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LINE ADMITTANCE AT THE JUNCTION OF TWO PLATES WITH AND WITHOUT FLUID LOADING

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A time harmonic line force or moment is applied at the junction of two plates in welded contact, with or without unilateral fluid loading. The objective is the 2×2 matrix of admittances relating the applied force and moment to the deflection and rotation at the load. The structural asymmetry leads to coupling between force and rotation, and between moment and deflection, even in the absence of fluid loading. The impedance matrix is derived for sources with linear phase variation of the form $e^{ik_y y}$ for real k_y. The dry plate problem is addressed first, and displays the possibility of a resonance when the drive is in phase with a flexural Stoneley wave, which is defined here. The fluid-loaded problem is attacked by expressing the vibration of either plate and the acoustic response in the fluid as transforms, the integrands of which are derived using the Wiener–Hopf technique. It is found that the wet response shows the same general behaviour as for the dry plates, but without the possibility of a structural resonance. Numerical examples are presented showing the frequency dependence of the admittance matrix for joined steel plates in water. The general theory for distinct plates under fluid loading also provides a new formula for the line admittance of a uniform plate in a fluid.

1. INTRODUCTION

The line admittance function for a fluid-loaded structure is one of the basic elements required for understanding and computing the interaction of acoustic and structure-borne sound. Over 20 years ago Nayak [1] published numerical computations for a steel plate in water. Subsequently, these results were complemented by Crighton [2], who gave explicit analytical solutions for the force and moment admittance, and also provided useful asymptotic approximations valid for heavy fluid loading. Both authors considered the classical thin plate model for flexural waves. Subsequent papers [3–6] provided alternative derivations of Crighton's [2] basic result. However, of these only Smith [5] provided any significant new insights. He showed that the Wiener–Hopf analysis of Crighton [2] can be easily circumvented using contour integration, and he also discussed generalizations of Crighton's formulae.

This paper deals with the analogous problem for a line force and moment applied at the junction of a pair of plates. The situation is depicted in Figure 1. Both plates are modelled by classical bending theory, and are assumed to be in welded contact at the join. The plates may be dry (*in vacuo*) or wet, i.e., fluid loaded on one side, $z > 0$ in Figure 1. We sill consider the dry and wet problems separately, and derive the response at the drive. The method of solution is quite different in each case, and far more involved for the latter. However, there are some interesting features of the dry plate problem that are not present in the fluid-loaded situation. We will see that resonances associated with interfacial flexural

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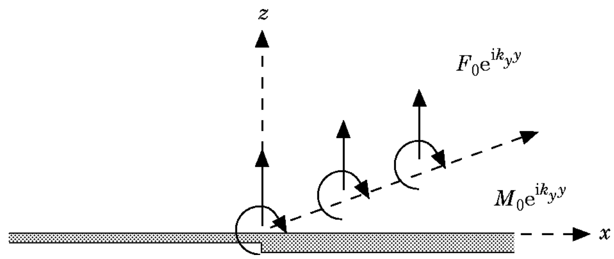

Figure 1. The plate junction, applied loading, and co-ordinate system.

waves are possible. The related but simpler problem of a phased line load on a uniform plate under fluid loading has recently been solved by Photiadis [7].

The main results of the paper concern the admittance matrix for a pair of plates in fluid. This solution is fundamental to problems involving acoustic and structural wave interaction with junctions. For instance, the scattering of a flexural wave from a three-member junction comprising the junction in Figure 1 with an internal member attached at $x=0$ can be solved in a relatively simple manner once the admittance matrix is known. For example, Photiadis [7] has used the solution for a uniform plate to find the acoustic and structural response for a flexural wave obliquely incident on a rib. Fluid-loaded problems of the type depicted in Figure 1 are necessarily tackled using the Wiener–Hopf technique [8]. For instance, Brazier-Smith [9] considered the diffraction of acoustic and structure-borne waves at plate junctions, and obtained numerical results using the Wiener–Hopf analysis. His method of solution relies upon numerical evaluation of many unknown constants, and is not readily adapted to finding the deflection and rotation at a line load. The procedure followed here is more direct than Brazier-Smith's, and leads to semi-explicit formulae for the admittance matrix relating the deflection and rotation to the force and moment. Unfortunately, the contour integration method of Smith [5] does not lend itself to the present problem. The method of solution used here is closely related to that of Norris and Wickham [10] who obtained a semi-analytic solution for the problem considered by Brazier-Smith [9]. There are two central steps involved in this approach. First, the Wiener–Hopf factorization of the dispersion functions is obtained as a finite and easily computed integral. The factorization also leads to explicit expressions for the plate displacement near the edge. Relatively simple matrix equations can then be derived, depending on the explicit edge constraints. In this paper we deal with a concentrated loading at the junction itself. We also include the possibility of a phase term $e^{ik_y y}$ along the junction $-\infty < y < \infty$. The previous analyses [9, 10] were for the case of "normal" incidence, $k_y = 0$.

The fluid loaded problem is defined in section 2, and the simpler case of a pair of plates *in vacuo* is considered and solved in section 3. The general solution for fluid loading is developed in section 4, with the more mathematical details relegated to appendices. Several interesting limiting cases are discussed in section 5, including light and heavy fluid loading.

2. FORMULATION OF THE PROBLEM

2.1. DYNAMIC EQUATIONS

The two plates may have different densities, elastic properties, and thicknesses, but each is uniform and its motion is modelled by the classical theory of dynamic flexure. The plates lie in the plane $z=0$ and meet along $x=0$. All motion is time harmonic motion of frequency $\omega > 0$, with the term $\mathrm{Re}\{\cdot e^{-i\omega t}\}$ understood but suppressed. The plate equations are

$$B_j \nabla^2 \nabla^2 w(x, y) - m_j \omega^2 w(x, y) = -p(x, y, 0), \qquad \begin{cases} x < 0, & j = 1, \\ x > 0, & j = 2, \end{cases} \tag{1}$$

where $w(x, y)$ is the plate deflection in the z-direction, and $p(x, y, z)$ is the acoustic pressure in the fluid, which occupies the half-space $0 < z < \infty$. Also, $m_{1,2}$ are the areal mass densities, and $B_{1,2}$ the bending stiffnesses of the distinct plates. The pressure and deflection are related by the continuity condition

$$\rho\omega^2 w(x, y) = \frac{\partial p}{\partial z}(x, y, 0), \qquad -\infty < x, y < \infty, \tag{2}$$

where ρ is the fluid mass density per unit volume. Finally, the pressure satisfies the Helmholtz equation

$$\nabla^2 p + k^2 p = 0, \qquad -\infty < x, y < \infty, \qquad 0 < z < \infty, \tag{3}$$

where $k = \omega/c$ is the acoustic wavenumber, and c is the fluid sound speed.

Elimination of $w(x, y)$ between the two boundary conditions (1) and (2) gives a single equation for the pressure,

$$\mathscr{L}_1 p(x, y, 0) = 0, \qquad x < 0, \qquad -\infty < y < \infty, \tag{4a}$$

$$\mathscr{L}_2 p(x, y, 0) = 0, \qquad x > 0, \qquad -\infty < y < \infty, \tag{4b}$$

where the operators are defined as

$$\mathscr{L}_j = 1 + a_j \left[\kappa_j^{-4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 - 1 \right] \frac{\partial}{\partial z}, \qquad j = 1, 2, \tag{5}$$

with

$$\kappa_j^4 = \omega^2 m_j / B_j, \qquad a_j = m_j/\rho, \qquad j = 1, 2. \tag{6}$$

Thus, $\kappa_{1,2}$ are the flexural wavenumbers of the plates, and $a_{1,2}$ are the "null frequency" lengths. We are interested in a line source situated right at the junction of the two plates. Equations (4) hold for all non-zero values of x, but not at $x = 0$, where certain conditions need to be imposed. First, the deflection, w, and the rotation, $w_{,x} \equiv \partial w(x, y)/\partial x$, are both continuous at the junction. We consider an applied phased line force in the positive z-direction, $F_0\, e^{ik_y y}$, and a phase line moment about the y-axis (in a right-handed sense), $M_0 e^{ik_y y}$, such that

$$M_x(0+) - M_x(0-) = -M_0\, e^{ik_y y}, \tag{7a}$$

$$[S_x(0+) - M_{xy,y}(0+)] - [S_x(0-) - M_{xy,y}(0-)] = -F_0\, e^{ik_y y}. \tag{7b}$$

The bending and twisting moments and the shear force on either plate are given by the classical relations

$$M_x(x, y) = -B_j[w_{,xx}(x, y) + \nu_j w_{,yy}(x, y)], \tag{8a}$$

$$M_{xy}(x, y) = B_j(1 - \nu_j) w_{,xy}(x, y), \qquad S_x(x) = M_{x,x} - M_{xy,y}, \tag{8b, c}$$

where ν_j is the Poisson's ratio and $j = 1$ and 2 for $x < 0$ and > 0, respectively. We note that $S_x - M_{xy,y}$ is the effective shear force acting in the positive z direction, or the Kelvin–Kirchhoff force [11]. Equation (7b), for example, may be derived by considering a free body diagram for a segment $-\epsilon/2 \leqslant x \leqslant \epsilon/2$, and letting $\epsilon \to 0$. In this limit the inertial effects of the plate and the pressure loading from the fluid, $-m\omega^2 w(0)\epsilon$ and $p(0, 0)\epsilon$, respectively, both tend to zero, leaving the force balance given. This argument implicitly assumes that the pressure is bounded at the junction, which can be verified *a posteriori*.

Finally, we note that the pressure, in addition to satisfying the boundary conditions on the plates, and the Helmholtz equation (3) in the fluid, is subject to a radiation condition as $\sqrt{x^2+y^2+z^2}\to\infty$. The plate deflection also satisfies a radiation condition as $|x|\to\infty$. The problem then has a unique and physically meaningful solution.

2.2. TRANSFORMED PROBLEM

The equations can be simplified by removing the multiplicative factor $e^{ik_y y}$ from all quantities. The problem is then essentially two-dimensional. Thus, we assume that p and w have the explicit dependence

$$p(x, y, z) = \bar{p}(x, z)e^{ik_y y}, \qquad w(x, y) = \bar{w}(x)\, e^{ik_y y}. \tag{9}$$

The boundary equations (4) for pressure become

$$\bar{\mathscr{L}}_1 \bar{p}(x, 0) = 0, \qquad x < 0; \qquad \bar{\mathscr{L}}_2 \bar{p}(x, 0) = 0, \qquad x > 0; \tag{10}$$

where the operators are now defined as

$$\bar{\mathscr{L}}_j = 1 + a_j \left[\kappa_j^{-4} \left(\frac{\partial^2}{\partial x^2} - k_y^2 \right)^2 - 1 \right] \frac{\partial}{\partial z}, \qquad j = 1, 2. \tag{11}$$

The Helmholtz equation (3) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \bar{k}^2 \right) \bar{p} = 0, \qquad -\infty < x < \infty, \qquad 0 < z < \infty, \tag{12}$$

where

$$\bar{k}^2 = k^2 - k_y^2. \tag{13}$$

The jump conditions at the plate junctions (7) and (8) can be written as

$$B_2[\bar{w}_{,xx}(0+) - (1-\eta_2)k_y^2 \bar{w}(0+)] - B_1[\bar{w}_{,xx}(0-) - (1-\eta_1)k_y^2 \bar{w}(0-)] = M_0, \tag{14a}$$

$$B_2[\bar{w}_{,xxx}(0+) - (1+\eta_2)k_y^2 \bar{w}_{,x}(0+)] - B_1[\bar{w}_{,xxx}(0-) - (1+\eta_1)k_y^2 \bar{w}_{,x}(0-)] = F_0, \tag{14b}$$

where

$$\eta_j = 1 - \nu_j, \qquad \text{for } j = 1, 2. \tag{15}$$

Also note that the deflection $\bar{w}$ and its first derivative $\bar{w}' = d\bar{w}/dx$ are both continuous at the junction.

