

Longitudinal and shear wave diffraction at the junction of two fluid loaded, doubly curved shells

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Abstract

A perturbation solution is developed to study diffracted membrane waves, both longitudinal and shear, originating from the junction of two doubly curved shells which are joined so that their tangent is continuous. The excitation is an acoustic wave incident obliquely to the tangent line of the junction. The diffracted membrane wavefield is dependent upon or forced by an outer solution which characterizes the acoustic interaction between two joined, flat plates and the surrounding fluid. To illustrate the simplicity of the approach, the outer solution is approximated by the acoustic diffraction from two massive plates. Explicit expressions are derived for the membrane wave diffraction coefficients which are related, by reciprocity, to the analogous membrane-to-acoustic diffraction coefficients.

1. Introduction

Two curved, thin elastic shells are joined together in such a manner that their surface tangent vector is continuous across the junction. Each shell can have different structural and material properties, including different radii of curvature perpendicular to the junction. The radii of curvature along the junction are the same. The problem is to characterize the interaction between the joint and the acoustic fluid, which loads the plates unilaterally. We are specifically interested in the coupling between the acoustic and structural waves. In this paper we present a simplified method of solution which ignores flexural effects, and focuses on the acoustic-membrane wave interaction.

A precise description of the dynamics at the fluid-loaded junction of dissimilar shells leads to considerable analytical complexity [1]. A simplification was proposed in an earlier paper [2] in which we considered a two-dimensional model of acoustic and longitudinal wave diffraction from plate junctions. A perturbation scheme was used to separate the background response from that of the membrane wavefield [3]. In essence, the coupling between the acoustic and membrane wavefields is weak for frequencies at which the radii of curvature are large in comparison with the wavelengths. The normal motion and acoustic response can then be

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determined separately, giving the “background solution” which then drives the in-plane displacements. Thus, the scattered longitudinal wavefields on the plates are found by a two-step process [2].

Our focus here is on the generalization of the 2D model [2] to account for three-dimensional effects, such as oblique incidence and curvature out of the plane. This introduces the possibility of acoustic coupling to both longitudinal and shear waves, and *vice versa*, in addition to membrane-membrane wave coupling at the junction. The 2D model of Norris and Rebinsky [2] ignores flexure, which is a gross simplification and probably quite unrealistic (although the acid test of the model is its comparison with the “correct” solution, which is the ultimate goal of this work). Flexural effects can be included, and to leading order they influence the background response. Their inclusion, however, complicates the picture somewhat and will be ignored in this preliminary description. We emphasize that this exclusion does not alter the solution process for the diffracted membrane waves. Basically, the background or outer solution is simply an input forcing parameter for the membrane dynamics. In calculating the forcing we will assume that the plates are massive with no stiffness so that a simple impedance boundary condition describes the acoustic diffraction from the curved shells [2]. Flexural effects will be included in a subsequent paper.

The paper proceeds as follows. The structural acoustics scattering problem is defined in Section 2. The perturbation scheme outlined in Section 2 is used to reduce the full problem to a localized problem in the tangent plane at the junction, described in Section 3. The background solution is also defined there. The formal analysis of the forced in-plane motion is discussed in Section 4 with the full solution given in Section 5. The in-plane wavefields are determined by solving a vector Wiener–Hopf problem with a simple kernel containing the membrane wave poles. Their solution is dependent on the scattered acoustic wavefield generated, in this instance, from two massive, flat plates. This involves another Wiener–Hopf problem which was previously evaluated for the 2D case [2] and its extension to the 3D configuration is described in the Appendix. The physically significant acoustic-membrane diffraction coefficients are discussed in Section 6 along with two limiting cases of interest. Finally, in Section 7, we show how the complementary problem of incident membrane waves can be solved using the principle of reciprocity and the acoustic solution.

2. Equations of motion and asymptotic scalings

2.1. Shell equations

Let u and v be the in-surface shell displacements in the x and y directions respectively, where x and y are orthogonal coordinates along the directions of principal curvature. The normal displacement into the fluid is w , and p is the total acoustic pressure in the fluid, which loads the shells on one side only. We consider time harmonic motion of frequency $\omega > 0$, the factor $e^{-i\omega t}$ understood but suppressed. The curved shells are modeled using the Donnell–Yu shell theory [4,5], giving

$$\sigma_{x,x} + \tau_{xy,y} + m\omega^2 u = 0, \quad (1a)$$

$$\sigma_{y,y} + \tau_{xy,x} + m\omega^2 v = 0, \quad (1b)$$

$$\frac{\sigma_x}{R_x} + \frac{\sigma_y}{R_y} + B \nabla^2 \nabla^2 w - m\omega^2 w = -p, \quad (1c)$$

where $B = Eh^3/12(1 - \nu^2)$ is the bending stiffness, m is the areal density of the shell, R_x and R_y are the radii of curvature in the x and y directions. The stresses in the shell are

$$\sigma_x = C \left[u_{,x} + \frac{w}{R_x} + \nu \left(v_{,y} + \frac{w}{R_y} \right) \right], \quad (2a)$$

$$\sigma_y = C \left[v_{,y} + \frac{w}{R_y} + \nu \left(u_{,x} + \frac{w}{R_x} \right) \right], \quad (2b)$$

$$\tau_{xy} = \frac{1-\nu}{2} C (v_{,x} + u_{,y}), \quad (2c)$$

and C is the extensional stiffness given by $C = Eh/(1 - \nu^2)$, where h , E , and ν are the thickness, Young's modulus, and Poisson's ratio, respectively. The equation of kinematic continuity between the curved elastic shell and the surrounding fluid is

$$\rho_f \omega^2 w = \frac{\partial p}{\partial n}. \quad (3)$$

The problem statement is completed by noting that the pressure satisfies the Helmholtz equation

$$\nabla^2 p + k_f^2 p = 0, \quad (4)$$

in the surrounding fluid. The fluid is described by its density ρ_f , wavespeed c_f , and wavenumber $k_f = \omega/c_f$. Finally, we note that the scattered part of the pressure is subject to a radiation condition as $(x^2 + y^2 + z^2)^{1/2} \rightarrow \infty$. The problem then has a unique and physically meaningful solution.

2.2. The tangent plane approximation

We assume the junction is the line $x = 0$, and consider a local region about the joint where the shell surfaces are approximated by $z \approx -x^2/(2R_x) - y^2/(2R_y)$. The x and y surface coordinates can then be considered as the Cartesian coordinates in the tangent plane at $x = y = 0$. For simplicity, we have aligned the principal radii of curvature R_x and R_y with the surface directions x and y , respectively. Also, R_y , the radius of curvature parallel to the junction, is the same for both shells. It is explicitly assumed that $\epsilon \ll 1$, where

$$\epsilon = \frac{1}{k_f R}, \quad R = \min\{R_x, R_y\}. \quad (5)$$

Hence, all the radii of curvature are much larger than the fluid wavelength. The local region can therefore be identified with a Fresnel zone [2], and the local representation may be called a shell flattening approximation. It does not, however, reduce the equations to those of a fluid-loaded flat plate, because the curvature terms are retained in the equations of motion. These terms provide the only mechanism by which in-surface membrane waves can couple to the acoustic medium, and they are of prime importance in this study. We also note that the incident acoustic wavefield can be approximated in the same manner, yielding an equivalent acoustic field on the tangent plane [3]. Therefore, the diffraction boundary value problem defined by the doubly curved surface can be mapped to an equivalent problem along the tangent plane, as long as the asymptotic argument $\epsilon \ll 1$ is valid [3].

