ C_2^+ \\ C_1^- \\ C_2^- \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ 0 \\ 0 \end{pmatrix}, \quad (4.22)$$

giving the following expressions for $C_1^\pm$:

$$C_1^\pm = \frac{\pm g_1 E_{21}^\mp \mp g_2 E_{11}^\mp}{E_{11}^+ E_{21}^- - E_{21}^+ E_{11}^-}. \quad (4.23)$$

Here, $g_{1,2}$ and $E_{mn}^\pm$, $m, n = 1, 2$, are specified by relations (4.11) and (4.13).

Once the constants $C_n^\pm$, $n = 1, 2$, have been determined from equations (4.21), (4.23), we obtain an accurate and unique solution to the diffraction problem (2.2)–(2.6) formulated above, in the form of the Sommerfeld integral (3.1) where

$$S(\alpha) = \Psi(\alpha) \left(\frac{\sigma(\alpha, \phi_0)}{\Psi(\phi_0)} + \Lambda(\alpha) \right), \quad (4.24)$$

and

$$\Lambda(\alpha) = C_1^+ \Lambda_1^+(\alpha) + C_1^- \Lambda_1^-(\alpha). \quad (4.25)$$

Here, $\sigma(\alpha, \phi_0)$ can be found from (3.36) everywhere in the complex α -plane. The functions $\Psi(\alpha)$, $\Lambda(\alpha)$ are given by (3.27) and (3.41) if $\alpha \in \Pi_0$. Extension of these functions outside the strip Π_0 is achieved by repeated application of the functional equations (3.8) and (3.9), and analytic continuation of $S(\alpha)$ follows from the system (3.6): for instance,

$$S(\alpha) = \frac{L_\pm(-\alpha \pm \Phi)}{L_\pm(\alpha \mp \Phi)} S(-\alpha \pm 2\Phi) + 2C_1^\pm \frac{\sin(\alpha \mp \Phi)}{L_\pm(\alpha \mp \Phi)}, \quad \alpha \in \Pi_{\pm 1}. \quad (4.26)$$

5. Far-field representations

(a) Diffraction coefficients

For $kr \gg 1$, the integral (3.1) can be evaluated asymptotically by deformation of the contour γ into a pair of steepest descent paths $\gamma(\pm\pi) = \{\alpha : \operatorname{Re} \alpha = \pm\pi - \operatorname{gd}(\operatorname{Im} \alpha)\}$ (figure 2) passing through the saddle points $\alpha = \pm\pi$, respectively (Malyuzhinets 1958c), where

$$\operatorname{gd}(x) = -\frac{1}{2}\pi + 2 \arctan(e^x).$$

This yields a representation

$$p(r, \phi) = p_g(r, \phi) + p_d(r, \phi) + p_+(r, \phi) + p_-(r, \phi), \quad (5.1)$$

in which

$$p_d(r, \phi) = \frac{P_0}{2\pi i} \int_{\gamma(\pi) \cup \gamma(-\pi)} e^{-ikr \cos \alpha} \Psi(\alpha + \phi) \left(\frac{\sigma(\alpha + \phi, \phi_0)}{\Psi(\phi_0)} + \Lambda(\alpha + \phi) \right) d\alpha, \quad (5.2)$$

whereas the other terms result from residues at poles of the integrand located in the region

$$\Pi^{SDP} = \{\alpha : -\pi - \operatorname{gd}(\operatorname{Im} \alpha) < \operatorname{Re} \alpha < \pi - \operatorname{gd}(\operatorname{Im} \alpha), -\infty < \operatorname{Im} \alpha < +\infty\}.$$

The total number of poles captured depends on the value of the angle Φ , increasing rapidly with a decrease in it (see equations (4.1), (4.3) and (4.4)). In this paper we have restricted ourselves to the case $\pi/2 \leq \Phi \leq \pi$; the corresponding treatment for smaller values of Φ presents no crucial difficulties involving only more algebraic manipulations (Osipov 1990b).

