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ON ACOUSTIC INTERACTION BETWEEN TWO THIN ELASTIC PLATES THROUGH AN ANGULAR WELDED JOINT

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Contact conditions are deduced for a pair of thin plates in welded contact at an angle and under fluid loading. The conditions take into account the interaction between the plates through not only flexural but also longitudinal motion, and include the possibility of an applied force and moment acting at the junction. The two-dimensional conditions are shown to be consistent with the principle of acoustic reciprocity for the configuration, and they guarantee unique solutions if certain constraints are satisfied by the plate longitudinal impedances. The conditions also have implications for the junction admittance matrix. Simple examples are presented of the application of the contact conditions to acoustic interaction with plate configurations that are almost flat.

1. INTRODUCTION

Let two thin elastic plates be joined at an angle so that they coincide with the directions $\phi = \pm\Phi$ in the cylindrical co-ordinate system (r, ϕ, z), with the Oz-axis aligned with the straight edge of the junction. The joint is assumed to be welded, that is, the kinematic and dynamic quantities are continuous there. The junction may be subject to an applied force and moment, as indicated in Figure 1, which could arise from an internally connected structure. An acoustic medium of density ρ and sound velocity c fills the region $|\phi| < \Phi$. The kinematic and dynamic conditions at a plate junction of this type are well known, but they have to date been applied only to dry configurations, see for example references [1–3]. Fluid-loaded structures with rectangular [4, 5] and arbitrary angle geometries [6] have been analyzed neglecting the longitudinal motion in plates. In these cases the acoustic diffraction problem can be formulated entirely in terms of the fluid pressure (or potential) and the transverse displacement of the structure, which is proportional to the normal derivative of the pressure. A proper description of the problem must account for the existence of in-plane longitudinal and shear motions in the structure, which are coupled to the transverse displacement only at the junction.

The goal of this paper is to deduce the correct contact conditions required to complete the formulation of diffraction and radiation problems for fluid-loaded configurations in which the plates interact through not only flexural but also longitudinal and shear wave

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motion. The desired conditions should involve only the pressure or the transverse displacement, and its tangential derivatives, and should be independent of in-plane displacements and forces. For simplicity, attention is restricted to the two-dimensional configuration of Figure 1, for which all shear motion in the plates is decoupled from the acoustic field.

Consideration is given for scattering of incident waves and radiation emanating from applied loads at the joint. In either case the acoustic pressure in the fluid, $p(r,\,\phi)$, satisfies the Helmholtz equation,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2}+k^2p=0, \qquad 0<r<\infty, \qquad -\Phi<\phi<\Phi, \tag{1}$$

where $k=\omega/c$, and the plates are described classically via the boundary conditions

$$((\partial^4/\partial r^4)-\kappa_\pm^4)\xi_\pm(r)+(1/D_\pm)p(r,\,\pm\Phi)=0. \tag{2}$$

