Acoustoelasticity theory and applications for fluid-saturated porous media

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(Received 17 April 1995; accepted for publication 6 May 1996)

The general theory for small dynamic motion superimposed upon large static deformation, or acoustoelasticity, is developed for isotropic fluid-filled poroelastic solids. Formulas are obtained for the change in acoustic wave speeds for arbitrary loading, both on the frame and the pore fluid. Specific experiments are proposed to find the complete set of third-order elastic moduli for an isotropic poroelastic medium. Because of the larger number of third-order moduli involved, seven as compared with three for a simple elastic medium, experiments combining open-pore, closed-pore, jacketed, and unjacketed configurations are required. The details for each type of loading are presented, and a set of possible experiments is discussed. The present theory is applicable to fluid-saturated, biconnected porous solids, such as sandstones or consolidated granular media.

PACS numbers: 43.25.Dc, 43.25.Ba, 43.20.Jr [MAB]

INTRODUCTION

The linear theory of poroelastic fluid-saturated media is a mature subject, with the classic studies of Frenkel[1] and especially of Biot[2] standing as the significant achievements in its modern development. Several different methods now exist for arriving at governing equations, including the theory of interpenetrating continua[3] and the method of ''homogenization'' based on two-scale asymptotic expansions.[4,5] Despite the fact that these approaches do not always yield the same governing equations and that further general theoretical studies are required, they have certainly proved useful in several branches of civil and geo-engineering and the Earth sciences,[6–8] and also in unexpected fields, such as low temperature physics[9] and pattern formation in polymer gels.[10] However, there are definite limitations to the linear theory. For example, a proper analysis of the dynamics of large amplitude sound in sediments or of small amplitude sound in rocks under large confining stress is obviously impossible within the framework of a linear theory. Even seismic waves of very small intensity at a large distance from their epicenter should be studied in the framework of a nonlinear theory because of the well-known effect of the accumulation of nonlinear distortions leading to the ''gradient catastrophe'' phenomenon.[11,12]

There is renewed interest in the nonlinear acoustics of rocks that is based partly on several observations of distinct nonlinear effects. These include the change of the *in situ* velocities of seismic waves[13–15] and direct measurement of harmonic distortion.[16,17] These experiments have clearly demonstrated that the *in situ* nonlinear elastic moduli of rocks, soil, and sediments are much greater than the linear ones. The nonlinear behavior of fluid-filled poroelastic solids is not as well understood, and further progress depends upon accurate experimental measurement of the effective nonlinear moduli of poroelastic media.

The determination of the linear moduli has been the subject of numerous experimental and theoretical studies.[7,18,19] An isotropic poroelastic medium has four static moduli that may be measured by a combination of stress-strain experiments on ''jacketed'' and ''unjacketed'' samples. The role of the jacketing is to constrain either the fluid pressure or its mass. The nonlinear moduli, in contrast, should be determined by measurement of essentially nonlinear effects. One of the simplest nonlinear phenomena is the elastoacoustic effect, whereby the speeds of small amplitude waves are changed by applying stress or strain. The theory of acoustoelasticity has been thoroughly discussed for purely elastic materials,[20–22] and is now commonly used in ultrasonics;[23–25] it has also been adapted to multiphase materials.[26]

The subject of this article is the acoustoelastic effect within the context of poroelasticity. The theory is based on a nonlinear generalization of the classic Biot theory. Biot himself discussed thermodynamic aspects of a nonlinear theory of poroelastic media, but did not specify any nonlinear stress–strain behavior (see papers 15, 16, 18, and 19 in Ref. 2). There has been some work on nonlinear poroelasticity within the framework of the theory of interpenetrating continua,[3] and also using the method of two-scale homogenization.[27,28] Three sources of nonlinearity are traditionally distinguished in elasticity: (i) the physical nonlinearity, that is, the nonlinearity of the constitutive relations relating the stress and the displacement gradient; (ii) nonlinearity of the universal equations, such as the equations of conservation of mass, momentum, etc.; (iii) geometric nonlinearity, which results from the nonlinear relationship between the deformation gradient and the tensor of finite deformations.

In order to keep the analysis as simple as possible, in this article we ignore all dissipative effects and concentrate on the first kind of nonlinearity only because it appears to be the most significant for geophysical materials. We begin with a discussion of acceleration waves in nonlinear poroelastic media. These are exact solutions independent of the state of prestress in the medium, and they reduce to the well-known

Biot fast and slow waves for unstressed isotropic materials. These general results serve as the basis for considering the principal body waves propagating in a slightly prestressed isotropic nonlinear saturated poroelastic substance. Finally, we discuss application of the general formulas for the stress dependence of the wave speeds to several specifically designed experimental configurations.

