



ROUGH ELASTIC SPHERES IN CONTACT: MEMORY EFFECTS AND THE TRANSVERSE FORCE

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ABSTRACT

A pair of identical elastic spheres is simultaneously pressed against one another and sheared sideways causing both normal and tangential relative displacement of their centers. The contacting surfaces are assumed to be rough and the contact zone is modeled using the theories of Hertz and Mindlin. Because we focus our attention on the case of perfect adhesion, we are able, for the first time, to derive a rule for the variation in tangential force T as a function of the displacement components. T displays a memory effect through its dependence upon a , the radius of the contact zone which is uniquely related to the normal displacement, and s , the tangential displacement. Thus, $\Delta T = C_1 a \Delta s$ if a is not decreasing, and C_1 is a material parameter. Otherwise, $\Delta T = C_1 \Delta(as) - C_1 s'' \Delta a$ where s'' is the value of s when the contact zone was most recently at the same radius. The consequences for work and energy are explored in detail. We derive a simple analytical formula for the work done in going around an arbitrary closed path. Applications to acoustic wave propagation in a granular medium are discussed. © 1997 Elsevier Science Ltd

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1. INTRODUCTION

The theoretical treatment of the elastic forces between two grains that are in contact at a point of asperity has a long history. Hertz showed that if the grains are compressed along the axis between centers of curvature there is a contact zone, circular in the case of spherical grains, which increases as the compression, w , is increased. An accessible derivation of these results can be found in Landau and Lifschitz (1986). This changing contact zone gives rise to a restoring force, N , normal to the contact zone, which is a highly nonlinear function of w . Subsequently, Mindlin (1949) considered the transverse force, ΔT , that results from an additional transverse displacement of the spheres, Δs . The combination of these results gives rise to a transverse force, T , which is a path dependent functional of the deformation history, $s(w)$; the normal force, N , is a simple function of w alone. It is the purpose of the present article to present new results for the path dependent T , with special attention given to the role of decreasing normal compression, w ; this result is presented in eqn (14). We derive a simple formula, eqn (25), for the work done in going around an arbitrary closed loop in the w - s plane.

According to the Hertz–Mindlin (H–M) theory, if w is increased, holding s fixed,

T does not change. It is more or less intuitively obvious that if w is decreased at constant s , then in general T should also decrease because the zone of contact has diminished. Nonetheless, one finds in the literature applications of H-M theory that T is assumed not to change when w decreases. Elata (1995) has shown that a strict application of these models violates conservation of energy in the sense that it is possible to go around a loop in the w - s plane and get energy out of the system with every single circuit. Moreover, the value of T changes by a fixed amount with each circuit of the loop. We show that this does not, of course, really happen; with one very specific and narrow exception the work done by an external force is always positive for any closed loop.

The Mindlin result shows that the traction on the contact zone diverges near the edges. Therefore, unless the coefficient of static friction is infinite, there will be a slip zone. Mindlin and Deresiewicz (1953) have derived the consequences of a finite value of the coefficient of static friction, μ . Because their model is now quite complicated, they are of necessity limited to a consideration of relatively simple trajectories in the w - s plane, which they must consider on a case-by-case basis. In the present article, because we consider the simpler problem of no-slip on the contact zone, we are able to derive results which are valid for an arbitrary path of deformation. To the best of our knowledge, nothing like this exists in the current literature. In this context, Elata (1995) has shown that as long as $dT/dN < \mu$ there will be no tendency to slip. Therefore our results are directly applicable to certain deformation paths even if μ is finite.

The organization of the article is as follows. We review the salient features of the H-M theory in Section 2. In Section 3 we derive a formula for the transverse force, T , which is valid for an arbitrary deformation path, including decreasing w . We derive a result for the work done when the system is taken around a closed loop in Section 4. With the possible exception of the first loop, the work done is always positive, is the same regardless of the direction in which the loop is taken, and is the same with every successive circuit; the "first loop" exception is clearly delineated. We give a simple application of our results to acoustic waves in Section 5 and we summarize our results in Section 6.

