ELSEVIER

Wave Motion 21 (1995) 127-147

WAVE MOTION

Rays, beams and quasimodes on thin shell structures

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Received 3 December 1993; revised 7 September 1994

Abstract

Dynamic ray equations for membrane and flexural waves on arbitrarily curved and inhomogeneous smooth thin shells are derived from the equations of motion. The ray paths are geodesics only if the wave speed is constant along the ray. The evolution equations for the wavefront curvature depends upon both the local Gaussian curvature along the ray, and the transverse derivative of the wave speed. The propagation of gaussian beams is discussed and applied to the problem of finding the quasimodes associated with closed ray paths. The ray equations are applied to a specific example of a smooth structure with piecewise constant curvature which exhibits both stable and unstable ray paths.

1. Introduction

Geometrical optics methods offer the possibility of describing the motion of very large thin shell structures of arbitrary, smooth shape [1-3]. However, many of the elements required have not been sufficiently developed to date, such as the equations for the evolution of wavefront curvature on regions of variable curvature and wave speed. This paper is an attempt to address some of these deficiencies. Such issues as how the simultaneous presence of surface curvature and inhomogeneity conspire to affect the ray equations are addressed. The basic parameters that enter into the description of gaussian beams and quasimodes are also outlined in some detail, with the view to applications on finite, smooth shells.

Separate ray theories for bending waves and membrane waves (longitudinal and shear) on thin shells have been described by Steele [1] and Chien and Steele [2]. Their starting point was the Lagrangian-variational formulation of Whitham [4], which circumvents the need to consider the equations for motion explicitly. Alternatively, using the more conventional approach, and starting from the general equations for thin shells of arbitrary curvature, Pierce [3] has shown that one can find a local dispersion relation for short wavelength disturbances that includes both flexural and membrane waves. Applications to cylindrical shells were given by Pierce and Kil [5]. In a recent paper, Norris and Rebinsky [6] presented simpler form of Pierce's equation, and also derived separate dispersion relations for membrane and flexural waves on arbitrarily shaped shells. The flexural relation is similar to one found previously by Germogenova [7] and Steele [1], and also by Pierce [3] as a limit of his general dispersion relation. The separate relations of Norris and Rebinsky [6] are based upon distinct asymptotic scalings for the membrane and flexural waves, where the scaling involves a non-dimensional curvature parameter, $\epsilon \ll 1$. A similar approach will be adopted here. The basic idea stems from noting that in the absence of curvature at all points (i.e., a flat plate) the equations for in-surface displacements decouple

0165-2125/95/$09.50
SSDI 0165-2125(94)00048-4

from the equation for the normal displacement. It is well-known that in this limit the decoupled equations support membrane and flexural waves, respectively, where the membrane wave types are longitudinal or shear, depending upon the displacement polarization. In the presence of curvature it is therefore natural to use the small parameter

$$\epsilon = \sqrt{h/R}, \tag{1}$$

where R is an arbitrary length, on the order of the minimum radius of curvature, and h is the shell thickness. We will develop asymptotic solutions for ϵ small but non-zero.

The equations of motion of arbitrarily curved thin shells are summarized in Section 2, where the notation is also defined. Membrane and flexural waves possess different dispersive properties and are discussed separately. Membrane waves are considered first, in Section 3, and the ray and transport equations are derived in Section 4. The parameters required to define the propagation of gaussian beams are discussed in Section 5, and some general properties are derived. For example, it is shown that a gaussian beam does not possess singularities at caustics, but evolves smoothly at all points. It is possible to obtain periodic short wavelength solutions associated with ray paths which are closed and stable. These "quasimodes" are gaussian beam solutions that are periodic and concentrated near periodic ray paths [8,9]. They are defined and constructed in Section 6, where a discussion is also given concerning their relationship to structural modes. A detailed example illustrating how to apply the beam and quasimode results is presented in Section 7 for a "complex structure" which exhibits stable and unstable periodic rays. Flexural waves are discussed in Section 8.