Each plate is now described by the equations of motion for a thin elastic shell, Eqs. (1), that have been "flattened" using the procedure discussed above. We consider an incident plane acoustic wave whose angle of incidence with respect to the plane is assumed to differ from the critical angles of all supersonic membrane waves. Incidence near one of the critical angles requires a unified treatment that combines the present analysis with that of Norris and Rebinsky [3], and will be discussed elsewhere. Under the present assumptions the in-surface displacements u and v are small compared to the normal displacement, and the following ansatz [3] is assumed,

$$(p, w) = (p^{(0)}, w^{(0)}) + \epsilon (p^{(1)}, w^{(1)}) + \dots, \quad (6a)$$

$$(u, v) = \epsilon^\alpha (u^{(0)}, v^{(0)}) + \epsilon^{2\alpha} (u^{(1)}, v^{(1)}) + \dots, \quad (6b)$$

and the stresses follow as

$$(\sigma_x, \sigma_y, \tau_{xy}) = \epsilon^\alpha (\sigma_x^{(0)}, \sigma_y^{(0)}, \tau_{xy}^{(0)}) + \epsilon^{2\alpha} (\sigma_x^{(1)}, \sigma_y^{(1)}, \tau_{xy}^{(1)}) + \dots \quad (7)$$

The asymptotic parameter ϵ^α is used to scale the equations, but in practice we only deal with the leading order terms in the expansions. Therefore, in all subsequent equations we have reset $\alpha \equiv 0$ to avoid confusion, and the superscript (0) indicates the leading order terms. Thus, $p^{(0)}$ and $w^{(0)}$ define the background solution.

3. The two decoupled problems

3.1. Forced in-plane equations of motion

The first consequence of the ansatz given by Eqs. (6) and (7) is that the w equation of motion (1) decouples from the u and v equations. The solution of the w equation, combined with the continuity condition (3), yields the outer or background wavefield, given by $w^{(0)}$ or $p^{(0)}$, which is discussed below. We note at this stage that the background field does not depend upon the curvature but only on the possible discontinuities in the plate densities and bending stiffness. The solution of the equations describing the in-plane motion can be considered as an inner solution, in terms of the local approximation, that is forced by the outer or background response. We assume that the background wavefield is known (see the discussion of this aspect in Section 1 and below) and concentrate on analyzing the in-plane behavior.

We consider straight-crested waves of the form $e^{i(k_x x + k_y y + i k_z z - \omega t)}$. Since the joint connecting the “flattened” shells is along the y -axis, the y -dependence, which is contained in the multiplicative factor $e^{i k_y y}$, can be removed from the equations of motion (1) and the constitutive relations (2). In doing so, the first two equations of (1) become

$$\bar{\sigma}_{x,x}^{(0)} + i k_y \bar{\tau}_{xy}^{(0)} + m \omega^2 \bar{u}^{(0)} = 0, \quad (8a)$$

$$i k_y \bar{\sigma}_y^{(0)} + \bar{\tau}_{xy,x}^{(0)} + m \omega^2 \bar{v}^{(0)} = 0, \quad (8b)$$

and the constitutive relations (2) are now

$$\bar{\sigma}_x^{(0)} = C \left[\bar{u}_{,x}^{(0)} + \frac{\bar{w}^{(0)}}{R_x} + \nu \left(i k_y \bar{v}^{(0)} + \frac{\bar{w}^{(0)}}{R_y} \right) \right], \quad (9a)$$

$$\bar{\sigma}_y^{(0)} = C \left[i k_y \bar{v}^{(0)} + \frac{\bar{w}^{(0)}}{R_y} + \nu \left(\bar{u}_{,x}^{(0)} + \frac{\bar{w}^{(0)}}{R_x} \right) \right], \quad (9b)$$

$$\bar{\tau}_{xy}^{(0)} = \frac{1-\nu}{2} C \left(\bar{v}_{,x}^{(0)} + i k_y \bar{u}^{(0)} \right), \quad (9c)$$

where variables with an overbar are related to those without through multiplication by the phase factor $e^{i k_y y}$.

The junction conditions for a welded joint between the two “flattened” shells involve continuity of $\bar{u}^{(0)}$, $\bar{v}^{(0)}$, $\bar{\sigma}_x^{(0)}$, and $\bar{\tau}_{xy}^{(0)}$. In the simpler two dimensional problem [2] we wrote the equation of motion in terms of the longitudinal stress, which allowed the plate junction continuity condition to be simply applied. After much consideration, it appears that the analogous procedure for the present problem is to write the in-plane equations of motion (8) solely in terms of the stress $\bar{\sigma}_x^{(0)}$ and the displacement $\bar{v}^{(0)}$. It will transpire that these fundamental variables are somehow optimal, and the rearranged equations of motion allow the joint conditions to be satisfied simply. After several straightforward algebraic steps, the transformed $\bar{u}^{(0)}$ and $\bar{v}^{(0)}$ equations of motion become

$$\bar{\sigma}_{x,xx}^{(0)} + (k^2 + \nu k_y^2) \bar{\sigma}_x^{(0)} - i k_y (1-\nu) C [k^2 - (1-\nu) k_y^2] \bar{v}^{(0)}$$

$$= C \left[\frac{k^2}{R_x} + \frac{\nu k^2 - (1 - \nu^2) k_y^2}{R_y} \right] \bar{w}^{(0)}, \quad (10a)$$

$$\bar{v}_{,xx}^{(0)} + \left[\frac{2}{1 - \nu} k^2 - (2 + \nu) k_y^2 \right] \bar{v}^{(0)} + \frac{ik_y}{C} \left(\frac{1 + \nu}{1 - \nu} \right) \bar{\sigma}_x^{(0)} = ik_y \left(\frac{1}{R_x} - \frac{2 + \nu}{R_y} \right) \bar{w}^{(0)}, \quad (10b)$$

where k is the longitudinal wavenumber for a flat plate, defined by

$$k^2 = m\omega^2 / C. \quad (11)$$

The in-plane Eqs. (10) show how the quantities $\bar{\sigma}_x^{(0)}$ and $\bar{v}^{(0)}$, are now explicitly dependent upon, or forced by, the background solution given, in this instance, by the normal displacement $\bar{w}^{(0)}$. Before attempting to solve for the membrane fields we first need to determine $\bar{w}^{(0)}$.

3.2. The outer or background solution

The leading order contribution from the w equation of motion (1) is

$$B \nabla^2 \nabla^2 w^{(0)} - m\omega^2 w^{(0)} = -p^{(0)}, \quad z = 0. \quad (12)$$

Substituting from the continuity condition (3) into the above, yields a pressure boundary condition,

$$(1 - \kappa^{-4} \nabla^2 \nabla^2) \frac{\partial p^{(0)}}{\partial z} = a p^{(0)}, \quad z = 0. \quad (13)$$

where the flexural wavenumber, κ , and the impedance wavenumber, a , are

$$\kappa^4 = m\omega^2 / B, \quad a = \rho_f / m. \quad (14)$$

The impedance wavenumber is related to the null frequency, $\omega_{\text{null}} = ac_f$, which provides a rough demarcation between low and high frequency regimes, $k_f < a$ and $k_f > a$, respectively. Equation (13) describes the dynamic interaction between an acoustic fluid and two joined, flat plates governed by the classical theory of flexure. The strictly 2D version of this problem ($k_y = 0$) was first considered by Brazier-Smith [6], and more recently Norris and Wickham [7] developed a semi-analytic solution of the same problem.