2. REVIEW OF THE THEORIES OF HERTZ AND MINDLIN

Consider two elastic grains in contact at points of asperity. For simplicity, we assume the two principal radii of curvature for each grain are equal to a common radius, R , and we assume the elastic constants of the two grains are the same. We imagine that the grains are compressed along their common centers by a relative amount $2w$ and are sheared by a relative amount $2s$. The resultant forces and the work done are crucially dependent upon the order in which this is done, that is to say, upon the path taken: $w = w(t)$, $s = s(t)$ where t is some convenient parameter.

The component of the force which is normal to the contact circle, N , depends only upon the normal displacement, w . Hertz showed that the change in N with w is

$$\Delta N = C_n a(w) \Delta w \quad (1)$$

where C_n is given in terms of the elastic constants of the material, $C_n = 4\mu/(1-\nu)$ and $a(w) = \sqrt{Rw}$ is the actual radius of the contact circle. Therefore $N(w) = \frac{2}{3}C_n\sqrt{Rw^3}$. This part of the force is not only path independent but it is conservative as well; we need consider it no further.

Mindlin (1949) has considered the elasticity problem of a transverse displacement under the assumption that there is perfect adhesion between the grains within the contact circle and perfect slip outside. He has shown that the application of an additional transverse displacement acts independently of the normal in the sense that N does not change if w is held fixed. Basically, the problem is reduced to the elastic deformation under a transverse far-field displacement of two half-spaces which are in welded contact over a circle of radius a , the size of which is determined from the application of a normal force (above). Let the contact circle lie in the x - y plane and let the transverse displacement be in the x -direction. One has

$$u_x(x, y, \pm \infty) = \pm \Delta s, \quad (2)$$

where Δs is the size of the transverse displacement, in addition to any pre-existing transverse displacement. Mindlin utilizes the key observation that the symmetry of the problem requires that

$$u_x(x, y, z) = -u_x(x, y, -z), \quad (3)$$

and therefore continuity of displacement requires that the points on the contact circle do not move:

$$u_x(x, y, 0) = 0. \quad (4)$$

(Outside the contact circle there is displacement, which is discontinuous across $z = 0$.) This, plus the condition that the surface outside the contact circle is traction free, is enough to guarantee a unique solution. The result is that the traction on the surface everywhere points in the x -direction with a magnitude which is axially symmetric:

$$\Delta \tau(r) = \begin{cases} \frac{C_t}{2\pi} \frac{\Delta s}{\sqrt{a^2 - r^2}} & r < a, \\ 0 & r > a, \end{cases} \quad (5)$$

where $C_t = 8\mu/(2-\nu)$. The resultant additional transverse force, ΔT , is simply the integral of (5) over the area of the contact circle:

$$\Delta T = C_t a(w) \Delta s \quad (\text{for fixed } w). \quad (6)$$

Observing that eqn (5) predicts a traction which is divergent, Mindlin and Deresiewicz (1953) considered the effects of a finite coefficient of static friction. The resultant traction does not, of course, diverge. In the present article we consider only the special case of purely elastic forces which correspond to an infinite coefficient of friction. Our model assumes that once adjacent points come in contact they cannot slip past each other. Adjacent points that are not in contact, either because they have never been in contact or because the normal force has been reduced to the point where they now lie outside the contact zone, are free to slip. With this simplification we are able to derive simple expressions for the traction as a function of path-dependent deformation, $s(t)$,

$w(t)$. We note that this is not possible for the more general Mindlin–Deresiewicz theory; each path must be calculated anew on a case-by-case basis.

We hasten to mention that our assumptions are somewhat overly restrictive. Elata (1995) has shown that if s and w are simultaneously changed then there is no tendency to slip if $dT/dN < \mu$ where μ is the coefficient of static friction.