2.3. ADMITTANCE AND IMPEDANCE MATRICES

The main focus of this paper is to determine the *admittance matrix*, $\mathbf{Y}^{(p)}$, which relates the forces and velocities according to

$$\begin{bmatrix} -i\omega\bar{w}(0) \\ i\omega\bar{w}'(0) \end{bmatrix} = \begin{bmatrix} Y_{11}^{(p)} & Y_{12}^{(p)} \\ Y_{21}^{(p)} & Y_{22}^{(p)} \end{bmatrix} \begin{bmatrix} F_0 \\ M_0 \end{bmatrix}, \tag{16}$$

or equivalently, the *impedance matrix*, defined by

$$\begin{bmatrix} F_0 \\ M_0 \end{bmatrix} = \begin{bmatrix} Z_{11}^{(p)} & Z_{12}^{(p)} \\ Z_{21}^{(p)} & Z_{22}^{(p)} \end{bmatrix} \begin{bmatrix} -i\omega\bar{w}(0) \\ i\omega\bar{w}'(0) \end{bmatrix}, \tag{17}$$

and therefore

$$\mathbf{Z}^{(p)} = (\mathbf{Y}^{(p)})^{-1}. \tag{18}$$

The superscript (p) is used to denote the matrices as "physical", i.e., in the sense that they are normally understood and defined. Equation (16) can be written succinctly as $v_i = \Sigma_j Y_{ij}^{(p)} f_j$, where i and j take the values 1 and 2. The average power that the drive puts out over a single cycle is $\frac{1}{2}\mathrm{Re}\, \Sigma_{i,j}\, (f_i^* Y_{ij}^{(p)} f_j)$, where * denotes the complex conjugate. This must be non-negative for arbitrary complex values of f_1 and f_2, implying that the elements of the admittance matrix satisfy the inequalities,

$$\mathrm{Re}\; Y_{11}^{(p)} \geqslant 0, \qquad \mathrm{Re}\; Y_{22}^{(p)} \geqslant 0, \tag{19a, b}$$

$$\mathrm{Re}\; Y_{11}^{(p)} \mathrm{Re}\; Y_{22}^{(p)} - \mathrm{Re}\; Y_{12}^{(p)} \mathrm{Re}\; Y_{21}^{(p)} \geqslant 0, \tag{19c}$$

$$\mathrm{Re}\; Y_{11}^{(p)} \mathrm{Re}\; Y_{22}^{(p)} + \mathrm{Im}\; Y_{12}^{(p)} \mathrm{Im}\; Y_{21}^{(p)} \geqslant 0. \tag{19d}$$

We also note that the off-diagonal elements of the matrices are related by

$$Y_{21}^{(p)} = Y_{12}^{(p)}, \qquad Z_{21}^{(p)} = Z_{12}^{(p)}, \tag{20}$$

which can be derived from the principle of mechanical reciprocity as follows. Consider an effective shear force F_0 applied at the junction. The resulting horizontal displacement at points through the thickness section at $x = 0$ then depends upon the rotation, or $Y_{21}^{(p)} F_0$ from equation (16). Now consider a pair of equal and opposite horizontal forces applied at the junction, equivalent to a couple M_0. The resulting deflection of the junction is proportional to $Y_{12}^{(p)} M_0$ from equation (16). General reciprocity in linear elasticity implies that the displacement in the $\mathbf{n}_2$ direction at $\mathbf{x}_2$ due to a force F applied at $\mathbf{x}_1$ in the $\mathbf{n}_1$ direction is the same as the displacement in the $\mathbf{n}_1$ direction at $\mathbf{x}_1$ due to a force F applied at $\mathbf{x}_2$ in the $\mathbf{n}_2$ direction. Application of this to the pair of loadings immediately implies equation (20).

It is more convenient for our purposes to define related matrices $\mathbf{Y}$ and $\mathbf{Z}$ such that

$$\begin{bmatrix} \bar{w}(0) \\ -i\bar{w}'(0) \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} -M_0 \\ iF_0 \end{bmatrix}, \tag{21}$$

and

$$\mathbf{Z} = \mathbf{Y}^{-1}. \tag{22}$$

The two sets of matrices are easily related to one another as, for example,

$$\begin{bmatrix} Y_{11}^{(p)} & Y_{12}^{(p)} \\ Y_{21}^{(p)} & Y_{22}^{(p)} \end{bmatrix} = \omega \begin{bmatrix} Y_{12} & iY_{11} \\ -iY_{22} & Y_{21} \end{bmatrix}, \qquad \begin{bmatrix} Z_{11}^{(p)} & Z_{12}^{(p)} \\ Z_{21}^{(p)} & Z_{22}^{(p)} \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} Z_{21} & iZ_{22} \\ -iZ_{11} & Z_{12} \end{bmatrix}. \tag{23a, b}$$

The reciprocity identities (20) imply that the alternative form for the admittance satisfies

$$Y_{22} = -Y_{11} \qquad \Leftrightarrow \mathbf{Y}^{-1} = -(\det \mathbf{Y})^{-1} \mathbf{Y}, \tag{24}$$

and hence the matrices $\mathbf{Y}$ and $\mathbf{Z}$ are proportionate.

The remainder of the paper will deal with the question of finding $\mathbf{Y}$ and $\mathbf{Z}$. We begin with the simpler case of dry plates, which will be used later for comparison with the wet results.

3. THE DRY JUNCTION IMPEDANCE

3.1. GENERAL SOLUTION

It is both instructive and useful in its own right to consider the admittance matrix for line forces and line moments applied at the junction of two dry plates, i.e., *in vacuo*. In the absence of the fluid loading the only waves of relevance are the propagating and evanescent flexural waves on either plate. The deflection $w(x)$ can therefore be expressed as a sum of the wave types which satisfy the radiation condition of propagation away from the junction. Also, the deflection and its first derivative are continuous at $x=0$.

The dry plate equations are given by equation (1) with $p\equiv 0$. The dispersion relation for straight-crested waves of the form $e^{i(\xi x+k_y y)}$ on uniform and infinite plates of either material is

$$(\xi_j^2+k_y^2)^2-\kappa_j^4=0, \qquad j=1,2. \tag{25}$$

The complex ξ-plane comprises two infinite subsets $\mathscr{H}^+$ and $\mathscr{H}^-$, which are defined to cover the upper and lower halves of the plane, respectively. Their intersection is a thin strip about the real line which by definition excludes all the material wavenumbers, such as κ_1 and κ_2. This can be realized by allowing ω to have a small positive imaginary part. Let ξ_{j1} and ξ_{j2} be the two roots in $\mathscr{H}^+$, such that

$$\xi_{j1}^2+\xi_{j2}^2=-2k_y^2, \qquad \xi_{j1}^2\xi_{j2}^2=k_y^4-\kappa_j^4. \tag{26}$$

It is then a simple matter to see that the general solution must be of the form

$$\bar{w}(x)=\frac{\xi_{12}\bar{w}(0)-i\bar{w}'(0)}{\xi_{12}-\xi_{11}}\,e^{-i\xi_{11}x}+\frac{\xi_{11}\bar{w}(0)-i\bar{w}'(0)}{\xi_{11}-\xi_{12}}\,e^{-i\xi_{12}x}, \qquad x<0, \tag{27a}$$

$$\bar{w}(x)=\frac{\xi_{22}\bar{w}(0)+i\bar{w}'(0)}{\xi_{22}-\xi_{21}}\,e^{i\xi_{21}x}+\frac{\xi_{21}\bar{w}(0)+i\bar{w}'(0)}{\xi_{21}-\xi_{22}}\,e^{i\xi_{22}x}, \qquad x>0. \tag{27b}$$

Substituting these into the joint conditions given by equations (7) or (14) yields a pair of coupled equations for the unknowns $\bar{w}(0)$ and $\bar{w}'(0)$ in terms of the applied line force F_0 and moment M_0. The coefficients in these linear equations immediately give us the elements of the impedance matrix $\mathbf{Z}$. Thus, using the identities (26),

$$\mathbf{Z}^{(dry)}=\begin{bmatrix} B_1(\xi_{11}\xi_{12}-\nu_1k_y^2)-B_2(\xi_{22}\xi_{21}-\nu_2k_y^2) & B_1(\xi_{11}+\xi_{12})+B_2(\xi_{22}+\xi_{21}) \\ -B_1(\xi_{11}+\xi_{12})\xi_{11}\xi_{12}-B_2(\xi_{21}+\xi_{22})\xi_{22}\xi_{21} & B_2(\xi_{22}\xi_{21}-\nu_2k_y^2)-B_1(\xi_{11}\xi_{12}-\nu_1k_y^2) \end{bmatrix}. \tag{28}$$

The admittance then follows from the fact that $\mathbf{Y}$ and $\mathbf{Z}$ are proportionate (see equation (24)), or

$$\mathbf{Y}^{(dry)}=-(\det \mathbf{Z}^{(dry)})^{-1}\mathbf{Z}^{(dry)}. \tag{29}$$

3.2. TWO-DIMENSIONAL LIMIT

The excitation of beams corresponds to the strictly two-dimensional limit of the present theory. When $k_y=0$ the roots become $\xi_{j1}=\kappa_j$ and $\xi_{j2}=i\kappa_j$, and the dry admittance and impedance matrices are

$$\mathbf{Y}^{(dry)}=\frac{(B_1\kappa_1^2+B_2\kappa_2^2)^{-1}}{(1+q)^2-(1+q^2)r^2}\,\mathbf{S}, \qquad \mathbf{Z}^{(dry)}=\frac{1}{2q}\,(B_1\kappa_1^2+B_2\kappa_2^2)\mathbf{S}, \tag{30}$$

where

$$\mathbf{S}=\begin{bmatrix} -i2qr & (1+i)\kappa_1^{-1}[1+q+(1-q)r] \\ (1-i)\kappa_2[1+q-(1-q)r] & i2qr \end{bmatrix}, \tag{31}$$

and

$$q\equiv\kappa_2/\kappa_1=(\alpha/\beta)^{1/4}, \qquad r\equiv(\beta q^2-1)/(\beta q^2+1). \tag{32a, b}$$

The dimensionless parameters α and β are defined as the density and stiffness ratios,

$$\alpha=m_2/m_1, \qquad \beta=B_2/B_1. \tag{33a, b}$$

We will find these parameters useful in the general theory for wet plates.

