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HAMILTONIAN AND ONSAGERISTIC APPROACHES IN THE NONLINEAR THEORY OF FLUID-PERMEABLE ELASTIC CONTINUA

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Abstract—A new system of equations governing nonlinear dynamics of fluid-permeable poro-elastic media is derived on the basis of Hamilton's principle for reversible effects and the Onsager–Sedov approach for irreversible effects. The equations are in Eulerian variables, suitable for dealing with fluid and solid phenomena simultaneously. The classical (Murnaghan-like) equations of nonlinear elasticity, as well as the governing equations of the ideal and Navier–Stokes fluids are shown to be special cases of the general governing system. It is well-known that the Navier–Stokes equations of compressible fluid provide a correct self-consistent basis for studying a wide variety of nonlinear effects in fluids. The authors believe that the governing system proposed here provides the same opportunities for various nonlinear effects in poro-elastic fluid-penetrable media. Copyright ©1996 Elsevier Science Ltd

1. INTRODUCTION

It is common practice to formulate nonlinear equations of solids in reference or Lagrangian coordinates. Fluid dynamics, on the other hand, is more naturally considered in current or Eulerian coordinates. The reasons are clear: in solids the material particles adhere to one another, so that the reference description always provides a continuous mapping to the current configuration of particles. Not so for fluids, where large scale shearing can separate neighboring particles in the reference system. The same is true of fluid–solid composite media, because the fluid can slide past solid particles. The use of Eulerian coordinates for such media therefore appears to be the most convenient procedure, and may be the only feasible one.

The purpose of this paper is to establish the governing equations for fluid–solid media in Eulerian coordinates by means of Hamilton's principle for conservative media and the Onsager–Sedov technique for irreversible effects. The idea is to start with a well defined Lagrangian density, from which all equations should follow. Although this approach is potentially the most reliable and flexible its usage for the Eulerian description of continuous media demands some care. However, we believe this is the most general and consistent approach to formulating nonlinear equations of fluid–solid continua. In particular, it allows one to study the internal structure of shock waves and flutter-like phenomena, among others.

We note that there are many excellent articles on the fundamental equations of fluid-permeable solids. Nonlinear treatments are usually framed in terms of a mixture theory [1,2] for which the starting point is a set of balance laws for mass, momentum, etc. Here we take one step back and invoke a stationarity principle, that of least action, to generate the equilibrium equations. Conservation of mass is considered fundamental for each species, but no other balance laws are invoked. Mixture theories can also be formulated from an energy stand point [3], and nonlinear poroelasticity has been developed from a thermodynamic point of view [4]. However, we are unaware of a treatment based on the Hamiltonian principle of least action as developed here.

In many cases discrete and distributed mechanical systems show a behavior which can be treated as conservative. It seems reasonable to study such behavior using a proper generalization of the Hamilton variational principle. Following some preliminary definitions in Section 2 we begin in Section 3 by considering the procedure of [5–8] for performing variations in Eulerian variables. The remainder of Section 3 describes the derivation of the governing equations for a conservative system from Hamilton's principle. We build upon these results in Section 4 and include dissipative or irreversible effects, associated with fluid viscosity for instance. The starting point is a generalization of Hamilton's principle using the Onsager–Sedov method. Some general results concerning energy are discussed. Finally in Section 5 we provide a variational derivation of the dynamic equations of classical hydrodynamics and of Murnaghan's equations of nonlinear elasticity [9]. We also present three examples of the application of the governing equations to two component media including fluid-filled poroelastic solids.

2. NOTATION AND THE EULERIAN DESCRIPTION OF TWO-COMPONENT CONTINUA

The use of the Eulerian description has several advantages for analyzing two-component media, and fluid-permeable solids in particular. In the Eulerian description time t and the coordinates z^i (Latin indices like i, j, k, l take the values 1, 2, 3) are used as independent variables. For the sake of simplicity the reader can think that the system is referred to the Cartesian coordinates, although we prefer to use a covariant form of the equations (in particular, this form makes general equations much more eloquent and expressive, and it allows one to switch easily between geometrically different coordinate systems). We use the notation z_{ij}, z^{ij}, ∇_i for the co- and contra-variant metrics and covariant differentiation (in the Cartesian system $z_{ij} = z^{ij} = \delta^i_j$ and co- and contra-variant components appear to be equal while covariant differentiation reduces to the partial $\nabla_i = \partial/\partial z^i$). In order to distinguish between different characteristics of the two components we use extra-subscripts a, b taking values 1 or 2 or the subscripts s or f when emphasizing the solid or fluid nature of the component. In particular, we use the notation $u^i_a(z,t)$, $\rho_a(z,t)$, and $v^i_a(z,t)$ for the displacements, densities and velocities of finite deformations of the components. Introducing the 'material' time derivative $\mathrm{D}_a/\mathrm{D}t = \partial/\partial t + v^i_a\nabla_i$ associated with the ath constituent we get by definition

$$v^i_a(z,t) = \frac{\mathrm{D}_a u^i_a}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + v^j_a\nabla_j\right)u^i_a. \tag{1}$$

The following pair of identities are immediate consequences of equation (1):

$$\frac{\partial u^i_a(z,t)}{\partial t} = v^j_a A^i_{.ja}, \qquad v^i_a(z,t) = \frac{\partial u^j_a(z,t)}{\partial t} B^i_{.ja}, \tag{2}$$

where $A^i_{.ja}(z,t) = \delta^i_j - \nabla_j u^i_a$ and $B^i_{.ja}(z,t)$ is the inverse of $A^i_{.ja}$, i.e. $B^i_{.ja}\,A^j_{.ka} = \delta^i_k$.