I. NONLINEAR POROELASTICITY

A. Governing equations

All further analysis is based on the following governing equations of the nonviscous, fluid-filled poroelastic medium:

$$\rho \frac{\partial^2 u^i}{\partial t^2} + \rho_f \frac{\partial^2 w^i}{\partial t^2} = \tau^{ij}_{,j}, \tag{1a}$$

$$\rho_f \frac{\partial^2 w^i}{\partial t^2} + \rho_f K^{ij} \frac{\partial^2 u_j}{\partial t^2} = -K^{ij} p_{,j}, \tag{1b}$$

where

$$\tau^{ij} = \frac{\partial W}{\partial u_{i,j}} (\zeta, u_{m,n}), \quad p = \frac{\partial W}{\partial \zeta} (\zeta, u_{m,n}). \tag{2}$$

Here, u^i is the displacement of the solid skeleton, w^i is the relative displacement of the fluid, τ^{ij} are the stresses, p is the fluid pressure, $W(\zeta, u_{m,n})$ is the poroelastic potential, ρ and ρ_f are the averaged density and the density of the fluid, respectively, $-\zeta = w^i_{,i}$ is a divergence of the relative displacement of the fluid, and K^{ij} is the ''instantaneous'' magnitude of a symmetric tensor of permeability (the value of the permeability hereditary operator at $t=0$). Also, x^i are the spatial coordinates, the Latin indices take the values 1, 2, and 3; summation over repeated indices is implied, and a comma followed by a Latin suffix symbolizes partial differentiation. We refer to the Refs. 2, 4, 27, and particularly Ref. 28 and for the motivation behind Eqs. (1) and (2).

By inserting the value of ζ in terms of w^i we can rewrite the equations of motion as

$$\rho \frac{\partial^2 u^i}{\partial t^2} + \rho_f \frac{\partial^2 w^i}{\partial t^2} = W^{ijkl} u_{k,jl} - W^{ij}_{\zeta} w^k_{,jk}, \tag{3a}$$

$$\rho_f \frac{\partial^2 w^i}{\partial t^2} + \rho_f K^{ij} \frac{\partial^2 u_j}{\partial t^2} = -K^{ij} (W^{kl}_{\zeta} u_{k,lj} - W_{\zeta\zeta} w^k_{,kj}). \tag{3b}$$

In these and subsequent equations we use the following notation for the derivatives of the potential energy function W: $W^{ij} = \partial W / \partial u_{i,j}$, $W_{\zeta\zeta} = \partial^2 W / \partial \zeta^2$, $W^{ij}_{\zeta} = \partial^2 W / \partial u_{i,j}\, \partial \zeta$, $W^{ijkl} = \partial^2 W / \partial u_{i,j}\, \partial u_{k,l}$, etc.

B. Acceleration waves

An acceleration wavefront is a propagating surface defined such that displacements and their first derivatives are continuous across this surface, but the second and higher derivatives and, in particular, the accelerations possess finite jumps. Acceleration waves are the most convenient object of theoretical study for both linear and nonlinear dynamics. They have been investigated thoroughly by Hadamard[29] and by Thomas,[30] while Chen[31] provides a more recent review for elastic materials. Continuity of the first derivatives imposes geometric and kinematic constraints on possible jumps of the second derivatives since the ''tangential'' components of the second derivatives are continuous. These constraints are known as compatibility conditions and are, essentially, geometric conditions originally developed by outstanding geometers like Hadamard,[29] Levi-Civita,[32] and Thomas[30] (fortunately all three were outstanding mathematical physicists so their works are accessible to physicists). In particular, the jumps of second derivatives across acceleration wavefronts satisfy[31,33]

$$[u_{i,jk}]^+_- = h_i n_j n_k, \quad \left[\frac{\partial^2 u_i}{\partial t^2}\right]^+_- = h_i c^2, \tag{4a}$$

$$[w_{i,jk}]^+_- = H_i n_j n_k, \quad \left[\frac{\partial^2 w_i}{\partial t^2}\right]^+_- = H_i c^2, \tag{4b}$$

where $h_k = [u_{i,pq}]^+_- n^p n^q$ and $H_k = [w_{i,pq}]^+_- n^p n^q$ are the second-order amplitude vectors of discontinuity, n^j is the unit normal to the wavefront, and c is the velocity of the front. The compatibility conditions indicate that the two vector functions h_k and H_k completely define the jumps of 60 partial derivatives of the displacements.

In order to determine the acceleration vectors themselves one has to extract some additional dynamic information from the governing Eqs. (3). These equations are not valid at the wavefront since the second derivatives are undefined at the front. By definition, only one-sided second derivatives are defined at the front, and these one-sided limits satisfy Eqs. (3). We first subtract, termwise, Eq. (3a) for the two one-sided limits taking into account continuity of the first derivatives and then do the same operation with Eq. (3b). Then, evaluating the jumps of the second derivatives using the second order compatibility conditions of Eqs. (4), we get the following linear algebraic system for the amplitude vectors:

$$(\rho c^2 \delta^{ik} - W^{ijkl} n_j n_l) h_k + (\rho_f c^2 \delta^{ik} + W^{ij}_{\zeta} n_j n^k) H_k = 0, \tag{5a}$$

$$(\rho_f c^2 \delta^{ik} + W^{kl}_{\zeta} n_l n^i) h_k + (\rho_f c^2 K^{ik}_{\text{inv}} - W_{\zeta\zeta} n^i n^k) H_k = 0, \tag{5b}$$

where K^{ij}_{inv} is the inverse of the permeability matrix K^{ij}, which is assumed to be invertible.