3.3. FLEXURAL STONELEY WAVES AND A RESONANCE PHENOMENON

The dry plate system can exhibit a resonance if the line is driven at a wavenumber k_y for which interfacial flexural waves can exist. These are solutions to the homogeneous equations of motion (no applied forces or moments) that are exponentially decaying on either side of the junction of the plates. Therefore, if they exist all four wavenumbers ξ_{ij} must be positive imaginary. Alternatively, the wavenumber k_y must exceed both κ_1 and κ_2 in magnitude. The wave speed along the y-axis is thus slower than either flexural wave speed. The wave is analogous to a Stoneley interface wave at the boundary joining two homogeneous elastic half spaces, and for that reason we call it a *flexural Stoneley wave*.

The condition for the existence of a flexural Stoneley wave can be expressed in terms of the impedance matrix. We require that equation (17) is satisfied with $F_0=M_0=0$, but with either or both of $\bar{w}(0)$ and $\bar{w}'(0)$ non-zero. The condition is therefore simply:

$$\det \mathbf{Z}^{(dry)}=0, \qquad \Leftrightarrow \text{flexural Stoneley wave.} \tag{34}$$

If the driving wavenumber k_y coincides with a flexural Stoneley wavenumber then the admittance is singular. The singularity is a resonance phenomenon and occurs because the forcing has precisely the phase of an interface wave propagating in the y-direction. Alternatively, the plates are driven by a line force which phase matches to a travelling wave, and hence the resonance. The resonance is pure (infinite Q) because there is no damping in the system, and disappears with the introduction of realistic dissipation.

As a specific illustration, consider the case when the plates are identical. Then the 11 and 22 elements of $\mathbf{Z}^{(dry)}$ in equation (28) vanish, implying that both $\mathbf{Z}^{(p,dry)}$ and $\mathbf{Y}^{(p,dry)}$ are diagonal. For instance, the "physical" admittance matrix is

$$\mathbf{Y}^{(p,dry)}=\frac{\omega}{2B(\xi_1+\xi_2)}\begin{bmatrix} -1/\xi_1\xi_2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{identical dry plates,} \tag{35}$$

where ξ_1 and ξ_2 are now the two roots of equation (26) in $\mathscr{H}^+$. The sum $\xi_1+\xi_2$ is non-zero for all values of k_y, however, the product $\xi_1\xi_2$ vanishes at $k_y=\kappa$, and consequently the force admittance $Y_{11}^{(p,dry)}$ is singular at this value of k_y. The moment admittance is always finite. This example is degenerate in the sense that the flexural Stoneley wave coincides with the simple flexural wave on the uniform plate, and does not decay away from the "junction".

In general, when the plates are dissimilar the existence of flexural Stoneley waves depends upon whether or not equation (34) has real roots for k_y. The question of existence can be simplified if we *assume* that there is at most a single propagating root. This implies that the sign of det $\mathbf{Z}^{(dry)}$ changes at most once for k_y in the range max $\{\kappa_1, \kappa_2\}<k_y<\infty$.

It is easily checked that

$$\det \mathbf{Z}^{(dry)} = [(3+\nu_1)B_1 + (1-\nu_2)B_2][(3+\nu_2)B_2 + (1-\nu_1)B_1]k_y^4 + O(k_y^2), \qquad k_y \to \infty. \tag{36}$$

The existence condition can therefore be expressed as

$$\det \mathbf{Z}^{(dry)} < 0, \qquad k_y = \max\{\kappa_1, \kappa_2\}, \qquad \Leftrightarrow \text{flexural Stoneley wave.} \tag{37}$$

With no loss in generality, let $\kappa_1 > \kappa_2$, then using the dimensionless parameters $0 < q < 1$ and β of equations (32a) and (33b), the condition (37) becomes

$$\begin{aligned} -\nu_1^2 + [(\sqrt{2}\sqrt{1-q^2} + \sqrt{2}\sqrt{1+q^2} + 2\nu_1)\sqrt{1-q^4} + 2\nu_1\nu_2]\beta \\ + [1 - q^4 + 2(1-\nu_2)\sqrt{1-q^4} - \nu_2^2]\beta^2 < 0, \qquad \Leftrightarrow \text{flexural Stonley wave.} \end{aligned} \tag{38}$$

This condition is satisfied if β is small with all the other parameters held fixed. Therefore, there are clearly circumstances under which flexural Stoneley waves can exist, and resonance is possible. However, for other combinations of plate parameters the condition (38) is not met and resonance will not occur. For example, the real and imaginary parts of the determinant of $\mathbf{Z}^{(dry)}$ of equation (28) are plotted as a function of the non-dimensional frequency parameter,

$$\Omega \equiv k^2/\kappa_1^2, \tag{39}$$

in Figure 2 for identical steel plates. The presence of the zero is apparent. The same plots for a pair of dissimilar thickness steel plates ($\alpha = 2$) in Figure 3 shows that there is no zero for the imaginary part, and hence no resonance. This is in agreement with the fact that the condition (38) is not met by the plate parameters of Figure 3. It is interesting to note that flexural edge waves along the free edge of a plate are always possible [12, 13], just as their analog in bulk elasticity always exists, i.e., the Rayleigh wave.

4. THE FLUID-LOADED SOLUTION

The procedure for developing the solution is to first represent it as a transform, and then apply the jump conditions (14). For uniform plates with line loads one can write down the transform by inspection, however, the major difficulty here is finding the transform itself. Once it is found one can perform asymptotic approximations to look at specific

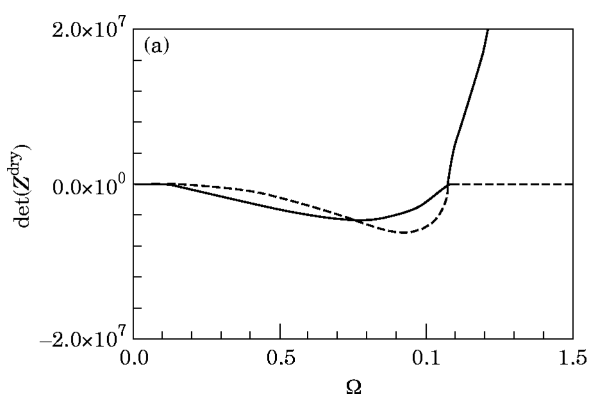

Figure 2. The determinant of the impedance matrix $\mathbf{Z}^{(dry)}$, equation (28), for a uniform steel plate where the phase of the loading is given by $k_y = k \sin 75°$. ——, Real part; - - -, imaginary part.

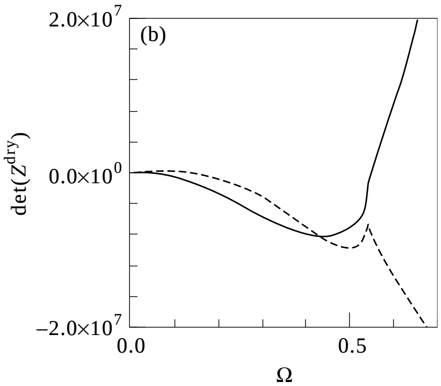

Figure 3. The frequency dependence of det $\mathbf{Z}^{(dry)}$ for a pair of steel plates with $\alpha=2$ which denotes a thickness change of 100% with the same phase of loading given in Figure 2.

physical contributions. We begin with a formal expression for the transform, and then deal with the jump conditions at the junction.

4.1. FORMAL SOLUTION

We first represent the pressure as a transform, or more precisely, we assume the solution has the form

$$\bar{p}(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{p}(\xi)\mathrm{e}^{[\mathrm{i}\xi x-\gamma(\xi)z]}\,\mathrm{d}\xi, \qquad \gamma(\xi)=(\xi^2-\bar{k}^2)^{1/2}. \tag{40}$$

$\gamma(\xi)$ is defined as an analytic function in the complex ξ-plane cut so that its real part is non-negative and on the real axis, $\gamma(\xi)=-i\sqrt{\bar{k}^2-\xi^2}$, or $\gamma(\xi)=\sqrt{\xi^2-\bar{k}^2}$, depending on whether $\xi^2<\bar{k}^2$ or $\xi^2>\bar{k}^2$, respectively. The transform solves equation (10) if $\tilde{p}(\xi)$ satisfies the dual equations

$$\int_{-\infty}^{\infty} D_1(\xi)\tilde{p}(\xi)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi=0, \qquad x<0; \qquad \int_{-\infty}^{\infty} D_2(\xi)\tilde{p}(\xi)\,\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi=0, \qquad x>0, \tag{41}$$

where

$$D_{1,2}(\xi)=1-\gamma(\xi)V_{1,2}(\xi), \qquad V_{1,2}(\xi)=a_{1,2}[\kappa_{1,2}^{-4}(\xi^2+k_y^2)^2-1]. \tag{42}$$

Define the quotient function K as

$$K(\xi)=D_1(\xi)/D_2(\xi) \tag{43}$$

and let $K^{\pm}(\xi)$ be the unique Wiener–Hopf factors, such that

$$K(\xi)=K^-(\xi)/K^+(\xi), \qquad K^-(-\xi)=1/K^+(\xi). \tag{44}$$

Thus, by definition, $K^{\pm}(\xi)$ are analytic in the half-planes $\mathscr{H}^{\pm}$. Explicit formulae for $K^+(\xi)$ are given in Appendix A, based upon a factorization method recently developed by Norris, Relinsky and Wickham [14]. Also, define the generalized dispersion function,

$$G(\xi)\equiv D_2(\xi)/K^+(\xi)=D_1(\xi)/K^-(\xi). \tag{45}$$

The general homogeneous solution of the dual integral equations (41) must be of the form

$$\tilde{p}(\xi)=A(\xi)/G(\xi), \tag{46}$$

where $A(\xi)$ is any polynomial, of degree q, say. For every polynomial we thus have a q-parameter family of outgoing pressure fields consistent with Helmholtz's equation and the plate boundary conditions (4). The precise form of the polynomial remains to be determined, and as one might expect, it depends upon the proper application of the junction conditions at $x=0$.