The fields $\rho_a(z,t)$ and $v^j_a(z,t)$ obey the well-known formula of mass conservation for each phase:

$$\frac{\partial \rho_a}{\partial t} + \nabla_j(\rho_a v^j_a) = 0. \tag{3}$$

The differential form of the mass conservation result (3) is equivalent to the following algebraic one:

$$\rho_a = \rho^\circ_a \det|A^i_{.ja}|, \tag{4}$$

where ρ°_a is the mass density in the reference configuration. Differentiating equation (4) we get the following relationship:

$$\frac{\partial \rho_a}{\partial \nabla_i u^j_{.a}} = -\rho_a B^i_{.ja}. \tag{5}$$

3. THE HAMILTONIAN APPROACH FOR TWO-COMPONENT MEDIA IN THE EULERIAN VARIABLES

Assuming the absence of external mass and surface forces we choose the Action S corresponding to a 'cold' multi-component substance in the following standard form:

$$S = \int_{t_0}^{t_1} dt \int_{\Omega} d\Omega\, \rho(z,t) L, \tag{6}$$

where $L = T - E$ and T and E are the kinetic and internal (elastic) energies of the substance per unit mass, and $\rho(z,t)$ is the mass density of the substance, which is equal to the sum of the partial densities $\rho_a(z,t)$ of the constituents:

$$\rho(z,t) = \sum_a \rho_a(z,t). \tag{7}$$

In what follows we assume that the kinetic energy density is an algebraic function of velocities of the constituents $v_a^i(z,t)$ while the internal energy density is a function of the displacement gradients $\nabla_j u_a^i(z,t)$ and the partial densities $\rho_a(z,t)$:

$$T = T(v_a^i), \quad E = E(\nabla_j u_a^i, \rho_a), \quad L = L(\rho_a, v_a^i, \nabla_j u^i). \tag{8}$$

The present analysis applies to any number of constituents, $a = 1, 2, \ldots$, but for practical purposes we will be concerned with two-component media.

Dynamic equations and natural boundary conditions of the actual motion can be obtained as the conditions of vanishing of the first variation of the Action S for all 'admissible' trajectories $(\rho_a(z,t), v_a^i(z,t), u_a^i(z,t))$. In addition to some natural demands of sufficient smoothness, all admissible fields have to obey certain 'mechanical' constraints. First of all, as always, all admissible dynamical fields have to coincide with the actual fields in the initial (at $t = t_0$) and final (at $t = t_1$) configurations. Other crucial constraints follow from the fact that the fields (ρ_a, v_a^i, u_a^i) are mutually dependent since, say, the two former fields can be found from the latter by means of spatial and time differentiation. This fact implies certain linear differential relationships for the variations of these fields, and the independent variations should be carefully extracted in the first variation of the Action δS. In order to make this point more transparent let us consider a one-parameter family of the admissible fields

$$\rho_a = \rho_a(z,t,\tau), \quad v_a^i = v_a^i(z,t,\tau), \quad u_a^i = u_a^i(z,t,\tau), \tag{9}$$

where τ is the parameter of variation. By definition, the Eulerian variations of these fields are the following functions of z and t:

$$\begin{aligned}
\delta\rho_a(z,t) &= \frac{\partial\rho_a}{\partial\tau}(z,t,\tau)\Big|_{\tau=0} = R_a(z,t) \\
\delta v_a^i(z,t) &= \frac{\partial v_a^i}{\partial\tau}(z,t,\tau)\Big|_{\tau=0} = G_a^i(z,t) \\
\delta u_a^i(z,t) &= \frac{\partial u_a^i}{\partial\tau}(z,t,\tau)\Big|_{\tau=0} = Q_a^i(z,t).
\end{aligned} \tag{10}$$

With the help of equations (2), (4) and (5) one can easily establish the following explicit formulae for the variations $R_a(z,t)$, $G_a^k(z,t)$ in terms of the variation $Q_a^i(z,t)$ and its derivatives:

$$G_a^k(z,t) = B^k_{.ma}\left(\frac{\partial Q_a^m}{\partial t} + \frac{\partial u_a^j}{\partial t} B^n_{.ja} \nabla_n Q_a^m\right), \qquad R_a(z,t) = -\rho_a B^j_{.ia} \nabla_j Q_a^i. \tag{11}$$

In order to establish a system of governing equations one ought to substitute the one-dimensional fields (9) into (6), differentiate the integral, and then extract the independent variation Q_a^i using the relationships (11) and integration by parts (in space and time). Then, equating to zero the coefficients of Q_a^i in the integrand we arrive at the desired master system. However, this approach, although absolutely correct conceptually, requires very laborious computation when the Eulerian description is used. In what follows we use a somewhat different technique which seems to be much more convenient when dealing with the Eulerian description. In this technique the virtual velocities $f_a^i(z,t,\tau)$ of the constituents are treated as independent variations rather than the functions $\delta u_a^i(z,t)$. The virtual velocities $f_a^i(z,t,\tau)$ are defined implicitly by the relationship

$$f_a^i(z,t,\tau) = \frac{\mathrm{D}_a}{\mathrm{D}\tau} u_a^i(z,t,\tau), \tag{12}$$

where $\mathrm{D}_a/\mathrm{D}\tau$ is the material derivative associated with the parameter $\tau : \mathrm{D}_a/\mathrm{D}\tau = \partial/\partial\tau + f_a^i\nabla_i$.

The following statement is crucial for overcoming future computational obstacles.

LEMMA 1. The operators $\mathrm{D}_a/\mathrm{D}t$ and $\mathrm{D}_a/\mathrm{D}\tau$ commute on $u_a^k(z,t,\tau)$:

$$\left[\frac{\mathrm{D}_a}{\mathrm{D}t}, \frac{\mathrm{D}_a}{\mathrm{D}\tau}\right] u_a^i = \left(\frac{\mathrm{D}_a}{\mathrm{D}t}\frac{\mathrm{D}_a}{\mathrm{D}\tau} - \frac{\mathrm{D}_a}{\mathrm{D}\tau}\frac{\mathrm{D}_a}{\mathrm{D}t}\right) u_a^i = \frac{\mathrm{D}_a f_a^i}{\mathrm{D}t} - \frac{\mathrm{D}_a v_a^i}{\mathrm{D}\tau} = 0. \tag{13}$$

Proof. Differentiating equation (2) we get

$$\frac{\mathrm{D}_a v_a^i}{\mathrm{D}\tau} = \frac{\mathrm{D}_a}{\mathrm{D}\tau}\left(\frac{\partial u_a^l}{\partial t} B^i_{.la}\right) = \frac{\mathrm{D}_a}{\mathrm{D}\tau}\left(\frac{\partial u_a^l}{\partial t}\right) B^i_{.la} + \frac{\partial u_a^l}{\partial t}\frac{\mathrm{D}_a B^i_{.la}}{\mathrm{D}\tau}. \tag{14}$$