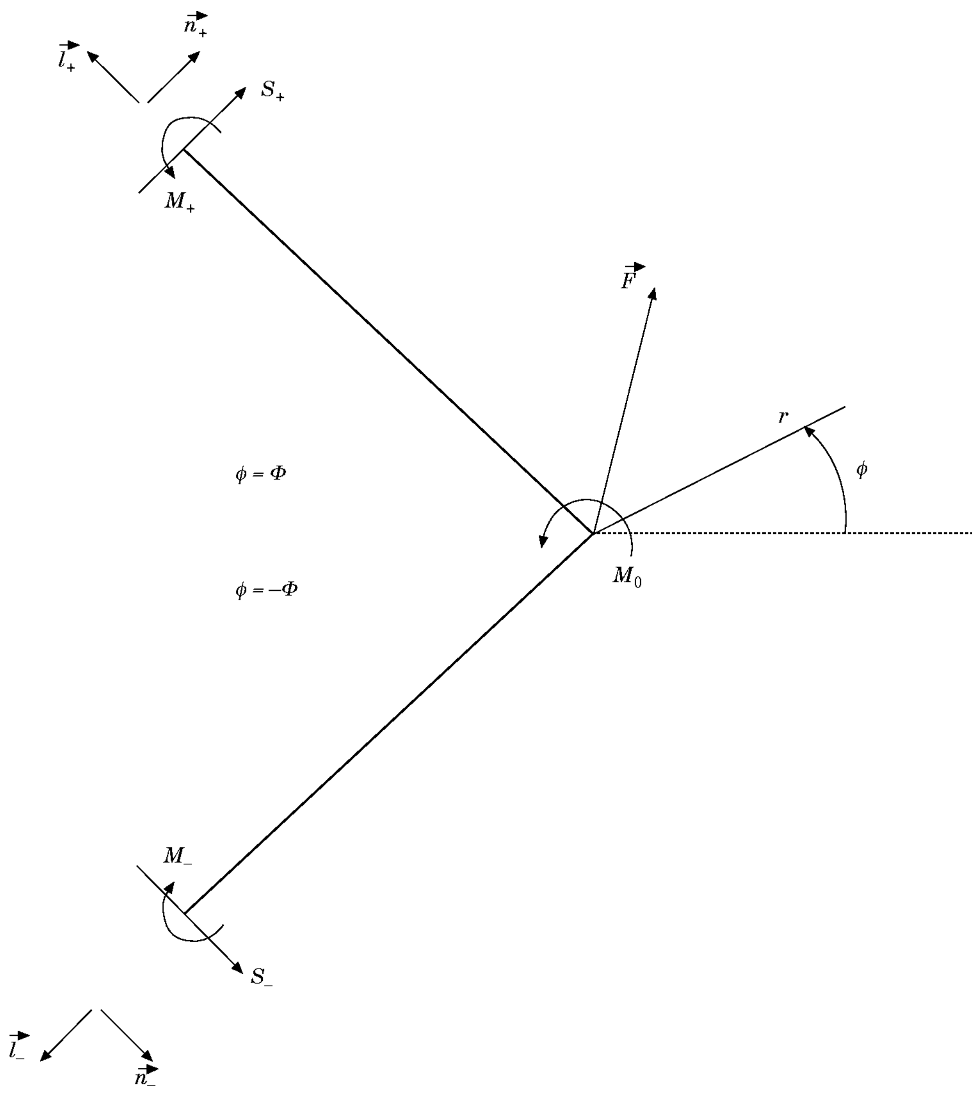

Figure 1. Geometrical and physical parameters at the welded junction.

Here $\xi_\pm(r) = \mp 1/(\omega^2 \rho r)(\partial p/\partial\phi)(r, \pm\Phi)$ denote the displacements of the plates, $D_\pm$ are the bending stiffness of the plates, and $\kappa_\pm$ identify the wave numbers associated with flexural motion of the plates in vacuum. The plus/minus subscripts relate the qunatity to the upper or lower boundary, respectively. The dependence $e^{-i\omega t}$ for time harmonic motion is understood throughout the paper. With no loss in generality consideration is restricted to $\omega > 0$, because the Fourier transform $f(\omega)$ of any physical parameter which is a real-valued function of t must satisfy $f(-\omega) = \bar{f}(\omega)$, where the bar denotes the complex conjugate quantity.

The basic question addressed here is how to express the contact conditions for the welded joint in terms of the fundamental variables $\xi_\pm$ and their derivatives with respect to r. These conditions supplement equations (1) and (2), and together with the incident field they complete the problem definition. One begins with the derivation of the contact conditions. The implications of the conditions regarding reciprocity, uniqueness and energy will be explored, and some simple applications to almost flat configurations occupy the remainder of the paper.

2. DERIVATION OF CONTACT CONDITIONS

2.1. KINEMATIC CONDITIONS

The transverse and in-surface plate velocity components are $w_\pm(r)$ and $u_\pm(r)$, defined such that the total velocities on either plate are

$$\vec{v}_\pm(r) = w_\pm(r)\vec{n}_\pm + u_\pm(r)\vec{l}_\pm. \tag{3}$$

Here, $\vec{n}_\pm = \mp\vec{e}_\phi|_{\phi=\pm\Phi}$ denote the unit vectors normal to the boundaries and directed towards the fluid, whereas $\vec{l}_\pm = \vec{e}_r|_{\phi=\pm\Phi}$ are the unit vectors aimed lengthwise. Let $\zeta_\pm(r)$ be the longitudinal displacements, then the velocities and displacements are simply related to one another by the formulae

$$w_\pm(r) = -i\omega\xi_\pm(r), \qquad u_\pm(r) = -i\omega\zeta_\pm(r). \tag{4}$$

The kinematic constraint at the junction of the welded plates requires that the total velocity and the rotational velocity are continuous there, implying respectively

$$\vec{v}_\pm(0) = \vec{v}_-(0), \qquad w'_+(0) = -w'_-(0), \tag{5, 6}$$

where the prime denotes the derivative with respect to r. This completes the kinematic conditions at the junction.

