

Nonlinear Elasticity of Granular Media

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The finite and incremental elasticity of a random packing of identical spheres is derived using energy methods. We consider different models for the contact forces between spheres, all of which are based upon or related to the fundamental Hertz theory; we consider only the special cases of perfect friction (no tangential slip) or no tangential friction. The existence of a strain energy function for the medium depends critically upon the type of contact. If the tangential contact stiffness is independent of the normal force, then the energy is well defined for all values of the macroscopic strain. Otherwise, the strain energy of the system is path dependent, in general. However, the concept of a quadratic strain energy function is always well defined for incremental motion superimposed on large confining stress and strain. For all models considered, we derive the changes in wave speeds due to incremental strains. For the models based upon an energy function we derive expressions for the third-order elastic constants as a function of confining pressure.

1 Introduction

Consider a random packing of identical elastic spheres, which is a simple model for a granular medium. There are three basic problems of interest. First, we would like to know the dependence of the macroscopic stress, σ_{ij} , on the macroscopic strain, $\sigma_{ij} = \sigma_{ij}(e_{mn})$. We would also like the inverse relationship, $e_{ij} = e_{ij}(\sigma_{mn})$. This is linked with the issue of finding the finite forces between the particles. The second task is to calculate the incremental moduli for small motion superimposed upon the initial deformation. Finally, we would like to know how the associated speeds of small amplitude waves change upon the application of small additional stresses (Schwartz et al., 1994). The concept of strain energy density for a hyperelastic material allows one, in principle, to calculate all of these quantities. Thus, the large stress/strain behavior is governed by the finite elastic response. Incremental motion is then governed by the second-order elasticity theory, or the elastic stiffnesses C_{ijkl} at the prevailing finite strain e_{ij} . Finally, the linear variations in wave speeds for subsequent static deformation, Δe_{ij} , are linearly dependent on the third-order elasticities, C_{ijklmn} (Toupin and Bernstein, 1961).

However, the existence of a strain energy function is not always guaranteed nor even necessary for an elastic medium. Following Truesdell and Noll (1965) we use the term *hyperelastic* to refer to a medium for which there is a unique strain energy which is a function only of the current value of the strain tensor. We consider a medium to be *elastic* if there is a strain energy, which may be path dependent but which is not hysteretic along that path. In this paper we explore the concept of a strain energy function for the granular medium, both for finite deformation and for incremental motion. The key to the problem lies in the mechanics of a typical representative contact. Consider the following thought experiment: two solid spheres touch at a point but are otherwise separated. A normal force is applied, resulting in a finite contact zone, and a simultaneous approach of the two spheres. Then a shear is applied to the pair of contacting

spheres, causing little or no further approach of their centers, but giving a lateral shift of the centers. Now consider the same initial configuration, subject to the same displacements, but in the reverse order. The energy expended in the two experiments may or may not be the same, even though the solid material is considered to be perfectly elastic (local deformations are reversible). The crux of the matter lies in the kinematic differences caused by the contact of the spheres. Thus, the energies are equal if there is no frictional resistance to the shear load. However, when the contact zone is rough, it induces a resistance to shearing, and the second experiment requires less energy than the first. This type of micromechanical picture must have an impact on the microscopic energy of an assemblage of spheres. In particular, it should be clear that frictional contacts will result in a macroscopic strain energy which depends upon the loading path, and is therefore not hyperelastic. The simple ideas exhibited in the two-grain experiment will be expanded upon in this paper, and the consequences discussed. Path-dependent effects are well known in granular media. For example, Deresiewicz (1958a, b) has discussed the effect in the context of simple cubic arrays of particles.

In Section 2 we review some general properties of the various contact forces which we use in Section 3 to derive expressions for the deformation energy of a single contact and, from that, the deformation energy of the ensemble of spheres. In this article we consider only the special cases of perfect friction (no tangential slip) or no tangential friction (perfect slip), although we briefly touch upon finite friction in the Appendix. For those models of contact forces which are derivable from a potential energy function, we are able to obtain explicit expressions for the stress, the second-order elastic constants, and the third-order elastic constants as a function of macroscopic strain. We restrict our consideration to those systems for which the change in stiffness is a much larger effect than the change in dimension, which is to say, systems for which the third-order constants are much larger than the second. This is a characteristic feature of weakly consolidated granular media, and is discussed further at the end of Section 3. For the contact forces which are not derivable from a potential energy function, we show, in Section 4, that the second-order elastic constants are still well defined, but not the third (or higher) order. Strictly speaking, for these elastic but not hyperelastic systems, one can still define third-order elastic constants simply as the second derivative of stress with respect to strain taken along a given path, and this, in effect, is what we do in Eq. (67). We derive the changes in wave speeds due to an incremental strain in Section 5 and we show an explicit

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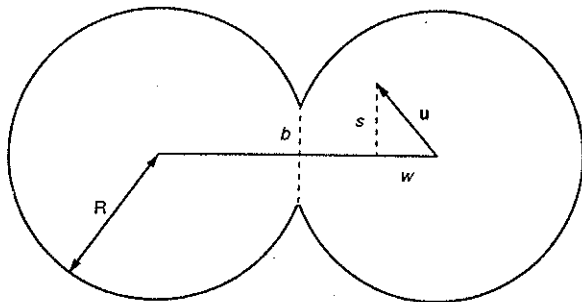


Fig. 1 Schematic of two spheres, each of radius R , in contact with each other. The radius of the initial unstressed circle of contact, b , may or may not be zero, depending on the model being considered. The center of one sphere is displaced relative to the other by a vector u which is resolved into the component along the line of centers, $2w$, and that perpendicular to it, $2s$. This deformation gives rise to a restoring force F (not shown) with component N along the line of centers, and T perpendicular to it.

example of hydrostatic confining pressure in Section 6. We present some numerical examples compared against experimental data in Section 7. Our concluding remarks appear in Section 8.

2 The Contact Forces

2.1 Contact Models. Consider two identical spheres each of radius R pressed together. The force between the spheres acts over a contact zone the radius of which is small compared with R . Let $2w(\geq 0)$ be the relative approach of the two spheres along the line joining their centers. The relative tangential displacement between the two spheres is $2s$. See Figure 1. Similarly, the force may be decomposed into a normal force N and tangential force T . We are interested in the finite and infinitesimal elasticity of an ensemble of spheres, for which the single pair in contact under normal and tangential loads describes the fundamental mechanics.

Our starting point is the incremental relations between the forces and the displacements,

$$\Delta N = D_n(w)\Delta w, \quad \Delta T = D_t(w)\Delta s, \quad (1)$$

where D_n and D_t are contact stiffnesses, in the notation of Digby (1981) and Winkler (1983). These are of the form

$$D_n = C_n a_n(w), \quad D_t = C_t a_t(w), \quad (2)$$

where C_n and C_t are actual stiffnesses (with units of pressure)

$$C_n = \frac{4\mu}{1-\nu} = \frac{8\mu(\lambda + \mu)}{\lambda + 2\mu}, \quad (3)$$

$$C_t = \frac{8\mu}{2-\nu} = \frac{16\mu(\lambda + \mu)}{3\lambda + 4\mu},$$

λ , μ are the Lamé constants of the spheres and the Poisson's ratio is ν . The lengths a_n and a_t depend upon the specific type of contact but do not depend upon the material properties of the spheres. Several models are discussed in the Appendix, and the functional forms for a_n and a_t are summarized in Table 1. All models considered in the present article either have smooth contacts with reversible slip or rough contacts with no relative slip. Both lengths are defined by the current value of w , but are independent of s . Note also that the properties of each contact, though calculated within the approximations of ordinary linear elasticity theory, lead to nonlinear restoring forces. As we shall see, these nonlinear forces can, in turn, lead to extremely large nonlinear elastic constants for the aggregate media.

All the models in Table 1 have $a_t \leq a_n$, with equality for the infinitely rough contacts in models Ib, Iib, and Iiib. When

$a_t = a_n$, the ratio D_n/D_t reduces to $C_n/C_t = 1 + \nu/(2 - 2\nu)$, which is approximately 1.17 for rocks ($\nu = 1/4$). This value is not consistent with the velocity data of Domenico (1977) on unconsolidated glass beads under pressure. Winkler (1983) demonstrated that Domenico's data yields a value for the ratio D_n/D_t , ranging from 1.79 to 3.36, which is consistent with a contact model with $a_t < a_n$. This is one possible justification for a contact model that allows for frictionless sliding in tangential deformation, as in the Digby model IIa for which $a_t = 0$.