One can easily derive the following relationships:

$$\frac{\mathrm{D}_a B^i_{.la}}{\mathrm{D}\tau} = -B^i_{.na} B^m_{.la} \frac{\mathrm{D}_a A^n_{.ma}}{\mathrm{D}\tau},$$

$$\frac{\mathrm{D}_a A^n_{.ma}}{\mathrm{D}\tau} = -\frac{\mathrm{D}_a \nabla_m u_a^n}{\mathrm{D}\tau} = -A^n_{.ja}\nabla_m f_a^j,$$

$$\frac{\mathrm{D}_a}{\mathrm{D}\tau}\frac{\partial u_a^l}{\partial t} = \frac{\partial}{\partial t}\frac{\mathrm{D}_a u_a^l}{\mathrm{D}\tau} - \frac{\partial f_a^j}{\partial t}\nabla_j u_a^l = \frac{\partial f_a^j}{\partial t} A^l_{.ja}.$$

Inserting these into the RHS of (14) we arrive at the desired result:

$$\frac{\mathrm{D}_a v^i_{.a}}{\mathrm{D}\tau} = \frac{\partial f^i_{.a}}{\partial t} + (\nabla_m f^i_{.a})\, B^m_{.la}\frac{\partial u^l_{.a}}{\partial t} = \frac{\partial f^i_{.a}}{\partial t} + v_a^m \nabla_m f_a^i = \frac{\mathrm{D}_a f_a^i}{\mathrm{D}t}.$$

□

We remind the reader of the following formulae for a derivative of the integral over a domain occupied by a moving body consisting of particles of the same material:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega \mathrm{d}\Omega\, \rho_a L = \int_\Omega \mathrm{d}\Omega\, \rho_a \frac{D_a L}{D\tau}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{d}\Omega\, \rho_a L = \int_\Omega \mathrm{d}\Omega\, \rho_a \frac{\mathrm{D}_a L}{\mathrm{D}t}. \tag{15}$$

Combining (6) and (15) we get the following formula:

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \int_{t_0}^{t_1} \mathrm{d}t \frac{\partial}{\partial\tau}\int_\Omega \mathrm{d}\Omega \sum_a \rho_a L = \int_{t_0}^{t_1} \mathrm{d}t \int_\Omega \mathrm{d}\Omega \sum_a \rho_a \frac{\mathrm{D}_a L}{\mathrm{D}\tau}. \tag{16}$$

The fundamental equations of the system follow by equating the latter to zero, which we will now proceed to do. First, we note the identity

$$\frac{\mathrm{D}_a}{\mathrm{D}\tau} = \frac{\mathrm{D}_b}{\mathrm{D}\tau} - (f_b^k - f_a^k)\nabla_k, \tag{17}$$

which combined with (13) yields the relationships

$$\begin{aligned} \frac{\mathrm{D}_a v_b^i}{\mathrm{D}\tau} &= \frac{\mathrm{D}_b f_b^i}{\mathrm{D}t} - (f_b^k - f_a^k)\nabla_k v_b^i \\ \frac{\mathrm{D}_a \rho_b}{\mathrm{D}\tau} &= -\rho_b \nabla_k f_b^k - (f_b^k - f_a^k)\nabla_k \rho_b \end{aligned} \tag{18}$$

$$\frac{\mathrm{D}_a}{\mathrm{D}\tau}\nabla_p u_{qb} = A^{\cdot}_{qrb}\nabla_p f^r_b - (f^k_b - f^k_a)\nabla_k\nabla_p u_{qb}.$$

Hence, the integrand of equation (16) becomes

$$\sum_a \rho_a \frac{\mathrm{D}_a L}{\mathrm{D}\tau} = \sum_a\sum_b \rho_a \left[L_{v^i_b}\left(\frac{\mathrm{D}_b f^i_b}{\mathrm{D}t} - (f^k_b - f^k_a)\nabla_k v^i_b\right) - L_{\rho_b}\left(\rho_b\nabla_k f^k_b + (f^k_b - f^k_a)\nabla_k\rho_b\right)\right.$$
$$\left. + L_{\nabla_p u_{qb}}\left(A^{\cdot}_{qrb}\nabla_p f^r_b - (f^k_b - f^k_a)\nabla_k\nabla_p u_{qb}\right)\right]. \tag{19}$$

The individual terms can be simplified, yielding sequentially

$$\begin{aligned}
\sum_a\sum_b \rho_a L_{v^i_b}\frac{\mathrm{D}_b f^i_b}{\mathrm{D}t} &= \sum_b \rho L_{v^i_b}\frac{\mathrm{D}_b f^i_b}{\mathrm{D}t} \\
&= \sum_b \rho_b \frac{\rho}{\rho_b} L_{v^i_b}\frac{\mathrm{D}_b f^i_b}{\mathrm{D}t} \\
&= \sum_b \rho_b \frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b} L_{v^i_b} f^i_b\right) - \sum_b \rho_b f^i_b \frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b} L_{v^i_b}\right) \\
\sum_a\sum_b \rho_a L_{\rho_b}\rho_b\nabla_k f^k_b &= \sum_b\left[\nabla_k\left(\rho\rho_b L_{\rho_b}\nabla_k f^k_b\right) - f^k_b\nabla_k\left(\rho\rho_b L_{\rho_b}\right)\right] \\
\sum_a\sum_b \rho_a L_{\nabla_p u_{qb}} A^{\cdot}_{qrb}\nabla_p f^r_b &= \sum_b\left[\nabla_p\left(\rho L_{\nabla_p u_{qb}} A^{\cdot}_{qrb} f^r_b\right) - f^r_b\nabla_p\left(\rho L_{\nabla_p u_{qb}} A^{\cdot}_{qrb}\right)\right] \\
\sum_a\sum_b \rho_a Z_{kb}(f^k_b - f^k_a) &= \sum_b \rho Z_{kb}(f^k_b - F^k)
\end{aligned} \tag{20}$$

where Z_{kb} are arbitrary, and

$$F^i = \frac{1}{\rho}\sum_a \rho_a f^i_a$$

is the 'mean' virtual velocity. Combining (19) and (20) we get the relationship

$$\sum_a \rho_a\frac{\mathrm{D}_a L}{\mathrm{D}\tau} = \sum_b\left[\rho_b\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}f^i_b\right) - \rho_b f^i_b\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}\right) - \rho L_{v^i_b}\nabla_k v^i_b(f^k_b - F^k)\right.$$
$$+ f^k_b\nabla_k(\rho\rho_b L_{\rho_b}) - \rho L_{\nabla_p u_{qb}}\nabla_k\nabla_p u_{qb}(f^k_b - F^k) - \rho L_{\rho_b}\nabla_k\rho_b(f^k_b - F^k)$$
$$\left. - f^k_b\nabla_p(\rho L_{\nabla_p u_{qb}} A^{\cdot}_{qrb}) + \nabla_p(\rho L_{\nabla_p u_{qb}} A^{\cdot}_{qrb} f^r_b) - \nabla_k(\rho\rho_b L_{\rho_b} f^k_b)\right]. \tag{21}$$