The pressure p is continuous across the wavefront, but its gradient is not, specifically,

$$[p_{,i}]^+_- = q n_i. \tag{6}$$

At the same time, it follows from the Eqs. (2) and (4) that

$$[p_{,i}]^+_- = W^{kl}_{\zeta} n_i n_l h_k - W_{\zeta\zeta} n_i n^k H_k. \tag{7}$$

Combining the previous two equations allows us to eliminate the quantity $n^k H_k$ in favor of the jump in the pressure gradient, q. Then Eq. (5b) provides an expression for H^i,

$$H^i = -K^{ij} \left(h_j + \frac{q}{\rho_f c^2} n_j \right). \tag{8}$$

After some manipulation, Eqs. (5) imply that

$$(\tilde{\rho}^{ik} c^2 - \tilde{W}^{ijkl} n_j n_l) h_k - \tilde{K}^{ij} n_j q = 0, \tag{9a}$$

$$\widetilde{K}^{kl} n_l h_k + \left(\frac{K^{ij} n_i n_j}{\rho_f c^2} - \frac{1}{W_{\zeta\zeta}} \right) q = 0, \tag{9b}$$

where

$$\widetilde{\rho}^{ik} = \rho \delta^{ik} - \rho_f K^{ik}, \tag{10a}$$

$$\widetilde{W}^{ijkl} = W^{ijkl} - W_{\zeta}^{ij} W_{\zeta}^{kl} / W_{\zeta\zeta}, \tag{10b}$$

$$\widetilde{K}^{ik} = K^{ik} + W_{\zeta}^{ik} / W_{\zeta\zeta}. \tag{10c}$$

Equations (9) form a system of four linear algebraic equations with respect to the four unknowns h_k and q. This system has a nonzero solution when the determinant vanishes, leading to a fourth order polynomial equation in c^2. The coefficients of the polynomial depend on $u_{k,l}$, ζ, and the orientation of the unit normal n_k, and thus the same is true for the velocities c. Each real root generates an associated nonzero solution (h_k, q). Further discussion of the fourth-order system can be found in Ref. 34. On the other hand, the system defined by Eqs. (5) is sixth order, because it involves the unknowns h_i and H_j. The fourth and sixth order systems are connected by the fact that the latter possesses the root $c^2=0$ of multiplicity 2. Thus, when $c^2=0$, the system of Eqs. (5) has infinitely many solutions of the form $h_k=0$, $H_k=R_k$, where R_k is an arbitrary vector in the two-dimensional (2-D) space orthogonal to the wave normal n_k. In summary, the propagating waves and the nonzero wave speeds can be found from either the fourth-order or the sixth-order systems.

C. Principal directions

A principal direction is defined as one in which an eigenvector, such as h^i, is aligned with or orthogonal to the wave normal n^i. For example, let us consider a longitudinal wave in the direction of n_i, that is, a solution of the form

$$h_k = h n_k, \quad H_k = H n_k. \tag{11}$$

Substituting from Eq. (11) into Eqs. (5) and contracting with n_i gives

$$(\rho c^2 - W^{ijkl} n_i n_j n_k n_l) h + (\rho_f c^2 + W_{\zeta}^{il} n_i n_l) H = 0, \tag{12a}$$

$$(\rho_f c^2 + W_{\zeta}^{il} n_i n_l) h + (\rho_f c^2 K_{\text{inv}}^{ik} n_i n_k - W_{\zeta\zeta}) H = 0. \tag{12b}$$

It follows immediately from Eqs. (12) that the corresponding eigenvalues c^2 are defined by a biquadratic characteristic equation, whereas the amplitudes h and H are connected by the relation

$$\frac{\rho c^2 - W^{ijkl} n_i n_j n_k n_l}{\rho_f c^2 K_{\text{inv}}^{ik} n_i n_k - W_{\zeta\zeta}} = \frac{H^2}{h^2}. \tag{13}$$

A transverse wave, on the other hand, satisfies $h_k = h e_k$, $H_k = H e_k$, where e^i is a polarization vector with $n^k e_k = 0$ and $e^k e_k = 1$. Under these circumstances, Eqs. (5) become

$$(\rho c^2 \delta^{ik} - W^{ijkl} n_j n_l) e_k h + \rho_f c^2 e^i H = 0, \tag{14a}$$

$$(\rho_f c^2 \delta^{ik} + W_{\zeta}^{kl} n_l n^i) e_k h + \rho_f c^2 K_{\text{inv}}^{ik} e_k H = 0, \tag{14b}$$

implying that the transverse wave velocity is given by

$$c_t^2 = [\rho - \rho_f (K_{\text{inv}}^{pq} e_p e_q)^{-1}]^{-1} W^{ijkl} e_i e_k n_j n_l. \tag{15}$$

When the material is isotropic then all directions are principal directions, i.e., all acceleration waves are either longitudinal or transverse in nature.