2.2. PLATE DYNAMICS

The internal forces and moments in the plates are illustrated in Figure 1. The shear forces $S_\pm(r)$ are the resultants of the stress components $\vec{n}_\pm, \vec{\vec{\sigma}}\vec{l}_\pm$, the longitudinal forces $T_\pm(r)$ are the resultants of the stress components $\vec{l}_\pm; \sigma\vec{l}_\pm$, and $M_\pm(r)\vec{n}_\pm \times \vec{l}_\pm$ are the resultant moments. The constitutive relations are

$$T_\pm = C_\pm\zeta'_\pm, \qquad M_\pm = -D_\pm\xi''_\pm, \qquad S_\pm = M'_\pm, \tag{7–9}$$

where $C_\pm$ are the longitudinal stiffnesses. The classical theory gives $C_\pm = E_\pm h_\pm/(1 - v_\pm^2)$, and $D_\pm = C_\pm h_\pm^2/12$, where $E_\pm$, $v_\pm$ and $h_\pm$ are plate Young's modulus, Poisson's ratio, and thickness, respectively.

The governing equation for the transverse momentum yields equation (2) with $\kappa_\pm = \omega^{1/2}(m_\pm/D_\pm)^{1/4}$, where $m_\pm = \rho_\pm h_\pm$ are the plate densities per unit spanwise length and $\rho_\pm$ are the material densities, while the in-plane momentum balance implies

$$C_\pm \zeta_\pm'' + \omega^2 m_\pm \zeta_\pm = 0. \tag{10}$$

Thus, the in-plane variables $\zeta_\pm$, $u_\pm$ and $T_\pm$ are dynamically uncoupled from the transverse variables $\xi_\pm$, $w_\pm$, $M_\pm$, $S_\pm$, and the fluid pressure p. The only coupling occurs at the junction. Furthermore, the in-plane quantities satisfy the one-dimensional wave equations with the general solution

$$\zeta_\pm(r) = \zeta_\pm^{(1)} e^{\mathrm{i}\omega r/c_\pm} + \zeta_\pm^{(2)} e^{-\mathrm{i}\omega r/c_\pm}, \tag{11}$$

where $c_\pm = \sqrt{C_\pm/m_\pm}$ and $\zeta_\pm^{(1)}$ and $\zeta_\pm^{(2)}$ are constants. The total in-surface forces and velocities are

$$T_\pm = T_\pm^{(in)} + T_\pm^{(sc)}, \qquad u_\pm = u_\pm^{(in)} + u_\pm^{(sc)}, \tag{12}$$

where $T_\pm^{(in)}$ and $u_\pm^{(in)}$ are introduced to allow for the possibility of excitation from incoming longitudinal waves on either plate, and $T_\pm^{(sc)}$ and $u_\pm^{(sc)}$ are the scattered or outgoing waves. Thus, using equations (4), (7), and (11), produces the relations

$$T_\pm^{(in)} - Z_\pm u_\pm^{(in)} = 0, \qquad T_\pm^{(sc)} + Z_\pm u_\pm^{(sc)} = 0, \tag{13}$$

where $Z_\pm = m_\pm c_\pm$ are the longitudinal wave impedances.

The possibility of a force $\vec{F}$ and moment M_0 is considered acting at the junction, as depicted in Figure 1. The equilibrium conditions at the junction require that the total force and moment acting there are zero,

$$S_+\vec{n}_+ + T_+\vec{l}_+ + S_-\vec{n}_- + T_-\vec{l}_- = -\vec{F}, \qquad M_+(0) - M_-(0) = -M_0. \tag{14, 15}$$

2.3. SIMPLIFICATION OF THE JUNCTION CONDITIONS

For the purposes of the scattering and radiation problems one desires junction conditions defined in terms of the transverse displacements and their derivatives in r. The vector equations (5) and (14) may be simplified by introducing components relative to an orthonormal basis. For example, when projected onto orthogonal directions $\vec{e}_r|_{\phi=0}$ and $\vec{e}_\phi|_{\phi=0}$ the condition (5) is

$$\begin{pmatrix} \sin\Phi & \cos\Phi \\ -\cos\Phi & \sin\Phi \end{pmatrix}\begin{pmatrix} w_+(0) \\ u_+(0) \end{pmatrix} = \begin{pmatrix} \sin\Phi & \cos\Phi \\ \cos\Phi & -\sin\Phi \end{pmatrix}\begin{pmatrix} w_-(0) \\ u_-(0) \end{pmatrix}. \tag{16}$$