2. Thin shell equations

The shell equations employed here are similar to those of Green and Zerna [10], and are discussed in more detail by Pierce [3] and by Norris and Rebinsky [6]. The curvilinear coordinates on the shell are θ^1 and θ^2, with corresponding direction vectors $\mathbf{a}_\alpha \equiv \mathbf{x}_{,\alpha}$, $\alpha = 1, 2$, and surface normal $\mathbf{a}_3 = \mathbf{a}_1 \wedge \mathbf{a}_2/|\mathbf{a}_1 \wedge \mathbf{a}_2|$. Greek sub- or superscripts assume the values 1 or 2, and the suffix $,\alpha$ denotes differentiation with respect to θ^α. The surface curvature tensor and the surface metric tensor, both symmetric, are $d_{\alpha\beta} = \mathbf{a}_\alpha \bullet \mathbf{a}_{3,\beta}$ and $a_{\alpha\beta} = \mathbf{a}_\alpha \bullet \mathbf{a}_\beta$, respectively. Vectors are defined relative to these tangent vectors by their components, e.g., $\mathbf{p} = p^\alpha \mathbf{a}_\alpha$. The covariant derivative is denoted by $D_\beta v^\alpha = v^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma$, where the Christoffel symbols are $\Gamma^\alpha_{\beta\gamma} = a^{\alpha\lambda}\, \mathbf{a}_\lambda \bullet \mathbf{a}_{\beta,\gamma}$. Finally, the principal directions on the surface are numbered I and II such that $(\mathbf{e}_{\mathrm{I}}, \mathbf{e}_{\mathrm{II}}, \mathbf{a}_3)$ form an orthonormal triad.

The displacement of a point originally on the middle surface is denoted $\mathbf{u} = v^\alpha \mathbf{a}_\alpha + w\,\mathbf{a}_3$. The equations of motion for a shell composed of isotropic material are

$$\rho h \frac{\partial^2 v^\alpha}{\partial t^2} = D_\beta \left\{ C \left[(1-\nu) e^{\alpha\beta} + \nu e^\gamma_\gamma a^{\alpha\beta} \right] \right\}, \quad \alpha = 1, 2, \tag{2a}$$

$$\rho h \frac{\partial^2 w}{\partial t^2} = -D_\alpha D_\beta \left[B(1-\nu) D^\alpha D^\beta w \right] - D_\alpha D^\alpha \left[B\nu D_\beta D^\beta w \right] - C \left[(1-\nu) d^\alpha_\beta e^\beta_\alpha + \nu d^\alpha_\alpha e^\beta_\beta \right]. \tag{2b}$$

The in-surface strains are

$$e_{\alpha\beta} = \frac{1}{2} \left(a_{\beta\gamma} D_\alpha v^\gamma + a_{\alpha\gamma} D_\beta v^\gamma \right) + d_{\alpha\beta}\, w, \tag{3}$$

h is the thickness, ρ the mass density of the shell material, and $C = Eh/(1-\nu^2)$ and $B = Eh^3/12(1-\nu^2)$ are the extensional and bending stiffness, respectively, where E is the Young's modulus. The wave speeds for membrane waves are c_{p} and c_{s}, where $c_{\mathrm{p}}^2 = E/(1-\nu^2)\rho$ for longitudinal waves, and $c_{\mathrm{s}}^2 = E/2(1+\nu)\rho$ for shear waves.

3. Membrane waves

Membrane waves are solutions that are short wavelength relative to the radius of curvature, but long wavelength relative to the thickness. In order to emphasize this scaling, we consider time-harmonic solutions of the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}^{(\epsilon)}(\mathbf{x}) \exp\left\{ i\epsilon^{-1}\omega\left(\phi(\mathbf{x}) - t\right)\right\}, \tag{4}$$

where ϵ is defined in (1). The scaling in (4) is specific to membrane waves, and assumes that the frequency ω is an O(1) parameter in terms of the length scale R. That is, $\omega|\nabla\phi|R = \mathrm{O}(1)$, and the wavenumber $\mathbf{k}$ associated with (4) then satisfies $kh \ll 1$ and $kR \gg 1$. Thus, the wavelength is large compared with h, as it should be for thin shell theory, and small relative to R, so that it approximates the flat plate limit. A similar scaling scheme was adopted by Logan [11] in considering edge effects on thin shells. The "conventional" procedure is to take ω as a large parameter, and is recovered by setting $\epsilon = 1$, or equivalently, choosing $R = h$ since it is an arbitrary parameter. In summary, the introduction of the scaling parameter ϵ is purely to assist in the asymptotics. After performing the asymptotic scaling we may set $\epsilon \to 1$, so that ω is then the actual frequency and $\phi(\mathbf{x})$ the phase function.