The pressure and the transverse plate deflection may now be expressed as

$$\bar{p}(x,z)=A\left(-i\frac{\partial}{\partial x}\right)p_0(x,z), \qquad \rho\omega^2\bar{w}(x)=A\left(-i\frac{\mathrm{d}}{\mathrm{d}x}\right)w_0(x), \tag{47a, b}$$

where the two fundamental potentials $\bar{p}_0(x,z)$, and $\bar{w}_0(x)$ are

$$p_0(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{(\mathrm{i}\xi x-\gamma z)}}{G(\xi)}\,\mathrm{d}\xi, \qquad w_0(x)=\frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma(\xi)}{G(\xi)}\,\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi. \tag{48}$$

4.2. BEHAVIOR OF w_0 NEAR THE JOIN

Before dealing with the junction conditions we need to know how $w_0(x)$, and hence $\bar{w}(x)$, behaves on either side of the join. In particular, some of its higher derivatives may be discontinuous. We will demonstrate in this subsection that

$$w_0(x)=\sum_{n=0}^{7}\lambda_n^{\pm}\frac{(ix)^n}{n!}+O(x^8\log|x|), \qquad x\gtrless 0, \tag{49}$$

where the terms $\lambda_n^{\pm}$ are given in Appendix B. We start with $\tilde{w}_0^+$, $\tilde{w}_0^-$, the half-line transforms of w_0, defined as

$$\tilde{w}_0^+(\xi)=\int_{-\infty}^{0}w_0(x)\,\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x, \qquad \tilde{w}_0^-(\xi)=\int_{0}^{\infty}w_0(x)\,\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x. \tag{50}$$

These are also the analytic partitions of the full transform, $\tilde{w}_0$, which follows from equation (48), with

$$\tilde{w}_0(\xi)=\tilde{w}_0^+(\xi)+\tilde{w}_0^-(\xi), \tag{51}$$

and such that $\tilde{w}_0^+(\xi)$ is analytic in $\mathscr{H}^+$ and $\tilde{w}_0^-(\xi)$ is analytic in $\mathscr{H}^-$. The inverse transforms associated with $\tilde{w}_0^+$ and $\tilde{w}_0^-$ vanish for $x>0$ and $x<0$, respectively. Hence, these can be used to find the behavior of $\tilde{w}_0$ on both sides of $x=0$.

In order to find the half-transforms we first write $\tilde{w}_0$ as

$$\tilde{w}_0(\xi)=-\gamma(\xi)/G(\xi)=[K^+(\xi)-K^-(\xi)]/P^*(\xi), \tag{52}$$

where $P^*(\xi)$ is the quartic

$$P^*(\xi)=V_2(\xi)-V_1(\xi)=P_0^*[(\xi^2+k_y^2)^2-\zeta_0^4], \tag{53}$$

with

$$P_0^*=\frac{B_2-B_1}{\rho\omega^2}, \qquad \zeta_0^4=\frac{\Delta m}{\Delta B}\,\omega^2=\kappa_1^4\left(\frac{\alpha-1}{\beta-1}\right), \tag{54}$$

and α and β are defined in equation (33). The identity (52) follows from the two equations (45) by elimination of γ. The zeros of $P^*(\xi)$ are $\pm\zeta_1$ and $\pm\zeta_2$, where

$$\zeta_1^2+\zeta_2^2=-2k_y^2, \qquad \zeta_1^2\,\zeta_2^2=k_y^4-\zeta_0^4. \tag{55}$$

Note the similarity to equations (26) for the four "dry" wavenumbers. Many of the subsequent equations are simplified by definite choices for the roots. We therefore define them as

$$\zeta_1^2 = \zeta_0^2 - k_y^2, \qquad \zeta_2^2 = -\zeta_0^2 - k_y^2. \tag{56}$$

The roots depend on the wavenumbers ζ_0 and k_y, and they are distinct ($\zeta_1 \neq \zeta_2$) as long as $\zeta_0^4 \neq 0$. We assume this to be the case, for simplicity.

We can now determine the partition functions from equation (52) by adding and subtracting poles at the roots of $P^*(\xi)$, with suitable residues. Thus,

$$\tilde{w}_0^{\pm}(\xi) = \pm \frac{K^{\pm}(\xi)}{P^*(\xi)} \mp \sum_{n=1}^{2} \left[\frac{u_n^+}{\xi - \zeta_n} + \frac{u_n^-}{\xi + \zeta_n} \right], \tag{57}$$

where we have assumed, with no loss in generality, that the roots ζ_1 and ζ_2 of $P^* = 0$ lie in $\mathscr{H}^+$, and consequently the remaining pair of roots, $-\zeta_1$ and $-\zeta_2$, are in $\mathscr{H}^-$. Also,

$$u_n^{\pm} = \text{residue of} \left[\frac{K^{\pm}(\xi)}{P^*(\xi)} \right]_{\xi = \pm\zeta_n}, \qquad n = 1, 2, \tag{58}$$

which can be evaluated as

$$u_n^{\pm} = \pm [4\zeta_n(\zeta_n^2 + k_y^2) P_0^*]^{-1} \, \mathrm{e}^{\pm\sigma_n}, \qquad n = 1, 2. \tag{59}$$

The numbers $\sigma_{1,2}$ are defined by

$$K^+(\zeta_n) = e^{\sigma_n}, \qquad n = 1, 2, \tag{60}$$

and the fact that $K(\pm\zeta_n) = 1$ has been used in reducing $u_n^{\pm}$. Equations (57) can now be expanded about the point at infinity. Omitting the details, which are given in Appendix B, we find

$$\tilde{w}_0^{\pm}(\xi) = \pm i \sum_{n=0}^{7} \lambda_n^{\mp} \xi^{-(n+1)} + O(\xi^{-9} \log \xi), \qquad |\xi| \to \infty. \tag{61}$$

The first five coefficients in these expansions are given in equation (B9). It is then a simple matter to relate these expansions to the asymptotic expansions of $w_0(x)$ at $x = \pm 0$, *viz*, equation (49). We note that $\lambda_n^+ = \lambda_n^-$ for $n = 0$ to $n = 2$, and hence $w_0(x)$ and its first two derivatives are continuous at $x = 0$.

4.3. DETERMINATION OF A(ξ)

The fact that there are four edge conditions to be satisfied, i.e., continuity of $\bar{w}$ and $\bar{w}'$ and equations (14), suggests that we try

$$A(\xi) = \sum_{n=0}^{3} A_n \xi^n. \tag{62}$$

The displacement near the origin therefore follows from equations (47b), (49), and (62), as

$$\rho\omega^2 \bar{w}(x) = \sum_{n=0}^{4} \left(\sum_{k=0}^{3} \lambda_{j+n}^{\pm} A_j \right) \frac{(ix)^n}{n!} + O(x^5 \log |\mathrm{x}|), \qquad x \gtrless 0. \tag{63}$$

Kinematic continuity at the plate junction required that both $\bar{w}(x)$ and its first derivative be continuous across $x = 0$, implying

$$\sum_{j=0}^{3} (\lambda_j^+ - \lambda_j^-) A_j = 0, \qquad \sum_{j=0}^{3} (\lambda_{j+1}^+ - \lambda_{j+1}^-) A_j = 0, \tag{64}$$

respectively. However, the fact that $\lambda_j^+ = \lambda_j^-$ for $j=0$, 1, and 2, implies $A_2 = A_3 = 0$, and we therefore have

$$A(\xi) = A_0 + A_1 \xi. \tag{65}$$

We may now apply the jump conditions at the junction. Thus, equations (14) become, respectively,

$$\begin{aligned}&\{B_2\lambda_2^+ - B_1\lambda_2^- + k_y^2[B_2(1-\eta_2)\lambda_0^+ - B_1(1-\eta_1)\lambda_0^-]\}A_0\\ &\quad + \{B_2\lambda_3^+ - B_1\lambda_3^- + k_y^2[B_2(1-\eta_2)\lambda_1^+ - B_1(1-\eta_1)\lambda_1^-]\}A_1 = -\rho\omega^2 M_0,\end{aligned} \tag{66a}$$

$$\begin{aligned}&\{B_2\lambda_3^+ - B_1\lambda_3^- + k_y^2[B_2(1+\eta_2)\lambda_1^+ - B_1(1+\eta_1)\lambda_1^-]\}A_0\\ &\quad + \{B_2\lambda_4^+ - B_1\lambda_4^- + k_y^2[B_2(1+\eta_2)\lambda_2^+ - B_1(1+\eta_1)\lambda_2^-]\}A_1 = i\rho\omega^2 F_0,\end{aligned} \tag{66b}$$

These can be simplified using the explicit expressions given in equation (B9), yielding the matrix system

$$(\mathbf{N}_1 + \mathbf{N}_2)\begin{bmatrix} A_1 \\ A_0 \end{bmatrix} = -2i\begin{bmatrix} -M_0 \\ iF_0 \end{bmatrix}, \tag{67}$$

where

$$N_n(\vartheta) = \begin{bmatrix} \left(1 - \dfrac{\vartheta k_y^2}{k_y^2 + \zeta_n^2}\right)\cosh\sigma_n & \left(1 - \dfrac{\vartheta k_y^2}{k_y^2+\zeta_n^2}\right)\zeta_n^{-1}\sinh\sigma_n \\ \left(1 + \dfrac{\vartheta k_y^2}{k_y^2+\zeta_n^2}\right)\zeta_n \sinh\sigma_n & \left(1 + \dfrac{\vartheta k_y^2}{k_y^2+\zeta_n^2}\right)\cosh\sigma_n \end{bmatrix} = \mathbf{J}_n(\vartheta)\mathbf{M}_n, \tag{68}$$

and

$$\mathbf{J}_n(\vartheta) = \begin{bmatrix} 1 - \dfrac{\vartheta k_y^2}{k_y^2+\zeta_n^2} & 0 \\ 0 & 1 + \dfrac{\vartheta k_y^2}{k_y^2+\zeta_n^2} \end{bmatrix}, \qquad \mathbf{M}_n^{\pm 1} = \begin{bmatrix} \cosh\sigma_n & \pm\zeta_n^{-1}\sinh\sigma_n \\ \pm\zeta_n\sinh\sigma_n & \cosh\sigma_n \end{bmatrix}, \tag{69}$$

for $n=1$ and 2. The number ϑ defines an averaged difference in the material properties of the two plates:

$$\vartheta \equiv (\eta_2 B_2 - \eta_1 B_1)/(B_2 - B_1) = 1 - (v_2 B_2 - v_1 B_1)/(B_2 - B_1). \tag{70}$$

Note that the determinant of $\mathbf{M}_n$ is unity

$$\det \mathbf{N}_1(\vartheta) = \det \mathbf{N}_2(\vartheta) = 1 - \vartheta^2 k_y^4/\zeta_0^4, \tag{71}$$

and also that

$$\mathbf{J}_2(\vartheta) = \mathbf{J}_1(-\vartheta). \tag{72}$$

Solving for the coefficients A_0 and A_1 using equations (67) and (72) gives

$$\begin{bmatrix} A_1 \\ A_0 \end{bmatrix} = -2i(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\begin{bmatrix} -M_0 \\ iF_0 \end{bmatrix} = -2i\,\frac{[\mathbf{M}_1^{-1}\mathbf{J}_2(\vartheta) + \mathbf{M}_2^{-1}\mathbf{J}_1(\vartheta)]}{\det[\mathbf{N}_1(\vartheta) + \mathbf{N}_2(\vartheta)]}\begin{bmatrix} -M_0 \\ iF_0 \end{bmatrix}. \tag{73}$$