Hence,

$$u_\pm(0) = (\sin 2\Phi)^{-1}(w_\mp(0) + w_\pm(0)\cos 2\Phi). \tag{17}$$

For any given incident longitudinal wave motion, that is, known values of $T_\pm^{(in)}$ or $u_\pm^{(in)}$, the scattered in-plane velocities and stresses at the junction, $u_\pm^{(sc)}(0)$ and $T_\pm^{(sc)}(0)$, now follow directly from $w_\pm(0)$ using equations (13) and (17). These radiate away from the junction according to the one-dimensional wave solutions of equation (11) with $\zeta_\pm^{(1)} = (-\mathrm{i}\omega)^{-1}u_\pm(0)$ and $\zeta_\pm^{(2)} = 0$.

Using the same projection method the force condition equation (14) may be rewritten as

$$S_\pm(0) = -(\sin 2\Phi)^{-1}(T_\mp(0) + T_\pm(0)\cos 2\Phi + F_\mp), \tag{18}$$

where $F_\pm = \vec{F}\vec{l}_\pm$ are the components of the applied force along each plate. Alternatively, equations (13), (17), and (18) give the shear forces as

$$S_\pm(0) = (Z_+ + Z_-)(\sin 2\Phi)^{-2}(w_\pm(0) + w_\mp(0)\cos 2\Phi) - Z_\pm w_\pm(0) - (\sin 2\Phi)^{-1}F_\mp^{(eff)}, \quad (19)$$

where the effective force $\vec{F}^{(eff)}$ is defined as

$$\vec{F}^{(eff)} = \vec{F} + 2Z_+u_+^{(in)}(0)\vec{l}_+ + 2Z_-u_-^{(in)}(0)\vec{l}_-, \quad (20)$$

and $F_\pm^{(eff)} = \vec{F}^{(eff)}\vec{l}_\pm$, or

$$F_\pm^{(eff)} = F_\pm + 2Z_\pm u_\pm^{(in)}(0) + 2Z_\mp u_\mp^{(in)}(0)\cos 2\Phi. \quad (21)$$

Eliminating the shear forces from equation (19) using equations (8) and (9) yields relations between the transverse displacements and velocities. Together with equations (6) and (15), these provide the four contact conditions sought which are now presented in terms of the transverse deflection only:

$$\xi_+'(0) + \xi_-'(0) = 0, \qquad D_+\xi_+''(0) - D_-\xi_-''(0) = M_0, \quad (22, 23)$$

$$D_\pm/(\mathrm{i}\omega)\xi_\pm'''(0) + Z_\pm\xi_\pm(0) - [(Z_+ + Z_-)/\sin^2 2\Phi](\xi_\pm(0) + \xi_\mp(0)\cos 2\Phi) = F_\mp^{(eff)}/(\mathrm{i}\omega\sin 2\Phi). \quad (24)$$

The final pair of equations account for the longitudinal interaction of the plates through the welded joint. In summary, one now has the same number of tip conditions, four, that one obtains in the absence of longitudinal wave effects. The four conditions only involve the transverse displacements and their derivatives.

3. GENERAL PROPERTIES OF THE CONTACT CONDITIONS

3.1. RECIPROCITY

First the consistency between the contact conditions, equations (22)–(24), and reciprocity is analyzed. This requires that the pressure is invariant under the interchange of the acoustic source and receiver directions, assuming plane wave incidence and far-field observation in the absence of any applied loads, i.e. for $M_0 = 0$, $\vec{F}^{(eff)} = 0$ (reciprocity does not hold with loads present). Assume that the functions $\xi_\pm(r)$ and $\eta_\pm(r)$ are the plate flexural displacements induced by an acoustic plane wave incident from two different directions $\phi_0 = \phi_1$ and $\phi_0 = \phi_2$, respectively. It has been shown [6] that in order to ensure the reciprocal solution the contact conditions must be consistent with the general relation

$$J_+(\phi_1, \phi_2) + J_-(\phi_1, \phi_2) = 0, \quad (25)$$

where

$$J_\pm(\phi_1, \phi_2) = D_\pm(\xi_\pm(0)\eta_\pm'''(0) - \xi_\pm'(0)\eta_\pm''(0) + \xi_\pm''(0)\eta_\pm'(0) - \xi_\pm'''(0)\eta_\pm(0)). \quad (26)$$

Using equations (22)–(24) one can verify that equation (25) does indeed hold. This means that the contact conditions (22)–(24) with $M_0 = 0$ and $F_\pm^{(eff)} = 0$ will guarantee the reciprocal solution regardless of the specific values of both material and geometrical parameters of the configuration considered.