Neglecting the flexural term in (13) gives a simple impedance boundary condition for each plate of the form

$$\frac{\partial \bar{p}^{(0)}}{\partial z} = a_j \bar{p}^{(0)}, \quad z = 0, \quad (15)$$

where $j = 1$ for $x < 0$ and $j = 2$ for $x > 0$. This describes the acoustic diffraction from the junction of two massive, flat plates. This problem has been studied for a two-dimensional geometry in considerable detail by Norris and Rebinsky [2] and its extension to the present 3D problem is summarized in the Appendix. In this paper we will use the background wavefield calculated from the two joined, massive plates as our forcing because of its simplicity. The corresponding normal displacement, which enters into the membrane equations, is given by Eq. (12) without the flexural term,

$$m\omega^2 \bar{w}^{(0)} = \bar{p}^{(0)}, \quad z = 0. \quad (16)$$

In summary, **any** suitable background solution $\bar{w}^{(0)}$ may be used to describe the forcing of the in-plane motion. We choose the simple solution based upon the acoustic diffraction from two massive, flat plates. The more exact solution which includes flexural effects is itself extremely difficult to obtain. The following analysis, although based upon the simpler form of $\bar{w}^{(0)}$, could be easily adapted to the precise form. This is the subject of a future article.

4. Analysis of the in-plane problem

We are now ready to tackle the in-plane equations (10) using the background response for two joined, massive plates with no bending stiffness. We consider an acoustic plane wave obliquely incident on the shell junction $x = 0$, $z = 0$. The background solution which satisfies Eq. (15) for all x is a generalization of the 2D solution discussed in [2], and the required details can be found in the Appendix. With the background presumed known, we can now derive the longitudinal and shear waves on each shell.

4.1. The scattered membrane wavefield

Assume that for plane wave incidence the total wavefields, $\bar{\sigma}_x^{(0)}$ and $\bar{v}^{(0)}$, can be written in terms of specular and scattered components as

$$\mathbf{S}^{(0)}(x) = \mathbf{U}_1 e^{ik_x^0 x} + \mathbf{S}^s(x), \quad (17)$$

where k_x^0 is the x -component of the incoming acoustic wavenumber and

$$\mathbf{S}^{(0)}(x) = \begin{pmatrix} \bar{\sigma}_x^{(0)}(x) \\ \bar{v}^{(0)}(x) \end{pmatrix}, \quad \mathbf{S}^s(x) = \begin{pmatrix} \bar{\sigma}_x^s(x) \\ \bar{v}^s(x) \end{pmatrix}, \quad \mathbf{U}_1 = \begin{pmatrix} T_1 \\ V_1 \end{pmatrix}. \quad (18)$$

The superscript (0) refers to the total quantity while the superscript s denotes that portion scattered from the joint along $x = 0$. The first term of Eq. (17) is the specular solution with amplitude $\mathbf{U}_1$ based upon the hypothesis of the curved plate system being entirely composed of plate 1. The specular inner solution follows from Eqs. (10), (16), and (A.4), as

$$\mathbf{U}_1 = (1 + \mathcal{R}_1) \mathbf{K}_1^{-1}(k_x^0) \mathbf{R}_1, \quad (19)$$

where $\mathcal{R}_1$ is the acoustic plane wave reflection coefficient for plate 1, given by Eq. (A.6). The matrices $\mathbf{K}_{1,2}$, which define the membrane wave dispersion on each plate, are

$$\mathbf{K}_j(k_x) = \begin{bmatrix} k_j^2 + \nu_j k_y^2 - k_x^2 & -ik_y(1 + \nu_j)C_j[k_j^2 - (1 - \nu_j)k_y^2] \\ ik_y(1 - \nu_j)/[(1 - \nu_j)C_j] & 2k_j^2/(1 - \nu_j) - (2 + \nu_j)k_y^2 - k_x^2 \end{bmatrix}, \quad (20)$$

and the curvature 2-vector $\mathbf{R}_j$ is

$$\mathbf{R}_j = \begin{pmatrix} 1/R_{x(j)} + [\nu_j k_j^2 - (1 - \nu_j)k_y^2]/k_j^2 R_y \\ ik_y [1/R_{x(j)} - (2 + \nu_j)/R_y]/m_j \omega^2 \end{pmatrix}, \quad (21)$$

where $j = 1, 2$ denotes the plate of concern.

The remaining part of the total membrane field, $\mathbf{S}^s$, depends upon the junction. We represent it by an outgoing Fourier superposition of plane waves of the form

$$\mathbf{S}^s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{U}(k_x) e^{ik_x x} dk_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{U}_{\pm}(k_x) e^{ik_x x} dk_x, \quad x \begin{matrix} < \\ > \end{matrix} 0. \quad (22)$$

The vector functions $\mathbf{U}_+$, $\mathbf{U}_-$ are the analytic partitions of $\mathbf{U}$ such that

$$\mathbf{U}(k_x) = \mathbf{U}_+(k_x) + \mathbf{U}_-(k_x), \quad (23)$$

and $\mathbf{U}_\pm$ are analytic in the upper $\mathcal{H}^+$ and lower $\mathcal{H}^-$ halves of the complex k_x -plane, respectively. The functions $\mathbf{U}_\pm$ are half transforms of $\mathbf{S}^s(x)$ along each plate separately.

4.2. Transformed equations of motion

Substituting Eq. (17) into the in-plane equations of motion (10) and applying the Fourier half-transforms from Eqs. (22) and (A.8), we obtain a system of equations for the unknowns $\mathbf{U}_\pm$,

$$\mathbf{K}_1 \mathbf{U}_+ = \mathbf{R}_1 P_+ - [\mathbf{S}_{,x}^s(0-) + ik_x \mathbf{S}^s(0-)] , \quad (24a)$$

$$\mathbf{K}_2 \mathbf{U}_- = \mathbf{R}_2 P_- + [\mathbf{S}_{,x}^s(0+) + ik_x \mathbf{S}^s(0+)] - \frac{i\Gamma}{k_x - k_x^0} . \quad (24b)$$

The half transforms $P_\pm$ are the analytic decompositions of the Wiener–Hopf problem for the background wavefield. They are associated with the background pressure solution on $-\infty < x < \infty$, $y = 0$. Progress in solving Eqs. (24) requires explicit versions of these. As discussed above, we will only consider the simplified case where the forcing is given by the massive plate boundary condition (16). The corresponding pressure solution is given in the Appendix, where explicit forms are presented for P_+ and P_- .

The vector Γ is the specular component of the acoustic response on plate 2, which remains because we initially subtracted out the response for an infinitely extended plate 1, see Eq. (17). A straightforward calculation yields

$$\Gamma = (1 + \mathcal{R}_1) [\mathbf{R}_2 - \mathbf{K}_2(k_x^0) \mathbf{K}_1^{-1}(k_x^0) \mathbf{R}_1] . \quad (25)$$