If $\Phi \geq \pi/2$, three poles of $\sigma(\alpha + \phi, \phi_0)$

$$\alpha = -\phi + \begin{cases} \phi_0, \\ 2\Phi - \phi_0, \\ -2\Phi - \phi_0, \end{cases} \quad (5.3)$$

may fall within Π^{SDP} . The residues at these poles, given by (4.2), are easily evaluated via the functional equations (3.8) for $\Psi(\alpha)$

$$\frac{\Psi(\pm 2\Phi - \phi_0)}{\Psi(\phi_0)} = -R_{\pm}(\Phi \mp \phi_0), \quad (5.4)$$

where $R_{\pm}(\chi)$ represent the reflection coefficients (3.12), giving therefore the geometrical acoustics field

$$\begin{aligned} p_g(r, \phi) &= H(\pi - |\phi - \phi_0|) P_0 e^{-ikr \cos(\phi - \phi_0)} \\ &\quad + H(\pi - |2\Phi - \phi - \phi_0|) P_0 R_+(\Phi - \phi_0) e^{-ikr \cos(\phi + \phi_0 - 2\Phi)} \\ &\quad + H(\pi - |2\Phi + \phi + \phi_0|) P_0 R_-(\Phi + \phi_0) e^{-ikr \cos(\phi + \phi_0 + 2\Phi)}, \end{aligned} \quad (5.5)$$

and $H(x) = 1$ if $x > 0$, 0 if $x < 0$.

The terms $p_{\pm}(r, \phi)$ denote the residues at captured poles of the function $\Psi(\alpha + \phi)$. These poles coincide with the roots of the equation

$$L_{\pm}(\mp \Phi + \phi + \alpha) = 0, \quad (5.6)$$

which lie within Π^{SDP} :

$$\alpha = -\phi \pm \Phi + \begin{cases} \mp \theta_2^\pm, \\ \pm \theta_2^\pm \pm \pi, \\ \pm \theta_1^\pm \pm \pi. \end{cases} \quad (5.7)$$

The poles $\alpha = \pm \Phi - \phi \mp \theta_2^\pm$ are the forbidden ones (see equations (4.5)), so they do not contribute to the terms $p_\pm(r, \phi)$ owing to the regularity conditions (4.9). The points $\alpha = \pm \Phi - \phi \pm \pi \pm \theta_2^\pm$ from (5.7), because of the condition $\text{Re } \theta_2^\pm < 0$, are localized within the area Π^- , and therefore produce residues which decrease exponentially as $kr \rightarrow +\infty$ and thus can be neglected. The dominant contributions to $p_\pm(r, \phi)$ are due to the residues at poles $\alpha = \pm \Phi - \phi \pm \pi \pm \theta_1^\pm$ which describe the membrane waves excited at the tip by the incident field and then traveling outwards along both sides of the wedge without dissipation if the material parameters of the membrane are assumed to be entirely real, that is $\text{Re } \theta_1^\pm = 0$. The corresponding expressions may be summarized as follows,

$$p_\pm(r, \phi) = H(\pm\phi - \Phi - \text{Re } \theta_1^\pm - \text{gd}(\text{Im } \theta_1^\pm)) P_0 A_\pm e^{ikr \cos(\Phi \mp \phi + \theta_1^\pm)}, \quad (5.8)$$

where

$$A_\pm = \frac{\pm 2S(\pm \Phi \mp \pi \mp \theta_1^\pm) + 2C_1^\pm \sin \theta_1^\pm}{ika_\pm \cos \theta_1^\pm (3M_\pm^2 \sin^2 \theta_1^\pm + 1 - M_\pm^2)}. \quad (5.9)$$

Note that for more convenience the formula above is written in such a way that the arguments of the functions $\Psi(\pm \Phi \mp \pi \mp \theta_1^\pm)$ and $\Lambda(\pm \Phi \mp \pi \mp \theta_1^\pm)$ in $S(\pm \Phi \mp \pi \mp \theta_1^\pm)$ take their values from Π_0 .

The integrals (5.2) can be evaluated by virtue of the steepest descent method. Retaining the leading terms of the asymptotic expansion yields

$$p_d(r, \phi) \approx P_0 D(\phi, \phi_0) \frac{\exp(ikr - i3\pi/4)}{\sqrt{2\pi kr}}, \quad (5.10)$$

where

$$D(\phi, \phi_0) = S(\phi + \pi) - S(\phi - \pi), \quad (5.11)$$

is the diffraction coefficient, and

$$S(\phi \pm \pi) = \Psi(\phi \pm \pi) \left(\frac{\sigma(\phi \pm \pi, \phi_0)}{\Psi(\phi_0)} + \Lambda(\phi \pm \pi) \right).$$