II. THE ACOUSTOELASTIC EFFECT

A. Third-order moduli for poroelasticity

We now consider the effects of prestress on the longitudinal and transverse waves in an isotropic medium. The permeability for an isotropic poroelastic medium is $K^{ij} = \kappa \delta^{ij}$, while the potential W is a function of ζ and the principal invariants I_M of the symmetric strain $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$. Thus

$$W = W(I_1, I_2, I_3, \zeta), \tag{16}$$

where

$$\begin{aligned} &I_1 = \delta^{ij} \varepsilon_{ij}, \quad I_2 = \tfrac{1}{2} (\delta^{qj} \delta^{rk} - \delta^{rj} \delta^{qk}) \varepsilon_{jq} \varepsilon_{kr}, \\ &I_3 = \tfrac{1}{6} \epsilon^{ijk} \epsilon^{pqr} \varepsilon_{ip} \varepsilon_{jq} \varepsilon_{kr}, \end{aligned} \tag{17}$$

and ϵ^{ijk} is the third-order alternating tensor. To within third-order terms in ε_{ij} and ζ the potential W can be approximated by the polynomial

$$\begin{aligned} W = {} & \Gamma_{11} I_1^2 + \Gamma_2 I_2 + \Gamma_{1\zeta} I_1 \zeta + \Gamma_{\zeta\zeta} \zeta^2 + \Gamma_{111} I_1^3 + \Gamma_{12} I_1 I_2 \\ & + \Gamma_3 I_3 + \Gamma_{1\zeta\zeta} I_1 \zeta^2 + \Gamma_{11\zeta} I_1^2 \zeta + \Gamma_{2\zeta} I_2 \zeta + \Gamma_{\zeta\zeta\zeta} \zeta^3. \end{aligned} \tag{18}$$

We consider the following field of perturbations with respect to the reference (unstressed) configuration:

$$\delta u_i = \left(\alpha_N n_j x^j n_i + \sum_{A=1,2} \alpha_A x^j e_{Aj} \right) \varepsilon, \quad \delta \zeta = \varepsilon \Delta, \tag{19}$$

where e_{1i} and e_{2i} are two mutually orthogonal transverse vectors, i.e., both are orthogonal to the n^i direction. The perturbation magnitude is defined by ε, with $|\varepsilon| \ll 1$ by assumption, and the parameters α_N, α_A, $A=1,2$, and Δ are $O(1)$ and independent. The prestrain is, from Eq. (19),

$$\varepsilon_{ij} = \left(\alpha_N n_j n_i + \sum_{A=1,2} \alpha_A e_{Ai} e_{Aj} \right) \varepsilon. \tag{20}$$

Differentiating Eq. (18) in the vicinity of the reference configuration $(\varepsilon=0)$ we obtain the second order elasticity tensors in terms of the elastic moduli:

$$\begin{aligned} &W^{ijkl} = \lambda_c \delta^{ij} \delta^{kl} + \mu (\delta^{il} \delta^{jk} + \delta^{ik} \delta^{jl}), \\ &W_{\zeta}^{ij} = -\alpha M \delta^{ij}, \quad W_{\zeta\zeta} = M, \end{aligned} \tag{21}$$

where λ_c, α, and M are the Biot parameters.[35] In fact, Biot defines the linearized stress–strain relations in terms of a quadratic energy potential similar to Eq. (18). Comparison of the quadratic terms in the latter with Eq. (3.4) of Biot[35] immediately implies that

$$\begin{aligned} &\lambda_c = 2\Gamma_{11} + \Gamma_2, \quad \mu = -\tfrac{1}{2} \Gamma_2, \quad M = 2\Gamma_{\zeta\zeta}, \\ &\alpha = -\Gamma_{1\zeta} / (2\Gamma_{\zeta\zeta}). \end{aligned} \tag{22}$$

There are many different notations used for describing the linear response of saturated porous media, and even Biot's own notation evolved from the time of his early work on consolidation (see paper 1 of Ref. 2) to his later work. His 1962 paper[35] provides a good comparison of the notations used.

For the third-order quantities we get

$$\begin{aligned}
W_\zeta^{ijkl}&=(\Gamma_{2\zeta}+2\Gamma_{11\zeta})\delta^{ij}\delta^{kl}-\tfrac{1}{2}\Gamma_{2\zeta}(\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}),\\
W_{\zeta\zeta}^{ij}&=2\Gamma_{1\zeta\zeta}\delta^{ij},\quad W_{\zeta\zeta\zeta}=6\Gamma_{\zeta\zeta\zeta},\\
W^{ijklmn}&=(6\Gamma_{111}+3\Gamma_{12})\delta_{ij}\delta_{kl}\delta_{mn}+\tfrac{1}{4}\Gamma_3(\epsilon^{ikm}\epsilon^{jln}\\
&\quad+\epsilon^{jkm}\epsilon^{iln}+\epsilon^{ilm}\epsilon^{jkn}+\epsilon^{jlm}\epsilon^{ikn})\\
&\quad-\tfrac{1}{2}\Gamma_{12}(\delta^{ij}(\delta^{km}\delta^{ln}+\delta^{kn}\delta^{lm})+\delta^{kl}(\delta^{im}\delta^{jn}\\
&\quad+\delta^{in}\delta^{jm})+\delta^{mn}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})).
\end{aligned}\tag{23}$$