We note that equations (63), (65) and (B9) imply that

$$\begin{bmatrix} \bar{w}(0) \\ -i\bar{w}'(0) \end{bmatrix} = \frac{-i}{2\zeta_0^2}\left(\frac{\mathbf{M}_2 - \mathbf{M}_1}{B_2 - B_1}\right)\begin{bmatrix} A_1 \\ A_0 \end{bmatrix}, \tag{74}$$

and inverting equation (74) gives A_0 and A_1,

$$\begin{bmatrix} A_1 \\ A_0 \end{bmatrix} = i2\zeta_0^2(B_2 - B_1)[\det(\mathbf{M}_2 - \mathbf{M}_1)]^{-1}(\mathbf{M}_2^{-1} - \mathbf{M}_1^{-1}) \begin{bmatrix} \bar{w}(0) \\ -i\bar{w}'(0) \end{bmatrix}. \tag{75}$$

The admittance matrix $\mathbf{Y}$ of equation (21) now follows from equations (73) and (74), while the impedance can be obtained using equations (22), (67) and (75). Thus,

$$\mathbf{Y} = [\zeta_0^2(B_2 - B_1)\det[\mathbf{N}_1(\vartheta) + \mathbf{N}_2(\vartheta)]]^{-1}(\mathbf{M}_1 - \mathbf{M}_2)[\mathbf{M}_1^{-1}\mathbf{J}_2(\vartheta) + \mathbf{M}_2^{-1}\mathbf{J}_1(\vartheta)], \tag{76a}$$

$$\mathbf{Z} = \zeta_0^2(B_2 - B_1)[\det(\mathbf{M}_2 - \mathbf{M}_1)]^{-1}[\mathbf{J}_1(\vartheta)\mathbf{M}_1 + \mathbf{J}_2(\vartheta)\mathbf{M}_2](\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1}). \tag{76b}$$

Expanding these matrix products, we find after some algebra that $\mathbf{Y}$ and $\mathbf{Z}$ are in fact proportional to one another, as expected, and

$$\mathbf{Y} = [(B_2 - B_1)\tfrac{1}{2}\det[\mathbf{N}_1(\vartheta) + \mathbf{N}_2(\vartheta)]]^{-1}\mathbf{Q}(\vartheta), \tag{77a}$$

$$\mathbf{Z} = -\omega^2(m_2 - m_1)[\tfrac{1}{2}\det(\mathbf{M}_2 - \mathbf{M}_1)]^{-1}\mathbf{Q}(\vartheta), \tag{77b}$$

where

$$\mathbf{Q}(\vartheta) = \frac{1}{2\zeta_0^2}\{\mathbf{M}_1\mathbf{M}_2^{-1} - \mathbf{M}_2\mathbf{M}_1^{-1} + \tfrac{1}{2}\det(\mathbf{M}_2 - \mathbf{M}_1)[\mathbf{J}_2(\vartheta) - \mathbf{J}_1(\vartheta)]\}. \tag{78}$$

These expressions can be simplified using equations (56) and (68)–(69), yielding

$$\tfrac{1}{2}\det(\mathbf{M}_2 \pm \mathrm{M}_1) = 1 \pm \cosh\sigma_1\cosh\sigma_2 \pm \frac{k_y^2}{\zeta_1\zeta_2}\sinh\sigma_1\sinh\sigma_2, \tag{79a}$$

$$\det[\mathbf{N}_1(\vartheta) + \mathbf{N}_2(\vartheta)] = \det(\mathbf{M}_2 + \mathbf{M}_1) - \vartheta^2\frac{k_y^4}{\zeta_0^4}\det(\mathbf{M}_2 - \mathbf{M}_1) - \vartheta\frac{4k_y^2}{\zeta_1\zeta_2}\sinh\sigma_1\sinh\sigma_2. \tag{79b}$$

Therefore,

$$\mathbf{Q}(\vartheta) = \begin{bmatrix} \left(1 - \vartheta\dfrac{k_y^4}{\zeta_0^4}\right)\dfrac{\sinh\sigma_1\sinh\sigma_2}{\zeta_1\zeta_2} + \vartheta\dfrac{k_y^2(1 - \cosh\sigma_1\cosh\sigma_2)}{\zeta_0^4} & \dfrac{1}{\zeta_0^2}\left(\dfrac{\sinh\sigma_1\cosh\sigma_2}{\zeta_1} - \dfrac{\sinh\sigma_2\cosh\sigma_1}{\zeta_2}\right) \\ \dfrac{1}{\zeta_0^2}(\zeta_1\sinh\sigma_1\cosh\sigma_2 - \zeta_2\sinh\sigma_2\cosh\sigma_1) & -\left(1 - \vartheta\dfrac{k_y^4}{\zeta_0^4}\right)\dfrac{\sinh\sigma_1\sinh\sigma_2}{\zeta_1\zeta_2} - \vartheta\dfrac{k_y^2(1 - \cosh\sigma_1\cosh\sigma_2)}{\zeta_0^4} \end{bmatrix}. \tag{80}$$

It may then be checked by explicit calculation that the matrices $\mathbf{Y}$ and $\mathbf{Z}$ of equation (77) satisfy $\mathbf{YZ} = \mathbf{I}$.

5. DISCUSSION

5.1. LIGHT FLUID-LOADING

The general results for the fluid-loaded plates in equation (77) should reduce to the "dry" results of equation (30) in the appropriate limit. We can actually perform the limiting process and check that the equations agree by the following method. Consider the fluid density tending to zero while the fluid wave speed remains finite. The lengths $a_{1,2}$ are then much greater than all others. Thus, in the limit as $\rho \to 0$, we have $K \to V_1/V_2$ from equations (42) and (43). The analytic factorization is straightforward, with

$$K^{+}(\xi)=\sqrt{\beta}\left(\frac{\xi+\xi_{21}}{\xi+\xi_{11}}\right)\left(\frac{\xi+\xi_{22}}{\xi+\xi_{12}}\right), \qquad K^{-}(\xi)=\frac{1}{\sqrt{\beta}}\left(\frac{\xi-\xi_{11}}{\xi-\xi_{21}}\right)\left(\frac{\xi-\xi_{12}}{\xi-\xi_{22}}\right). \tag{81}$$

Then using the identity $K(\zeta_n)=1$, equations (26), (56), and some algebra give

$$\cosh\sigma_n=\frac{\sqrt{\beta}}{\zeta_0^4-\kappa_1^4}\{\zeta_n^4+[\xi_{11}\xi_{12}+\xi_{22}\xi_{21}-(\xi_{11}+\xi_{12})(\xi_{22}+\xi_{21})]\zeta_n^2+\xi_{11}\xi_{12}\xi_{22}\xi_{21}\}, \tag{82a}$$

$$\sinh\sigma_n=\frac{\sqrt{\beta}\zeta_n}{\zeta_0^4-\kappa_1^4}\{(\xi_{22}+\xi_{21}-\xi_{11}-\xi_{12})\zeta_n^2+(\xi_{22}+\xi_{21})\xi_{11}\xi_{12}-(\xi_{11}+\xi_{12})\xi_{22}\xi_{21}\}. \tag{82b}$$

Substitution of these identities into equations (77b), (79a) and (80) should yield the identity

$$\lim_{\rho\to 0}\mathbf{Z}=\mathbf{Z}^{(dry)}. \tag{83}$$

We have not been able to verify this owing to the excessive algebra involved. However, we checked the equivalence numerically by taking a wide range of all the parameters for the dry plates and the frequencies. The agreement found provides a confirmation of the validity of the general expression for $\mathbf{Z}$.

5.2. HEAVY FLUID-LOADING

The heavy fluid-loading limit for a two-dimensional configuration ($k_y=0$) occurs at low frequency when the fluid-loaded plate wavenumbers far exceed all others in magnitude. A similar limit applies to the present problem if we assume that the transverse wavenumber, k_y, scales linearly with frequency. For instance, if the excitation is acoustic, then $k_y=k\sin\theta$ for some $-\pi/2<\theta<\pi/2$. Under these circumstances the low frequency asymptotic limit reduces to the strictly two-dimensional heavy fluid-loading regime. Let B, a and κ be the parameters for either plate, then the low frequency, heavy fluid regime is defined by $\lambda \gg \kappa$, where

$$\lambda\equiv(\kappa^4/a)^{1/5}=(\rho\omega^2/B)^{1/5}. \tag{84}$$

Crighton [2] argued convincingly that this can be relaxed to the requirement $\kappa \gg k$. In this regime we have $D(\xi)\approx 1-\lambda^{-5}\xi^4(\xi^2)^{1/2}$, where the natural limit of the square root function γ is $(\xi^2)^{1/2}=\xi\,\mathrm{sgn}\,\mathrm{Re}\,\xi$ [15]. The equation $D(\xi)\bar{D}(\xi)=0$ (see Appendix A) reduces to $\xi^{10}=\lambda^{10}$. The roots in $\mathscr{H}^{+}$ are $\xi_n=\lambda z^{(n-1)}$, $n=1,\ldots,5$, where $z=e^{i\pi/5}$, and thus $n=1$, 3, and 4 correspond to zeros of $D(\xi)$ while $n=2$ and 5 arise from $\bar{D}(\xi)=0$. We reiterate that the present limit coincides with the strictly two-dimensional low frequency limit, but not with the general three-dimensional low frequency regime, in general. The dependence of k_y upon frequency is the crucial factor. Thus, the following asymptotics are not relevant to the oblique scattering of a low frequency flexural wave from a junction.

The limiting form of the split function $K^{+}(\xi)$ can be determined using some results of Crighton and Innes [15] for the factorization of $D(\xi)=D^{+}(\xi)D^{-}(\xi)$ in the same limit, where $D^-(-\xi) = D^+(\xi)$. Thus, for $\xi = O(\kappa)$ we have

$$D^+(\xi) = 1 + d_m \frac{\xi}{\lambda} + d_m^2 \frac{\xi^2}{2\lambda^2} + (d_m^3 + 2d_f)\frac{\xi^3}{6\lambda^3} + \cdots, \tag{85}$$

where

$$d_m = 1 - i \cot \pi/5, \qquad d_f = 1 + i \tan \pi/10. \tag{86}$$

Terms up to quadratic in equation (85) are given explicitly in equation (A33) of reference [15], while the cubic term follows from equations (A29)–(A32) of reference [15] after some manipulation. The expansion (85) may be rewritten:

$$D^+(\xi) = \left(1 + d_f \frac{\xi^3}{3\lambda^3} + \cdots\right) \mathrm{e}^{d_m \xi/\lambda}, \tag{87}$$

from which we deduce that

$$K^+(\xi) = \left(1 + d_f \frac{\xi^3}{3} \Delta(1/\lambda^3) + \cdots\right) \mathrm{e}^{d_m \xi \Delta(1/\lambda)}, \tag{88}$$

where $\Delta f \equiv f_2 - f_1$ is the difference in the parameter f for the two plates.