Now, let us insert equation (21) into (16) and eliminate all spatial divergence terms—viz. the final two terms in equation (21) (here and in the following we tacitly assume that all integrals over boundaries vanish: the reader can easily figure out several conservative boundary conditions of this sort), yielding

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \int_{t_0}^{t_1}\mathrm{d}t\int_\Omega \mathrm{d}\Omega\sum_b\left[\rho_b\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}f^i_b\right) - \rho_b f^i_b\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}\right)\right.$$
$$-\rho L_{v^i_b}\nabla_k v^i_b(f^k_b - F^k) + f^k_b\nabla_k(\rho\rho_b L_{\rho_b}) - \rho L_{\rho_b}\nabla_k\rho_b(f^k_b - F^k)$$
$$\left. - f^r_b\nabla_p(\rho L_{\nabla_p u_{qb}} A^{\cdot}_{qrb}) - \rho L_{\nabla_p u_{qb}}\nabla_k\nabla_p u_{qb}(f^k_b - F^k)\right]. \tag{22}$$

Using the second of the two relationships (15) we get

$$\int_{t_0}^{t_1}\mathrm{d}t\int_\Omega \mathrm{d}\Omega\sum_b \rho_b\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}f^i_b\right) = \sum_b\int_\Omega \mathrm{d}\Omega\,\rho L_{v^i_b}f^i_b\Big|_{t=t_1}^{t=t_0}. \tag{23}$$

The last substitution vanishes because the initial and final configurations of all admissible trajectories are the same. In view of (23) the first integral in equation (22) disappears. Now, by separating independent virtual velocities f_b^i in (22) and setting $dS/d\tau = 0$, we obtain the following governing system of evolution:

$$\rho_b \frac{D_b}{Dt}\left(\frac{\rho}{\rho_b} L_{v_b^i}\right) = \rho_b \sum_a \left(L_{v_a^j} \nabla_i v_a^j + L_{\rho_a} \nabla_i \rho_a\right) - \rho L_{v_b^j} \nabla_i v_b^j - \rho L_{\rho_b} \nabla_i \rho_b - \rho L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb}$$
$$+ \nabla_i(\rho \rho_b L_{\rho_b}) - \nabla_j(\rho L_{\nabla_j u_{kb}} A^{\cdot}_{kib}) + \rho_b \sum_a L_{\nabla_j u_{ka}} \nabla_i \nabla_j u_{ka},$$

which we can rewrite as

$$\rho_b \frac{D_b}{Dt}\left(\frac{\rho}{\rho_b} L_{v_b^i}\right) = \rho_b \nabla_i L - \rho \left(L_{v_b^j} \nabla_i v_b^j + L_{\rho_b} \nabla_i \rho_b + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb}\right)$$
$$+ \nabla_i(\rho \rho_b L_{\rho_b}) - \nabla_j(\rho L_{\nabla_j u_{kb}} A^{\cdot}_{kib}). \tag{24}$$

This is the first significant result of the paper. In the next section we will generalize this to include dissipation.

4. THE HAMILTON PRINCIPLE FOR IRREVERSIBLE PROCESSES IN TWO-COMPONENT MEDIA

4.1 The governing equations

Let us consider irreversible processes in two component media. In order to include into consideration the Darcy and Navier–Stokes dissipation we assume the presence of the viscous mass force $P^i_{a.vis}$ and of the viscous stress tensor $P^{ji}_{a.vis}$. Also we extend the Hamilton principle and formulate it in the following nonholonomic form:

$$\frac{d}{d\tau} \int_{t_0}^{t_1} dt \int_\Omega d\Omega \, \rho L + \int_{t_0}^{t_1} dt \int_\Omega d\Omega \sum_a \left(P^i_{a.vis} \frac{D_a u_{ia}}{D\tau} - P^{ji}_{a.vis} \nabla_j \frac{D_a u_{ia}}{D\tau}\right) = 0 \,. \tag{25}$$

Copying the derivation of the system (24) we arrive at the following generalization of the governing system:

$$\rho_b \frac{D_b}{Dt}\left(\frac{\rho}{\rho_b} L_{v_b^i}\right) = \rho_b \nabla_i L - \rho \left(L_{v_b^j} \nabla_i v_b^j + L_{\rho_b} \nabla_i \rho_b + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb}\right)$$
$$+ \nabla_i(\rho \rho_b L_{\rho_b}) - \nabla_j(\rho L_{\nabla_j u_{kb}} A^{\cdot}_{kib}) + P^i_{b.vis} + \nabla_j P^{ji}_{b.vis} \,. \tag{26}$$

This reduces to equation (24) for conservative systems.

4.2 Energy relations and general properties

For each constituent, a, define the elastic stress tensor

$$P^{j}_{.ia} = -\rho L_{\nabla_j u_{ka}} A^{\cdot}_{kia} + \rho \rho_a L_{\rho_a} \delta^j_i, \tag{27}$$

and the 'interaction' force, or momentum supply in the terminology of mixture theory [1,2]

$$G^i_a = \rho_a \nabla_i L - \rho \left(L_{v_a^j} \nabla_i v_a^j + L_{\rho_a} \nabla_i \rho_a + L_{\nabla_j u_{ka}} \nabla_i \nabla_j u_{ka}\right); \tag{28}$$

then the equilibrium conditions (26) become

$$\rho_b \frac{D_b}{Dt}\left(\frac{\rho}{\rho_b} L_{v_b^i}\right) = \nabla_j \left(P^{ji}_b + P^{ji}_{b.vis}\right) + G^i_b + P^i_{b.vis} \,. \tag{29}$$

Note that the sum of the interaction forces at a point is zero:

$$\sum_a G_a^i = 0. \tag{30}$$

The sum of the elastic and viscous stresses, $P_a^{ji} + P_{a.vis}^{ji}$, can be considered the total stress in phase a. We caution that these definitions are not unique because the vectors G_a^i could be changed by adding divergence-like terms.