3. PATH DEPENDENT TANGENTIAL FORCE

First, let us imagine that we are at some point w, s corresponding to a traction T . How do the forces change as the deformation evolves along some path $w(t), s(t)$? A change Δs_1 leads to a change ΔT_1 given by (6), with no change in N . A change in w , either positive or negative, leads to a change in N given by (1); if Δw is positive there is no change in T . At this point, if w has increased, the radius of the contact circle has also increased. Therefore, any further change in transverse displacement, Δs_2 , results in a change in traction given by (5) evaluated at the new, larger radius. Similarly for ΔT_2 , given by (6). Within the context of linear elasticity applied to this problem, the “old” traction due to any previous transverse displacements is still frozen in at the values of those smaller radii.

We consider an arbitrary path $w(t), s''(t)$ subject only to the constraint that w is a nondecreasing function of the conveniently chosen parameter t ; the obvious choices are $t = w$ or even $t = a$. It is convenient to view this process in the a – s plane (Fig. 1). We start at point A and proceed toward point B along the path $w(t), s''(t)$. We discretize the path in an obvious manner. The traction is

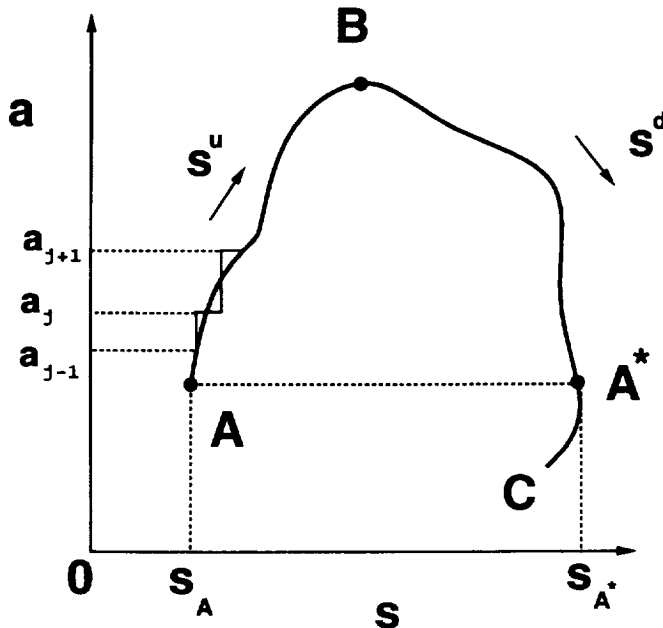


Fig. 1. Path in the a – s plane.

$$\tau_j^u(r) = \tau_A(r) + \frac{C_1}{2\pi} \sum_{i=1}^j \frac{\Delta s_i^u}{\sqrt{a_i^2 - r^2}}, \quad (7)$$

where we use the shorthand that any term for which $r > a_i$ is zero, as in (5). Also,

$$s^u(j) = s_A + \sum_{i=1}^j \Delta s_i^u \quad (8)$$

and

$$T^u(j) = T_A + C_1 \sum_{i=1}^j a_i \Delta s_i^u. \quad (9)$$

This last we write in integral form as

$$T^u(t) = T_A + C_1 \int_{t_A}^t a[w(t)] \frac{ds^u}{dt} dt. \quad (10)$$

We are now in a position to consider what happens when the normal compression decreases and the transverse displacement changes but the path taken down, $w(t)$, $s^d(t)$, is different than the path going up (see Fig. 1). We emphasize that the parameter t is defined such that $w(t)$ is single valued, the same for both trajectories, implying that t could be either w itself or a , but it is *not* an arc-length parameter. We have drawn the figure such that $s^d > s^u$ but this is not a necessary condition; the two curves could cross each other arbitrarily. Clearly, even if s is held fixed T must decrease because the contact circle has decreased. At any point, the traction consists of a sum of terms as in eqn (7). When w is decreased, holding s fixed, the contact radius also decreases, from $a(w)$ to $a(w')$, say. It is obvious that the traction must vanish in the annular region between the initial and current radii, by the assumptions of the model. It is not obvious, but true nonetheless, that any contribution to τ of the form $\alpha/\sqrt{a_i^2 - r^2}$ where $a(w') < a_i < a(w)$ is completely eliminated for all r and replaced by $\alpha/\sqrt{[a(w')]^2 - r^2}$. This is because the only solutions to the equations of elasticity with the boundary conditions (2) and (4) are of the form (5); there are no terms of the form $1/\sqrt{a_i^2 - r^2}$ limited to $0 < r < a(w')$, for example. The coefficient α is the same as before in order to guarantee that s does not change.