This term describes the cylindrical wave arising due to diffraction of the incident field by the tip. When calculating the diffraction coefficient in the expression for the edge-diffracted field (5.10), one has to account for $\pi + \phi \in \Pi_1$, $-\pi + \phi \in \Pi_0$ if $\phi \in (\pi - \Phi, \Phi)$, $\pi + \phi \in \Pi_1$, $-\pi + \phi \in \Pi_{-1}$ if $\phi \in (\Phi - \pi, \pi - \Phi)$, and $\pi + \phi \in \Pi_0$, $-\pi + \phi \in \Pi_{-1}$ if $\phi \in (-\Phi, \Phi - \pi)$, so as to choose the correct representation for the functions $S(\phi \pm \pi)$ which can be evaluated using the continuation formula (4.26).

The diffraction coefficient must be reciprocal under the interchange of the source and observation directions,

$$D(\phi_0, \phi) = D(\phi, \phi_0). \quad (5.12)$$

This symmetry can be exploited to simplify the structure of $D(\phi, \phi_0)$. Taking into account the dependence of $\Lambda(\phi)$ on ϕ_0 through (4.11), (4.23) and (4.25), $D(\phi, \phi_0)$

has the form,

$$D(\phi, \phi_0) = \frac{\cos(\mu\phi_0)}{\Psi(\phi)\Psi(\phi_0)} \left(\frac{f_1(\phi)}{\sin(\mu(\phi + \pi)) - \sin(\mu\phi_0)} + \frac{f_2(\phi)}{\sin(\mu(\phi - \pi)) - \sin(\mu\phi_0)} \right. \\ \left. + \frac{f_3(\phi)}{\cos(\mu\theta_2^+) - \sin(\mu\phi_0)} + \frac{f_4(\phi)}{\cos(\mu\theta_2^-) + \sin(\mu\phi_0)} \right), \quad (5.13)$$

where f_n , $n = 1, 2, 3, 4$ can be evaluated but the main point is they are functions of ϕ alone. Therefore, on account of reciprocity the diffraction coefficient must have the structure,

$$D(\phi, \phi_0) = \frac{\mu \cos(\mu\phi)}{\Psi(\phi)} \sum_{p=1}^4 \frac{D_p(\phi_0)}{\sin(\mu\phi) - \sin(\mu\phi_p)}, \quad (5.14)$$

where $\phi_1 = \phi_0 + \pi$, $\phi_2 = \phi_0 - \pi$, $\phi_3 = \Phi + \theta_2^+$, $\phi_4 = -\Phi - \theta_2^-$. The coefficients $D_p(\phi_0)$ are specified by equating both sides of (5.14) at the poles $\phi = \phi_p$, $p = 1, 2, 3, 4$. This gives

$$D_1(\phi_0) = -\Psi(\phi_1), \quad D_2(\phi_0) = \Psi(\phi_2), \quad D_3(\phi_0) = B_+ \Psi(\phi_3), \quad D_4(\phi_0) = B_- \Psi(\phi_4),$$

where

$$B_{\pm} = \frac{2S(\pm\Phi \mp \pi \mp \theta_2^{\pm}) \pm 2C_1^{\pm} \sin \theta_2^{\pm}}{ika_{\pm} \cos \theta_2^{\pm} (3M_{\pm}^2 \sin^2 \theta_2^{\pm} + 1 - M_{\pm}^2)}. \quad (5.15)$$

Furthermore, using functional properties of the Malyuzhinets functions (3.19) again implies that

$$\Psi(\phi) = \psi_{\Phi}^4(\frac{1}{2}\pi) \prod_{n=1}^3 Q_+(\phi, \theta_n^+) Q_-(\phi, \theta_n^-) \\ \times \left(\cos \left(\frac{\pi}{4\Phi} (\phi + \Phi + (-1)^n \theta_n^+) \right) \cos \left(\frac{\pi}{4\Phi} (\phi - \Phi - (-1)^n \theta_n^-) \right) \right)^{(-1)^{n-1}}, \quad (5.16)$$

where

$$Q_{\pm}(\phi, \theta) = \frac{\psi_{\Phi}(\phi \pm (\Phi - \frac{1}{2}\pi + \theta))}{\psi_{\Phi}(\phi \pm (\Phi - \frac{1}{2}\pi - \theta))}, \quad (5.17)$$