In the absence of poroelastic effects, the third-order elastic moduli W_{ijklmn} are equivalent to the standard third order moduli C_{ijklmn}, or C_{IJK} in the concise Voigt notation.[23] In order to compare the moduli here with the more common notation, we note the equivalences

$$\begin{aligned}
\Gamma_{111}&=\tfrac{1}{6}C_{111},\quad \Gamma_{12}=\tfrac{1}{2}(C_{112}-C_{111}),\\
\Gamma_3&=\tfrac{1}{2}(C_{111}-3C_{112}+2C_{123}).
\end{aligned}\tag{24}$$

The connection with other notations for third-order moduli can be inferred from the table in Green's review[23] that compares many different systems of notation. We note for future reference the following formulas:

$$W^{ijklmn}n_in_jn_kn_l=6\Gamma_{111}\delta^{mn}+2\Gamma_{12}(\delta^{mn}-n^mn^n),\tag{25a}$$

$$W^{ijklmn}e_in_je_kn_l=-\tfrac{1}{2}\Gamma_{12}\delta^{mn}-\tfrac{1}{2}\Gamma_3(\delta^{mn}-n^mn^n-e^me^n).\tag{25b}$$

B. Wave-speed dependence on the applied strain

For each deformed configuration there are three orthogonal directions of principal strain, and the small amplitude waves that propagate in these directions are either longitudinal or transverse. We will now derive approximative formulas for the incremental changes of the velocities of these principal waves to leading order in ε.

First, the variation in the velocity of the transverse wave with the polarization vector e_{Ci} follows from Eqs. (15) and (20) as

$$\begin{aligned}
\delta c_{tC}^2&=(\rho-\kappa\rho_f)^{-1}\delta W^{ijkl}e_{Ci}e_{Ck}n_jn_l\\
&=(\rho-\kappa\rho_f)^{-1}(W^{ijklmn}e_{Ci}e_{Ck}n_jn_l\varepsilon_{mn}\\
&\quad+W_\zeta^{ijkl}e_{Ci}e_{Ck}n_jn_l\varepsilon\Delta).
\end{aligned}\tag{26}$$

Note that the densities ρ and ρ_f are unchanged because we are using the reference or Lagrangian description as opposed to the current or Eulerian description. Conservation of mass requires that these densities are constant. The structure constant κ is also assumed to remain the same under the deformation. Combining Eqs. (20), (25), and (26) we arrive at the formula for the elastoacoustic effect for transverse waves,

$$\begin{aligned}
\delta c_{tC}^2&=\frac{-\varepsilon}{2(\rho-\kappa\rho_f)}((\alpha_1+\alpha_2+\alpha_N)\Gamma_{12}+(\alpha_1+\alpha_2\\
&\quad-\alpha_C)\Gamma_3+\Gamma_{2\zeta}\Delta).
\end{aligned}\tag{27}$$

In particular, the split in the velocities of two transverse waves with the same propagation direction but different polarizations is, from Eq. (27),

$$\frac{\delta c_{tA}^2-\delta c_{tB}^2}{\alpha_A-\alpha_B}=\frac{\varepsilon\Gamma_3}{2(\rho-\kappa\rho_f)}.\tag{28}$$

In order to establish analogous formulae for longitudinal principal waves we first write the system of Eqs. (5) as

$$\mathbf{AX}=c^2\mathbf{CX},\tag{29}$$

where $\mathbf{X}=(h,H)^T$, and the symmetric matrices $\mathbf{A}$, $\mathbf{C}$ are

$$\mathbf{A}=\begin{bmatrix} W^{ijkl}n_in_jn_kn_l & -W_\zeta^{il}n_in_l\\ -W_\zeta^{il}n_in_l & W_{\zeta\zeta}\end{bmatrix},\quad C=\begin{bmatrix}\rho & \rho_f\\ \rho_f & \kappa^{-1}\rho_f\end{bmatrix}.\tag{30}$$

The inertial matrix $\mathbf{C}$ is again unchanged by the initial deformation, and the new wave speeds depend upon the variation in $\mathbf{A}$. By making use of Eqs. (18)–(23) we can approximate this to within first order in ε,

$$\mathbf{A}=\mathbf{A}^0+\varepsilon\mathbf{A}^1,\tag{31}$$

where

$$\begin{aligned}
\mathbf{A}^0&=\begin{bmatrix}\lambda_c+2\mu & \alpha M\\ \alpha M & M\end{bmatrix},\\
\mathbf{A}^1&=\alpha_N\begin{bmatrix}6\Gamma_{111} & -2\Gamma_{11\zeta}\\ -2\Gamma_{11\zeta} & 2\Gamma_{1\zeta\zeta}\end{bmatrix}+(\alpha_1+\alpha_2)\\
&\quad\times\begin{bmatrix}6\Gamma_{111}+2\Gamma_{12} & -2\Gamma_{11\zeta}-\Gamma_{2\zeta}\\ -2\Gamma_{11\zeta}-\Gamma_{2\zeta} & 2\Gamma_{1\zeta\zeta}\end{bmatrix}\\
&\quad+\Delta\begin{bmatrix}2\Gamma_{11\zeta} & -2\Gamma_{1\zeta\zeta}\\ -2\Gamma_{1\zeta\zeta} & 6\Gamma_{\zeta\zeta\zeta}\end{bmatrix}.
\end{aligned}\tag{32}$$