The protocol for moving down from point B back towards point C along the curve w , s^d is as follows. When a is decreased from a_{i+1} to a_i , any contribution to τ such as $\alpha/\sqrt{a_{i+1}^2 - r^2}$ is replaced by $\alpha/\sqrt{a_i^2 - r^2}$ (contributions corresponding to a_{i+2} , a_{i+3} ... having already been eliminated). Then an additional transverse displacement Δs_i^d is applied which results in a contribution

$$\Delta \tau^d(r) = \begin{cases} \frac{C_1}{2\pi} \frac{\Delta s_i^d}{\sqrt{a_i^2 - r^2}} & r < a_i, \\ 0 & r > a_i \end{cases}. \quad (11)$$

This process is repeated going down the staircase, starting at point B. We find

$$\tau_j^d(r) = \tau_A(r) + \frac{C_t}{2\pi} \left[\frac{(s_j^d - s_j^u)}{\sqrt{a_j^2 - r^2}} + \sum_{i=1}^{j-1} \frac{\Delta s_i^u}{\sqrt{a_i^2 - r^2}} \right], \quad (12)$$

and

$$T^d(j) = T_A + C_t \left[a_j(s_j^d - s_j^u) + \sum_{i=1}^{j-1} a_i \Delta s_i^u \right]. \quad (13)$$

This last we write in integral form as

$$\begin{aligned} T^d(t) &= T_A + C_t \left[\int_{t_A}^t a[w(t)] \frac{ds^u}{dt} dt + a[w(t)](s^d(t) - s^u(t)) \right] \\ &= T^u(t) + C_t a[w(t)](s^d(t) - s^u(t)). \end{aligned} \quad (14)$$

This result is path dependent in an unusual way: it depends only upon the current value of s and upon the path going up to the current value of a , but not at all upon the path for values of a greater than the current value. The resultant transverse force is exactly the same as if one went up the path s^u and then changed s to s^d , holding w (and therefore a) fixed. We see immediately from (14) that at any point where the two paths cross, $s^d = s^u$, we have $T^d \equiv T^u$. The incremental change in T is modified from (6)

$$\Delta T = C_t \left[a(w) \Delta s + H(-\Delta w) \frac{da}{dw} [s - s^*] \Delta w \right], \quad (15)$$

where H is the Heaviside function and s^* is the most recent upward trajectory.

Equation (15) summarizes the contact law for rough elastic spheres, and can be considered the central result of the paper. It may be deduced in a more geometrical manner, as follows. As the point proceeds from A to the point A* with the same value of $a = a_A$ on the downward path, see Fig. 1, the contact zone is first increased and then decreased back to its initial state. The consequent change in T must therefore depend *only* on the value of a_A and the change in s between A and A*,

$$T_{A^*} = T_A + C_t a_A (s_{A^*} - s_A). \quad (16)$$

Our starting point is therefore equivalent to the realization that eqn (14) must hold, i.e. the closing and subsequent opening of the additional contact area along this path does not influence the change in T . However, according to the rule of eqn (6) for the upward path, the change in T depends upon the area under the curve in the a - s plane. Thus, the difference $T_B - T_A$ is equal to C_t times the area under the curve between A and B. Equation (16), on the other hand, implies that the change in T between A and A* is proportional to the rectangular area under the horizontal joining A with A*, see Fig. 1. The only way to reconcile these facts is that the change in T on the downward path must exactly cancel the area above the horizontal AA* and under the curve ABA*. This must hold for arbitrary s^u and s^d , and therefore the incremental areas at equal values of a must cancel, implying the incremental rule

$$\Delta T = C_t a \Delta s + C_t (s - s'') \Delta a \quad \text{for } \Delta a < 0. \quad (17)$$