2.2 Path Dependence. The incremental rule for ΔT in (1) does not hold for all deformation paths in the $w-s$ space when the contact is not perfectly smooth. Generally, the increment in ΔT depends upon whether w is increasing or decreasing, with different rules applicable in each case. However, it can be demonstrated (Johnson and Norris, 1995) that the distinction disappears for the models of Table 1 if the trajectory is self-repeating—that is, it retraces itself whenever w is decreasing. Therefore, to be specific, in this paper we only consider paths which are self-repeating in this sense.

Thus, T does not possess a unique functional form for models Ib, Iib, Iiib, and IV. Rather, it depends upon the path history of the loading, and is only defined along a given path in the $w-s$ space. We distinguish the "path-dependent" models, Ib, Iib, Iiib, and IV, from the remaining models for which T is uniquely defined at all values of w and s , i.e., it is a function of these parameters. The path dependence vanishes only for constant tangential stiffness. In order to integrate the force-displacement equations for a single contact with path dependence it is necessary to assume some relationship between w and $s = s(w)$. We can then rewrite the incremental force relations (1) as

$$dN = C_n a_n(w)dw, \quad dT = C_t a_t(w)ds. \quad (4)$$

The contact forces can be determined by integrating these three equations subject to the initial conditions $N = T = 0$ at $w = 0$, yielding

$$N = C_n A_n(w), \quad T = C_t \int_{\text{path}} a_t(\xi)ds \quad (5)$$

where A_n is a path-independent quantity having the dimension of area:

$$A_n(w) = \int_0^w a_n(\xi)d\xi. \quad (6)$$

The precise form of T therefore depends upon the path $s(w)$. For example, consider a linear relationship,

$$\frac{ds}{dw} = \text{constant}, \quad (7)$$

and the constant can vary from contact to contact. Equations (5) become

$$N = C_n A_n(w), \quad T = C_t a_t^{(1)}(w)s, \quad (8)$$

where $a_t^{(1)}$ also has the dimension of a length

$$a_t^{(1)}(w) = \frac{1}{w} \int_0^w a_t(\xi)d\xi. \quad (9)$$

We emphasize that w and s are not independent variables in Eq. (8). The constraint (7) implies that the spheres approach one another along a constantly directed line.

Alternatively, one could assume that the line of action of the force at each contact remains constant. That is,

$$\frac{dT}{dN} = \text{constant}, \quad (10)$$

and the constant can differ from contact to contact. The two constraints (7) and (10) are equivalent only if the ratio D_t/D_n is constant along the loading path. This is the case for the path-dependent models Ib, Iib, and Iiib, but not for IV. We will use

Table 1 The lengths a_n and a_t for different contact conditions. Models I, II, and III have two subcases, corresponding to (a) smooth contact with reversible slip; and (b) rough contact with no subsequent slip. See the Appendix for further details and definitions.

Contact model	Description	$a_n(w)$	$a_t(w)$	
			(a)	(b)
I	Hertzian contact	$(Rw)^{1/2}$	0	a_n
II	initial contact radius b (Digby)	$[R^2w^2 + \frac{b^4}{4}]^{1/2} + \frac{b^2}{2}$	b	a_n
III	ogival/spherical contact $d \approx R\alpha^2$ (Spence/Goddard)	$(Rw + d^2)^{1/2} - d$	0	a_n
IV	frictional sliding (special loading) (Mindlin and Deresiewicz)	$(Rw)^{1/2}$	$(\frac{e}{a_n} + \frac{1-e}{c})^{-1}$	

(7) later for specific examples, but emphasize that the subsequent analysis applies to arbitrary self-repeating paths $s = s(w)$.

The path-independent smooth models (Ia, IIa, and IIIa), do not require an assumption of the form $s = s(w)$, and can be integrated directly to give forces with the same functional form as in Eq. (8). Furthermore,

$$a_t^{(1)} = a_t = \begin{cases} 0 & \text{for Ia and IIIa,} \\ b & \text{for IIa.} \end{cases} \quad (11)$$

3 Finite Energy, Strain and Stress

3.1 A Single Contact. Let F be the total force exerted by a sphere at a single contact, and let u be the total displacement of the center of the sphere from its original position. The sphere then moves a further distance du . The work done by the sphere associated with that contact, assuming it does not rotate, is

$$dW = F \cdot du = NdW + T \cdot ds. \quad (12)$$

Alternatively, the potential energy for a single contact is $2dW$, when viewed in terms of the two spheres each contributing dW . We speak of a "potential energy" function for the path-dependent models with the understanding that we are restricting our discussion to motion on one specific path. The work done along that path is a conserved quantity; going down the path is the reverse of going up.

For either the path-independent models, or assuming (7) for the path-dependent ones, the explicit nature of N and T in (8) and (12) implies that

$$W = C_n V_n(w) + \frac{1}{2} C_t a_t^{(2)}(w) s^2, \quad (13)$$

where V_n is a volume and $a_t^{(2)}$ a length,

$$V_n(w) = \int_0^w d\zeta \int_0^\zeta a_n(\xi) d\xi, \\ a_t^{(2)}(w) = \frac{2}{w^2} \int_0^w d\zeta \int_0^\zeta a_t(\xi) d\xi. \quad (14)$$

We have set $W = 0$ at the initial point, with no loss in generality. Thus, W is one half of the energy stored in a single contact. The result (13) is valid for all contact models, with the understanding that w and $s = |s|$ are related for the path-dependent models. For the others, we have the simplification that

$$a_t^{(2)} = a_t = \begin{cases} 0 & \text{for Ia and IIIa,} \\ b & \text{for IIa.} \end{cases} \quad (15)$$

The length a_n , area A_n , and volume V_n are fundamental quanti-

ties, as is the dimensionless quantity $a'_n(w)$ which occurs later. Explicit formulae for $a'_n(w)$, A_n , and V_n are given in Table 2 for models I, II and III, (a) or (b).

3.2 The Ensemble of Spheres. We now turn to the random packing of spheres, and make the standard kinematic assumption relating the displacement of sphere m to the macroscopic deformation gradient f_{ij} (Digby, 1981; Walton, 1987; Jenkins, 1991). The displacement of the center of a given sphere is

$$u_i = f_{ij} X_j, \quad \text{or} \quad \mathbf{u} = \mathbf{f} \cdot \mathbf{X}, \quad (16)$$

where $\mathbf{X}$ is the position of the center of the sphere. Let $\hat{n}$ be the unit vector joining the centers of two contacting spheres, at $\mathbf{X}$ and $\mathbf{X} + 2R\hat{n}$. The associated displacements are $\mathbf{f} \cdot \mathbf{X}$ and $\mathbf{f} \cdot \mathbf{X} + 2R\mathbf{f} \cdot \hat{n}$, and so the components of the relative displacement are

$$w = -\hat{n} \cdot \mathbf{f} \cdot \hat{n} R, \quad s = \mathbf{P} \cdot \mathbf{f} \cdot \hat{n} R, \quad (17)$$

where $\hat{n} \cdot \mathbf{f} \cdot \hat{n} = n_i f_{ij} n_j$, and $\mathbf{P}$ projects onto the tangent plane,

$$P_{ij} \equiv \delta_{ij} - n_i n_j. \quad (18)$$

The total strain energy density per unit volume is

$$U = \frac{1}{V} \sum_{\text{contacts}} W = \frac{1}{V} \sum_{\text{contacts}} \int \mathbf{F} \cdot d\mathbf{u}, \quad (19)$$

where the sum is over all contacts on each sphere (each contact is counted twice since W is only half the contact energy), and V is the total volume of the sample. The effective medium approximation we employ for this article is that statistically all grains are "the same" and each may be replaced by its ensemble average: $\sum_{\text{contacts}} W(w_i, s_i) \approx Nn \langle W[w(\hat{n}), s(\hat{n})] \rangle$ where N is the number of grains in the volume V , n is the number of

Table 2 The quantities a'_n , A_n , and V_n for the three contact models I, II, and III; see Table 1.