3.2. UNIQUENESS

The question of uniqueness is now discussed. Let (p_1, ξ_1) and (p_2, ξ_2) be two solutions of the same scattering or radiation problem. The difference, (p, ξ) where $p = p_1 - p_2$, $\xi = \xi_1 - \xi_2$, must satisfy the same Helmholtz equation (1), boundary conditions (2), and junction condition (22). The difference satisfies the homogeneous versions of equations (23) and (24), that is, with $M_0 = 0$, $\vec{F}^{(eff)} = 0$, and also $u_\pm^{(in)} = 0$. One first needs to define the flexural and longitudinal powers,

$$P_\pm^{flex}(r) = w_\pm^* S_\pm^* - w_\pm'^* M_\pm^*, \qquad P_\pm^{long}(r) = u_\pm^* T_\pm^*, \tag{27, 28}$$

where $f^* = \operatorname{Re}(fe^{-i\omega t})$. It can be shown [6] that the difference solution satisfies the identity

$$\frac{\operatorname{Im} k^2}{2\omega\rho} \int_{-0}^{+\infty} \int_{-\Phi}^{\Phi} |p(r, \phi)|^2 r \, dr \, d\phi + \langle P_+^{flex}(0) + P_-^{flex}(0) \rangle = 0, \tag{29}$$

where $\langle g \rangle = (\omega/2\pi) \int_0^{2\pi/\omega} g(t) \, dt$ is the average over a cycle. Since $\operatorname{Im} k^2 > 0$ by assumption, the following condition is sufficient for a unique solution [6]:

$$\langle P_+^{flex}(0) + P_-^{flex}(0) \rangle \geqslant 0. \tag{30}$$

The average powers can be written

$$\langle P_\pm^{flex}(0) \rangle = \tfrac{1}{2} \operatorname{Re}(w_\pm(0)\bar{S}_\pm(0) - w_\pm''(0)\bar{M}_\pm(0)). \tag{31}$$

These can be simplified using equations (6) and (15) with $M_0 = 0$, and using equations (17) and (18) with $\vec{F}^{(eff)} = 0$ to express $w_\pm$ and $S_\pm$ in terms of $u_\pm$ and $T_\pm$, respectively. This yields

$$\langle P_+^{flex}(0) + P_-^{flex}(0) \rangle = -\langle P_+^{long}(0) + P_-^{long}(0) \rangle. \tag{32}$$

Equation (32) can be further reduced using equation (13), to

$$\langle P_+^{flex}(0) + P_-^{flex}(0) \rangle = \tfrac{1}{2}|u_+(0)|^2 \operatorname{Re} Z_+ + \tfrac{1}{2}|u_-(0)|^2 \operatorname{Re} Z_-. \tag{33}$$

Thus, contact conditions of the form (22) through (24), combined with the restrictions

$$\operatorname{Re} Z_\pm \geqslant 0, \tag{34}$$

are sufficient to ensure the unique solution of the diffraction and radiation problems for the configuration of interest.

3.3. AN ADMITTANCE RESULT

Consider the radiation problem with the loads M_0 and $\vec{F}$ present but no incident waves, giving the solution (p, ξ). Since there is no incident wave excitation one can use exactly the same procedure which led to equation (29) to obtain the same identity for the radiation solution. The distinction is that now the loads M_0 and $\vec{F}$ are arbitrary and also the solution p is non-zero. The latter now implies that

$$\langle P_+^{flex}(0) + P_-^{flex}(0) \rangle \leqslant 0. \tag{35}$$

Proceeding in the same manner as before, but with M_0 and $\vec{F}^{(eff)} = \vec{F}$ non-zero, one finds that

$$\begin{aligned}\langle P_+^{flex}(0) + P_-^{flex}(0) \rangle = &-\langle P_+^{long}(0) + P_-^{long}(0) \rangle + \tfrac{1}{2} \operatorname{Re} w_+'(0)\bar{M}_0 \\ &-(2 \sin 2\Phi)^{-1} \operatorname{Re}(w_+(0)\bar{F}_- + w_-(0)\bar{F}_+).\end{aligned} \tag{36}$$

It can be shown that the kinematic conditions imply that

$$(\sin 2\Phi)^{-1}(w_-(0)\vec{l}_+ + w_+(0)\vec{l}_-) = \vec{v}(0), \tag{37}$$

and hence,

$$\tfrac{1}{2} \operatorname{Re} \bar{v}(0)F - \tfrac{1}{2} \operatorname{Re} w_+'(0)\bar{M}_0 \geqslant \tfrac{1}{2}|u_+(0)|^2 \operatorname{Re} Z_+ + \tfrac{1}{2}|u_-(0)|^2 \operatorname{Re} Z_-. \tag{38}$$