Combining eqns (6) and (17) yields

$$\Delta T = C_t a \Delta s + H(-\Delta a) C_t (s - s'') \Delta a, \quad (18)$$

which is equivalent to (15).

4. WORK AND ENERGY

The form of the incremental rule (18) suggests that we consider the Legendre transform of T :

$$Q = C_t a s - T. \quad (19)$$

This has the property, implied by (18), that

$$\Delta Q = C_t s^* \Delta a, \quad (20)$$

where $s^* = s$ for $\Delta a > 0$, otherwise s^* is the value of s'' on the most recent section of the path with the same value of a where $\Delta a > 0$. The parameter Q helps simplify some general results for the tangential force T and the associated work. Thus, let us first consider a circuit in the w - s or a - s planes, see Fig. 2. If the circuit starts and finishes at the lowest point, $a = a_{\min}$, then the initial upward path defines the value of s^* for the downward part of the circuit. Consequently, the value of Q , and hence T returns to its initial value after the circuit is completed. This is true regardless of the direction

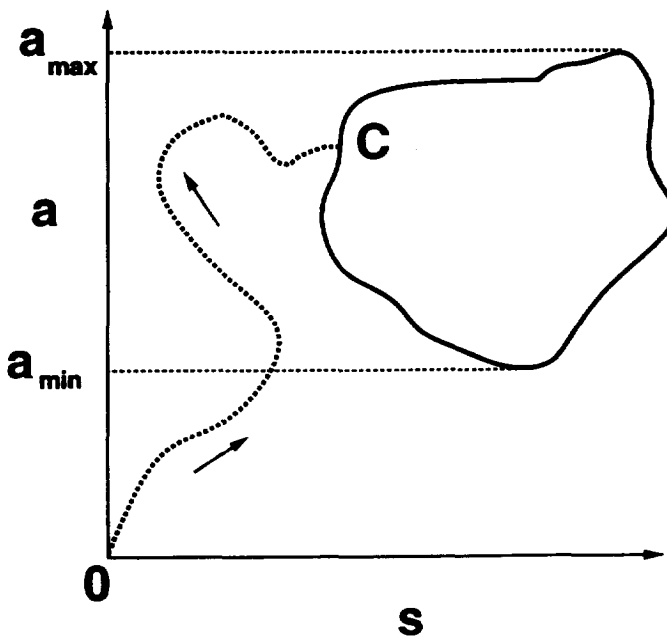


Fig. 2. Closed path or loop.

taken, clockwise or counterclockwise in Fig. 2. However, if the circuit starts at some other point, the value of T upon return to the starting point can differ from its initial value. This is the case, for example, if the starting point is C in Fig. 2 and that point has been reached along the path indicated. The value of T is different upon its first return to C, and different depending upon whether the circuit was traveled clockwise or counterclockwise, as follows from (14). For second and subsequent circuits of the loop the value of s^* on the downward leg has already been set up by the upward section of the loop and therefore T repeats itself at each and every point on the circuit.