All things being equal, as the frequency diminishes the special wavenumbers ζ_1 and ζ_2 scale with κ_1 and κ_2. Hence, $\zeta_n = O(\kappa)$ and we can use the previous asymptotic expressions to simplify quantities like cosh σ_n. The low frequency limit of the admittance matrix then follows from equations (60), (77a), and (88) as, to leading order,

$$\mathbf{Y} = \begin{bmatrix} \dfrac{d_m^2}{2(\rho\omega^2)^{2/5}} \dfrac{(\Delta B^{1/5})^2}{\Delta B} & \dfrac{1}{3(\rho\omega^2)^{3/5}}\left(d_\mathrm{f} \dfrac{\Delta \mathrm{B}^{3/5}}{\Delta \mathrm{B}} - d_\mathrm{m}^3 \dfrac{(\Delta \mathrm{B}^{1/5})^3}{\Delta \mathrm{B}}\right) \\ \dfrac{d_m}{(\rho\omega^2)^{1/5}} \dfrac{\Delta \mathrm{B}^{1/5}}{\Delta \mathrm{B}} & -\dfrac{d_m^2}{2(\rho\omega^2)^{2/5}} \dfrac{(\Delta B^{1/5})^2}{\Delta B} \end{bmatrix}, \qquad \omega \to 0. \tag{89}$$

Note that $\arg(-d_m^3) = \arg d_f$. and hence the argument of each term in $\mathbf{Y}$ in equation (89) is fixed, regardless of the plate properties.

The case of a uniform plate is a simple consequence of this result, by virtue of the limit $\Delta B^\alpha/\Delta B \to \alpha B^\alpha/B$ as $\Delta B \to 0$. We then recover Crighton's asymptotic approximations for a uniform plate for the same frequency limit [2], or using equation (23a),

$$\mathbf{Y}^{(p)} = \frac{\omega}{5B\lambda^2} \begin{bmatrix} d_f \lambda^{-1} & 0 \\ 0 & d_m \lambda \end{bmatrix}, \qquad \text{identical plates, } \omega \to 0. \tag{90}$$

5.3. PLATES OF EQUAL STIFFNESS

We next consider the limit of $B_2 \to B_1$ ($\beta \to 1$), *but assume that the areal densities remain distinct*, $m_1 - m_2 \neq 0$. In this limit the wavenumbers ζ_1 and ζ_2 become large relative to all others, and appropriate approximations can be made. Thus, equations (60) and (B1) imply that

$$\cosh \sigma_n = 1 + \frac{(\Delta\mu_1)^2}{2\zeta_n^2} + \frac{\Delta\mu_1 \Delta\mu_3 + (\Delta\mu_1)^4/24}{\zeta_n^4} + O(\zeta_n^{-6}), \tag{91a}$$

$$\frac{1}{\zeta_n} \sinh \sigma_n = \frac{\Delta\mu_1}{\zeta_n^2} + \frac{\Delta\mu_3 + (\Delta\mu_1)^3/6}{\zeta_n^4} + O(\zeta_n^{-5}), \tag{91b}$$

for $n=1$ and 2, where $\Delta\mu_1$ and $\Delta\mu_3$ are defined by equations (B2) and (B3). These asymptotic approximations allow us to take the limit of each element in the $\mathbf{Y}$ matrix of equation (77a), with the precise result

$$\mathbf{Y}=\frac{1}{\omega^2\Delta m}\begin{bmatrix} -\frac{1}{2}(\Delta\mu_1)^2 & \Delta\mu_1 \\ \Delta\mu_3-\frac{1}{3}(\Delta\mu_1)^3 & \frac{1}{2}(\Delta\mu_1)^2 \end{bmatrix}, \qquad \text{for } B_1=B_2. \tag{92}$$

The numbers μ_1 and μ_3 on either side of the join follow from equation (B3) as

$$\mu_j=\frac{1}{j}\sum_{n=1}^{5}(\xi_n)^j\left(\frac{1}{2}+\frac{s_n\theta_n}{\pi}\right), \qquad j=1 \text{ and } 3, \tag{93}$$

where $\xi_1, \ldots, \xi_5$ are the five zeros of $P(\xi)=D(\xi)\bar{D}(\xi)$ in $\mathscr{H}^+$, i.e., they solve

$$[(\xi^2+k_y^2)^2-\kappa^4]^2(\xi^2-\bar{k}^2)=\lambda^{10}. \tag{94}$$

The complex angles $\theta_1, \ldots, \theta_5$, in equation (93) are defined in accordance with equation (A6) as $\theta_n=\cos^{-1}(\xi_n/\bar{k})$ and $s_n=1$ or -1, depending if ξ_n is a zero of $D(\xi)$ or $\bar{D}(\xi)$, respectively. Thus,

$$s_n=\gamma(\xi_n)V(\xi_n)=\gamma(\xi_n)[(\xi_n^2+k_y^2)^2-\kappa^4]/\lambda^5. \tag{95}$$

5.4. A UNIFORM PLATE

We may now consider the further limit of two completely identical plates, i.e., a single uniform plate of infinite extent. This is found by letting $m_2 \to m_1 \equiv m$ in equation (92), yielding for the physical admittance for identical plates,

$$\mathbf{Y}^{(p)}=\begin{bmatrix} \omega^{-1}\partial\mu_1/\partial m & 0 \\ 0 & \omega^{-1}\partial\mu_3/\partial m \end{bmatrix}, \tag{96}$$

The derivative may be effected by partial differentiation, using equation (93) and the fact that each root ξ_n solves equation (94). Equations (94) and (95) yield the identities $\partial\xi_n/\partial m=-2s_n\gamma(\xi_n)/\rho P'(\xi_n)$ and $\partial\theta_n/\partial m=-2is_n/\rho P'(\xi_n)$, while the latter combined with

$$\sum_{n=1}^{5}(\xi_n)^m/P'(\xi_n)=0, \qquad \text{for } m=1, 3 \text{ and } 5, \tag{97}$$

which is a consequence of the polynomial nature of $P(\xi)$, implies that the terms involving $\partial\theta_k/\partial m$ cancel. We finally obtain

$$\begin{aligned}\frac{1}{\omega}\frac{\partial\mu_j}{\partial m}&=\frac{1}{\omega}\sum_{n=1}^{5}\left(\frac{1}{2}+\frac{s_n\theta_n}{\pi}\right)(\xi_n)^{j-1}\frac{\partial\xi_n}{\partial m}\\ &=\frac{\omega}{B}\sum_{n=1}^{5}(\xi_n)^{j-2}\left[\frac{1/2+s_n\theta_n/\pi}{4(\xi_n^2+k_y^2)+s_n\lambda^5/\gamma^3(\xi_n)}\right], \qquad j=1 \text{ and } 3.\end{aligned} \tag{98}$$

Crighton [2] derived similar formulae for the force and moment admittance for the strictly two-dimensional case, while Photiadis [7] has recently solved the phased line loading problem. The expressions (98) agree with those obtained by Photiadis, and both reduce to Crighton's formulae when $k_y=0$. In Figure 4, the frequency dependence of the force and moment admittances are shown for two cases: (a) $k_y=0$; and (b) $k_y=k\sin 75^\circ$.

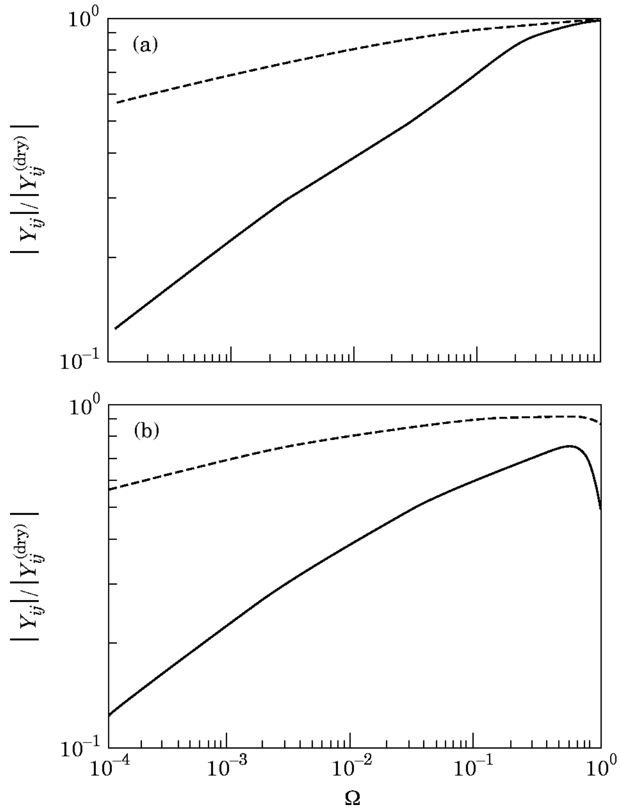

Figure 4. The fluid loaded admittances for a uniform steel plate in water as a function of the non-dimensional frequency parameter Ω of equation (39). The admittances are normalized with respect to their *in-vacuo* counterparts. (a) $k_y=0$; (b) $k_y=k\sin 75^\circ$. ——, Force admittance; - - -, moment.

We note that the numerical results of case (a) are in agreement with those obtained by Nayak [1].

5.5. TWO-DIMENSIONAL SIMPLIFICATION

The previous results simplify greatly when there is no y-dependence. Thus, setting $k_y=0$, equation (73) simplifies to

$$\begin{bmatrix} A_1 \\ A_0 \end{bmatrix} = -i(\cosh\sigma_1\cosh\sigma_2+1)^{-1}(\mathbf{M}_1^{-1}+\mathbf{M}_2^{-1})\begin{bmatrix} -M_0 \\ iF_0 \end{bmatrix}. \tag{99}$$

Hence, $A(\xi)$ easily follows from equations (65) and (99), as

$$A(\xi)=F_0\left[\frac{(\cosh\sigma_1+\cosh\sigma_2)-(\zeta_1^{-1}\sinh\sigma_1+\zeta_2^{-1}\sinh\sigma_2)\xi}{\cosh\sigma_1\cosh\sigma_2+1}\right].$$

$$+iM_0\left[\frac{(\cosh\sigma_1+\cosh\sigma_2)\xi-(\zeta_1\sinh\sigma_1+\zeta_2\sinh\sigma_2)}{\cosh\sigma_1\cosh\sigma_2+1}\right]. \tag{100}$$

Alternatively, another expression of $A(\xi)$ is obtained by using equation (75), which gives

$$A(\xi)=w'(0)(B_2-B_1)\left[\frac{(\zeta_1^2\cosh\sigma_1+\zeta_2^2\cosh\sigma_2)-(\zeta_1\sinh\sigma_1+\zeta_2\sinh\sigma_2)\xi}{\cosh\sigma_1\cosh\sigma_2-1}\right]$$
$$+iw(0)(B_2-B_1)\left[\frac{(\zeta_1^2\cosh\sigma_1+\zeta_2^2\cosh\sigma_2)\xi-(\zeta_1^3\sinh\sigma_1+\zeta_2^3\sinh\sigma_2)}{\cosh\sigma_1\cosh\sigma_2-1}\right]. \tag{101}$$

Finally, the admittance and impedance matrices for $ky=0$ are

$$\mathbf{Y}=[(B_2-B_1)(\cosh\sigma_1\cosh\sigma_2+1)]^{-1}\mathbf{Q}(0), \tag{102a}$$

$$\mathbf{Z}=\omega^2(m_2-m_1)[(\cosh\sigma_1\cosh\sigma_2-1)]^{-1}\mathbf{Q}(0), \tag{102b}$$

where $\mathbf{Q}(0)$ follows from equation (80).