Model	$a'_n(w)$	$A_n(w)$	$V_n(w)$
I	$\frac{R}{2a_n} = \frac{1}{2} R^{1/2} w^{-1/2}$	$\frac{2}{3} w a_n = \frac{2}{3} R^{1/2} w^{3/2}$	$\frac{4}{15} w^2 a_n = \frac{4}{15} R^{1/2} w^{5/2}$
II	$\frac{R(a_n^2 - b^2)^{1/2}}{2a_n^2 - b^2}$	$\frac{2}{3} w (a_n + \frac{b^2}{2a_n})$	$\frac{4}{15} w^2 (a_n + \frac{b^2}{2a_n}) - \frac{(a_n - b)^4}{15R^2}$
III	$\frac{R}{2(a_n + d)}$	$\frac{2}{3} a_n w - \frac{e^2 d}{3R}$	$\frac{2}{5} \frac{A_n}{R} (a_n + d)^2 - \frac{w^2 d}{10}$

contacts per grain, and $\langle \dots \rangle$ denotes average over the distribution of directions $\hat{n}$. Therefore,

$$U = \frac{n(1-\phi)}{V_0} \langle W[w(\hat{n}), s(\hat{n})] \rangle$$

$$= \frac{n(1-\phi)}{V_0} \left\langle \int \mathbf{F} \cdot d\mathbf{u} \right\rangle, \quad (20)$$

where ϕ is the porosity of the sample and V_0 is the volume of a single grain. In this article we explore the consequences of assuming an isotropic distribution $P(\hat{n}) = 1/4\pi$ although there is some evidence that this is not necessarily the case (Jenkins et al., 1989).

Before we compute U we must address the issue of whether or not each sphere rotates under the action of the applied strain (Schwartz et al., 1984). If the neighbors of a given sphere rotate, then there will be an energy cost unless that sphere rotates, too. We now show that, within the context of our effective medium approach, each sphere will not rotate if the deformation gradient is symmetric $f_{ij} = f_{ji}$. First, let us consider a rigid rotation of the system about an axis $\hat{\theta}$ by an amount θ . In terms of the rotation matrix $\mathbf{R}$ each coordinate moves to a new position $\mathbf{r}' = \mathbf{R} \cdot \mathbf{r}$ and the displacement is $\mathbf{u} = \mathbf{r}' - \mathbf{r}$. It is straightforward to show that

$$\nabla \times \mathbf{u} = 2 \sin \theta \hat{\theta}. \quad (21)$$

Such a rigid-body rotation is an example of Eq. (16) which gives

$$\nabla \times \mathbf{u} = (f_{32} - f_{23}, f_{13} - f_{31}, f_{21} - f_{12}). \quad (22)$$

Therefore it is clear that if the macroscopic deformation gradient is symmetric, there is no macroscopic rigid-body rotation. Under such a deformation we may assume that each sphere *does not* rotate so as to match the positions of its neighbors. The deformation gradient is then equal to the strain tensor. In order to make explicit this assumption we rewrite (17) as

$$\mathbf{w} = -\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}, \quad \mathbf{s} = \mathbf{P} \hat{n} \mathbf{R}, \quad (23)$$

where $\mathbf{e}$ is the symmetric part of the macroscopic strain,

$$\mathbf{e} = \frac{1}{2}(\mathbf{f} + \mathbf{f}^T), \quad (24)$$

and we shall write $\mathbf{P} \hat{n}$ as a shorthand for the vector $\mathbf{P} \cdot \mathbf{e} \cdot \hat{n}$ (w is unaffected). In summary, within the mean field approximation adopted here only the symmetric part of the macroscopic deformation gradient influences the internal strain energy and so we need consider only symmetric deformations.

We are now ready to calculate the deformation energy from Eq. (20). We assume there are N spheres with an average of n contacts per sphere. First, consider the case where each contact follows the linear path defined by Eq. (7), so that

$$\mathbf{F} = -C_n A_n (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) \hat{n} + C_t a_t^{(1)} (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) \mathbf{R} \mathbf{P} \hat{n},$$

$$d\mathbf{u} = R(d\mathbf{e}) \cdot \hat{n}. \quad (25)$$

The energy can now be obtained by substitution into Eq. (20) followed by integration, yielding

$$U = (1-\phi) \frac{n}{V_0} \{ C_n \langle V_n (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) \rangle$$

$$+ \frac{1}{2} C_t R^2 \langle a_t^{(2)} (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) | \mathbf{P} \hat{n} |^2 \rangle \}, \quad (26)$$

where $\langle \dots \rangle$ denotes the average over all directions $\hat{n}$, V_0 is the volume of a single sphere, and ϕ is the porosity of the sample,

$$V_0 = \frac{4}{3} \pi R^3, \quad 1 - \phi = \frac{NV_0}{V}. \quad (27)$$

Strictly speaking, Eqs. (27) do not apply to the Digby model

II because of the missing spherical caps, nor to the Spence/Goddard model III because the ogive volume is not the same as the sphere's. We note that the same formula for U in (26) may also be obtained more directly using Eqs. (13) and (23).

We may generalize (25) and (26) to the case of an arbitrary path in $w-s$ space. Let the path be parameterized by $w = \xi$, $s = s(\xi)$. The path may differ from contact to contact and is subject only to the constraints that it is self-repeating and the end points are given by Eq. (23). Then, using Eq. (5),

$$\mathbf{F} = -C_n A_n (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) \hat{n} + C_t \int_{\text{path}} a_t(\xi) ds(\xi),$$

$$d\mathbf{u} = R(d\mathbf{e}) \cdot \hat{n}, \quad (28)$$

and the energy U follows accordingly. In summary, the energy U is defined for arbitrary strain $\mathbf{e}$ for the models with constant tangential stiffness. However, for the path-dependent models U is acquired under the specific loading path chosen.

3.3 Elasticity for the Energy Models. The energy can also be defined by

$$U = \int_0^e \sigma_{ij} de_{ij}, \quad (29)$$

where $\sigma_{ij} = \sigma_{ij}(\mathbf{e})$ is the effective stress, which can be calculated separately by considering an average stress over many spheres. This is the approach taken by Walton (see Eq. (3.15) of Walton, 1987). We note at this stage that σ_{ij} cannot be obtained by simply differentiating U with respect to $\mathbf{e}$, because the energy is path specific in general. However, it is important to note that

$$\sigma_{ij} = \frac{\partial U}{\partial e_{ij}} \text{ for models Ia, IIa, and IIIa,} \quad (30)$$

and hence the strain energy function $U(\mathbf{e})$ of (26) is all that is required in order to determine the macroscopic stress for these models. Thus, from Eqs. (15), (26), and (30),

$$\sigma_{ij} = (1-\phi) \frac{nR}{V_0} \{ C_t a_t R \langle \frac{1}{2} (n_i P_{jk} + n_j P_{ik}) e_{kl} n_l \rangle$$

$$- C_n \langle A_n (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) n_i n_j \rangle \}. \quad (31)$$

This gives the finite stress at any strain $\mathbf{e}$ for models Ia, IIa, and IIIa.

If a further small strain, ϵ_{ij} , is superimposed upon $\mathbf{e}$ then the resulting stress can be found by simply substituting $e_{ij} + \epsilon_{ij}$ for e_{ij} in (31). Let the total stress be $\sigma_{ij} + \tau_{ij}$, then τ_{ij} is a nonlinear function of ϵ_{ij} , which can be expanded in a Taylor series about $\epsilon_{ij} = 0$. The linear coefficients are, by definition, the coefficients of linear elasticity at the finite strain $\mathbf{e}$. These can be obtained directly by inspection from the strain energy expansion. Thus,

$$U(\mathbf{e} + \epsilon) = U(\mathbf{e}) + \sigma_{ij} \epsilon_{ij} + \frac{1}{2} C_{ijkl}^* \epsilon_{ij} \epsilon_{kl}$$

$$+ \frac{1}{6} C_{ijklmn}^* \epsilon_{ij} \epsilon_{kl} \epsilon_{mn} + \dots, \quad (32)$$

where C_{ijkl}^* and C_{ijklmn}^* are the second and third order elastic moduli, respectively. They satisfy the usual symmetries associated with a material possessing a strain energy:

$$C_{ijkl}^* = C_{jikl}^*, \quad C_{ijkl}^* = C_{klij}^*, \quad (33)$$

$$C_{ijklmn}^* = C_{jiklmn}^*, \quad C_{ijklmn}^* = C_{mnlkij}^* = C_{mnklji}^*. \quad (34)$$

The first identity is a statement of the symmetry of the stress.