The amplitude functions are also expanded as asymptotic series in ϵ,

$$w^{(\epsilon)}(\mathbf{x}) = \epsilon W(\mathbf{x}) + \epsilon^2 \widehat{W}(\mathbf{x}) + \ldots, \qquad v^{(\epsilon)\alpha}(\mathbf{x}) = V^\alpha(\mathbf{x}) + \epsilon \widehat{V}^\alpha(\mathbf{x}) + \ldots . \tag{5}$$

Substitute (4) and (3) into (2) and keep only the leading order terms, of order unity. This implies that the normal displacement is decoupled and the in-plane components satisfy

$$\left[\frac{1}{2}(1-\nu)p^2 a^\alpha_\beta + \frac{1}{2}(1+\nu)p^\alpha p_\beta - c_{\mathrm{p}}^{-2} a^\alpha_\beta\right] V^\beta = 0. \tag{6}$$

All quantities in this equation are defined in Section 2 except $\mathbf{p}$, the slowness vector,

$$p_\alpha = D_\alpha \phi. \tag{7}$$

Also, $p^2 = p_\alpha p^\alpha$, and define the unit vector in the direction of the slowness vector as $\mathbf{n} = \mathbf{p}/p$, also known as the wave normal, and let $\mathbf{m}$ be the in-surface unit vector perpendicular to it, $\mathbf{m} = \mathbf{a}_3 \wedge \mathbf{n}$. The next in the asymptotic sequence of equations obtained from (2), (4) and (3) gives a pair of equations involving both V^α and $\widehat{V}^\alpha$. Contraction of these equations with V_α, and subsequent use of (6) gives a single equation, the *transport equation*,

$$D_\beta\left\{C\left[\frac{1}{2}(1-\nu)p^\beta V_\alpha V^\alpha + \frac{1}{2}(1+\nu)p^\alpha V_\alpha V^\beta\right]\right\} = 0. \tag{8}$$

Setting the determinant to zero in (6) and using the identity $a^\alpha_\beta = n^\alpha n_\beta + m^\alpha m_\beta$ to simplify, gives

$$\left(p^2 c_{\mathrm{p}}^2 - 1\right)\left(p^2 c_{\mathrm{s}}^2 - 1\right) = 0. \tag{9}$$

The possible values of the slowness are therefore $p = 1/c_{\mathrm{p}}$, or $p = 1/c_{\mathrm{s}}$ with eigenvectors (polarizations) $\mathbf{V} = V\mathbf{n}$, and $\mathbf{V} = V\mathbf{m}$. These are longitudinal and shear waves, respectively. Substituting for V^α into the transport equation (8), it reduces in either case to

$$D_\alpha\left(V^2 \rho h c n^\alpha\right) = 0, \tag{10}$$

where c is the appropriate speed, c_{p} or c_{s}. These equations reflect the conservation of energy flux for each wave type, where $\rho h c$ is the analog of the acoustic impedance.

4. Ray and transport equations

4.1. Summary of results

The main results for membrane waves are first summarized. The details of the derivations are in the next three subsections, followed by a discussion. The equations for ray paths are found to be

$$\frac{\mathrm{d}^2\theta^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}\theta^\beta}{\mathrm{d}s}\frac{\mathrm{d}\theta^\gamma}{\mathrm{d}s} + \left(a^{\alpha\beta} - \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s}\frac{\mathrm{d}\theta^\beta}{\mathrm{d}s}\right)\frac{c_{,\beta}}{c} = 0, \tag{11}$$

where s is arc length. Apart from the final term involving the surface derivative of the speed, this is the usual equation for geodesics on a surface [12,13]. Hence, the rays can diverge from the surface geodesics when c varies with position, which is similar to the departure of rays from straight lines in a 2 or 3 dimensional acoustic medium with variable sound speed.

The amplitude function, $V(s)$, for either type of membrane wave, depends upon the local wave impedance ρhc and a ray tube area parameter $A(s)$,

$$V(s) = V(0)\sqrt{\frac{(\rho hc\,A)(0)}{(\rho hc\,A)(s)}}. \tag{12}$$

The equation for the change in ray tube area is a modified Jacobi equation,

$$c\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{c}\frac{\mathrm{d}A}{\mathrm{d}s} + (K+L)\,A = 0, \tag{13}$$

where K is the Gaussian curvature, and L is related to the second derivative of the speed in the direction perpendicular to the ray direction, i.e.,

$$K = \frac{1}{R_{\mathrm{I}}R_{\mathrm{II}}}, \qquad L \equiv m^\alpha m^\beta\frac{D_\alpha D_\beta c}{c}. \tag{14}$$

The parameter A is also related to the second derivative of the phase function ϕ in the direction transverse to the ray. Let $\mathbf{M}$ be the matrix of second derivatives (the first derivative is simply $\mathbf{p}$) of the phase evaluated on the ray, i.e.

$$M_{\alpha\beta} = D_\alpha D_\beta\,\phi. \tag{15}$$

Then,

$$M_{\alpha\beta} = \frac{1}{Ac}\frac{\mathrm{d}A}{\mathrm{d}s}\,m_\alpha m_\beta - m^\lambda\frac{c_{,\lambda}}{c^2}\left(n_\alpha m_\beta + m_\alpha n_\beta\right) - n^\lambda\frac{c_{,\lambda}}{c^2}\,n_\alpha n_\beta. \tag{16}$$

This matrix is useful for paraxial approximations, which will be discussed in more detail later.