We now turn to the total work done in deforming the contact between the spheres. The total work done by the external agent is twice the individual work done on each sphere. Starting from the origin, $w = s = 0$, this is

$$\begin{aligned}
 W &= \int_0 \mathbf{F} \cdot d\mathbf{r} \\
 &= \int_0 N dw + \int_0 T ds \\
 &= \frac{4}{15} C_n R^{1/2} w^{5/2} + C_t \int_0 a s ds - \int_0 Q ds \\
 &= \frac{4}{15} C_n \frac{a^5}{R^2} + \frac{1}{2} C_t a s^2 - Qs + \frac{1}{2} C_t \int_0 (2s^* - s) s da. \quad (21)
 \end{aligned}$$

Eliminating Q , and separating the integral into up- and down-going contributions yields

$$W = \frac{4}{15} C_n \frac{a^5}{R^2} + Ts - \frac{1}{2} C_t a s^2 + \frac{1}{2} C_t \int_{\Delta a > 0} s^2 da + \frac{1}{2} C_t \int_{\Delta a < 0} (2s^* - s) s da, \quad (22)$$

where s^* refers to the most recent upward path. The work is clearly path dependent, although the path only affects the work done by the tangential force.

Let us now consider the work done by going completely around the closed loop of Fig. 2,

$$W_{\text{loop}} = W_{\text{final}} - W_i. \quad (23)$$

The normal force contribution to W_{loop} vanishes because of its path independence. If s^u and s^d define the up and down sections of the circuit, respectively, then (22) implies that

$$W_{\text{loop}} = \frac{1}{2} C_t \int_{a_{\min}}^{a_{\max}} (s^d - s^u)^2 da + C_t \int_{\Delta a < 0} (s^* - s^u) s^d da + (T_{\text{final}} - T_i) s_i, \quad (24)$$

where s_i is the value of s at the beginning (and end) of the loop. The value of W_{loop} depends, in general, upon the sense, clockwise (cw) or counterclockwise (ccw), in which the loop is traversed.

An interesting special case occurs if the loop commences at $a_{\min}$ or if it is the second

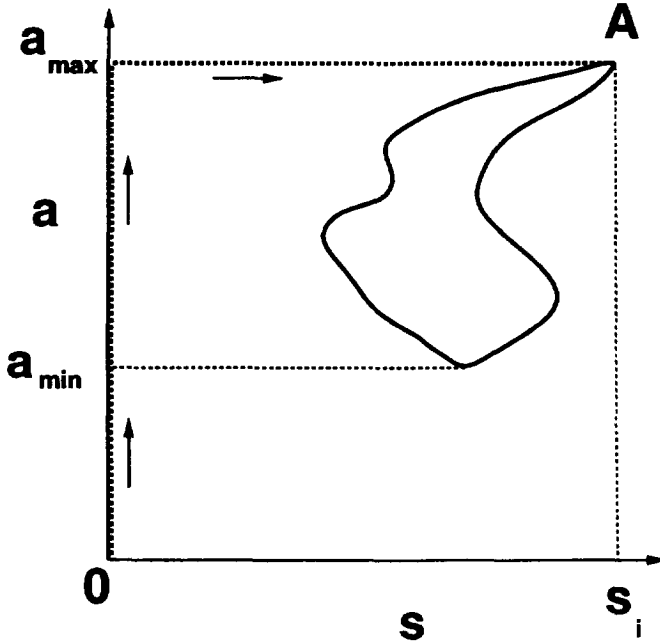


Fig. 3. Closed circuit starting at the point A which has been reached by the vertical and horizontal paths indicated. The work done in the first traversal of the loop is given by eqn (27). For this particular loop shown, the work is negative for a counterclockwise path because the initial value s_i exceeds s at all other points.

or subsequent circuit. Then $T_{\text{final}} = T_i$ and $s^* = s^u$, in which case we have the simple and general result

$$W_{\text{loop}} = \frac{1}{2} C_t \int_{a_{\min}}^{a_{\max}} (s^d - s^u)^2 da. \quad (25)$$

In particular, $W_{\text{loop}} > 0$, and the work done by the external agent is the same whether one goes around the path clockwise or counterclockwise. For subsequent circuits, we observe that (25) holds regardless where we define the "start" of the loop. Where does the work go? Referring to the staircase of Fig. 1, we observe that whenever w and a are decreased with no change in s there is a stress relief in T . The microscopic elastic deformation energy associated with $\tau(r)$ decreases in this process. Since this energy is not transferred to the external agent (because s does not change), it is radiated away as acoustic energy and ultimately transferred to the heat bath. The details depend upon the rate at which w is decreased and upon the attenuation mechanism of the acoustic waves.