5. Solution of the Wiener–Hopf equations

5.1. Plate dispersion relations

In order to evaluate the analytic functions $\mathbf{U}_\pm$, we multiply Eqs. (24a) and (24b) by the inverse of $\mathbf{K}_1$ and $\mathbf{K}_2$, respectively. These inverses of these matrices are analytic everywhere in the complex k_x -plane except for poles at the physical wavenumbers for membrane waves. Thus, the determinant of each $\mathbf{K}_j$ provides the dispersion relation for the membrane waves on each plate ($j = 1, 2$),

$$\det(\mathbf{K}_j) = (k_j^2 - k_x^2 - k_y^2) \left(\frac{2}{1 - \nu_j} k_j^2 - k_x^2 - k_y^2 \right) = [(k_x^{Lj})^2 - k_x^2] [(k_x^{Tj})^2 - k_x^2], \quad (26)$$

with the first bracketed term describing the longitudinal waves and the second the transverse or shear waves. The inverse of $\mathbf{K}_j$ can be written in terms of a cofactor expansion based upon the dyadic product of the left and right eigenvectors of $\mathbf{K}_j$ evaluated at $k_x^2 = (k_x^{Lj})^2$ and $(k_x^{Tj})^2$. We find

$$\mathbf{K}_j^{-1} = \frac{\mathbf{q}_j^L \otimes \mathbf{p}_j^L}{(k_x^{Lj})^2 - k_x^2} + \frac{\mathbf{q}_j^T \otimes \mathbf{p}_j^T}{(k_x^{Tj})^2 - k_x^2}, \quad (27)$$

where L and T signify longitudinal and transverse waves, respectively, and the polarization vectors are

$$\mathbf{q}_j^L = \begin{pmatrix} d_j/k_j^2 \\ -c_j/k_j^2 \end{pmatrix}, \quad \mathbf{p}_j^L = \begin{pmatrix} 1 \\ -b_j \end{pmatrix}, \quad \mathbf{q}_j^T = \begin{pmatrix} b_j \\ 1 \end{pmatrix}, \quad \mathbf{p}_j^T = \begin{pmatrix} c_j/k_j^2 \\ d_j/k_j^2 \end{pmatrix}. \quad (28)$$

The coefficients b_j, c_j , and d_j are

$$b_j = -ik_y(1 - \nu_j)C_j, \quad c_j = ik_y/C_j, \quad d_j = k_j^2 - (1 - \nu_j)k_y^2. \quad (29)$$

Explicit formulae for the transforms $U_{\pm}$ now follow from Eqs. (24) and (27). These expressions contain apparent poles that generate waves incoming to the joint at $x = 0$, in violation of the radiation condition for the diffracted solution requiring the fields to be outgoing. Cancelling the contributions of k_x^{L1} and k_x^{T1} on plate 1 ($x < 0$), and $-k_x^{L2}$ and $-k_x^{T2}$ on plate 2 ($x > 0$), yields four conditions. These are obtained by taking the dot product of the RHS of Eqs. (24) with the second eigenvector of each numerator of Eq. (27). This corresponds to equating the magnitude of each residue contribution to zero. Setting the contributions from k_x^{L1} and k_x^{T1} to zero in Eq. (24a) for U_- , gives

$$\mathbf{p}_1^L \cdot \left\{ \mathbf{R}_1 P_- - [\mathbf{S}_{,x}^s(0-) + ik_x \mathbf{S}^s(0-)] \right\} \Big|_{k_1=k_1^{L1}} = 0, \quad (30a)$$

$$\mathbf{p}_1^T \cdot \left\{ \mathbf{R}_1 P_- - [\mathbf{S}_{,x}^s(0-) + ik_x \mathbf{S}^s(0-)] \right\} \Big|_{k_1=k_1^{T1}} = 0. \quad (30b)$$

Similarly, the pole contributions from $-k_x^{L2}$ and $-k_x^{T2}$ are zero in the equation for U_- , implying the identities

$$\mathbf{p}_2^L \cdot \left\{ \mathbf{R}_2 P_- + [\mathbf{S}_{,x}^s(0+) + ik_x \mathbf{S}^s(0+)] - \frac{i\Gamma}{k_x - k_x^0} \right\} \Big|_{k_1=-k_1^{L2}} = 0, \quad (31a)$$

$$\mathbf{p}_2^T \cdot \left\{ \mathbf{R}_2 P_- + [\mathbf{S}_{,x}^s(0+) + ik_x \mathbf{S}^s(0+)] - \frac{i\Gamma}{k_x - k_x^0} \right\} \Big|_{k_1=-k_1^{T2}} = 0. \quad (31b)$$

The four conditions in Eqs. (30) and (31) can be solved in the following concise vector form

$$\mathbf{S}_{,x}^s(0-) + i\mathbf{M}_1 \mathbf{S}^s(0-) = P_- (k_x^{L1}) (\mathbf{p}_1^L \cdot \mathbf{R}_1) \mathbf{q}_1^L - P_+ (k_x^{T1}) (\mathbf{p}_1^T \cdot \mathbf{R}_1) \mathbf{q}_1^T \quad (32a)$$

$$\begin{aligned} -\mathbf{S}_{,x}^s(0+) + i\mathbf{M}_2 \mathbf{S}^s(0+) &= P_- (-k_x^{L2}) (\mathbf{p}_2^L \cdot \mathbf{R}_2) \mathbf{q}_2^L + P_+ (-k_x^{T2}) (\mathbf{p}_2^T \cdot \mathbf{R}_2) \mathbf{q}_2^T \\ &\quad + \frac{\mathbf{p}_2^L \cdot \Gamma}{k_x^{L2} + k_x^0} \mathbf{q}_2^L + \frac{\mathbf{p}_2^T \cdot \Gamma}{k_x^{T2} + k_x^0} \mathbf{q}_2^T. \end{aligned} \quad (32b)$$

The interaction matrix $\mathbf{M}_j$ is

$$\mathbf{M}_j = k_x^{Lj} \mathbf{q}_j^L \otimes \mathbf{p}_j^L - k_x^{Tj} \mathbf{q}_j^T \otimes \mathbf{p}_j^T, \quad (33)$$

and the polarization vectors were defined previously in (28).

5.2. Junction conditions

All that remains in determining the half-transforms $U_{\pm}$ is the evaluation of $\mathbf{S}^s$ and its derivatives at $x = \pm 0$. At this point we need to specify the junction conditions for the membrane waves. We consider welded joint conditions. This implies, first, that $\tilde{\sigma}_x^{(0)}$ and $\tilde{\tau}^{(0)}$ are continuous across the junction from $x = 0-$ to $x = 0+$. Using Eq. (18), these are simply

$$\mathbf{S}^{(0)}(0+) = \mathbf{S}^{(0)}(0-). \quad (34)$$

The remaining two conditions are for the displacement perpendicular to the joint $\tilde{u}^{(0)}$ and the shear force $\tilde{\tau}_{xy}^{(0)}$ acting on each plate edge,

$$\tilde{u}^{(0)}(0+) = \tilde{u}^{(0)}(0-), \quad \tilde{\tau}_{xy}^{(0)}(0+) = \tilde{\tau}_{xy}^{(0)}(0-). \quad (35)$$

Equations (35) can be related to $\mathbf{S}_{,x}^{(0)}(x)$ by using the $\bar{u}^{(0)}$ equation (8a) and the constitutive relation (9c) defining $\bar{\tau}_{xy}^{(0)}$, to obtain

$$\rho_f \omega^2 \begin{pmatrix} \bar{u}^{(0)}(x) \\ \bar{\tau}_{xy}^{(0)}(x) \end{pmatrix} = -\mathbf{A}_j \mathbf{S}_{,x}^{(0)}(x), \quad (36)$$

where

$$\mathbf{A}_j = a_j \frac{k_j^2}{(k_x^{Tj})^2} \begin{pmatrix} 2/(1-\nu_j) & ik_y C_j \\ ik_y C_j & -(C_j k_j)^2 \end{pmatrix}. \quad (37)$$

Also, a_j is the impedance wavenumber (14) for each plate, and as usual $j = 1$ for $x < 0$ and $j = 2$ for $x > 0$. The joint condition (35) can now be rewritten succinctly as

$$\mathbf{A}_1 \mathbf{S}_{,x}^{(0)}(0-) = \mathbf{A}_2 \mathbf{S}_{,x}^{(0)}(0+). \quad (38)$$