Define the total energy density per unit mass of the system

$$\mathcal{E} = \sum_b \frac{\partial L}{\partial v_b^k} v_b^k - L, \tag{31}$$

and define the flux vector for component a

$$\mathcal{F}_a^j = -v_a^i P_{.ia}^j + \rho\,(v_a^j - V^j) v_a^k L_{v_a^k}, \tag{32}$$

where

$$V^l = \frac{1}{\rho} \sum_b \rho_b v_b^l \tag{33}$$

is the mean velocity. Then it follows from equation (26), with the details in the Appendix, that

$$\sum_a \left(\rho_a \frac{\mathrm{D}_a \mathcal{E}}{\mathrm{D}t} + \nabla_j \mathcal{F}_a^j \right) = \sum_a \left(P_{a.vis}^i v_{ia} - P_{a.vis}^{ji} \nabla_j v_{ia} \right). \tag{34}$$

This is the pointwise energy dissipation relation. It is interesting to note that the flux vector of phase a contains a term proportional to the relative velocity of a with respect to the mean flow.

Integrating the balance law (34) over the domain Ω and using equation (15) leads us to the following identity:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathrm{d}\Omega\, \rho\, \mathcal{E} = \int_\Omega \mathrm{d}\Omega \sum_a \left(P_{a.vis}^i v_{ia} - P_{a.vis}^{ji} \nabla_j v_{ia} \right), \tag{35}$$

which is the energy-dissipation relation for the volume Ω. In the absence of dissipation we can establish the energy conservation equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathrm{d}\Omega\, \rho\, \mathcal{E} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathrm{d}\Omega\, \rho \left(\sum_b \frac{\partial L}{\partial v_b^k} v_b^k - L \right) = 0. \tag{36}$$

Alternatively, let $\mathcal{L}$ be the Lagrangian density per unit volume:

$$\mathcal{L} = \rho L, \tag{37}$$

in terms of which the governing equations (26) become

$$\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t} \left(\frac{\mathcal{L}_{v_b^i}}{\rho_b} \right) = \rho_b \nabla_i \mathcal{L}_{\rho_b} - \mathcal{L}_{v_b^j} \nabla_i v_b^j - \mathcal{L}_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} - \nabla_j (\mathcal{L}_{\nabla_j u_{kb}} A_{kib}^{\cdot}) + P_{b.vis}^i + \nabla_j P_{b.vis}^{ji}. \tag{38}$$

This form of the equilibrium equations is perhaps simpler. It is interesting to note that if $\mathcal{L}$ is additive in the following sense:

$$\mathcal{L} = \sum_a \rho_a L_a(\rho_a, v_a^i, \nabla_j u_{ka}), \tag{39}$$

then the equations (38) reduce to

$$\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t} L_{b,v_b^i} = \nabla_j \left(\rho_b^2 L_{b,\rho_b} \delta^{ij} - \rho_b L_{b,\nabla_j u_{kb}} A_{kib}^{\cdot} \right) + P_{b.vis}^i + \nabla_j P_{b.vis}^{ji}. \tag{40}$$

These equations are coupled only through the viscous terms. Thus, an additive volumetric Lagrangian density leads to decoupling.

4.3 Dissipation effects: constitutive relations

For the stresses $P^{ji}_{b.vis}$ we limit ourselves with the traditional choice of the rheology

$$P^{ji}_{b.vis} = D^{ijkl}_b \nabla_{(k} v_{lb)}, \tag{41}$$

where the tensor of viscous coefficients D^{ijkl}_b is symmetric with respect to the first and second pairs of indices and to interchange of these pairs. Also, based on equation (34), it should satisfy the following dissipation inequality:

$$D^{ijkl}_b \nabla_{(i} v_{jb)} \nabla_{(k} v_{lb)} > 0. \tag{42}$$

In order to establish a constitutive relation for the viscous body force $P^i_{b.vis}$ we impose the following 'natural' demands:

(a) Dissipativity

$$\sum_b P^i_{b.vis} v_{ib} < 0 \tag{43}$$

(b) Additivity

$$P^i_{b.vis} = \sum_a F^i_{ab} \tag{44}$$

(c) The 'action–reaction' equality

$$F^i_{ab} = -F^i_{ba} \tag{45}$$

(d) Linearity in the velocities

$$F^i_{ab} = \sum_c D^{ij}_{abc} v_{jc}. \tag{46}$$

Constraint (a) is a natural consequence of the energy identity (34), while the conditions (b) and (c) imply that

$$\sum_b P^i_{b.vis} = 0, \tag{47}$$

which is similar to the general result (30).

We now examine the implications of these constraints for two-component media. First, equation (45) implies that

$$F^i_{11} = F^i_{22} = 0, \quad F^i_{12} = -F^i_{21}. \tag{48}$$

Combining (46) and (48) we get

$$\begin{gathered} D^{ij}_{111} v_{j1} + D^{ij}_{112} v_{j2} = 0 \\ D^{ij}_{221} v_{j1} + D^{ij}_{222} v_{j2} = 0 \\ \left(D^{ij}_{121} + D^{ij}_{211}\right) v_{j1} + \left(D^{ij}_{122} + D^{ij}_{212}\right) v_{j2} = 0. \end{gathered} \tag{49}$$

Equations (49) lead us to the following formulas of the coefficients D^{ij}_{abc}:

$$\begin{gathered} D^{ij}_{111} = D^{ij}_{112} = 0, \quad D^{ij}_{221} = D^{ij}_{222} = 0 \\ D^{ij}_{121} = -D^{ij}_{211} = \chi^{ij}_1, \quad D^{ij}_{122} = -D^{ij}_{212} = \chi^{ij}_2. \end{gathered} \tag{50}$$