The work done by the external agent in a closed circuit is not necessarily positive for the first loop, as the following rather general example illustrates. Consider a starting point in the a - s plane reached by first increasing a with $s = 0$ and then s is changed with a held fixed. We assume the path then describes a circuit with a always less than or equal to the initial value and with only two values of s for $a_{\min} \leq a \leq a_{\max}$, as in Fig. 3. Let s^d and s^u be the down and up-going parts of the circuit, respectively. Therefore, because $s^* = 0$ for the first loop up to $a_{\max}$, eqn (14) implies

$$T_{\text{final}} - T_i = -C_t \int_{a_{\min}}^{a_{\max}} s'' da, \quad (26)$$

and consequently it follows from (24) that

$$W_{\text{loop}} = \frac{1}{2} C_t \int_{a_{\min}}^{a_{\max}} [(s^d)^2 + (s'')^2 - 2s''s_i] da. \quad (27)$$

In contrast to the previous result of (25) the work now depends upon the sense of the path. Thus, the difference between the value of (27) for clockwise (cw) versus counterclockwise (ccw) paths is

$$W_{\text{cw}} - W_{\text{ccw}} = C_t s_i \int_{a_{\min}}^{a_{\max}} (s_{>} - s_{<}) da \quad (28)$$

where $s_{>}$ and $s_{<}$ are the greater and lesser values of s for each value of a . Thus, $W_{\text{cw}} > W_{\text{ccw}}$ for the loop of Fig. 3. Regarding the sign of W_{loop} , consider, for example, a circuit in the region $s \geq 0$ with $s^d \leq s'' \leq s_i$. Then W_{loop} is negative if the circuit is followed in the counterclockwise sense, because then the integrand of (27) is everywhere negative. The work done in subsequent loops is given by (25) and is non-negative. The fact that $W_{\text{loop}} < 0$ for the initial circuit indicates energy is returned to the external agent from pent up forces, or internal stresses, in the contact. This energy is returned only in the first cycle, after which the external agent must provide energy to the system. Of course, the energy returned to the external agent in the first cycle is always less than the work done in first loading the system up to the start point.

5. EXAMPLE: ACOUSTIC WAVES IN A GRANULAR MEDIUM

What are the implications of these results for acoustics? We focus on two possible effects for steady state and transient motion in a compacted medium composed of rough spherical grains. Assuming a mean field approximation, the acoustically induced relative motion of an adjacent pair of spheres is defined in an affine manner by the local quasistatic strain field of the averaged medium. We first consider steady state or cyclic motion for which eqn (25) applies. Therefore for a linearly polarized wave there is no loss of energy due to this mechanism. Of course there are acoustic situations in which the local displacement is elliptically polarized, such as in a Rayleigh wave. Let us therefore consider a path in the a - s plane given by

$$\left(\frac{a-a_0}{\Delta a}\right)^2 + \left(\frac{s-s_0}{\Delta s}\right)^2 = 1. \quad (29)$$

It is straightforward to evaluate (25) using this path:

$$W(\text{elliptical loop}) = \frac{8}{3} C_t (\Delta s)^2 \Delta a. \quad (30)$$

Equation (30) is a specific illustration of the general issue that even if the path is not

linearly polarized, the dissipation per cycle is proportional to A^3 , where A is the amplitude of the wave. Since the stored energy is proportional to A^2 , this means that, in the same amplitude limit, linear acoustics are recovered with no (linear) dissipation. This result is true for any "small" closed loop. It is easy to see that such a traveling wave would decay, not exponentially with distance, but much more slowly, inversely with distance for large distances: $\lim_{x \rightarrow \infty} A \propto x^{-1}$.