The local behavior of the scattered stress $\bar{\sigma}_x^s$ and displacement $\bar{v}^s$, given by $\mathbf{S}^s$ at the joint, depend upon the expansion of the transform $\mathbf{U}_{\pm}$ at infinity according to

$$\pm \mathbf{U}_{\pm}(k_x) = \frac{i}{k_x} \mathbf{S}^s(0\mp) + \frac{1}{k_x^2} \mathbf{S}_{,x}^s(0\mp) + o(k_x^{-2}). \quad (39)$$

The junction conditions (34) and (38) are defined for the total fields, $\mathbf{S}^{(0)}(0)$ and $\mathbf{S}_{,x}^{(0)}(0)$, which can be expressed, using (17), as

$$\mathbf{S}^s(0) = \mathbf{S}^{(0)}(0) - \mathbf{U}_1, \quad \mathbf{S}_{,x}^s(0) = \mathbf{S}_{,x}^{(0)}(0) - ik_x^0 \mathbf{U}_1. \quad (40)$$

The second junction condition, (38), can now be written in terms of the unknown value of $\mathbf{S}^{(0)}(0-) = \mathbf{S}^{(0)}(0+) \equiv \mathbf{S}^{(0)}(0)$ by using Eqs. (39) and (40),

$$\lim_{|k_x| \rightarrow \infty} [k_x^2 (\mathbf{A}_1 \mathbf{U}_+ + \mathbf{A}_2 \mathbf{U}_-) + ik_x (\mathbf{A}_2 - \mathbf{A}_1) \mathbf{S}^s(0)] = ik_x^0 (\mathbf{A}_2 - \mathbf{A}_1) \mathbf{U}_1. \quad (41)$$

Substituting for $\mathbf{U}_+$ and $\mathbf{U}_-$ yields an explicit expression for $\mathbf{S}^s(0)$. The total normal stress and tangential displacement at the junction follow from Eqs. (24), (27), (32), and (33), and a good deal of algebra, as

$$\begin{aligned} \mathbf{S}^{(0)}(0) = p_0 (\mathbf{A}_1 \mathbf{M}_1 + \mathbf{A}_2 \mathbf{M}_2)^{-1} & \left\{ \mathbf{A}_1 \left[\frac{D_+(k_x^{L1})}{k_x^{L1} - k_x^0} (\mathbf{p}_1^L \cdot \mathbf{R}_1) \mathbf{q}_1^L + \frac{D_+(k_x^{T1})}{k_x^{T1} - k_x^0} (\mathbf{p}_1^T \cdot \mathbf{R}_1) \mathbf{q}_1^T \right] \right. \\ & \left. + \mathbf{A}_2 \left[\frac{D_-(-k_x^{L2})}{k_x^{L2} + k_x^0} (\mathbf{p}_2^L \cdot \mathbf{R}_2) \mathbf{q}_2^L + \frac{D_-(-k_x^{T2})}{k_x^{T2} - k_x^0} (\mathbf{p}_2^T \cdot \mathbf{R}_2) \mathbf{q}_2^T \right] \right\}. \end{aligned} \quad (42)$$

The junction pressure p_0 is given in Eq. (A.12), and $D_{\pm}$, defined in Eqs. (A.11) through (A.15), are the analytic decompositions of the quotient of acoustic dispersion relations for each plate, see Eq. (A.7).

The transforms for the scattered normal stress and tangential displacement finally follow from Eqs. (24), (32), and (42) as

$$\mathbf{U}_+ = \mathbf{K}_1^{-1} \mathbf{R}_1 P_- + i(\mathbf{M}_1 - k_x \mathbf{I}) \mathbf{S}^s(0) - \mathbf{K}_1^{-1} [(\mathbf{p}_1^L \cdot \mathbf{R}_1) P_+(k_x^{L1}) \mathbf{q}_1^L - (\mathbf{p}_1^T \cdot \mathbf{R}_1) P_+(k_x^{T1}) \mathbf{q}_1^T], \quad (43a)$$

$$\begin{aligned} \mathbf{U}_- = \mathbf{K}_2^{-1} \mathbf{R}_2 P_- + i(\mathbf{M}_2 + k_x \mathbf{I}) \mathbf{S}^s(0) - \mathbf{K}_2^{-1} & [(\mathbf{p}_2^L \cdot \mathbf{R}_2) P_-(-k_x^{L2}) \mathbf{q}_2^L + (\mathbf{p}_2^T \cdot \mathbf{R}_2) P_-(-k_x^{T2}) \mathbf{q}_2^T] \\ & - \left[\frac{\Gamma}{k_x - k_x^0} + \frac{\mathbf{p}_2^L \cdot \Gamma}{k_x^{L2} + k_x^0} \mathbf{q}_2^L + \frac{\mathbf{p}_2^T \cdot \Gamma}{k_x^{T2} + k_x^0} \mathbf{q}_2^T \right], \end{aligned} \quad (43b)$$

where $\mathbf{I}$ is the identity matrix.

6. Diffracted membrane wavefields

6.1. Longitudinal and transverse diffraction coefficients

The scattered solution $\mathbf{S}^{(0)}(x)$ is determined by evaluating Eq. (22) for $x < 0$ and $x > 0$, and therefore depends upon the functions $\mathbf{U}_+$ and $\mathbf{U}_-$ of Eqs. (43a) and (43b), respectively. The physically significant poles of the half-transforms are $k_x = -k_x^{L1}$, and $-k_x^{T1}$ for $x < 0$, and $k_x = k_x^0$, k_x^{L2} , and k_x^{T2} for $x > 0$. By isolating the residue contributions from these poles and using Eq. (17), we deduce that the total scattered wavefield $\mathbf{S}^{(0)}$ has the following form

$$\mathbf{S}^{(0)}(x) = \mathbf{U}_j e^{ik_x^{(j)}x} + \mathbf{D}_j^L e^{ik_x^{L(j)}|x|} + \mathbf{D}_j^T e^{ik_x^{T(j)}|x|} + \mathbf{S}_{near}(x), \quad (44)$$

where $j = 1$ or 2 corresponds to $x < 0$ and $x > 0$, respectively. The first term in the right member of (44) represents the field on each shell in the absence of the other, and the coefficients $\mathbf{D}_j^L$ and $\mathbf{D}_j^T$ give the diffracted longitudinal and shear wave amplitudes, respectively. The remainder $\mathbf{S}_{near}$ denotes the Fourier integral of a function with no physically significant poles. It therefore vanishes far from the joint, and is important only in the near field.