Combining equations (48)–(50) we obtain the relationships

$$F^i_{12} = -F^i_{21} = D^{ij}_{121}v_{j1} + D^{ij}_{122}v_{j2} = \chi^{ij}_1 v_{j1} + \chi^{ij}_2 v_{j2}$$
$$P^i_{1.vis} = F^i_{21}$$
$$P^i_{2.vis} = F^i_{12} = -F^i_{21} = -P^i_{1.vis}.$$

Hence,

$$P^i_{1.vis}v_{i1} + P^i_{2.vis}v_{i2} = F^i_{21}(v_{i1} - v_{i2}) = -(\chi^{ij}_1 v_{j1} + \chi^{ij}_2 v_{j2})(v_{i1} - v_{i2}).$$

The inequality (43) will be guaranteed provided $\chi^{ij}_1 = -\chi^{ij}_2 = D^{ij}$, a positively definite matrix. Indeed, in this case we get

$$P^i_{1.vis}v_{i1} + P^i_{2.vis}v_{i2} = -D^{ij}(v_{i1} - v_{i2})(v_{j1} - v_{j2}) < 0.$$

With the above-mentioned choice we obtain the following constitutive equation:

$$P^i_{1.vis} = -D^{ij}(v_{i1} - v_{i2}), \quad P^i_{2.vis} = -D^{ij}(v_{i2} - v_{i1}), \quad P^i_{a.vis} = -D^{ij}(v_{ia} - v_{ib}). \tag{51}$$

Inserting the constitutive relations (41) and (51) into (26) we arrive at the following master system:

$$\begin{aligned}\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}\right) &= \rho_b \nabla_i L - \rho\left(L_{v^j_b}\nabla_i v^j_b + L_{\rho_b}\nabla_i \rho_b + L_{\nabla_j u_{kb}}\nabla_i\nabla_j u_{kb}\right) + \nabla_i(\rho\rho_b L_{\rho_b}) \\ &\quad -\nabla_j(\rho L_{\nabla_j u_{kb}} A^{\cdot}_{kib}) - D^{ij}(v_{ib} - v_{ia}) + \nabla_j\left(D^{ijkl}_{b.vis}\nabla_k v_{lb}\right)\end{aligned} \tag{52}$$

or equivalently

$$\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}L_{v^i_b}\right) = \nabla_j P^{ji}_b + G^i_b - D^{ij}(v_{ib} - v_{ia}) + \nabla_j\left(D^{ijkl}_{b.vis}\nabla_k v_{lb}\right), \tag{53}$$

where the interaction force is

$$\begin{aligned}G^i_b = -G^i_a &= \rho_b\left(L_{v^j_a}\nabla_i v^j_a + L_{\rho_a}\nabla_i\rho_a + L_{\nabla_j u_{ka}}\nabla_i\nabla_j u_{ka}\right) \\ &\quad -\rho_a\left(L_{v^j_b}\nabla_i v^j_b + L_{\rho_b}\nabla_i\rho_b + L_{\nabla_j u_{kb}}\nabla_i\nabla_j u_{kb}\right).\end{aligned} \tag{54}$$

5. EXAMPLES

5.1 One-component models

Example 1. Nonlinear elastic solid. The governing system of a nonlinear one-component solid can be derived from the master system (53) if we neglect viscous forces and choose the Lagrangian

$$L = T - E = \frac{1}{2}v^i v_i - E(\nabla_j u_k). \tag{55}$$

In this case P^{ji} (from equation (27)) is the Cauchy stress tensor

$$P^{ji} = \rho\frac{\partial E}{\partial \nabla_j u_k}A^{\cdot i}_{k}$$

and there are no viscous or interaction terms. The equilibrium equations (53) are simply

$$\rho\frac{\mathrm{D}v^i}{\mathrm{D}t} = \nabla_j P^{ji}. \tag{56}$$

Example 2. The Navier–Stokes fluid. The governing system of a nonlinear one-component viscous fluid can be derived from the master system (52) if we ignore the drag force P^i_{vis} and choose the Lagrangian and the viscosity tensor D^{ijkl} in the following form:

$$L = T - E = \frac{1}{2}v^i v_i - E(\rho), \quad D^{ijkl} = \nu_v \delta^{ij}\delta^{kl} + \nu_s\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) \tag{57}$$

where ν_v, ν_s are the volume and shear viscosities, respectively. The elastic stress is hydrostatic, defined by a pressure p:

$$P^{ji} = -p\,\delta^{ji}, \quad p = \rho^2 \frac{\partial E}{\partial \rho}, \tag{58}$$

and the viscous stresses P^{ij}_{vis} are

$$P^{ij}_{vis} = \nu_v \nabla^k v_k \delta^{ij} + \nu_s\left(\nabla^i v^j + \nabla^j v^i\right). \tag{59}$$

The equilibrium equations (53) are now

$$\rho \frac{\mathrm{D}v^i}{\mathrm{D}t} = -\nabla^i p + \nabla_j P^{ij}_{vis}. \tag{60}$$

As always, when combined with the mass conservation equation (3) the system (59), (60) is closed in terms of ρ, v^i.

5.2 Two-component models

Let us assume that the kinetic energy T is a quadratic form of the velocities v^k_a:

$$T = \frac{1}{2}\sum_{a,b} K_{ijab} v^i_a v^j_b. \tag{61}$$

Using (61) we get

$$\frac{\partial T}{\partial v^l_c} = \frac{1}{2}\sum_b K_{ijab}\left(\delta^i_l \delta_{ac} v^j_b + v^i_a \delta^j_l \delta_{bc} v^j_b\right) = \frac{1}{2}\sum_b \left(K_{ljcb} + K_{jlbc}\right) v^j_b. \tag{62}$$

In the case of isotropic two-phase medium we get

$$K^{ij}_{ab} = \tilde{k}_{ab}\delta^{ij}, \quad D^{ij} = \Delta\delta^{ij}. \tag{63}$$