The eigenvalue problem for the matrix $\mathbf{C}^{-1}\mathbf{A}^0$ was first studied by Biot in 1956 (see paper 8 of Ref. 2) when he established the existence of two different longitudinal modes: the ''fast'' and ''slow'' waves with velocities c_f and c_s, respectively. The eigenvectors $\mathbf{X}_f$ and $\mathbf{X}_s$ associated with these velocities can be calculated by making use of the following equation implied by Eq. (13):

$$\frac{\rho c^2-(\lambda_c+2\mu)}{\kappa^{-1}\rho_fc^2-M}=\frac{H^2}{h^2}.\tag{33}$$

Corrections to the Biot fast and slow velocities due to the incremental deformations can now be calculated using standard perturbation theory, leading to the formula

$$\delta c^2=\varepsilon\,\frac{\mathbf{A}_{ij}^1\mathbf{X}^{0i}\mathbf{X}^{0j}}{\mathbf{C}_{ij}\mathbf{X}^{0i}\mathbf{X}^{0j}}.\tag{34}$$

Equations (27), (32), and (34) show that by measurement of elastoacoustic effects we can, in principle, find the third order elastic moduli by solving a system of linear algebraic equations. In Sec. III we shall examine some possible experimental configurations with this purpose in mind.

TABLE I. Different compressibilities of a porous medium, with definitions of K_c, K, and K_M. Here, p and p_c are the pore and confining pressure, respectively. Note that λ_c, μ, α, and M are defined in Eqs. (22), and $\lambda = \lambda_c - \alpha^2 M$ (Ref. 35). The relations between these and other moduli are explored in greater detail by Kümpel (Ref. 36), and also by Brown and Korringa (Ref. 37).

Compressibility—definition	In current notation	Terminology, notation, and references
$\frac{1}{K_c} = \frac{-1}{V} \frac{\partial V}{\partial p_c}\Big\vert_{\zeta}$	$\frac{1}{\lambda_c + \frac{2}{3}\mu}$	Closed compressibility (Ref. 35) undrained compressibility, c_u (Ref. 36)
$\frac{1}{K} = \frac{-1}{V} \frac{\partial V}{\partial p_c}\Big\vert_{p}$	$\frac{1}{\lambda + \frac{2}{3}\mu}$	Open or jacketed compressibility (Refs. 35 and 18) drained or matrix compressibility, c (Ref. 36)
$\frac{1}{K_M} = \frac{-1}{V} \frac{\partial V}{\partial p_c}\Big\vert_{p=p_c}$	$\frac{1-\alpha}{\lambda + \frac{2}{3}\mu}$	Grain or pore compressibility (Ref. 37), c_s (Ref. 36), unjacketed compressibility, δ (Ref. 18)

III. THE ELASTOACOUSTIC EFFECT IN EXPERIMENTS

A. Experimental nomenclature and moduli

The application of the previous ideas to a poroelastic sample requires the ability to impose various types of stress states if all seven third-order moduli are to be determined, i.e., $\{\Gamma_{111}, \Gamma_{12}, \Gamma_3, \Gamma_{1\zeta\zeta}, \Gamma_{11\zeta}, \Gamma_{2\zeta}$, We first discuss the types of measurements employed in determining the four second-order moduli $\{\Gamma_{11}, \Gamma_2, \Gamma_{1\zeta}, \Gamma_{\zeta\zeta}\}$ or $\{\lambda_c, \mu, \alpha, M\}$. The second-order moduli are related to one another via Eq. (22) and yield the classic Biot theory for linear poroelasticity[35] for which

$$\tau_{ij} = \lambda_c u^k_{,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \alpha M \zeta \delta_{ij}, \tag{35a}$$

$$p = -\alpha M u^k_{,k} + M\zeta. \tag{35b}$$

Note that p is the pore pressure, and τ_{ij} is the total or confining stress. The effective stress, $\tau_{ij} + \alpha p \delta_{ij}$, is also commonly used. For example, if a hydrostatic confining stress $\tau_{ij} = -p_c \delta_{ij}$ is applied, then the effective pressure is $p_e = p_c - \alpha p$. The main point is that there are two independent stress variables, or pressures, if the system is hydrostatically loaded. Table I lists and defines three distinct bulk moduli (inverse of compressibility), K_c, K, and K_M, associated with hydrostatic deformation of undrained, jacketed, and unjacketed samples, respectively. We also define the bulk modulus for the fluid alone, K_f, and the ''coefficient of fluid content'' γ,[18,35]

$$\frac{1}{K_f} = \frac{-1}{V_f} \frac{\partial V_f}{\partial p}, \quad \gamma = \phi\left(\frac{1}{K_f} - \frac{1}{K_M}\right), \tag{36}$$

in terms of which the Gassmann relation[35] is

$$\gamma = \frac{1}{M} - \frac{\alpha}{K_M}. \tag{37}$$

Different types of experiments have been used to determine the second-order elasticities of fluid-saturated poroelastic media. We shall discuss them briefly in order to avoid possible confusion in terminology. The terms ''open'' and ''closed,'' ''jacketed'' and ''unjacketed,'' and ''drained'' and ''undrained'' are all relevant to the problem, but have different connotations. We shall next describe the four states of deformation illustrated in Fig. 1, and then consider their mathematical description and their implications for elastoacoustic measurements.