This can be interpreted in the following manner. The energy flux for the radiated longitudinal wave in each plate is $\mathscr{F}_{\pm}(r) = -u_{\pm}^{*} T_{\pm}^{*}$. Based on equation (13) the averaged flux is $\langle \mathscr{F}_{\pm}(r) \rangle = \frac{1}{2}|u_{\pm}|^2 \operatorname{Re} Z_{\pm}$, which is independent of r. Define W^{in} as the rate of work done by the applied loads M_0 and $\vec{F}$. The inequality equation (38) says that the power input to the system must equal or exceed the power that is radiated as longitudinal energy, or

$$\langle W^{in} \rangle \geqslant \mathscr{F}_{+} + \mathscr{F}_{-}. \tag{39}$$

Alternatively, define the junction admittance matrix [**A**] such that

$$\{\vec{v}(0), -w_{+}'(0)\}^{\mathrm{T}} = [\mathbf{A}]\{\vec{F}, M_0\}^{\mathrm{T}}. \tag{40}$$

Then the inequality equation (38) implies that $[\mathbf{A}] + [\bar{\mathbf{A}}]^{\mathrm{T}}$ is positive semi-definite.

3.4. ALTERNATIVE FORMS OF THE JUNCTION CONDITIONS

The junction conditions have relatively simple forms in terms of the sum and differences w_s and w_d defined by

$$w_s = \tfrac{1}{2}(w_{+}(0) + w_{-}(0)), \qquad w_d = \tfrac{1}{2}(w_{+}(0) - w_{-}(0)). \tag{41}$$

Similar definitions are assumed for u_s, u_d, S_s, S_d, Z_s, Z_d, $F_s^{(eff)}$, and $F_d^{(eff)}$. Two alternative versions of equations (17) and (19) are now presented that are useful in the separate limiting cases when the wedge angle Φ is close to either $\pi/2$ or π.

First, the four identies in equations (17) and (19) are expressed as

$$\begin{gathered} w_d = \cot^2 \Phi (Z_s)^{-1}(Z_d w_s + S_d - (\sin 2\Phi)^{-1} F_d^{(eff)}), \\ S_s + (\sin 2\Phi)^{-1} F_s^{(eff)} = \cot^2 \Phi (Z_s)^{-1}(Z_{+}Z_{-} w_s - Z_d S_d + (\sin 2\Phi)^{-1} Z_d F_d^{(eff)}), \\ u_s = \cot \Phi w_s, \qquad u_d - (2Z_s \sin^2 \Phi)^{-1} F_d^{(eff)} = -\cot \Phi (Z_s)^{-1}(Z_d w_s + S_d). \end{gathered} \tag{42}$$

Thus, when $\Phi = \pi/2$ the right members of equations (42) vanish, implying that the contact conditions are identically $w_d = 0$, $S_s = -\frac{1}{2}\vec{F}^{(eff)}\vec{e}_r|_{\phi=0}$, $u_s = 0$, and $u_d = (2Z_s)^{-1}\vec{F}^{(eff)}\vec{e}_{\phi}|_{\phi=0}$, as expected for the flat junction of two plates.

Alternatively, equations (42) may be written as

$$\begin{gathered} w_s = \tan^2 \Phi (Z_s)^{-1}(Z_d w_d + S_s + (\sin 2\Phi)^{-1} F_s^{(eff)}), \\ S_d - (\sin 2\Phi)^{-1} F_d^{(eff)} = \tan^2 \Phi (Z_s)^{-1}(Z_{+}Z_{-} w_d - Z_d S_s - (\sin 2\Phi)^{-1} Z_d F_s^{(eff)}), \\ u_d = -\tan \Phi w_d, \qquad u_s - (2Z_s \cos^2 \Phi)^{-1} F_s^{(eff)} = \tan \Phi (Z_s)^{-1}(Z_d w_d + S_s). \end{gathered} \tag{43}$$

The limiting case of $\Phi = \pi$ corresponds to two parallel semi-infinite plates jointed only at their ends, and equations (43) imply that the junction conditions reduce to $w_s = 0$, $S_d = -\frac{1}{2}\vec{F}^{(eff)}\vec{e}_{\phi}|_{\phi=0}$, $u_d = 0$, and $u_s = (2Z_s)^{-1}\vec{F}^{(eff)}\vec{e}_r|_{\phi=0}$.