The diffraction coefficient vectors can be written as

$$\mathbf{D}_j^L = \mathcal{L}_j \mathbf{q}_j^L, \quad \mathbf{D}_j^T = \mathcal{T}_j \mathbf{q}_j^T, \quad (45)$$

using the polarization vectors $\mathbf{q}_j^{L,T}$ given by (28). Evaluating the residues and simplifying, yields for plate 1,

$$\mathcal{L}_1 = \mathbf{p}_1^L \cdot \left\{ \mathbf{S}^{(0)}(0) - \frac{p_0}{2k_x^{L1}} \mathbf{R}_1 \left[\frac{D_+(k_x^{L1})}{k_x^{L1} - k_x^0} + \frac{D_+(-k_x^{L1})}{k_x^{L1} + k_x^0} \right] \right\}, \quad (46a)$$

$$\mathcal{T}_1 = \mathbf{p}_1^T \cdot \left\{ \mathbf{S}^{(0)}(0) - \frac{p_0}{2k_x^{T1}} \mathbf{R}_1 \left[\frac{D_+(k_x^{T1})}{k_x^{T1} - k_x^0} + \frac{D_+(-k_x^{T1})}{k_x^{T1} + k_x^0} \right] \right\}, \quad (46b)$$

and for plate 2,

$$\mathcal{L}_2 = \mathbf{p}_2^L \cdot \left\{ \mathbf{S}^{(0)}(0) - \frac{p_0}{2k_x^{L2}} \mathbf{R}_2 \left[\frac{D_-(k_x^{L2})}{k_x^{L2} - k_x^0} + \frac{D_-(-k_x^{L2})}{k_x^{L2} + k_x^0} \right] \right\}, \quad (47a)$$

$$\mathcal{T}_2 = \mathbf{p}_2^T \cdot \left\{ \mathbf{S}^{(0)}(0) - \frac{p_0}{2k_x^{T2}} \mathbf{R}_2 \left[\frac{D_-(k_x^{T2})}{k_x^{T2} - k_x^0} + \frac{D_-(-k_x^{T2})}{k_x^{T2} + k_x^0} \right] \right\}, \quad (47b)$$

where the quantities associated with the background solution, *i.e.* the junction pressure p_0 , and the split functions $D_{\pm}$, are explained in the Appendix. We observe that the diffraction coefficients depend upon $\mathbf{S}^{(0)}(0)$ given by Eq. (42), which consists of the normal stress $\bar{\sigma}_x^{(0)}$ and the $\bar{v}^{(0)}$ displacement, and also on the pressure p_0 at the joint given by Eq. (A.12). All the diffraction coefficients vanish when both shells are flat, because there is no coupling of the normal motion to the in-plane motion in this limit. However, if one shell is flat with the other having curvature $1/R_x$, then longitudinal and shear waves are still diffracted onto both shells. The amplitude of the diffracted waves on the shell which is flat are simply just the dot product of the representative wave type polarization vector $\mathbf{p}_{1,2}^{L,T}$ with the total field $\mathbf{S}^{(0)}(0)$ at the junction, in Eq. (42).

6.2. The limiting case of $a_1 = a_2$

The general solution simplifies considerably when both plates have equal mass densities per unit area, $m_1 = m_2$ or equivalently, $a_1 = a_2$. The outer solution, which we have simplified as a diffracted response from a discontinuity in the value of a at the junction, is now trivial. The change in plate properties is reflected in

the fact that $\mathbf{A}_1 \neq \mathbf{A}_2$. Equation (44) still applies, but the near-field $\mathbf{S}_{near}$ is identically zero, and the diffracted amplitudes are simply

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{T}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{p}_1^L \\ \mathbf{p}_1^T \end{pmatrix} \cdot [\mathbf{S}^{(0)}(0) - \mathbf{U}_1], \quad \begin{pmatrix} \mathcal{L}_2 \\ \mathcal{T}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{p}_2^L \\ \mathbf{p}_2^T \end{pmatrix} \cdot [\mathbf{S}^{(0)}(0) - \mathbf{U}_2], \quad (48)$$

where $\mathbf{S}^{(0)}(0)$ is now given by

$$\mathbf{S}^{(0)}(0) = (\mathbf{A}_1 \mathbf{M}_1 + \mathbf{A}_2 \mathbf{M}_2)^{-1} [\mathbf{A}_1 (\mathbf{M}_1 + k_x^0 \mathbf{I}) \mathbf{U}_1 + \mathbf{A}_2 (\mathbf{M}_2 - k_x^0 \mathbf{I}) \mathbf{U}_2]. \quad (49)$$

These results follow from the general solution of Eqs. (46) and (47) by noting that both kernel decompositions D_+ and D_- are unity in this limit (cf. the Appendix). We also note that if the plates have the same longitudinal and shear wave speeds in addition to having equal mass densities, then $\mathbf{A}_1 = \mathbf{A}_2$ and $\mathbf{M}_1 = \mathbf{M}_2$ and the diffracted amplitudes simplify further to

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{T}_1 \end{pmatrix} = \frac{p_0}{2} \begin{pmatrix} \mathbf{p}_1^L \\ \mathbf{p}_1^T \end{pmatrix} \cdot (\mathbf{I} + k_x^0 \mathbf{M}_1^{-1}) \mathbf{K}_1^{-1}(k_x^0) (\mathbf{R}_2 - \mathbf{R}_1), \quad (50a)$$

$$\begin{pmatrix} \mathcal{L}_2 \\ \mathcal{T}_2 \end{pmatrix} = \frac{p_0}{2} \begin{pmatrix} \mathbf{p}_1^L \\ \mathbf{p}_1^T \end{pmatrix} \cdot (\mathbf{I} - k_x^0 \mathbf{M}_1^{-1}) \mathbf{K}_1^{-1}(k_x^0) (\mathbf{R}_1 - \mathbf{R}_2). \quad (50b)$$

It evident from Eq. (50) that the membrane wave diffraction effect vanishes under these circumstances when the curvatures are identical ($\mathbf{R}_1 = \mathbf{R}_2$), as one would expect. In summary, Eq. (50) describes the acoustic-to-membrane diffraction for two shells with the same material properties and thickness but with different curvatures on either side of the junction line.

6.3. The limiting case of $k_y = 0$

We previously studied the related two-dimensional diffraction problem [2], which corresponds to the special case of $k_y = 0$ in the present problem. The limiting forms of Eqs. (46) are

$$\mathcal{L}_1 = \bar{\sigma}^{(0)}(0) - \frac{p_0}{2k_x^{L1}} \left(\frac{1}{R_{x(1)}} + \frac{\nu_1}{R_y} \right) \left[\frac{D_+(-k_x^{L1})}{k_x^{L1} + k_x^0} + \frac{D_+(k_x^{L1})}{k_x^{L1} - k_x^0} \right], \quad (51a)$$

$$\mathcal{L}_2 = \bar{\sigma}^{(0)}(0) - \frac{p_0}{2k_x^{L2}} \left(\frac{1}{R_{x(2)}} + \frac{\nu_2}{R_y} \right) \left[\frac{D_-(-k_x^{L2})}{k_x^{L2} + k_x^0} + \frac{D_-(k_x^{L2})}{k_x^{L2} - k_x^0} \right], \quad (51b)$$

where the junction stress is

$$\bar{\sigma}^{(0)}(0) = \frac{p_0}{a_1 k_x^{L1} + a_2 k_x^{L2}} \left[a_1 \left(\frac{1}{R_{x(1)}} + \frac{\nu_1}{R_y} \right) \frac{D_+(k_x^{L1})}{k_x^{L1} - k_x^0} + a_2 \left(\frac{1}{R_{x(2)}} + \frac{\nu_2}{R_y} \right) \frac{D_-(-k_x^{L2})}{k_x^{L2} + k_x^0} \right]. \quad (52)$$

The shear diffraction coefficients are found to be, using Eq. (42), $\mathcal{T}_1 = \mathcal{T}_2 = v^{(0)}(0) = 0$, as expected.

7. Membrane to acoustic diffraction and reciprocity

We now consider the reciprocal problem of an incoming membrane wave striking the junction of the shells. This interaction generates reflected and transmitted membrane wavefields in the shell and a diffracted acoustic

response in the fluid. We have discussed this case in some length previously for the two dimensional geometry [2], where we obtained the results by both direct calculation and through the use of reciprocity [2,8]. The principle of reciprocity offers a simple and logical means to address this problem and we will outline the procedure for applying it.