and $|Q_{\pm}(\phi, \theta)| = 1$ if θ is purely imaginary because of the analytic property $\psi_{\Phi}(\alpha^*) = \psi_{\Phi}^*(\alpha)$. If $0 < M_{\pm} \leq 1$ and $0 < k_{\pm} \leq k$, all six Brewster angles are purely imaginary, and consequently the magnitude of the diffraction coefficient viewed as a function of ϕ can be expressed in terms of trigonometric functions only,

$$|D(\phi, \phi_0)| = \frac{2\mu \cos(\mu\phi)}{\psi_{\Phi}^4(\pi/2)} \left| \sum_{p=1}^4 \frac{D_p(\phi_0)}{\sin(\mu\phi) - \sin(\mu\phi_p)} \right| \\ \times \prod_{n=1}^3 ((\cos(\mu\theta_n^+) - \sin(\mu\phi))^{1/2} (\cos(\mu\theta_n^-) + \sin(\mu\phi))^{1/2})^{(-1)^n}. \quad (5.18)$$

(b) Numerical examples

We consider the case of two identical membranes, implying $L_+(\alpha) = L_-(-\alpha)$, $\theta_n^+ = \theta_n^-$, and that $\Psi(\alpha)$ is an even function of its argument. The following identities

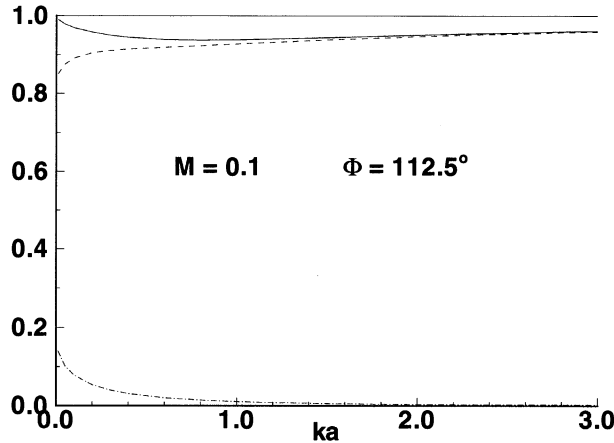


Figure 3. The redistribution of energy for a membrane wave incident on the corner separating identical membranes with $M = 0.1$, $\Phi = \frac{5}{8}\pi$. Reflected surface wave energy (---); transmitted surface wave energy (- · - · -); total surface wave energy (—). The horizontal line at unity indicates the entire energy in the system and the difference between it and the solid curve is the radiated acoustic energy.

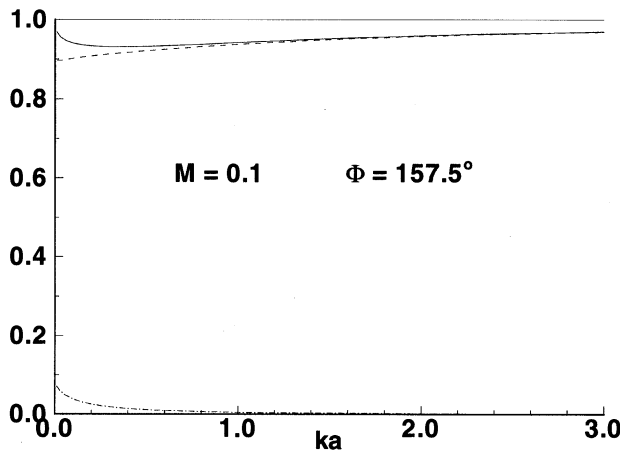


Figure 4. The same as figure 3 but for $\Phi = \frac{7}{8}\pi$.

are further consequences of the symmetric configuration,

$$E_{1n}^{\pm} + E_{2n}^{\mp} = 0, \quad (5.19)$$

which means that only two of the parameters $E_{mn}^{\pm}$ need to be numerically evaluated. It is possible in this case to interpret the solution in terms of symmetric and anti-symmetric parts relative to $\phi = 0$, but in general no such symmetry exists. The integrals (3.41) were numerically evaluated by reducing them to integrals over a semi-infinite range, as in (3.42), however, the principal value integrals were avoided by using a contour deformation described in Norris & Osipov (1997).