Explicit expansion of the strain energy, using Eqs. (15), (26), (30), and (32) yields

$$C_{ijkl}^* = (1-\phi) \frac{nR^2}{V_0} \{ \langle [C_n A_n (-\hat{n} \cdot \mathbf{e} \cdot \hat{n} \mathbf{R}) - C_t a_t] n_i n_j n_k n_l \rangle$$

$$+ C_t a_t \langle Q_{ijkl}(\hat{n}) \rangle \}, \quad (35)$$

where

$$Q_{ijk}(\hat{n}) = \frac{1}{4}(\delta_{ik}n_jn_l + \delta_{il}n_jn_k + \delta_{jl}n_in_k + \delta_{jk}n_in_l). \quad (36)$$

Note that the part of the moduli C_{ijk}^* attributable to the constant tangential stiffness $D_i = C_{ai}$ is isotropic, i.e.,

$$C_{ijk}^* = (1 - \phi) \frac{nR^2}{V_0} \left\{ C_n \langle a_n(-\hat{n} \cdot \mathbf{e} \cdot \hat{n} R) n_in_jn_kn_l \rangle + \frac{C_t a_t}{15} (3I_{ijkl} - \delta_{ij}\delta_{kl}) \right\}, \quad (37)$$

where $I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ is the identity tensor. Similarly, the third-order constants are given by

$$C_{ijklm}^* = -(1 - \phi) \frac{3n}{4\pi} C_n \langle a_n'(-\hat{n} \cdot \mathbf{e} \cdot \hat{n} R) n_in_jn_kn_ln_m \rangle. \quad (38)$$

The tangential stiffness does not contribute to the third-order moduli because it has been assumed to be a Hookean spring.

3.4 Finite Elasticity and Finite Strain. Here we make explicit our earlier remark that we neglect the change in sample dimensions as being a smaller effect compared to that due to the change(s) in stiffness(es). Consider the sample subject to a strain of order $\epsilon \ll 1$. The change in the sample dimension is $\Delta V/V = O(\epsilon)$, whereas the relative change in the stiffness of the system is of order $\epsilon C_{ijklmn}^*/C_{pqrs}^*$. The relative magnitude of the second and third-order moduli follows from Eqs. (37) and (38) and Tables 1 and 2:

$$\frac{\text{Second-order moduli}}{\text{Third-order moduli}} = O\left(\frac{a_n(w)}{Ra_n'(w)}\right) = O\left(\frac{a_n^2}{R^2}\right) \ll 1. \quad (39)$$

Hence, the relative stiffness change far exceeds the relative volume change. This is a fundamental characteristic of granular media with Hertzian contact forces. It is manifested in the relatively large change of elastic wave speeds as a function of confining stress, which we discuss in detail in Section 5.

We have so far omitted any mention of the finite strain tensor, defined as $\mathbf{E} = \mathbf{e} + \frac{1}{2}\mathbf{f}^T\mathbf{f}$. This is normally the fundamental quantity of finite elasticity, in particular, a hyperelastic material is a function of $\mathbf{E}$. The strain energy is assumed to have a power series expansion of symbolic form $U = U_0 + C_2EE + C_3EEE + \dots$, where C_2, C_3 represent second and third-order moduli. Alternatively, expanding in terms of $\mathbf{f}$, we have

$$U = U_0 + C_2eee + C_3eee + O(C_2eff, C_3eff). \quad (40)$$

The scaling of the moduli in (39) indicates that the correction term, C_2eff , is negligible in comparison with C_3eee , and hence it is entirely consistent to take the energy as $U = U_0 + C_2ee + C_3eee$, correct to third order in the finite strain. Consequently, there is no necessity to distinguish the finite strain $\mathbf{E}$ from the linear strain $\mathbf{e}$, even for the purpose of discussing nonlinear elastic effects (up to the order considered here).

4 Energy and Elasticity for Path-Dependent Models

The previous results for the effective moduli are not immediately applicable to the path-dependent models because there is no analog of Eq. (30) for determining σ_{ij} . This difficulty can be resolved by considering small departures from the path-dependent loading, which is physically reasonable for small wave motion or incremental deformation superimposed upon the finite deformation $\mathbf{e}$. The central difficulty is present at the

level of the individual contacts, so it makes sense to start there before considering the aggregate.

4.1 A Single Contact. The path-dependent strain energy for a single contact, W of (13), is not a function of w and s , and hence cannot be differentiated arbitrarily. It is, however, possible to consider variations in W for arbitrary unconstrained changes in w and s about their equilibrium values, w_0 and s_0 . Let $w = w_0 + w_1$ and $s = s_0 + s_1$, where $|w_1| \ll |w_0|$ and $|s_1| \ll |s_0|$, and the extra displacements w_1 and s_1 are, in general, unrelated to one another. We first note that

$$dW = N(w, s)dw + \mathbf{T}(w, s) \cdot d\mathbf{s} \quad (41)$$

and hence

$$W = W_0(w_0, s_0) + N_0(w_0, s_0)w_1 + \mathbf{T}_0(w_0, s_0) \cdot \mathbf{s}_1 + \int_0^{w_1} N_1 dw_1 + \int_0^{s_1} \mathbf{T}_1 \cdot d\mathbf{s}_1, \quad (42)$$

where $N = N_0 + N_1$ and $\mathbf{T} = \mathbf{T}_0 + \mathbf{T}_1$. The values of $W_0, \mathbf{T}_0, N_0$ may depend on the loading history $s(w)$ to that point. The incremental force equations are, from (4),

$$dN_1 = C_n a_n(w_0 + w_1)dw_1, \quad d\mathbf{T}_1 = C_t a_t(w_0 + w_1)d\mathbf{s}_1. \quad (43)$$

Expanding and integrating gives

$$N_1(w_0, s_0; w_1, s_1) = C_n [a_n(w_0)w_1 + \frac{1}{2}a_n'(w_0)w_1^2 + \dots], \quad (44)$$

$$\mathbf{T}_1(w_0, s_0; w_1, s_1) = C_t [a_t(w_0)\mathbf{s}_1 + a_t'(w_0) \int_0^{s_1} w_1 ds + \dots]. \quad (45)$$

The form of N_1 is unambiguous because of the fact that N is always a function of w . On the contrary, the integral in $\mathbf{T}_1$ depends upon the loading path, or equivalently, upon the functional relationship between w_1 and s_1 , if any. This integral is second order in the additional displacements; the first-order term $C_t a_t(w_0)\mathbf{s}_1$ does not depend on the loading history of (w_1, s_1) . Substituting for N_1 and $\mathbf{T}_1$ into (42) we see that the incremental strain energy is defined only up to second order in the incremental displacements, but not to third or higher order, i.e.,

$$W(w_0 + w_1, s_0 + s_1) = W_0(w_0, s_0) + C_n A_n(w_0)w_1 + C_t a_t^{(1)}(w_0)s_0 \cdot s_1 + \frac{1}{2}C_n a_n'(w_0)w_1^2 + \frac{1}{2}C_t a_t'(w_0)s_1^2 + \text{path-dependent third-order terms in } (w_1, s_1). \quad (46)$$

The third-order terms depend upon the path taken from $(0, 0)$ to (w_1, s_1) . This distinguishes the path-dependent models Ib, IIb, IIIb, and IV from the others which possess well-defined strain energy functions. The concept of second-order elasticity is valid for path-dependent models but we cannot use or define third-order elasticity. We will now see how this functional restriction translates into the macroscopic strain energy, and also the effective moduli for the granular medium.