As a linearly polarized wave passes a pair of contacting grains the work done in each cycle is zero with the positive exception of the first half-cycle when a decreases below its static value. Let us assume, for simplicity, that the initial state was reached by compressional followed by tangential shear, similar to the state of affairs at point A in Fig. 3. We consider single frequency motion starting at $t = 0$:

$$\frac{a - a_0}{\Delta a} = \frac{s - s_0}{\Delta s} = -\sin(\omega t)H(t), \quad (31)$$

where $\Delta a > 0$ with no loss in generality. Therefore eqn (27) applies in the first half of the first period of the motion, with $a_{\min} = a_0 - \Delta a$, $a_{\max} = a_0$, and $s^d = s_u = s_0 + (a - a_0)/\Delta a \Delta s$. Evaluation of (27) yields to leading order

$$\begin{aligned} W(\text{first half-cycle}) &= -\frac{1}{2} C_{ts_0} \Delta s \Delta a \\ &= -\frac{T_0}{2a_0} \Delta s \Delta a, \end{aligned} \quad (32)$$

where T_0 is the initial value of the tangential force. Equation (32) can be positive or negative, indicating work put into the system, or energy released from the initially stressed contact.

Consider two possible paths caused by the passage of a small amplitude wave as indicated in Fig. 4. The work done is positive for a path such as BB because $\Delta s < 0$ in that case, and similarly the work is negative for the small motion AA. We note that the energy is released or dissipated only on the first half-cycle, and the motion is conservative for subsequent cycles. Also, if the granular medium contains a random orientation of contacts then the net sum of all these energetic effects will be reduced because they will tend to cancel one another on average.

6. SUMMARY

We have considered the static elastic deformation of spherical grain-grain contacts within the context of linear elasticity, as originally delineated by Hertz and by Mindlin. Our focus has been on examining arbitrary transverse (s) and normal (w) relative displacements under the assumption that there is no slip of the grains when they are in contact but the path of deformation $s(w)$ is arbitrary. We have derived simple rules for the path dependent transverse force, T ; we have shown that when w decreases, the value of T depends upon the current value of s and upon that part of the path taken when w was increased to its current value, but not at all upon the path taken for values of w greater than the current one: eqn (14). We have derived a simple

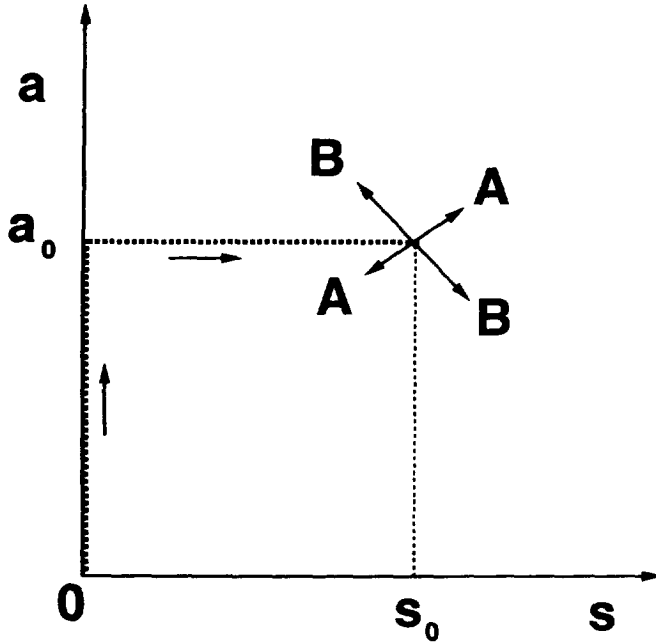


Fig. 4. Small amplitude acoustic wave induces motion along the self repeating path AA or BB. Work is done only on the part of the first cycle with $a < a_0$, and the value is given by eqn (32). Hence, $W < 0$ for the path AA and $W > 0$ for BB.

integral formula, eqn (25), for the work done per cycle by an external agent which is valid for "most" closed loops when the path of w increasing is different from that of w decreasing.

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