In state (a) the closed-pore system corresponds to constancy of the fluid content or mass, and in the geometrically linear approximation this implies that $\zeta = 0$. The closed-pore experiment is the simplest for theoretical analysis but is rather difficult to realize in practice, at least for a high level of wetting. Conceptually, it requires that the sample is covered by an impervious closed deformable jacket; see Fig. 1(a). (b) Alternatively, the open porous system, in the terminology of Biot and Willis,[18] corresponds to the case of constant fluid pressure, or $p = 0$. However, it is more consistent to refer to this as a ''drained'' test and to treat it as a special case of a more general open configuration with p = const. Such an experiment is sketched in Fig. 1(b) and is regarded as the jacketed test. The amount of absorbed liquid is not fixed in this experiment since it can leave the sample through the tube connected with the air. The unjacketed tests are

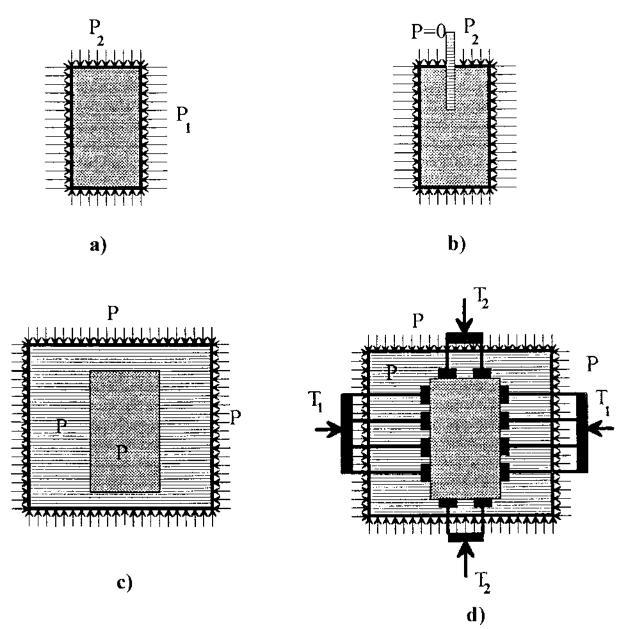

FIG. 1. Different schemes of loading: (a) closed-pore jacketed test, (b) open-pore jacketed test, (c) conventional unjacketed test, and (d) triaxial unjacketed test.

sketched in Fig. 1(c) and Fig. 1(d). Although the system as a whole is again contained within an impervious jacket, the porous solid sample can now exchange fluid with the surrounding free fluid in the reservoir. (c) A ''conventional'' unjacketed test is shown in Fig. 1(c). In this case the hydrostatic stress on the solid is caused by pressurizing the surrounding fluid, and therefore both the skeleton and the fluid are hydrostatically stressed with the same pressure, $\tau_{ij} = -p_c \delta_{ij}$ and $p = p_c$. This type of conventional unjacketed test does not permit triaxial deformation of the sample, and is thus too restrictive for the acoustoelastic effect. (d) The experiment sketched in Fig. 1(d) might be helpful in avoiding this drawback, although it is presumably rather difficult to realize practically. In what follows we call it a ''triaxial unjacketed'' test, and it is characterized by a constant pressure p in the pore fluid, and a triaxial state of stress in the skeleton, implying three independent elements for τ_{ij}.

The four states outlined above are as follows. First, for the ''closed-pore'' jacketed test we have (see Table I for definitions)

$$\delta\zeta = 0 \Leftrightarrow p_c / K_c = p_e / K. \tag{38}$$

In the ''open-pore'' jacketed test, on the other hand,

$$p = 0 \Leftrightarrow \delta\zeta = -\alpha p_c / K. \tag{39}$$

In the ''conventional'' unjacketed test the skeleton is in a state of hydrostatic stress, experiencing the same pressure as the fluid, i.e., $p_c = p$. In the case at hand the skeleton deformation is purely dilatational, and Eqs. (35) then imply

$$\delta u_{i,j} = -(p/3K_M)\delta_{ij}, \quad \delta\zeta = \gamma p, \tag{40}$$

where γ is defined in Eq. (36). Finally, in the triaxial unjacketed test both the fluid pressure and the deformation of the skeleton are simultaneously kept under control. In this case we can express the fluid dilatation parameter in terms of these as

$$\delta\zeta = \alpha\, \delta u^k_{,k} + p/M. \tag{41}$$

B. Wave speeds for specific experimental configurations

The above formulas together imply specific relations for the change in the velocities of the three wave types: transverse, and fast and slow longitudinal waves. For the first three cases below, the loading is hydrostatic, defined by the confining pressure p_c. The change in the transverse speed is therefore independent of its polarization direction.