4.2 The Granular Medium. We now consider a departure from the stress/strain state $(\sigma, \mathbf{e})$, such that the total strain and stress are $\mathbf{e} + \epsilon$ and $\sigma + \tau$, respectively, where $|\epsilon| \ll |\mathbf{e}|$ and $|\tau| \ll |\sigma|$. The force at a contact becomes $\mathbf{F} + \mathbf{G}$, where $|\mathbf{G}| \ll |\mathbf{F}|$, and the associated incremental displacement $2\mathbf{v}$ of the pair of spheres relative to one another is given by

$$\mathbf{v} = R\epsilon \cdot \hat{\mathbf{n}}. \quad (47)$$

The increment in energy for a single contact, along a specific but arbitrary path, is therefore $dW = (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{v}$, implying that

$$W = W_0 + \mathbf{F} \cdot \mathbf{v} + \int \mathbf{G} \cdot d\mathbf{v}, \quad (48)$$

where W_0 is the (in general, path-dependent) energy for $\epsilon = 0$, as given by Eqs. (13) and (17), for example. The total energy density of the aggregate is

$$U = U_0 + \frac{R}{V} \sum_{\text{contacts}} \mathbf{F} \cdot \epsilon \cdot \hat{\mathbf{n}} + \frac{R}{V} \sum_{\text{contacts}} \int \mathbf{G} \cdot (d\epsilon) \cdot \hat{\mathbf{n}}, \quad (49)$$

where U_0 is the strain energy at $\epsilon = 0$, given by (26).

We can now take advantage of the arbitrariness in ϵ_{ij} to express the total stress as

$$\sigma_{ij} + \tau_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}. \quad (50)$$

The effective stress at strain ϵ is therefore

$$\sigma_{ij} = \left. \frac{\partial U}{\partial \epsilon_{ij}} \right|_{\epsilon=0}, \quad (51)$$

and the total energy density may be rewritten, from Eqs. (49) and (51),

$$U = U_0 + \sigma_{ij} \epsilon_{ij} + U_1, \quad (52)$$

where U_1 is defined as the ultimate term in (49). Finally, the incremental stress is

$$\tau_{ij} = \frac{\partial U_1}{\partial \epsilon_{ij}}. \quad (53)$$

The macroscopic stress follows from Eqs. (49) and (51) as

$$\sigma_{ij} = \frac{R}{2V} \sum_{\text{contacts}} (F_i n_j + F_j n_i). \quad (54)$$

This is an exact relation for point contacts, and it is commonly used to derive the stress (Digby, 1981; Walton, 1987; Jenkins, 1991). The main point of note here is that the same formula drops out quite naturally from the strain energy. Combining Eq. (54) with Eq. (28)₁ for $\mathbf{F}$ gives

$$\sigma_{ij} = (1 - \phi) \frac{nR}{V_0} \left\{ \frac{C_t}{2} \left\langle \int_{\text{path}} a_i(\xi) (n_i ds_j + n_j ds_i) \right\rangle - C_n \langle A_n (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) n_i n_j \rangle \right\}, \quad (55)$$

which is valid for arbitrary deformation paths, with the end condition (23)₂ for s . As a specific example, we consider the linear s trajectory of Eq. (7), for which (55) reduces to

$$\sigma_{ij} = (1 - \phi) \frac{nR}{V_0} \times \left\{ C_t R \left\langle a_i^{(1)} (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) \frac{1}{2} (n_i P_{jk} + n_j P_{ik}) e_{kl} n_l \right\rangle - C_n \langle A_n (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) n_i n_j \rangle \right\}. \quad (56)$$

When the length $a_i^{(1)}$ and the area A_n for the contact models Ia and Ib of Table 1 are used, Eq. (56) gives precisely the results derived separately by Walton (1987) for Hertzian contact with either smooth or infinitely rough contacts. Schwartz et al. (1994) have based their theory of stress-induced anisotropy upon this model. We also note that the stress σ_{ij} for the energy

models, Eq. (31), is clearly a special case of (56), and follows from the latter by simply replacing the variable length $a_i^{(1)}$ by its constant value appropriate to the energy models, see Eq. (11). Finally, by substituting (56) into (29) and integrating, it can be checked that the energy agrees with (26).

The incremental behavior depends upon the additional strain and stress in excess of ϵ and σ , the crucial quantity being the extra contact force $\mathbf{G}$. It follows from the preceding analysis for the single contact that $\mathbf{G}$ is only path independent to first order in the incremental strain. The associated linear form is

$$\mathbf{G} = -RC_n a_n (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) (-\hat{\mathbf{n}} \cdot \epsilon \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + RC_t a_t (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) \mathbf{P} \epsilon \hat{\mathbf{n}}. \quad (57)$$

The integral in (49) can now be evaluated, yielding

$$U_1 = (1 - \phi) \frac{nR^2}{2V_0} \{ C_n \langle a_n (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) (\hat{\mathbf{n}} \cdot \epsilon \cdot \hat{\mathbf{n}})^2 \rangle + C_t \langle a_t (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) |\mathbf{P} \epsilon \hat{\mathbf{n}}|^2 \rangle \}, \quad (58)$$

or

$$U_1 = \frac{1}{2} C_{ijkl}^* \epsilon_{ij} \epsilon_{kl}, \quad (59)$$

where

$$C_{ijkl}^*(\mathbf{e}) = (1 - \phi) \frac{nR^2}{V_0} \times \{ [C_n a_n (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) - C_t a_t (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R)] n_i n_j n_k n_l \} + \langle C_t a_t (-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) Q_{ijkl}(\hat{\mathbf{n}}) \rangle. \quad (60)$$

We wish to emphasize that Eq. (60) is always a well-defined path-independent function of the macroscopic strain, ϵ , even for the path-dependent models. Of course, the stress σ corresponding to that strain ϵ may well be path dependent. In the special case of Hertz-Mindlin contacts, for which $a_n(w) \equiv a_t(w) = (Rw)^{1/2}$, Eq. (60) reduces exactly to Eq. (4.6) of (Walton, 1987) but without the necessity of invoking his assumption that $s/w = \text{constant}$.

The incremental stress follows from Eqs. (53) and (59),

$$\tau_{ij} = C_{ijkl}^* \epsilon_{kl}. \quad (61)$$

Note that the moduli possess the usual symmetries associated with an elastic material, (33). The linear stress-strain relation for τ_{ij} may also be found without using the concept of strain energy, e.g. Walton (1987), but rather from a force balance on the macroscopic scale. This results in an identity which has the same form as (54) but with σ_{ij} , F_i replaced by τ_{ij} , G_i . It can be easily checked that this yields the same stress as (61).

5 Sensitivity of Wave Speeds to Confining Strain and Stress

The wave speeds for small motion superimposed upon the large strain ϵ are defined by the effective moduli C_{ijkl}^* and the effective density, $\rho^* = (1 - \phi)\rho$, where ρ is the granular density. If the wave or phase normal is $\mathbf{m}$, $|\mathbf{m}| = 1$, and the polarization direction is $\mathbf{p}$, $|\mathbf{p}| = 1$, then the wave speed v satisfies

$$\rho^* v^2 = C_{ijkl}^*(\mathbf{e}) m_j m_i p_l p_k. \quad (62)$$

Note that $\mathbf{m}$ and $\mathbf{p}$ are not independent, but must satisfy the eigenvector relation $C_{ijkl}^*(\mathbf{e}) m_j m_i p_l p_k = \rho^* v^2 p_l p_k$. We are interested in the incremental change in speed, Δv , when the strain is changed to $\epsilon + \Delta\epsilon$. The additional strain arises from a static deformation, and need not be proportional to the original, finite strain ϵ .

For the models with energy potentials we can use the standard theory of acoustoelasticity (Toupin and Bernstein, 1961),

$$\rho^* \Delta v^2 = C_{ijklmn}^* m_j m_i p_i p_k \Delta f_{mn} + (C_{ikqr}^* \Delta f_{qr} \delta_{jl} + C_{qjkl}^* \Delta f_{iq} + C_{iqkl}^* \Delta f_{jq} + C_{ijkl}^* \Delta f_{kq} + C_{ijkq}^* \Delta f_{iq}) p_i p_j m_j m_k, \quad (63)$$

where Δf is the incremental deformation gradient and $C_{ijklmn}^*(\mathbf{e})$ are the third-order moduli. Based on Eq. (39) we can safely and consistently ignore the terms in (63) involving the second-order moduli as being of a smaller order than the third-order moduli terms for the granular medium. Furthermore, using the symmetries of C_{ijklmn}^* in (38) implies

$$\rho^* \Delta v^2 = C_{ijklmn}^*(\mathbf{e}) m_j m_i p_i p_k \Delta e_{mn}, \quad (64)$$

that is, the change in speed depends only upon the symmetric strain increment defined in accordance with Eq. (24).