Combining (62) and (63) we get

$$\frac{\partial T}{\partial v^l_c} = \sum_a k_{ac} v_{la} \tag{64}$$

where the symmetric matrix k_{ac} is defined as

$$k_{ac} = \frac{1}{2}\left(\tilde{k}_{ac} + \tilde{k}_{ca}\right).$$

Example 3. Liquid two-component mixture. We obtain the simplest possible model of a two-component liquid mixture by choosing the Lagrangian and viscosity tensors in the following form:

$$\begin{aligned} L = T - E &= \frac{1}{2}\sum_{ab} k_{ab} v^i_a v_{ib} - E(\rho_a) \\ D^{ij} &= \Delta\delta^{ij} \\ D^{ijkl}_a &= \nu_{va}\delta^{ij}\delta^{kl} + \nu_{sa}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right). \end{aligned} \tag{65}$$

The partial stresses are hydrostatic:

$$P^{ji}_b = -p_b\,\delta^{ji}, \quad p_b = \rho\rho_b \frac{\partial E}{\partial \rho_b}, \tag{66}$$

and the interaction forces are given by

$$G_b^i = \rho_b \nabla_i \left(\frac{1}{2} \sum_{ac} k_{ac} v_a^i v_{ic} - E \right) - \rho \sum_a k_{ab} v_{ja} \nabla_i v_b^j + \rho E_{\rho_b} \nabla_i \rho_b$$
$$= \sum_c v_{ic} \left(\rho_b k_{ca} \nabla_i v_a^j - \rho_a k_{cb} \nabla_i v_b^j \right) + \rho_a E_{\rho_b} \nabla_i \rho_b - \rho_b E_{\rho_a} \nabla_i \rho_a. \tag{67}$$

Inserting these into the master equations (53) we obtain the following system of equilibrium equations for the two-liquid mixture:

$$\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t} \left(\frac{\rho}{\rho_b} \sum_a k_{ab} v_{ia} \right) = \sum_c v_{ic} \left(\rho_b k_{ca} \nabla_i v_a^j - \rho_a k_{cb} \nabla_i v_b^j \right) + \rho_a E_{\rho_b} \nabla_i \rho_b - \rho_b E_{\rho_a} \nabla_i \rho_a - \nabla_i (\rho \rho_b E_{\rho_b})$$
$$-\Delta(v_{ib} - v_{ia}) + \nabla_j \left[v_{vb} \delta_i^j \nabla_k v_b^k + v_{sb} \left(\nabla_i v_b^j + \nabla^j v_{ib} \right) \right]. \tag{68}$$

Example 4. Solid two-component mixture. The simplest model of a two-component solid mixture corresponds to a Lagrangian of the form

$$L = T - E = \frac{1}{2} \sum_{ab} k_{ab} v_a^i v_{ib} - E(\nabla_i u_{ja}). \tag{69}$$

Inserting (69) into (53) we obtain the following master system for a two-component solid elastic mixture:

$$\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t} \left(\frac{\rho}{\rho_b} \sum_a k_{ab} v_{ia} \right) - \rho_b \nabla_i \left(\frac{1}{2} \sum_{ac} k_{ac} v_a^i v_{ic} \right) + \rho \sum_a k_{ab} v_{jb} \nabla_i v_b^j$$
$$= -\rho_b \nabla_i E + \rho E_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} + \nabla_j (\rho E_{\nabla_j u_{kb}} A_{kib}) - D^{ij}(v_{ib} - v_{ia}) + \nabla_j \left(D_{b.\mathrm{vis}}^{ijkl} \nabla_k v_{la} \right). \tag{70}$$

Example 5. Fluid-permeable elastic solid. We choose subscript s for the solid and f for the liquid component. The simplest model of two-component solid mixture is obtained by choosing the Lagrangian in the following form:

$$L = T - E = \frac{1}{2} \left(k_{ss} v_s^i v_{is} + 2 k_{sf} v_s^i v_{if} + k_{ff} v_f^i v_{if} \right) - E(\nabla_i u_{js}, \rho_f). \tag{71}$$

Inserting (71) into (53) yields the following master system:

$$\rho_s \frac{\mathrm{D}_s}{\mathrm{D}t} \left(\frac{\rho}{\rho_s} \left(k_{ss} v_{is} + k_{sf} v_{if} \right) \right) = \rho_s \left(k_{fs} v_{js} + k_{ff} v_{jf} \right) \nabla_i v_f^j - \rho_f \left(k_{ss} v_{js} + k_{sf} v_{jf} \right) \nabla_i v_s^j$$
$$-\rho_s \nabla_i E + \rho E_{\nabla_j u_{ks}} \nabla_i \nabla_j u_{ks} + \nabla_j (\rho E_{\nabla_j u_{ks}} A_{kis})$$
$$-D^{ij}(v_{js} - v_{jf}) + \nabla_j \left(D_{s.\mathrm{vis}}^{ijkl} \nabla_k v_{ls} \right) \tag{72}$$

$$\rho_f \frac{\mathrm{D}_f}{\mathrm{D}t} \left(\frac{\rho}{\rho_f} \left(k_{ff} v_{if} + k_{sf} v_{is} \right) \right) = \rho_f \left(k_{ss} v_{js} + k_{sf} v_{jf} \right) \nabla_i v_s^j - \rho_s \left(k_{fs} v_{js} + k_{ff} v_{jf} \right) \nabla_i v_f^j$$
$$-\rho_f \nabla_i E + \rho E_{\rho_f} \nabla_i \rho_f - \nabla_i (\rho \rho_f E_{\rho_f})$$
$$-D^{ij}(v_{jf} - v_{js}) + \nabla_j \left(D_{f.\mathrm{vis}}^{ijkl} \nabla_k v_{lf} \right). \tag{73}$$

6. CONCLUSION

We have established a general framework for dealing with fully nonlinear problems for two-component media. The central result is equation (26) which gives the equilibrium equations in Eulerian coordinates for an arbitrary Lagrangian density. These governing equations are a direct consequence of the generalized Hamilton's principle in the presence of dissipative effects, i.e. the Onsager–Sedov principle of equation (25). We have illustrated the application of the general governing equations to specific examples of single and two-phase continua. In particular, the standard theories of solids and fluids are included. The full potential of the theory is its applicability to multi-phase continua.