7.1. General solution

An obliquely incident membrane wave, either longitudinal or shear, incoming from plate 1 and striking the joint at $x = 0$, generates both reflected and transmitted membrane waves and a diffracted pressure wavefield. However, in keeping with the decoupling arguments of Section 3, the plate wavefields can be approximated as if the fluid was absent. Thus, to leading order the fluid loading does not influence the membrane-membrane scattering. The reflected and transmission coefficients are then calculated using the joint conditions given by Eqs. (34) and (35) for the shells and junction *in vacuo*.

Applying the same perturbation formalism of Section 2, it turns out that the diffracted acoustic pressure is found by considering this leading order membrane wavefield as the driving mechanism for the leading order pressure, $\bar{p}^{(0)}$. The coupling is *via* the w -equation of (1c), neglecting the bending term as discussed before. The leading order acoustic field can then be obtained by solving this system with the forcing given by the *in-vacuo* membrane wavefield, combined with the Helmholtz equation (4) and the radiation condition. This approach is as cumbersome, in terms of algebra, as the solution derived in Sections 4 through 6, and will not be pursued. We note, however, that the analogous 2D solution for membrane incidence is discussed in some detail by Norris and Rebinsky [2].

7.2. Reciprocity

The principle of acoustic reciprocity for arbitrary fluid-structure interaction is as follows. Let $\bar{\mathbf{u}}^{(1)}$ be the structural displacement resulting from a monopole of strength $\rho_f \omega^2$ at position $\mathbf{x}^{(1)}$ in the fluid, and $\bar{p}^{(2)}$ the acoustic pressure caused by a unit force directed along $\mathbf{e}$ at the point $\mathbf{x}^{(2)}$ on the structure. Then [8],

$$\mathbf{e} \cdot \bar{\mathbf{u}}^{(1)}(\mathbf{x}^{(2)}) = \bar{p}^{(2)}(\mathbf{x}^{(1)}) . \quad (53)$$

This principle will now be applied to show that knowledge of the acoustic-to-membrane diffraction coefficients allows one to obtain the membrane-to-acoustic diffracted response.

The left member in Eq. (53) follows from the analysis of the previous Sections. Thus, the $\bar{v}$ -displacement associated with the diffracted membrane waves is given by the second component of $\mathbf{S}^s$ in (18), and the $\bar{u}$ -displacement follows from the first component of the relation (36). The scattered membrane displacement is therefore, on plate 1 for example,

$$\begin{pmatrix} \bar{u}^s \\ \bar{v}^s \end{pmatrix} = \frac{\bar{p}^{(1)}(0,0)}{m_1 \omega^2} \begin{bmatrix} 1 \\ i k_x^{L1} \\ -k_y/k_x^{L1} \end{bmatrix} \mathcal{L}_1 e^{i k_x^{L1} |x|} + m_1 \omega^2 \begin{pmatrix} k_y/k_x^{T1} \\ 1 \end{pmatrix} \mathcal{T}_1 e^{i k_x^{T1} |x|} , \quad (54)$$

where $\bar{p}^{(1)}(0,0) = (\rho_f \omega^2) (-i/4) H_0^{(1)}(\bar{k}_f r^{(1)})$ is the incident pressure at the junction induced by the monopole in the fluid, and $\bar{k}_f$ is defined in Eq. (A.3).

In order to determine the right member in the reciprocal identity (53) we need to solve for a line force in the plate equations governing the membrane motion. Again, the leading order contribution to this Green's function follows from the *in-vacuo* equations, and the curvature effects can be neglected. Thus, referring to Eqs. (1) and (2), we need to solve

$$\bar{u}_{,xx} + i k_y \frac{(1 + \nu_1)}{2} \bar{v}_{,x} + \left[k_1^2 - \frac{(1 - \nu_1)}{2} k_y^2 \right] \bar{u} = e_1 C_1^{-1} \delta(x - x') , \quad (55a)$$

$$\frac{(1-\nu_1)}{2} \bar{v}_{,xx} + ik_y \frac{(1+\nu_1)}{2} \bar{u}_{,x} + (k_1^2 - k_y^2) \bar{v} = e_2 C_1^{-1} \delta(x-x') . \quad (55b)$$

The solution for $x > x'$, for example, is easily determined as

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \left\{ e^{ik_x^{(1)}(x-x')} \begin{pmatrix} 1 \\ k_y/k_x^{(1)} \end{pmatrix} \begin{pmatrix} k_x^{(1)} \\ k_y \end{pmatrix}^T + e^{ik_x^{(1)}(x-x')} \begin{pmatrix} -k_y/k_x^{(1)} \\ 1 \end{pmatrix} \begin{pmatrix} -k_y \\ k_x^{(1)} \end{pmatrix}^T \right\} \cdot \frac{\mathbf{e}}{2im_1\omega^2} . \quad (56)$$

The line force generates both longitudinal and shear waves, as expected. We now define membrane to acoustic diffraction coefficients, $\mathcal{D}_1^L$ and $\mathcal{D}_1^T$, which couple longitudinal and shear waves of unit amplitude displacement, respectively, to the acoustic field. The contribution to the diffracted far-field acoustic pressure is defined as

$$p^s = \mathcal{D}_1(\theta) \sqrt{\frac{2}{\pi k_{\text{tr}}}} e^{-i\pi/4} e^{ik_{\text{tr}}r} , \quad \bar{k}_{\text{tr}}r \rightarrow \infty , \quad 0 < \theta \leq \pi , \quad (57)$$

where $\mathcal{D}_1$ is either $\mathcal{D}_1^L$ or $\mathcal{D}_1^T$.

We are now in a position to determine both sides of the reciprocity relation (53). The relationship is simplified by choosing two different direction vectors $\mathbf{e}$ to isolate either the longitudinal or shear wave. Thus, $\mathbf{e}_1^L = (1, -k_y/k_x^{(1)})$ and $\mathbf{e}_1^T = (k_y/k_x^{(1)}, 1)$ generate only longitudinal and transverse waves, respectively. Using these direction vectors in (53) with the displacement generated by a monopole in the fluid $\bar{\mathbf{u}}^{(1)}(\mathbf{x}^{(2)})$ given by (54) and the acoustic pressure $\bar{p}^{(2)}(\mathbf{x}^{(1)})$ generated by a force directed along $\mathbf{e}$ defined by (57), one then obtains for an incident longitudinal wave on plate 1

$$\mathcal{D}_1^L(\theta) = \frac{1}{4} k_x^{(1)} a_1 \left(1 + \frac{k_y^2}{k_x^{(1)} k_x^{(1)}} \right) \mathcal{L}_1(\pi - \theta) , \quad (58)$$

and for an incident shear wave on plate 1

$$\mathcal{D}_1^T(\theta) = \frac{1}{4} k_y a_1 \left(1 + \frac{k_y^2}{k_x^{(1)} k_x^{(1)}} \right) \left(\frac{m_1 \omega^2}{ik_y} \right) \mathcal{T}_1(\pi - \theta) , \quad (59)$$

where θ is defined by $k_x = \bar{k}_{\text{tr}} \cos \theta$ (cf. Eq. (A.3)). Also note that the Hankel function appearing in $\bar{p}^{(1)}(0,0)$ of (54) was approximated for large argument. Similar reciprocity relationships exist for the diffraction coefficients $\mathcal{D}_2^{L,T}$ and the corresponding coefficients $\mathcal{L}_2, \mathcal{T}_2$ for longitudinal or shear wave incidence on plate 2.