Finally, one notes that if there are no incident longitudinal waves, then the longitudinal wave energy is purely radiative, and it follows from equation (42) that

$$\langle \mathscr{F}_{+} \rangle + \langle \mathscr{F}_{-} \rangle = \cot^2 \Phi (Z_s)^{-1}(|S_d - (\sin 2\Phi)^{-1} F_d|^2 + Z_{+}Z_{-}|w_s|^2). \tag{44}$$

Alternatively, using equation (43) gives

$$\langle \mathscr{F}_{+} \rangle + \langle \mathscr{F}_{-} \rangle = \tan^2 \Phi (Z_s)^{-1}(|S_s + (\sin 2\Phi)^{-1} F_s|^2 + Z_{+}Z_{-}|w_d|^2). \tag{45}$$

These imply, respectively, that the longitudinal energy flux vanishes in the absence of any applied force for the special cases of $\Phi = \pi/2$ and $\Phi = \pi$.

4. APPLICATIONS TO ACOUSTICAL AND FLEXURAL WAVE SCATTERING

4.1. PERTURBATION PROCEDURE FOR A SLIGHTLY KINKED CONFIGURATION

The four contact conditions of equations (22)–(24) serve as the starting point for attacking acoustic and structural wave scattering problems for the wedge configuration. A consistent procedure for solving these types of problems will be presented in a separate paper. However, as an example of how the contact conditions can be applied, the case of an "almost" flat junction is considered subject to incident acoustic or flexural waves with no applied loads at the junction and no incident longitudinal waves. One is interested in how the slight kink effects the interaction between acoustic and longitudinal energy for acoustic incidence, and between the flexural and longitudinal waves for flexural incidence. Both interactions vanish for a perfectly flat junction, and therefore it is of some use to estimate the leading order effects for slight kinks.

Perturbation methods are used based on the small parameter $\epsilon = \cot \Phi$, for $|\epsilon| \ll 1$. One assumes the asymptotic expansions

$$w_\pm = w_\pm^{(0)} + \epsilon w_\pm^{(1)} + \epsilon^2 w_\pm^{(2)} + \cdots, \qquad u_\pm = u_\pm^{(0)} + \epsilon u_\pm^{(1)} + \epsilon^2 u_\pm^{(2)} + \cdots,$$
$$p = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots, \cdot\cdot \tag{46}$$

The junction conditions (42) imply a sequence of identities, the first few being

$$w_d^{(0)} = 0, \qquad S_s^{(0)} = 0; \qquad w_d^{(1)} = 0, \qquad S_s^{(1)} = 0;$$
$$w_d^{(2)} = (Z_d w_s^{(0)} + S_d^{(0)})/Z_s, \qquad S_s^{(2)} = (Z_+ Z_- w_s^{(0)} - Z_d S_d^{(0)})/Z_s, \tag{47}$$

and

$$u_\pm^{(0)}(0) = 0, \qquad u_\pm^{(1)}(0) = (Z_\mp w_s^{(0)} \mp S_d^{(0)})/Z_s, \cdots \tag{48}$$

Let $\mu = \pi/(2\Phi)$, and $\theta = \mu\phi$, then equation (1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + k^2 p = \frac{(1-\mu^2)}{r^2}\frac{\partial^2 p}{\partial \theta^2}, \qquad 0 < r < \infty, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{49}$$

The parameter $(1 - \mu^2)$ is small, of order ϵ:

$$1 - \mu^2 = -(4/\pi)\epsilon - (12/\pi^2)\epsilon^2 + \cdots \tag{50}$$

Thus, one again obtains a sequence of equations, the first two being

$$\nabla^2 p^{(0)} + k^2 p^{(0)} = 0, \qquad \nabla^2 p^{(1)} + k^2 p^{(1)} = -(4/\pi r^2)(\partial^2 p^{(0)}/\partial\theta^2). \tag{51}$$

Here, ∇^2 denotes the 2D Laplacian, and the equations are all valid in the half space $-\infty < x < \infty$, $0 < y < \infty$, where $\vec{e}_x = -\vec{e}_\theta|_{\theta=0}$, $\vec{e}_y = \vec{e}_r|_{\theta=0}$.

The Helmholtz equation (51) and the junction conditions (47), combined with the leading order versions of the continuity condition and of equations (6) and (15), implies that the leading order fields $p^{(0)}$, $w_\pm^{(0)}$, $S_\pm^{(0)}$, etc., are those for a pair of flat joined plates. This can be "easily solved", in principle. For example, if the plates are identical and the excitation is from an incident acoustic plane wave then the zeroth solution is quite trivial. Even when the plates are dissimilar, the general solution can be found in semi-analytic form [7].