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APPENDIX A

Here we derive the energy identity (34). We start with

$$\sum_a \rho_a \frac{\mathrm{D}_a \mathcal{E}}{\mathrm{D}t} = \sum_a \rho_a \frac{\mathrm{D}_a}{\mathrm{D}t}\left(\sum_b \frac{\partial L}{\partial v_b^k} v_b^k - L\right)$$
$$= \sum_a \rho_a \left(\sum_b v_b^k \frac{\mathrm{D}_a}{\mathrm{D}t}\left(\frac{\partial L}{\partial v_b^k}\right) - \sum_b L_\rho \frac{\mathrm{D}_a \rho_b}{\mathrm{D}t} - \sum_b L_{\nabla_j u_{lb}} \frac{\mathrm{D}_a}{\mathrm{D}t} \nabla_j u_{lb}\right). \tag{A.1}$$

We next simplify each of these terms in turn. First, using (17),

$$\sum_b v_b^k \frac{\mathrm{D}_a}{\mathrm{D}t}\left(\frac{\partial L}{\partial v_b^k}\right) = \sum_b v_b^k \frac{\mathrm{D}_b}{\mathrm{D}t}\frac{\partial L}{\partial v_b^k} - \sum_b v_b^k\left(v_b^l - v_a^l\right)\nabla_l \frac{\partial L}{\partial v_b^k}$$
$$= \sum_b v_b^k\left[\frac{\rho_b}{\rho}\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}\frac{\partial L}{\partial v_b^k}\right) - \frac{\partial L}{\partial v_b^k}\frac{\rho_b}{\rho}\frac{\mathrm{D}_b}{\mathrm{D}t}\frac{\rho}{\rho_b} - \left(v_b^l - v_a^l\right)\nabla_l\frac{\partial L}{\partial v_b^k}\right],$$

which implies that

$$\sum_a \rho_a \sum_b v_b^k \frac{\mathrm{D}_a}{\mathrm{D}t}\left(\frac{\partial L}{\partial v_b^k}\right) = \sum_b v_b^k \rho_b\left[\frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}\frac{\partial L}{\partial v_b^k}\right) - \frac{\partial L}{\partial v_b^k}\frac{\mathrm{D}_b}{\mathrm{D}t}\frac{\rho}{\rho_b}\right] - \rho\sum_b v_b^k (v_b^l - V^l)\nabla_l \frac{\partial L}{\partial v_b^k}, \tag{A.2}$$

and V^l is defined in (33). By making use of the governing system (26) we get the following result:

$$\sum_b v_b^k \rho_b \frac{\mathrm{D}_b}{\mathrm{D}t}\left(\frac{\rho}{\rho_b}\frac{\partial L}{\partial v_b^k}\right) = \sum_b v_b^i\left[\rho_b \nabla_i L - \rho\left(L_{v_b^k}\nabla_i v_b^k + L_{\nabla_j u_{kb}}\nabla_i\nabla_j u_{kb} + L_{\rho_b}\nabla_i \rho_b\right)\right.$$
$$\left. - \nabla_j\left(\rho L_{\nabla_j u_{kb}} A_{kib}^{\cdot}\right) + \nabla_i(\rho\rho_b L_{\rho_b}) + P_{vis.b}^i + \nabla_j P_{vis.b}^{ji}\right]. \tag{A.3}$$

Use of equation (3) gives us

$$\rho_b \frac{\mathrm{D}_b}{\mathrm{D}t}\frac{\rho}{\rho_b} = \sum_a \frac{\mathrm{D}_b \rho_a}{\mathrm{D}t} - \frac{\rho}{\rho_b}\frac{\mathrm{D}_b \rho_b}{\mathrm{D}t}$$
$$= \sum_a\left[-\rho_a \nabla_l v_a^l - (v_a^l - v_b^l)\nabla_l \rho_a + \rho_a \nabla_l v_b^l\right]$$
$$= \nabla_l\left(\rho(v_b^l - V^l)\right),$$

implying

$$\sum_b v_b^k \rho_b \frac{\partial L}{\partial v_b^k}\frac{\mathrm{D}_b}{\mathrm{D}t}\frac{\rho}{\rho_b} = \sum_b v_b^k \frac{\partial L}{\partial v_b^k}\nabla_l\left(\rho\,(v_b^l - V^l)\right). \tag{A.4}$$

Combining equations (A.2), (A.3), and (A.4) gives us the first sum in the left member of equation (A.1):

$$\sum_a \sum_b \rho_a v_b^k \frac{D_a}{Dt}\left(\frac{\partial L}{\partial v_b^k}\right) = \rho V^i \nabla_i L$$

$$+ \sum_b v_b^i \Big[-\rho \left(L_{v_b^k} \nabla_i v_b^k + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} + L_{\rho_b} \nabla_i \rho_b \right) - \nabla_j \left(\rho L_{\nabla_j u_{kb}} A_{kib} \right)$$

$$+ \nabla_i (\rho \rho_b L_{\rho_b}) + P^i_{vis.b} + \nabla_j P^{ji}_{vis.b} + \nabla_l \left(\rho L_{v_b^i} (V^l - v_b^l) \right) \Big]. \tag{A.5}$$

Now consider the other terms in (A.1). Using the mass conservation equation (3) again gives

$$\sum_a \sum_b \rho_a L_{\rho_b} \frac{D_a \rho_b}{Dt} = \sum_a \sum_b \rho_a L_{\rho_b} \left[-\rho_b \nabla_k v_b^k - (v_b^k - v_a^k) \nabla_k \rho_b \right]$$

$$= \sum_b \rho L_{\rho_b} \left[V^k \nabla_k \rho_b - \nabla_k (\rho_b v_b^k) \right]. \tag{A.6}$$

Also, by using of the last part of equation (18) (with the change of the parameter τ for t) we get the following equation:

$$\sum_a \sum_b \rho_a L_{\nabla_j u_{lb}} \frac{D_a}{Dt} \nabla_j u_{lb} = \sum_a \sum_b \rho_a L_{\nabla_j u_{lb}} \left[A_{lrb} \nabla_j v_b^r - (v_b^k - v_a^k) \nabla_k \nabla_j u_{lb} \right]$$

$$= \sum_b \rho L_{\nabla_j u_{lb}} \left[A_{lrb} \nabla_j v_b^r - v_b^k \nabla_k \nabla_j u_{lb} + V^k \nabla_k \nabla_j u_{lb} \right]. \tag{A.7}$$

By combining (A.1), (A.5), (A.6), and (A.7) we obtain the relation (34), where the flux vectors are defined in (32).