The analogous result for the path-dependent models can be obtained by returning to the fundamental relation for the incremental strain energy of a single contact, (46). At issue is how the coefficients of w_1^2 and s_1^2 are altered as we change w_0 to $w_0 + \Delta w_0$, and s_0 to $s_0 + \Delta s_0$. The change in the terms W_0 and $a^{(1)}(w_0)$ in (46) are path dependent, and require that the increment in s_0 be related to that for w_0 . However, this path dependence does not effect the terms of interest, i.e., we simply replace the arguments of a_n and a_t with $w_0 + \Delta w_0$ in the quadratic terms. The increment in W involving the quadratic small strain is therefore

$$\Delta W = \dots + \frac{1}{2} (C_n a_n'(w_0) w_1^2 + C_t a_t'(w_0) s_1^2) \Delta w_0. \quad (65)$$

When we translate this result to the aggregate, it is clear that the change in the wave speed is of the form

$$\rho^* \Delta v^2 = B_{ijklmn}^*(\mathbf{e}) m_j m_i p_i p_k \Delta e_{mn}, \quad (66)$$

where B_{ijklmn}^* are simply the derivatives of $C_{ijkl}^*(\mathbf{e})$, given by Eq. (60),

$$B_{ijklmn}^*(\mathbf{e}) = \frac{\partial C_{ijkl}^*(\mathbf{e})}{\partial e_{mn}}. \quad (67)$$

The explicit form follows from Eqs. (60) and (67),

$$B_{ijklmn}^* = -(1 - \phi) \frac{3n}{4\pi} \times \{ [C_n a_n'(-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) - C_t a_t'(-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R)] n_i n_j n_k n_m n_n \} + C_t a_t'(-\hat{\mathbf{n}} \cdot \mathbf{e} \cdot \hat{\mathbf{n}} R) Q_{ijkl} n_m n_n \}. \quad (68)$$

Comparing Eqs. (38) and (68) we see that

$$B_{ijklmn}^* = C_{ijklmn}^* \text{ if and only if } a_t \text{ is constant.} \quad (69)$$

In general, the third-order tensor B_{ijklmn}^* does not have all the symmetries of the third-order elastic moduli C_{ijklmn}^* in (33). Thus,

$$B_{ijklmn}^* = B_{jiklmn}^*, \quad B_{klijmn}^* = B_{ijklmn}^*, \quad B_{ijklmn}^* = B_{ijklmn}^*. \quad (70)$$

6 Example: Hydrostatic Confining Pressure

Consider a hydrostatic strain, $e_{ij} = e \delta_{ij}$, $e < 0$. The macroscopic stress σ of (56) is hydrostatic, $\sigma_{ij} = -p \delta_{ij}$, and the confining pressure is

$$p = (1 - \phi) \frac{nR}{3V_0} C_n A_n(-eR). \quad (71)$$

This is true for both the energy models and the path-dependent models, because it is independent of the tangential stiffness. The effective moduli are isotropic with two second-order and

(when applicable) three third-order moduli. The Lamé moduli, for all models considered in this paper, are

$$\left. \begin{matrix} \lambda^* \\ \mu^* \\ K^* \end{matrix} \right\} = (1 - \phi) \frac{n}{20\pi R} \times \begin{cases} C_n a_n'(-eR) - C_t a_t'(-eR), \\ C_n a_n'(-eR) + \frac{3}{2} C_t a_t'(-eR) \\ \frac{5}{3} C_n a_n'(-eR). \end{cases} \quad (72)$$

If the model possesses a unique potential energy then the third order moduli are

$$\left. \begin{matrix} C_{111}^* \\ C_{112}^* \\ C_{123}^* \end{matrix} \right\} = \frac{-(1 - \phi)n}{140\pi} C_n a_n'(-eR) \times \begin{cases} 15, \\ 3, \\ 1. \end{cases} \quad (73)$$

Note that these are all negative and

$$C_{111}^* : C_{112}^* : C_{123}^* = 15:3:1. \quad (74)$$

All other nonzero elastic constants are simply related to these three (Green, 1973). There are many alternative systems of notation for the third-order moduli, and Eqs. (73) and (74) can be converted accordingly. Green (1973) has provided a very useful table for converting from one system of notation to another. For example, Eq. (74) implies that the moduli of Toupin and Bernstein (1961) are identical, i.e., $\nu_1 = \nu_2 = \nu_3$, while those of Landau and Lifshitz (1986) satisfy $A:B:C = 8:2:1$, viz:

$$\left. \begin{matrix} A \\ B \\ C \end{matrix} \right\} = \frac{-(1 - \phi)n}{140\pi} C_n a_n'(-eR) \times \begin{cases} 4, \\ 1, \\ 1/2. \end{cases} \quad (75)$$

Departures from linear elasticity for the path-dependent models requires the moduli B_{ijklmn}^* , which are of the form

$$B_{ijklmn}^* = B_{ijklmn} + \tilde{B} I_{ijkl} \delta_{mn}. \quad (76)$$

Here, B_{ijklmn} satisfy all the symmetries of third-order elastic moduli, viz. Eq. (33), and

$$\left. \begin{matrix} B_{111} \\ B_{112} \\ B_{123} \\ \tilde{B} \end{matrix} \right\} = \frac{-(1 - \phi)n}{140\pi} \times \begin{cases} 15 C_n a_n'(-eR) - C_t a_t'(-eR), \\ 3 C_n a_t'(-eR) - 3 C_t a_t'(-eR), \\ C_n a_n'(-eR) - C_t a_t'(-eR), \\ 7 C_t a_t'(-eR). \end{cases} \quad (77)$$

We can now test the general expression (66) for the change in wave speeds. We consider the change in the speed of longitudinal and transverse waves, v_e , and v_s , for which $\mathbf{p}$ is parallel to $\mathbf{m}$ and $\mathbf{p}$ is perpendicular to $\mathbf{m}$, respectively. Noting that $dp/de = -3K^*$, where $K^* \equiv \lambda^* + 2\mu^*/3$ is the effective bulk modulus, it follows from Eq. (66) that

$$\begin{aligned} \rho^* \frac{dv_e^2}{dp} &= - \frac{(B_{111}^* + 2B_{112}^*)}{3K^*}, \\ \rho^* \frac{dv_s^2}{dp} &= - \frac{(B_{441}^* + 2B_{442}^*)}{3K^*}, \end{aligned} \quad (78)$$

where (Green, 1973) $B_{441}^* = (B_{112} - B_{123} + \tilde{B})/2$, $B_{442}^* = (B_{111} - B_{112} + 2\tilde{B})/4$. The results for the elastic models simplify because $a_t' \equiv 0$. We have $B_{111}^* + 2B_{112}^* = 21B$ and $B_{441}^* + 2B_{442}^* = 7B$.

The speeds of propagation of small-amplitude compressional and shear waves are

$$v_c = \left(\frac{\lambda^* + 2\mu^*}{\rho^*} \right)^{1/2}, \quad v_s = \left(\frac{\mu^*}{\rho^*} \right)^{1/2}, \quad (79)$$

respectively. The derivatives of the speeds with respect to pressure can then be worked out directly from the definitions of the effective moduli in Eq. (72). It is then simply a matter of some algebra to check that these do in fact agree with the expressions in Eq. (78).

Note that $C_{11}a'_n(-eR)$ is the modulus determining all the third-order moduli in (73) and (75). The additional modulus $C_{12}a'_n(-eR)$ is required when the length a , governing shear deformation at the granular level is not constant, but depends upon the loading. In the path-dependent models Ib, Iib, and IIb, this parameter is the same as a_n . The behavior of a'_n is quite different for each of models I, II, and III. Thus, as the applied strain e tends to zero, we have $C_{11}a'_n(-eR)$ tending to ∞ , 0, and the finite value $(2/\alpha\pi)C_n$, respectively, for models I, II, and III.