An incident surface wave is converted at the corner into diffracted acoustic energy plus outgoing surface waves on either membrane. Figures 3 through 8 illustrate this redistribution of surface energy for a range of parameters: $M = 0.1, 1, 5$, $\Phi = \frac{5}{8}\pi, \frac{7}{8}\pi$, and $0 < ka \leq 3$. At least 90% of the energy is reflected from the corner for small M with very little transmission when $ka > 1$, see figures 3 and 4. Figures 3 through

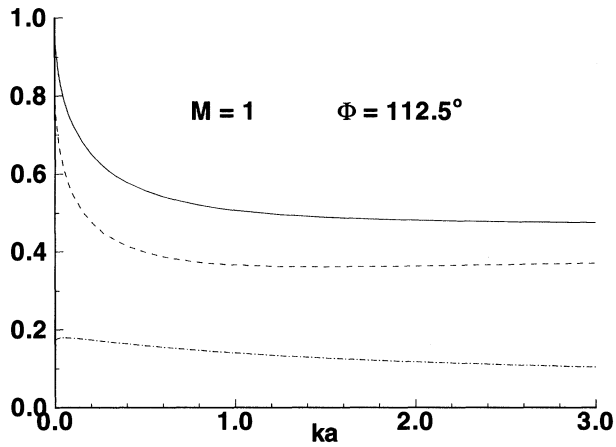


Figure 5. The same as figure 3 but for $M = 1$.

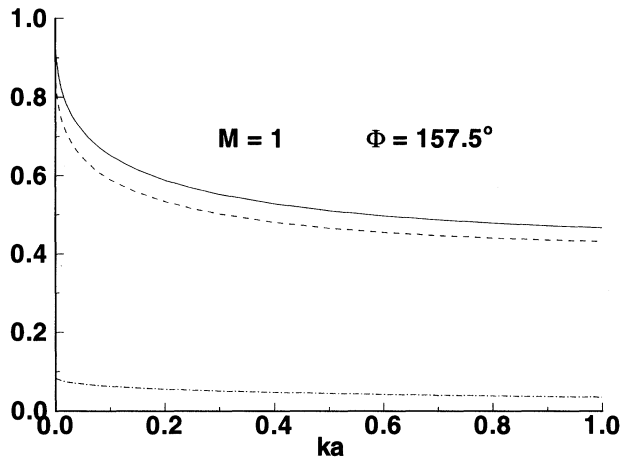


Figure 6. The same as figure 5 but for $\Phi = \frac{7}{8}\pi$.

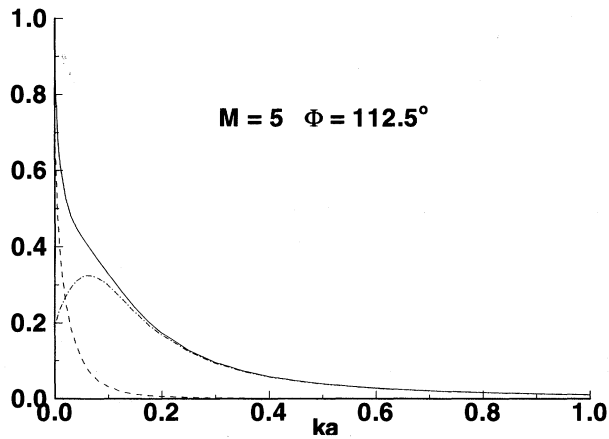
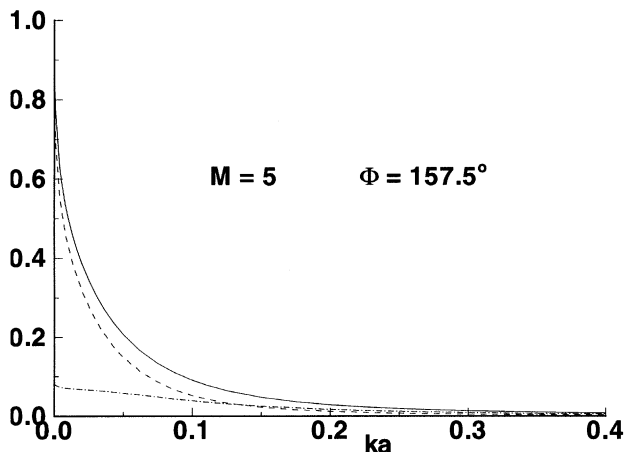
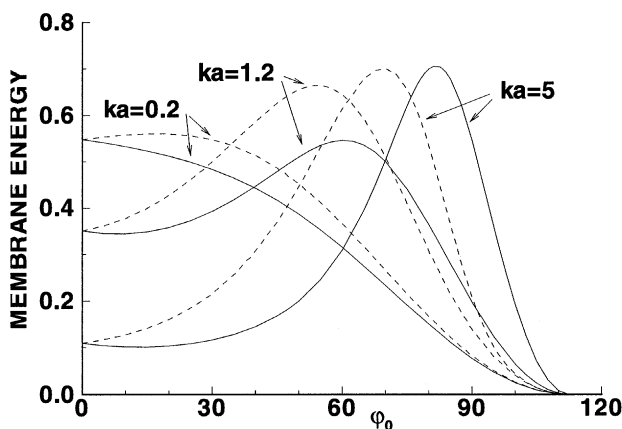