8. Conclusions

We have proposed a perturbation scheme to obtain approximate but explicit expressions for the acoustic-to-membrane diffraction coefficients which describe the wavefield in two joined shells. The perturbation procedure implies a leading order decoupling of the membrane or in-surface dynamics from the background or outer solution. The latter depends upon the inertial response of the shells, and may itself require solving a rather difficult diffraction problem. In this study we have neglected flexural effects in order to obtain the simplest background wavefield, and thus illustrate the method of solution.

We have shown how the diffraction coefficients for the longitudinal and shear plate waves can be obtained once the background wavefield is known. The diffraction coefficients for acoustic plane wave incidence are given in Eqs. (46) and (47). These formulae involve the leading order junction values of the normal stress

σ_x , the y -displacement v , and the acoustic pressure p . If one plate is flat then its diffracted response depends simply upon the total field at the junction. The results reported here provide estimates of the coupling of an acoustic signal to longitudinal and shear waves at junctions between two curved shells. In a future study we will incorporate the proper background solution appropriate to a junction of two flat plates in flexure.

Appendix A. Diffraction from two massive plates

The background solution for the normal displacement $\bar{w}^{(0)}$ and the pressure $\bar{p}^{(0)}$ is defined by the diffraction from two massive plates of an acoustic wave incident at an oblique angle to the junction. The two-dimensional version of this Wiener–Hopf problem ($k_y = 0$) was discussed in a previous paper [2]. The two plates have different masses, implying the impedance boundary conditions, see Eq. (15),

$$\frac{\partial \bar{p}^{(0)}}{\partial z} = a_1 \bar{p}^{(0)}, \quad x < 0, \quad z = 0, \quad (\text{A.1a})$$

$$\frac{\partial \bar{p}^{(0)}}{\partial z} = a_2 \bar{p}^{(0)}, \quad x > 0, \quad z = 0. \quad (\text{A.1b})$$

The y -dependence has been removed so that $p^{(0)} = \bar{p}^{(0)} e^{ik_y y}$. The problem statement is completed by noting that the pressure satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_f^2 - k_y^2 \right) \bar{p}^{(0)} = 0, \quad (\text{A.2})$$

plus the radiation condition for the scattered field in the fluid.

The solution for three-dimensional incidence (oblique relative to the junction) can be determined from the two-dimensional problem by replacing k_f with

$$\bar{k}_f \equiv \sqrt{k_f^2 - k_y^2}, \quad \forall k_y, \quad (\text{A.3})$$

in the 2D solution. The total pressure, $\bar{p}^{(0)}$, is then given by

$$\bar{p}^{(0)} = \bar{p}^i + \bar{p}^r + \bar{p}^s = e^{ik_x^0 x + k_z^0 z} + \mathcal{R}_1(k_x^0) e^{ik_x^0 x - k_z^0 z} + \bar{p}^s(x, z), \quad (\text{A.4})$$

where

$$k_z = (k_x^2 - \bar{k}_f^2)^{1/2}, \quad (\text{A.5})$$

and k_z is defined as an analytic function in the complex k_x -plane so that along the real axis $k_z = -i(\bar{k}_f^2 - k_x^2)^{1/2}$ for $|k_x| < \bar{k}_f$ and $k_z = (k_x^2 - \bar{k}_f^2)^{1/2}$ for $|k_x| > \bar{k}_f$. The wavenumber $k_z^0 = k_z(k_x^0)$ where k_x^0 is the x -component of the incoming acoustic plane wave and is assumed to lie in the upper half of the complex k_x -plane. The acoustic reflection coefficients are

$$\mathcal{R}_j(k_x) = -\tilde{D}_j(k_x)/D_j(k_x), \quad (\text{A.6})$$

where $j = 1$ for $x < 0$ and $j = 2$ for $x > 0$, and the fluid loaded dispersion relations

$$D_j(k_x) = a_j - k_z, \quad \tilde{D}_j(k_x) = a_j + k_z, \quad (\text{A.7})$$

satisfy Eq. (15) for outgoing wave solutions and for incoming solutions, respectively. We assume that $D_{1,2}(k_x) \neq 0$, $k_x \in \mathcal{H}^+ \cap \mathcal{H}^-$, where $\mathcal{H}^\pm$ are the upper and lower halves of the complex k_x -plane. Also,

we further assume for later convenience that $\tilde{k}_f$ has a small positive imaginary part i.e., $\tilde{k}_f = |\tilde{k}_f|e^{i\delta}$, where $0 < \delta \ll 1$.

The scattered component of the wavefield $\bar{p}^s$ is written as a Fourier spectrum partitioned into two distinct half-transforms each of which vanishes for either $x < 0$ or $x > 0$. Thus,

$$\bar{p}^s(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(k_x) e^{ik_x x - k_z z} dk_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{\pm}(k_x) e^{ik_x x - k_z z} dk_x, x \begin{matrix} < \\ > \end{matrix} 0, \quad (\text{A.8})$$

where $P_{\pm}$ are defined such that

$$P(k_x) = P_+(k_x) + P_-(k_x), \quad (\text{A.9})$$

and $P_{\pm}$ are analytic in $\mathcal{H}^{\pm}$, respectively. Borrowing from the analysis in ref. [2], we can write the scattered pressure transform $P(k_x)$ as

$$P(k_x) = \frac{i p_0}{k_x - k_x^0} [D_+(k_x) - D_-(k_x)], \quad (\text{A.10})$$

where the functions $D_{\pm}(k_x)$ are the analytic decompositions in the upper $\mathcal{H}^+$ and lower $\mathcal{H}^-$ complex k_x -plane, respectively, such that

$$D(k_x) = \frac{D_1(k_x)}{D_2(k_x)} = \frac{D_-(k_x)}{D_+(k_x)}. \quad (\text{A.11})$$

Also, p_0 is the total pressure at the joint,

$$p_0 = \frac{1 + \mathcal{R}_1(k_x^0)}{D_+(k_x^0)} = \frac{1 + \mathcal{R}_2(k_x^0)}{D_-(k_x^0)}, \quad (\text{A.12})$$

and $\mathcal{R}_{1,2}$ are the plane wave reflection coefficients for each plate. The function D_- can be written in semi-explicit form

$$D_-(k_x) = [D(k_x)]^{1/2} \left[\frac{(\alpha_1 - k_x)(\alpha_2 + k_x)}{(\alpha_1 + k_x)(\alpha_2 - k_x)} \right]^{1/4} e^{i[I(a_1, k_x) - I(a_2, k_x)]}, \quad (\text{A.13})$$

where

$$I(a, k_x) = \frac{1}{\pi} \int_0^{k_1} \left[\alpha \cosh^{-1} \left(\frac{\alpha}{\tilde{k}_f} \right) - \frac{i a t}{k_z(t)} \sin^{-1} \left(\frac{t}{\tilde{k}_f} \right) \right] \frac{dt}{\alpha^2 - t^2}. \quad (\text{A.14})$$

and the parameter $\alpha \equiv (\tilde{k}_f^2 + a^2)^{1/2}$. The function $D_+(k_x)$ follows from the symmetry property

$$D_+(k_x) = 1/D_-(-k_x), \quad (\text{A.15})$$

because D is an even function of k_x . Finally, the sum decomposition of P given by Eq. (A.10) is obtained by taking care of the offending pole at $k_x = k_x^0$, with the result

$$P_{\pm}(k_x) = \frac{\pm i p_0}{k_x - k_x^0} [D_{\pm}(k_x) - D_{\pm}(k_x^0)]. \quad (\text{A.16})$$

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