7 Numerical Examples and Experimental Data

In this section we present numerical calculations of the speeds and the moduli as a function of confining pressure. For the sake of specificity we limit our analysis to the Digby model, IIa. We note that there is a scale invariance which guarantees that the moduli, considered as functions of pressure, depend only on the ratio of b/R . We consider glass beads for which we assume the elastic constants and the density to be those measured by Johnson and Plona (1982). We take $n = 9$ as deduced by Bernal and Mason (1960). The calculated speeds are presented in Fig. 2. This figure is essentially the same as that of Fig 5 of Digby (1981). We note that when $b = 0$ the sound speeds are proportional to $p^{1/6}$ as discussed by Digby and others.

Also shown in Fig. 2 are the speeds of sound which Domenico (1977) has measured in glass beads under confining pressure. We have converted his values to more conventional units. Although it is not perfect, we see that the Digby model with $b = 0$ provides a reasonable description of the sound speeds, with no adjustable parameters. Winkler (1983) has pointed out that, in the context of contact models treated via the present effective medium theory, the Hertz-Mindlin model with $0 < a_i < a_n$ would provide a better description. Goddard (1990) has further critiqued the pressure dependence of the speeds. He has argued that in a disordered packing there is a variation in the number density of Hertzian contacts, which can lead to a sound speed which varies more rapidly than $p^{1/3}$, especially at lower pressures. Experimental and numerical data of Jenkins et al. (1989) and Cundall et al. (1989) at much lower confining pressures (< 140 kPa) than those considered here gave values for the shear modulus of about one third that predicted by Digby's model. They also found many redundant grains, with an average coordination number closer to five. The present model is not appropriate to this type of "loose" packing, where shear banding is common, and the mean field assumption needs to be re-examined. Thus, Jenkins et al. (1989) considered anisotropic orientation distribution functions for the contacts, and they also permitted the grains to rotate individually.

For the purposes of the present article, we consider the model to give a reasonable semi-quantitative description of the acoustic properties of granular media. In Fig. 3(a) we plot the pressure dependence of the second-order elastic constant $C_{11} = \lambda + 2\mu$ (the P -wave modulus) from Eq. (72). In Fig. 3(b), we plot the corresponding pressure dependence for the third-order constant, $-B$, using Eq. (75). The dimensionless rate of change of P -wave speed, $\rho^*(dV_p^2/dp)$ from Eq. (78), is plotted in Fig. 3(c). The following points emerge from these figures:

(a) When $b = 0$ the second-order elastic constants vanish, and the third-order constants diverge, in the limit $p \rightarrow 0$.

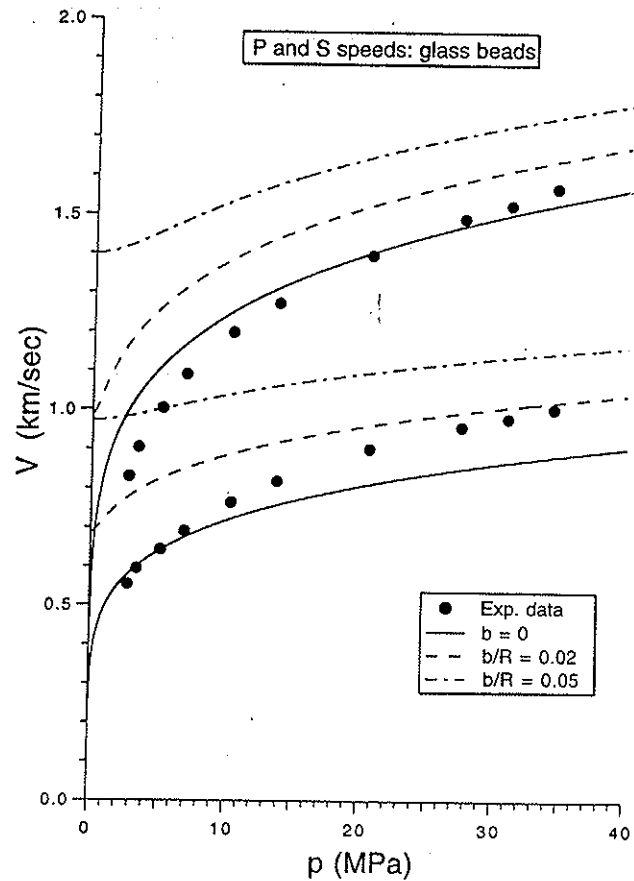


Fig. 2 Compressional and shear speeds in unsintered glass beads as a function of confining pressure. The continuous lines are calculated results based upon the Digby model of grain-grain contacts. We have used the values $n = 9$, $\mu = 49.9$ GPa, $\nu = 0.277$, and $\rho_s = 2.48$ gm $\cdot$ cm $^{-3}$. The symbols represent the experimental data of Domenico (1977) which should be compared against the $b = 0$ curves.

(b) When $b \neq 0$ the second-order elastic constants tend to a finite value and the third-order constants tend to zero, in the limit $p \rightarrow 0$. This is obvious from Eq. (13) and Table 2: V_n is an even function of w for the Digby model and so there is no third-order term in an expansion around $w = 0$.

(c) In the range of pressure over which there is experimental data, the third-order elastic coefficients are much larger than the second, thus justifying our neglect in the change of sample size as the pressure is increased. This approximation is also valid for small nonzero values of b/R .

(d) Typical values of $\rho^*(dV_p^2/dp)$ in Fig. 3(c) are much larger than the corresponding values for nongranular solids which are in the range 5–10. (See, for example, Johnson et al., (1994) and references therein.) They are, however, much less than values which have been seen in consolidated sedimentary rocks (Johnson et al., 1994). This may be due to the fact that the asperities in grain-grain contacts are much smaller than the average grain diameters and/or the ogival contact model is the more appropriate one.

8 Conclusion

We have developed a simple theory of macroscopic elasticity of granular systems based upon various models of the grain-grain contact forces which have appeared in the literature. For all models considered, the second-order elastic constants are well defined at any given static state of stress. Thus the speeds of sound are well defined at a given state of stress. The same cannot be said of third or higher-order elastic constants; in general, the change in sound speeds due to an incremental

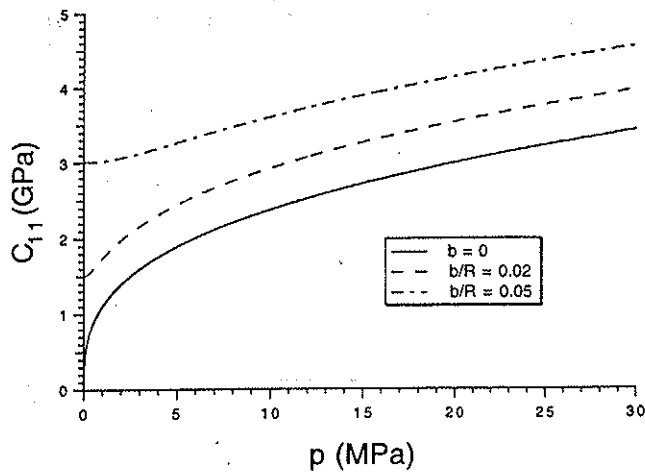


Fig. 3(a)

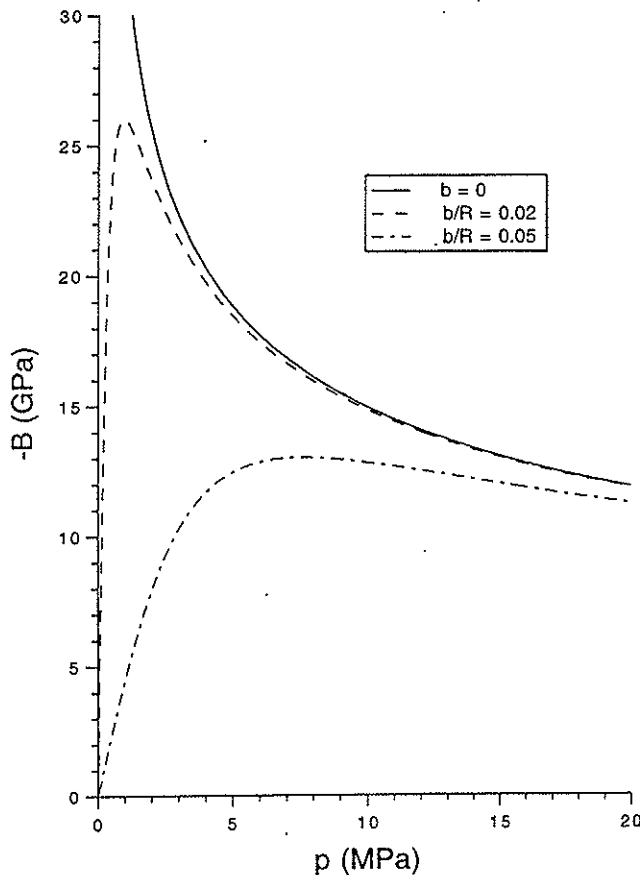


Fig. 3(b)