Figure 7. The same as figure 3 but for $M = 5$.

Figure 8. The same as figure 7 but for $\Phi = \frac{7}{8}\pi$.Figure 9. The membrane wave energies $|A_+|^2$ (—) and $|A_-|^2$ (---) for a plane wave of unit amplitude incident, with $\Phi = \frac{5}{8}\pi$ and $M = 1$.

8 show that the relative significance of acoustic radiation increases with M , at the cost of surface energy. For example, figure 8 indicates that at least 90% of the energy is lost to acoustic radiation for $ka > 0.1$ when $M = 5$ and $\Phi = \frac{7}{8}\pi$. The figures also suggest that the energy redistribution is not strongly dependent upon Φ , in the range $\frac{5}{8}\pi \leq \Phi \leq \frac{7}{8}\pi$ anyway.

A plane wave striking the corner of a system with $\Phi = \frac{5}{8}\pi$ and $M = 1$ is considered in figure 9 for three frequencies and a range of incidence angles. There is clearly asymmetry in the amount of energy diffracted onto either membrane for a given angle of incidence. It is useful to compare the results of figure 9 with those for a perfectly flat membrane with a point constrained from moving ($\Phi = \frac{1}{2}\pi$). The point constraint can be met by applying a point force normal to the membrane of magnitude sufficient to cancel the unconstrained motion. Hence, the diffracted membrane waves must be of equal amplitude. In contrast, the numerical results for the wedge geometry indicate unequal amplitudes, in general. Comparison of subsonic and the supersonic results in figure 10 shows that the diffracted membrane wave amplitudes display a greater asymmetry in the latter case.

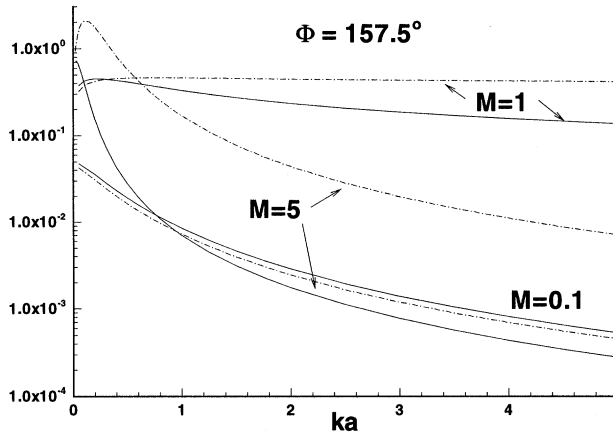


Figure 10. The membrane wave energies $|A_+|^2$ (—) and $|A_-|^2$ (---) for a plane wave of unit amplitude incident at $\phi_0 = 34^\circ$ for different values of M .

6. Conclusions

Using the modified Malyuzhinets technique, an accurate solution for the diffraction of a time harmonic plane sound wave by a sharp ridge on a fluid-loaded membrane pinned at its tip has been deduced in the form of a Sommerfeld integral. The solution is defined completely in terms of the well-known special function $\psi_\phi(\alpha)$ of a complex argument thus being accessible to both numerical and analytical calculations. In the case that the sector filled with fluid is wider than π , the far-field representation has been presented showing the physical validity of the solution. It is worth pointing out that the generalization of our approach to edge conditions other than (2.6) is quite straightforward. The only requirement is that they must be a pair of linearly independent relations, linear in terms of the edge values of the acoustic pressure, membrane displacements and rotational angles. This will produce a system of linear algebraic equations which differs from that for the pinned membrane (4.22) only in the coefficients of the two lower rows of the matrix operator, thus remaining accessible to solution in explicit form.

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