5.6. NUMERICAL EXAMPLES

Some typical numerical results for the admittance functions are shown in Figures 5 and 6. All the curves shown were calculated using the algorithm for the split function K^+ outlined in Appendix A. Figures 5 and 6 show the force and moment admittances Y_{12} and Y_{21}, respectively, for two joined steel plates of different thicknesses $\alpha=2$ for both fluid loaded and *in-vacuo* conditions. Note that plates of the same material but with different thicknesses in the ratio α of equation (33) have *in-vacuo* flexural wavenumbers related by $\kappa_2=\kappa_1/\sqrt{\alpha}$. Figure 5 shows the two-dimensional ($k_y=0$) results whereas Figure 6 illustrates the effect of phased loading with $k_y=k\sin 75^\circ$. Note that for $k_y=0$, the magnitude of the fluid-loaded admittances are decreased in comparison to the *in-vacuo* results. The same general observation applies to the fluid-loaded admittance as compared to *in-vacuo* for non-zero k_y in Figure 6, except near the frequency $\kappa_2=k_y$. As discussed in section 3.3, the *in-vacuo* solution can exhibit resonance-type behaviour in this frequency range. As noted in section 3.3, the resonance phenomenon for the dry structure does not occur for the parameters chosen in Figure 6 (see the discussion of Figure 3 in section 3.3). It is interesting that both the force and moment admittances exhibit a peak or valley, respectively, near this frequency. The quasi-resonance effect is most apparent in the rapid variation of the phase occuring near the resonance frequency, as shown in Figures 6(b) and 6(d). The resonance behaviour is also observed in the fluid-loaded admittances but it is heavily damped.

6. CONCLUSIONS

We have derived explicit expressions for both the dry and wet admittance matrices at the junction of two plates subject to a line force and moment. The dry admittance can be singular for certain combinations of plates if the line forcing is in phase with a flexural Stoneley wave travelling along the drive line. The resonance is not possible with fluid loading, but there is a remnant of the phenomenon at frequencies which are close to phase-matching with structural waves propagating down the line. In this paper we have emphasized the intrinsic properties of the admittance matrix. The full strength of the results rests in their potential for solving scattering and radiation problems. For example, in a separate paper we consider the scattering of a flexural wave from an internally attached rib at the junction of two plates. The admittance matrix derived here can be used to determine all the interaction mechanisms for waves incident on a three-member junction. Scattering coefficients can be found in a semi-analytic fashion, which are easy to compute.

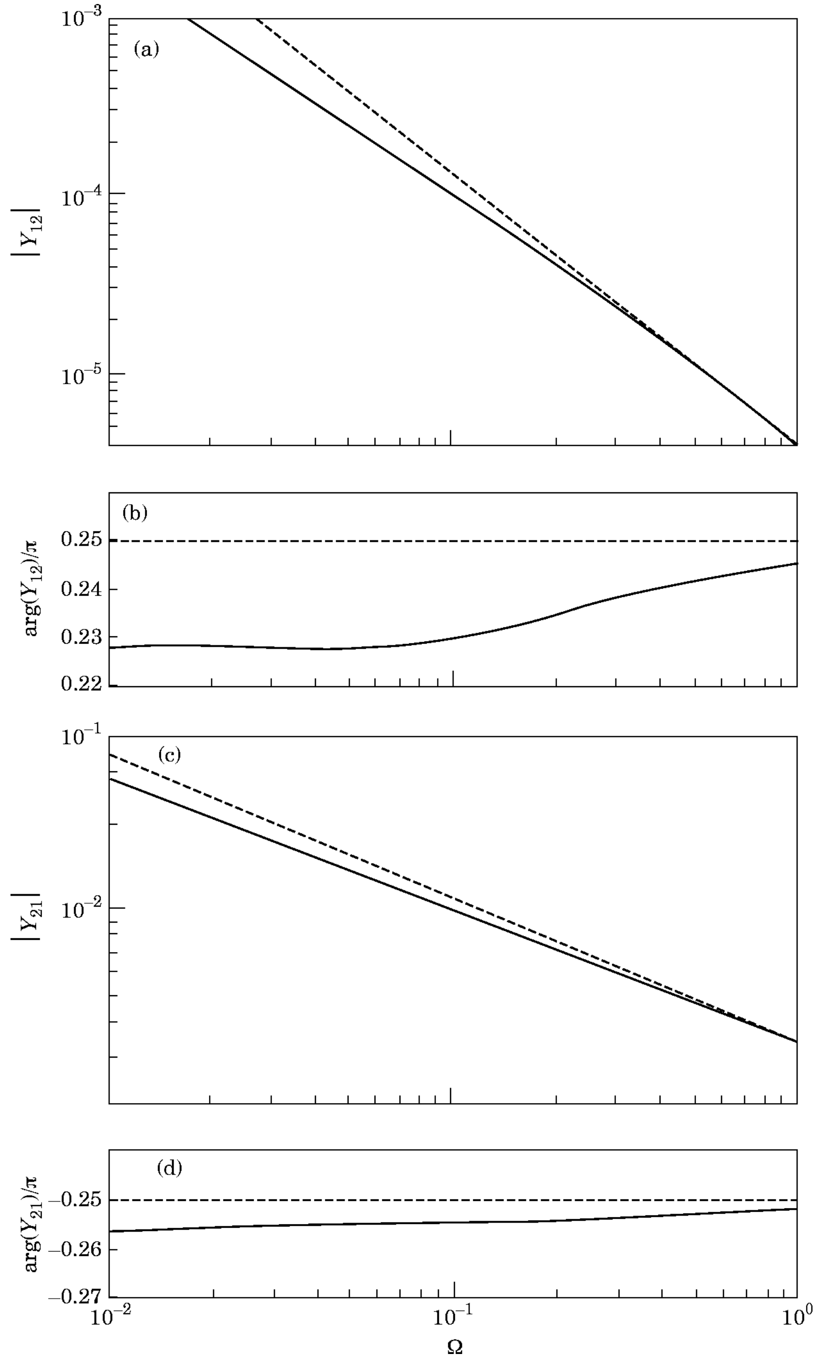

Figure 5. The frequency dependence of the force and moment admittances, **Y** of equation (21), for $k_y = 0$. The system comprises a pair of steel plates with $h_2 = 2h_1$. (a) $|Y_{12}|$, (b) arg (Y_{12}), (c) $|Y_{21}|$, and (d) arg (Y_{21}). ——, Fluid-loading (water); - - -, corresponding *in-vacuo* results.

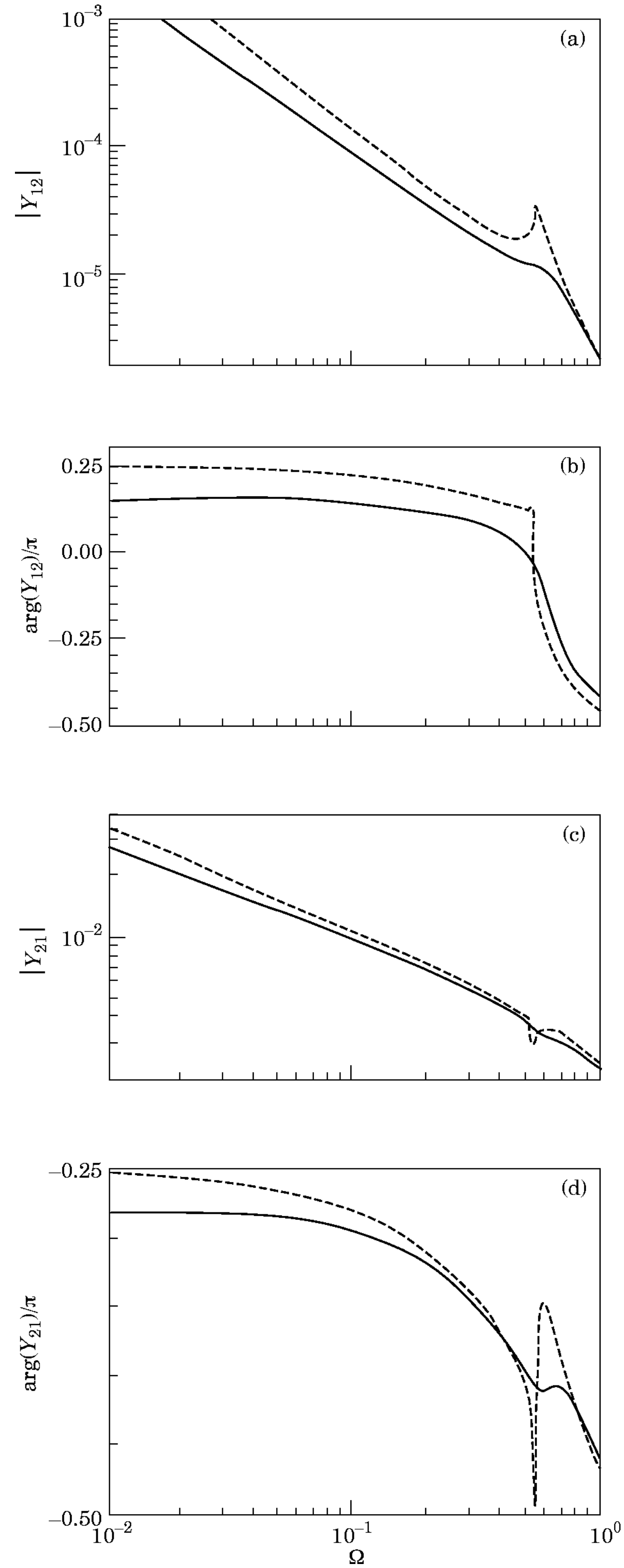

Figure 6. The same parameters as in Figure 5, but $k_y = k \sin 75^\circ$.