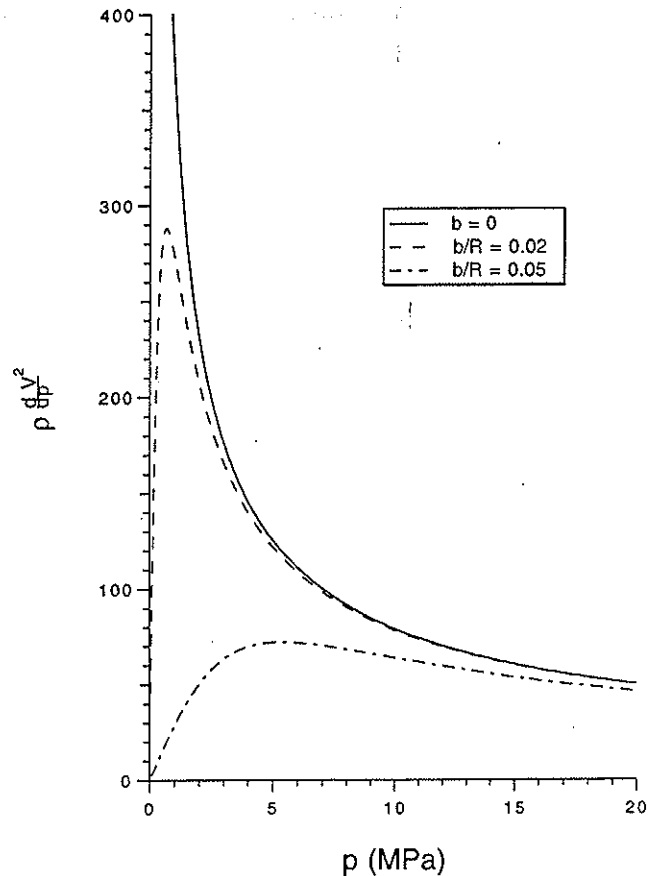


Fig. 3(c)

Fig. 3 Calculated elastic properties of granular media within the context of the Digby model. Same values of input parameters as in Fig. 2 with $\phi = 0.38$. (a) Second-order elastic constant C_{11} as a function of pressure. (b) Third-order elastic constant, B , as a function of confining pressure. (c) $\rho^*(dV^2/dp)$ as a function of confining pressure. Note the differing scales.

order constants. This theory can be extended to other states of stress, as well as to other models of the contact forces.

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change in state of stress is dependent on the history in which that change is applied. The exceptions occur for models of grain-grain contact forces which are derivable from a potential energy function, which guarantees that the solid is hyperelastic. Here, the third-order elastic constants are well defined, as are all higher order constants, independent of the path of the deformation.

We have illustrated our results with some calculations relevant to glass beads under confining pressure. The calculated speeds of sound are in semi-quantitative agreement with the existing data of Domenico (1977), as has been noted before by others. For such a system we believe this lends credence to the approximate validity of our calculations of the third-

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APPENDIX

Alternative Theories for Intergranular Forces

Oblique Loading on a Pair of Spheres. In a series of papers Mindlin and Deresiewicz (Mindlin, 1949; Mindlin and Deresiewicz, 1953; Deresiewicz, 1958) provided an exhaustive analysis of the mechanics near the contact region of two elastic spheres. This work extended the classical analysis of Hertz (who verified his contact theory by experiment) to account for tangential forces and oblique loading. Mindlin and Deresiewicz (1953) showed that the relation between the tangential force and the tangential displacement depends upon the loading history.

Consider two spheres forced together under a load N . This results in a circular contact zone of radius a and a normal displacement w ,

$$a = \left(\frac{3RN}{2C_n} \right)^{1/3}, \quad w = a^2/R. \quad (A1)$$

That is, the spheres approach twice the distance achieved by slicing off caps of radius a on each. Equations (A1) imply that

$$N = \frac{2}{3} C_n R^{1/2} w^{3/2}. \quad (A2)$$

Let a_0 be the initial contact radius for applied force N_0 . An oblique compressive force is now applied, resulting in a tangential force T and total normal force N . The additional force is applied incrementally in such a manner that its line

of action is constant, and it lies outside the friction cone, that is

$$\frac{dT}{dN} = \beta > f, \quad (A3)$$

where β is constant and f is the coefficient of friction between the surfaces. The contact zone then grows to

$$a = (N/N_0)^{1/3} a_0 = (1 + \theta L)^{1/3} a_0, \quad (A4)$$

where

$$\theta = \frac{f}{\beta}, \quad L = \frac{T}{fN_0}. \quad (A5)$$

Note that $0 \leq \theta \leq 1$, and the upper limit corresponds to infinite roughness at the contact, and hence no slip. As the contact zone grows a slip zone in the shape of a circular annulus also grows inward, so that the adhered region is of radius c where

$$c = (1 - T/fN)^{1/3} a = [1 - (1 - \theta)L]^{1/3} a_0. \quad (A6)$$

Thus, $c \leq a_0$, as expected, with equality only for infinitely rough surfaces.

The incremental change in the relative tangential displacement between the two spheres, $2s$, is related to the applied load by the differential relation (Eq. (82) of Mindlin and Deresiewicz (1953))

$$\frac{dT}{ds} = C_t \left(\frac{\theta}{a} + \frac{1 - \theta}{c} \right)^{-1}. \quad (A7)$$

There are three limits of this general relation that are of interest.

(i) First, if the surfaces are frictionless, then $c \equiv 0$ identically (there is no adhered region) and case Ia of Table 1 is obtained.

(ii) Second, if the friction is infinite, or $\theta = 1$, then T satisfies (1)₂ with $a_n = a_0$. If we integrate this relation with the initial condition that $N_0 = 0$ (which only makes sense for infinite roughness), we recover Walton's result (Eq. (2.5)₁ of Walton (1987))

$$T = \frac{2}{3} C_t a s. \quad (A8)$$

Thus, Walton's model corresponds to homothetic loading from zero confining stress. This is model Ib in Table 1. The more general relation in Eq. (A7) allows us to consider different load paths and finite friction. However, we will use Walton's formula in practice.

(iii) Alternatively, if the normal load is kept fixed as the tangential force is applied, then $\theta = 0$, and we have instead

$$dT = C_t c ds. \quad (A9)$$

This is the case considered by Mindlin in Eq. (103) of Mindlin (1949), and is model IV in Table 1.

Normal Loading of a Pair of Spheres in Contact. The models of the previous subsection can be generalized to account for initial contact. Digby (1981) considered the case of a circular zone of radius b of initial contact between the spheres at zero confining force. He showed that the contact zone for a compressive normal force N is a circular region of radius a given by the formula

$$a(a^2 - b^2)^{1/2} = R w. \quad (A10)$$

This gives the formula for $a_n = a$ in Table 1, for model II.

The Spence/Goddard Contact Model. An *ogival* indenter, composed of a sphere with a conical tip of interior angle $\pi - 2\alpha$, $0 \leq \alpha \leq 1$, is pressed into a sphere of the same radius. This type of contact geometry was discussed by Goddard (1990), and is based on a class of solutions generated by Spence (1968) for frictionless indentation of self-similar shapes on planar surfaces. Thus, the normal force required to press the ogival body a distance w against a planar surface of the same material is

$$N = C_n \frac{a^2}{2} \left(B_0 - \frac{\pi}{4} B_1 - \frac{2}{3} B_2 \right), \quad (A11)$$

where a is the contact radius, $w = B_0 a$, $B_0 = (\pi/2)B_1 + 2B_2$, and $B_1 = \alpha$, $B_2 = a/2R$. These follow from Appendix D of Spence (1968). The analogous force for the ogive/sphere contact is obtained simply by the replacement $B_2 = a/R$, which combined with (A11) leads to the incremental form of the force equation, (1)₁ and (2)₁, where a_n is given in Table 1 under model III.