(a) The ''closed-pore jacketed'' tests: for the transverse wave,

$$\frac{dc_t^2}{dp_c} = \frac{\Gamma_{12} + \frac{1}{3}\Gamma_3}{2K_c(\rho - \kappa\rho_f)}, \tag{42}$$

and for the fast and slow waves,

$$\frac{dc^2}{dp_c} = \frac{\mathbf{B}_{ij}\mathbf{X}^{0i}\mathbf{X}^{0j}}{\mathbf{C}_{ij}\mathbf{X}^{0i}\mathbf{X}^{0j}}, \tag{43}$$

where

$$\mathbf{B} = \frac{1}{K_c}\begin{bmatrix} -6\Gamma_{111} - \frac{4}{3}\Gamma_{12} & 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} \\ 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} & -2\Gamma_{1\zeta\zeta} \end{bmatrix}. \tag{44}$$

(b) The ''open-pore jacketed'' tests:

$$\frac{dc_t^2}{dp_c} = \frac{\Gamma_{12} + \frac{1}{3}\Gamma_3 + \alpha\Gamma_{2\zeta}}{2K(\rho - \kappa\rho_f)}. \tag{45}$$

The variation in the fast and slow wave speeds is given by Eq. (43), where now

$$\mathbf{B} = \frac{1}{K}\begin{bmatrix} -6\Gamma_{111} - \frac{4}{3}\Gamma_{12} - 2\alpha\Gamma_{11\zeta} & 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\alpha\Gamma_{1\zeta\zeta} \\ 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\alpha\Gamma_{1\zeta\zeta} & -2\Gamma_{1\zeta\zeta} - 6\alpha\Gamma_{\zeta\zeta\zeta} \end{bmatrix}. \tag{46}$$

(c) The conventional ''unjacketed'' tests:

$$\frac{dc_t^2}{dp_c} = \frac{\Gamma_{12} + \frac{1}{3}\Gamma_3 - \gamma K_M\Gamma_{2\zeta}}{2K_M(\rho - \kappa\rho_f)}. \tag{47}$$

The fast and slow speed changes are given by Eq. (43) with

$$\mathbf{B} = \frac{1}{K_M}\begin{bmatrix} -6\Gamma_{111} - \frac{4}{3}\Gamma_{12} - 2\gamma K_M\Gamma_{11\zeta} & 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\gamma K_M\Gamma_{1\zeta\zeta} \\ 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\gamma K_M\Gamma_{1\zeta\zeta} & -2\Gamma_{1\zeta\zeta} - 6\gamma K_M\Gamma_{\zeta\zeta\zeta} \end{bmatrix}. \tag{48}$$

(d) The ''unjacketed'' triaxial tests: The three principal strains α_N, α_1, α_2, and the pore pressure $p \equiv \varepsilon P$ are now independent parameters, and, according to Eq. (41),

$$\Delta = \alpha(\alpha_N + \alpha_1 + \alpha_2) + P/M. \tag{49}$$

The changes in the three wave speeds are then given by the general formulas in Eqs. (27), (32), and (34).

IV. DISCUSSION AND CONCLUSIONS

The acoustoelastic effect is the simplest nonlinear elastic effect that can be used to determine nonlinear elastic moduli. To this end one should measure changes of velocities of acoustic waves induced by prestrain. Although it is of higher order as compared to linear effects, acoustoelasticity can be reliably detected in sedimentary materials since the nonlinear elastic moduli are of several orders higher that their linear moduli.[16] The experiments can be performed either on core samples or *in situ* as, for instance, by experimental measurement of strain-induced changes in seismic waves velocities due to the solid Earth tides.[13] In fact, it is found that the *in situ* nonlinearity is as large or larger than that observed in the laboratory, and may be due to the large scale heterogeneity in the Earth.[13] Such measurements can provide valuable information about the Earth's interior, just as measurements of the velocities themselves allow one to determine linear moduli.

In the present article we have established relations for the strain-induced changes of acoustic wave speeds in fluid-filled poroelastic media. We have concentrated on the longitudinal and transverse waves in isotropic poroelastic substances, which become slightly anisotropic under triaxial prestrain. These changes depend not only on the prestrain of the solid skeleton but also on the drainage regime of the fluid component. Thus, the acoustoelastic effect is different for drained, undrained, jacketed, and unjacketed tests, and explicit formulas for each configuration are described.

The seven nonlinear elastic moduli characterizing an isotropic poroelastic substance can be found, in principle, by a sequence of acoustoelastic tests and by using the formulas presented here. Thus, the nonlinear elastic modulus Γ_3 may be determined directly using the rather simple formula of Eq. (28) for the split in the velocities of two transverse waves having the same direction of propagation but different polarizations. Then, using the formulae in Eqs. (42) and (45), one can determine the nonlinear moduli Γ_{12} and $\Gamma_{2\zeta}$ by making shear wave measurements under uniform (hydrostatic) prestrain with closed-pore jacketed and open-pore unjacketed conditions. The same information can be extracted using shear wave data from a conventional unjacketed test and Eq. (47). Compatibility of the numerical results for the moduli determined through different experiments can be used as the indication of acceptability of the model of an isotropic poroelastic substance. The determination of the remaining four nonlinear moduli can be based on measurements of the fast and slow longitudinal waves, using Eqs. (34), (43), (44), (46), and (48). Again, the fact that there are more relations than moduli allows one to check the admissibility of the model of an isotropic poroelastic medium.

ACKNOWLEDGMENT

This work was supported by the United States Office of Naval Research.

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