4.2. EXAMPLES

Assuming the plates are identical, so that $Z_+ = Z_- \equiv Z$, etc. one lets $c_0 = c_\pm$. The leading order wave field is assumed to have the x-dependence $\exp(ik_x x)$, so that $w_s^{(0)} = w_\pm^{(0)} \equiv w_0$ is the leading order transverse velocity at the kink, and

$$S_d^{(0)} = \omega^{-1} k_x^3 D w_0. \tag{52}$$

As a first example consider a plane acoustic wave incident at the angle ϕ_0, with $k_x = k \sin \phi_0$. The leading order longitudinal motion is therefore, from equations (46) and (48),

$$u_\pm(0) \approx \epsilon w_0(1 \mp (c/c_0)(\Omega^2/M^3)), \tag{53}$$

where $M = 1/\sin \phi_0 > 1$ is the supersonic surface Mach number of the acoustic wave, $\Omega = \omega/\omega_c$ and $\omega_c = c^2\sqrt{m/D}$ is the coincidence frequency. Hence, at low frequencies relative to the coincidence frequency, one has simply $u_\pm(0) \approx \epsilon w_0$. This is exactly what one might expect purely on the basis of quasi-static kinematics of the kink junction.

As a second example a sub-sonic flexural wave incident with horizontal wavenumber k_{flex} is considered. Then by the same analysis one finds that $u_\pm(0)$ are of exactly the same form as equation (53), where now $M = k/k_{flex} < 1$ is the subsonic Mach number. At low frequencies or under heavy fluid loading, equation (2) implies that

$$Dk_{flex}^5 - \rho\omega^2 \approx 0, \tag{54}$$

or $M \approx \epsilon_c^{-1/5}\Omega^{3/5}$, where $\epsilon_c = \rho c/(\omega_c m)$ is a fluid loading parameter. Note that the coincidence frequency ω_c depends upon the plate thickness, but that ϵ_c is a function only of the material parameters. Equation (53) then becomes,

$$u_\pm(0) \approx \epsilon w_0(1 \mp (c/c_0)\epsilon_c^{3/5}\Omega^{4/5}) \approx \epsilon w_0. \tag{55}$$

The behavior of the flexural wave in the low frequency regime is therefore similar to that for the acoustic wave.

In summary, the leading order interaction with the longitudinal motion is given by equation (53) for acoustic or flexural wave incidence. In either case the converted longitudinal energy can be easily determined as

$$\langle \mathscr{F}_+ \rangle + \langle \mathscr{F}_- \rangle \approx \epsilon^2 |w_0|^2 (1 + (c^2/c_0^2)(\Omega^4/M^6)) \operatorname{Re} Z, \tag{56}$$

and at low frequencies equation (55) implies that the converted longitudinal energy is distributed equally between the two plates.

Finally, we note that the related problem of estimating the leading order (in ϵ) conversion between acoustic and flexural or longitudinal and flexural motions is considerably more complicated. The problem lies with equations (47) which define the junction conditions for the leading order correction for the transverse motion, $w_d^{(2)}$ and $S_d^{(2)}$. In order to find these we must solve the fluid loaded flat plate radiation problem with given discontinuities in the transverse displacement and shear force at $x = 0$. This amounts to finding the generalized Green's function for the fluid loaded flat plate, which can be achieved using transform techniques, but the analysis is not as simple or elegant as the example considered here.

5. CONCLUSIONS

The contact conditions which account for both the flexural and longitudinal interaction between two thin elastic plates through a welded junction have been examined. It has been shown that equations (22)–(24) will provide reciprocal solutions to diffraction problems involving an angular joint of thin plates in contact with an acoustic medium. A set of constraints (34) on the longitudinal wave impedances $Z_\pm$ has been deduced which is sufficient (but not *necessary!*) to guarantee a unique solution to a related diffraction problem. Generally, these constraints are satisfied by the impedances associated with the plates. Some simple applications of the contact conditions to almost flat plates under acoustic and flexural wave incidence shows that the leading order longitudinal velocity is $\cot(\Phi)$ times the transverse velocity on a perfectly flat junction. This leads to the estimate that the fraction of energy converted from acoustic or flexural motion into longitudinal motion is $O((\Phi - (\pi/2))^2)$ for nearly flat junctions.

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