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REFERENCES

1. P. R. NAYAK 1970 *Journal of the Acoustical Society of America* **47**, 191–201. Line admittance of infinite isotropic fluid-loaded plates.
2. D. G. CRIGHTON 1972 *Journal of Sound and Vibration* **20**, 209–218. Force and moment admittance of plates under arbitrary fluid loading.
3. G. W. BENTHIEN, M. A. GONZALEZ and J. W. YOUNG 1976 *Journal of Sound and Vibration* **47**, 136–137. Alternative analytic method for calculating fluid-loaded plate admittances.
4. D. G. CRIGHTON 1976 *Journal of Sound and Vibration* **47**, 137–138. Author's reply.
5. P. W. SMITH 1976 *Journal of Sound and Vibration* **48**, 45–56. Line input admittance of infinite isotropic fluid-loaded plates.
6. P. FILIPPI 1985 *Journal of Sound and Vibration* **100**, 69–81. Sound radiation by baffled plates and related boundary integral equations.
7. D. M. PHOTIADIS 1995 *Journal of the Acoustical Society of America* **97,** 1404–1414. Transport of energy across a discontinuity by fluid-loading.
8. B. NOBLE 1958 *Methods based on the Wiener–Hopf Technique*. New York: Pergamon.
9. P. R. BRAZIER-SMITH 1987 *Proceedings of the Royal Society of London*, **A409**, 115–139. The acoustic properties of two co-planar half-plane plates.
10. A. N. NORRIS and G. WICKHAM 1995 *Proceedings of the Royal Society of London*, **A451,** 631–656. Acoustic diffraction from the junction of two flat plates.
11. S. TIMOSHENKO 1940 *Theory of Plates and Shells*. New York: McGraw-Hill.
12. R. N. THURSTON and J. MCKENNA 1974 *IEEE Transactions on Sonics and Ultrasonics* **21**, 296–297. Flexural acoustic waves along the edge of a plate.
13. A. N. NORRIS 1994 *Journal of Sound and Vibration* **171**, 571–573. Flexural edge waves.
14. A. N. NORRIS, D. A. REBINSKY and G. WICKHAM (to be submitted). Acoustic diffraction from the junction of two curved shells.
15. D. G. CRIGHTON and D. INNES 1984 *Philosophical Transactions of the Royal Society of London* **A312**, 295–341. The modes, resonances and forced response of elastic structures under heavy fluid loading.

APPENDIX A: FACTORIZATION OF THE KERNEL

Factorization of K is simplified using the following functions for each plate:

$$R_j(\xi) = -\bar{D}_j/D_j, \qquad P_j(\xi) = \bar{D}_j D_j, \tag{A1}$$

where $j=1$ or 2, and

$$\bar{D}_j(\xi) = 1 + \gamma(\xi) V_j(\xi), \tag{A2}$$

is the "unphysical" dispersion relation corresponding to $D_j(\xi)$. Thus, for $j=1$ and 2, R_j is the acoustic reflection coefficient for a uniform plate and P_j is the rationalized form of the dispersion relation, given by equation (A1) or,

$$P_j(\xi) = 1 - a_j^2(\xi^2 - \bar{k}^2)[\kappa_{1,2}^{-4}(\xi^2 + k_y^2)^2 - 1]^2, \tag{A3}$$

and $\bar{k}$ is defined by equation (13). Let $\xi = \pm\xi_n^{(1,2)}$, $n=1, 2, \ldots, 5$, be the zeros of $P_{1,2}(\xi)$ such that $\xi_n^{(1,2)}$ are in $\mathscr{H}^+$, with no loss in generality. Following the procedure of Norris, Relinsky and Wickham [10, 14], we can derive two alternative expressions for K^+:

$$K^+(\xi)=\sqrt{\beta}\prod_{n=1}^{5}\left(\frac{\xi+\xi_n^{(2)}}{\xi+\xi_n^{(1)}}\right)^{1/2}\times\begin{cases}\left[\dfrac{R_1(0)}{R_2(0)}\right]^{1/4}\exp\left\{\displaystyle\int_0^{\xi}[\Phi_1(s)-\Phi_2(s)]\,\mathrm{d}s\right\},\\ \exp\left\{-\displaystyle\int_{\xi}^{\infty}[\Phi_1(s)-\Phi_2(s)]\,\mathrm{d}s\right\},\end{cases}\tag{A4}$$

where

$$\Phi_j(\xi)=\frac{1}{2\pi}\left[[\log R_j(\xi)]'\cos^{-1}(\xi/\bar{k})+\sum_{n=1}^{5}\frac{2s_n^{(j)}\xi_n^{(j)}\theta_n^{(j)}}{\xi^2-(\xi_n^{(j)})^2}\right],\qquad j=1,\ 2,\tag{A5}$$

and

$$\theta_n^{(j)}=\cos^{-1}(\xi_n^{(j)}/\bar{k}),\qquad s_n^{(j)}=\begin{cases}-1, & \text{if }\ \bar{D}_j(\xi_n^{(j)})=0,\\ +1, & \text{if }\ D_j(\xi_n^{(j)})=0.\end{cases}\tag{A6}$$

The branch of the inverse cosine is

$$\cos^{-1}(\xi/\bar{k})=\mathrm{i}\log[\xi/\bar{k}+\gamma(\xi)/\bar{k}],\tag{A7}$$

where the principal branch of the logarithm is taken, $-\pi<\operatorname{Im}\log(\cdot)<\pi$. Note that $\Phi_{1,2}$ possess no poles in the upper half-plane, $\mathscr{H}^+$, because the poles of $R_{1,2}$ are exactly cancelled by the poles in the second term in equation (A5).

The alternative formulae for K^+ in equation (A4) are designed for values of ξ near 0 and near ∞, respectively. The latter is used to derive the asymptotic behavior in Appendix B, while the former can be simplified further. Thus, following Norris and Wickham [10], we express the first term of equation (A4) in the integrand in partial fractions, and obtain (dropping the suffix j)

$$\Phi(\xi)=\frac{2}{\pi}\int_{\pi/2}^{\cos^{-1}(\xi/\bar{k})}\sum_{n=1}^{5}\frac{\theta\cos\theta\sin\theta_n-\theta_n\cos\theta_n\sin\theta}{\cos^2\theta-\cos^2\theta_n}s_n\,\mathrm{d}\theta.\tag{A8}$$

It is then a simple matter of rearrangement of terms to arrive at another form for K^+,

$$K^+(\xi)=\frac{\Pi'(1+\xi/\xi_n^{(2)})}{\Pi'(1+\xi/\xi_n^{(1)})}\left[\frac{D_2(0)}{D_1(0)}\right]^{1/2}\exp[\phi_1(\xi)-\phi_2(\xi)],\tag{A9}$$

where the products Π' are taken only over the three roots for which $s_n=1$, and

$$\phi(\xi)=\frac{1}{2\pi}\int_{\pi/2}^{\cos^{-1}(\xi/\bar{k})}\sum_{n=1}^{5}\left[\frac{\theta\sin\theta_n-\theta_n\sin\theta}{\cos\theta-\cos\theta_n}+\frac{\theta\sin\theta_n-(\pi-\theta_n)\sin\theta}{\cos\theta+\cos\theta_n}\right]s_n\,\mathrm{d}\theta.\tag{A10}$$

The form (A9) is used for practical calculations because it does not have any possibly ambiguous square root functions in the pre-exponent, and the integrand is smooth.

APPENDIX B: EXPANSION COEFFICIENTS

The asymptotic behavior of K^+ for large ξ follows from equations (A4) and (A5) as

$$K^+(\xi)=\sqrt{\beta}\exp\left[\sum_{j=1}^{4}\frac{\Delta\mu_j}{\xi^j}+O(\xi^{-5}\log\xi)\right],\tag{B1}$$

where

$$\Delta\mu_j=\mu_j^{(2)}-\mu_j^{(1)},\tag{B2}$$

and referring to Appendix A,

$$\mu_j^{(n)}=\frac{1}{2j}\sum_{k=1}^{5}\left\{\frac{1}{\pi}s_k^{(n)}\theta_k^{(n)}[(\xi_k^{(n)})^j-(-\xi_k^{(n)})^j]-(-\xi_k^{(n)})^j\right\}. \tag{B3}$$

Expanding the polynomials P_1 and P_2 of equation (A1), it can be shown that

$$\sum_{j=1}^{5}(\xi_j^{(n)})^2=\bar{k}^2-4k_y^2,\qquad \sum_{j=1}^{5}(\xi_j^{(n)})^4=\bar{k}^4+4k_y^4+4\kappa_n^4,\qquad n=1,\ 2, \tag{B4}$$

and hence,

$$\Delta\mu_2=0,\qquad \Delta\mu_4=\tfrac{1}{2}(\kappa_1^4-\kappa_2^4). \tag{B5}$$

The asymptotic expansions of $\tilde{w}_0^{\pm}$ are straightforward except for the terms involving $K^{\pm}(\xi)$; see equation (57). It follows from equations (53), (54), (B1) and (B5), that

$$\frac{K^{\pm}(\xi)}{P^*(\xi)}=\frac{\beta^{\pm 1/2}}{P_0^*}\sum_{j=0}^{4}\frac{\delta_j^{\pm}}{\xi^{(4+j)}}+O(\xi^{-9}\log\xi), \tag{B6}$$

where

$$\delta_0^{\pm}=1,\qquad \delta_1^{\pm}=\Delta\mu_1,\qquad \delta_2^{\pm}=\tfrac{1}{2}(\Delta\mu_1)^2-2k_y^2, \tag{B7a}$$

$$\delta_3^{\pm}=\Delta\mu_3+\tfrac{1}{6}(\Delta\mu_1)^3-2k_y^2\Delta\mu_1, \tag{B7b}$$

$$\delta_4^{\pm}=\tfrac{1}{24}(\Delta\mu_1)^4+\Delta\mu_1\Delta\mu_3+\zeta_0^4-k_y^2(\Delta\mu_1)^2+3k_y^4\pm\Delta\mu_4. \tag{B7c}$$

Hence, the expansions for $\bar{w}_0^{\pm}$ of equation (57) imply, using equations (49) and (61), that

$$\lambda_n^{\pm}=\sum_{m=1}^{2}i(\zeta_m)^n(u_m^{+}+(-1)^n u_m^{-})-\frac{i}{P_0^*}\beta^{\mp 1/2}\delta_{n-3}^{\mp}, \tag{B8}$$

for $n=0,\ldots,7$ and where $\delta_m^{\pm}=0$ for negative m. Then using equation (59) we deduce the following identities, which are all that we will need,

$$\lambda_0^{\pm}=\frac{i}{2\zeta_0^2P_0^*}\left(\frac{\sinh\sigma_1}{\zeta_1}-\frac{\sinh\sigma_2}{\zeta_2}\right),\qquad \lambda_1^{\pm}=\frac{i}{2\zeta_0^2P_0^*}(\cosh\sigma_1-\cosh\sigma_2), \tag{B9a, b}$$

$$\lambda_2^{\pm}=\frac{i}{2\zeta_0^2P_0^*}(\zeta_1\sinh\sigma_1-\zeta_2\sinh\sigma_2),\qquad \lambda_3^{\pm}=\frac{i}{2\zeta_0^2P_0^*}(\zeta_1^2\cosh\sigma_1-\zeta_2^2\cosh\sigma_2-2\beta^{\mp 1/2}), \tag{B9c, d}$$

$$\lambda_4^{\pm}=\frac{i}{2\zeta_0^2P_0^*}(\zeta_1^3\sinh\sigma_1-\zeta_2^3\sinh\sigma_2-2\Delta\mu_1\beta^{\mp 1/2}), \tag{B9e}$$