4.2. The ray equations

We first demonstrate that the rays of the eiconal equation for membrane waves are identical to the shell geodesics, modified by possible variations in wave speed with position. For simplicity, let the wave speed be $c = c_{\mathrm{p}}$ or c_{s}, then the eiconal equation which follows from (6) can be written $H = 1/2$, where

$$H \equiv \frac{1}{2}\,c^2 a^{\alpha\beta}D_\alpha\phi\,D_\beta\phi. \tag{17}$$

We now restrict the definition (7) to the value along a *ray*, which is defined as a solution to the equation $H = 1/2$ for some initial values of the position and the normal direction. Therefore, $p_\alpha p^\alpha = 1/c^2$ and $n_\alpha = cp_\alpha$ along the ray. We consider θ^α and p_α as conjugate variables, analogous to position and momentum, for the "Hamiltonian" $H = H(\theta^\alpha, \phi_{,\beta})$. This is independent of t, which could be taken as the parameter along the ray; however, it is preferable to use the arc-length, s, where $\mathrm{d}s = c\mathrm{d}t$. The ray or Hamilton–Jacobi equations are then

$$\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s} = \frac{1}{c}\frac{\partial H}{\partial p_\alpha}, \qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}s} = -\frac{1}{c}\frac{\partial H}{\partial \theta^\alpha}, \tag{18}$$

which immediately guarantee that $H = 1/2$ along the ray, because $\mathrm{d}H/\mathrm{d}s = 0$ from (18). Substituting for H from (17), the ray equations become

$$\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s} = n^\alpha, \qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}s} = -\frac{c_{,\alpha}}{c^2} - \frac{1}{2c} n_\beta n_\gamma \, a^{\beta\gamma}{}_{,\alpha}. \tag{19}$$

The differential equation for $\mathbf{p}$ can be converted into one for the wave normal $\mathbf{n}$, using Eq. (19), the relation $n^\alpha = c\, a^{\alpha\beta} p_\beta$, and the identities

$$a_{\beta\gamma,\alpha} = a_{\lambda\beta}\Gamma^\lambda_{\gamma\alpha} + a_{\lambda\gamma}\Gamma^\lambda_{\beta\alpha}, \qquad a^{\beta\gamma}{}_{,\alpha} = -a^{\lambda\beta}\Gamma^\gamma_{\lambda\alpha} - a^{\lambda\gamma}\Gamma^\beta_{\lambda\alpha}, \tag{20}$$

yielding the following alternate form of the ray equations,

$$\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s} = n^\alpha, \qquad \frac{\mathrm{d}n^\alpha}{\mathrm{d}s} = -\Gamma^\alpha_{\beta\gamma}\, n^\beta n^\gamma - m^\alpha m^\beta \frac{c_{,\beta}}{c}. \tag{21}$$

A ray is completely determined once an initial position and direction are given for θ^α and n^α. Combining the separate equations in (21) yields the single second order differential equation (11) for the position along the ray.

4.3. Variational ray equations

We next consider how the higher order derivatives of the phase function ϕ propagate along the rays. In particular, the second order derivatives are sufficient to determine both the wavefront curvature and the amplitude of disturbances with initially curved wavefronts and also gaussian beams.

We first note that the identity $D_\alpha H = 0$ gives

$$c_{,\alpha}\, a^{\lambda\gamma}(D_\lambda\phi)\,(D_\gamma\phi) + c\, a^{\lambda\gamma}(D_\lambda\phi)\, D_\alpha D_\gamma\phi = 0, \tag{22}$$

which, when evaluated on the ray, implies

$$M_{\alpha\beta}\, n^\beta = -c^{-2}\, c_{,\alpha}\,, \tag{23}$$

where $M_{\alpha\beta}$ are defined in (15). The remaining unknown element of $\mathbf{M}$ is

$$\mu \equiv m^\alpha m^\beta\, M_{\alpha\beta}. \tag{24}$$

In order to find μ, we first expand the identity $D_\beta\,(c^{-1}\, D_\alpha H) = 0$ and evaluate it on the ray, using (23) to simplify,

$$\frac{1}{c^2}\left(D_\alpha D_\beta c - 3\, c_{,\alpha} c_{,\beta}\right) + c\, M^\lambda_\alpha M_{\lambda\beta} + n^\lambda\, D_\beta D_\lambda D_\alpha\phi = 0. \tag{25}$$

Equations (16) and (25) together imply

$$m^\alpha m^\beta n^\lambda\, D_\beta D_\lambda D_\alpha \phi = \frac{2}{c^3}\,(m^\alpha c_{,\alpha})^2 - \frac{1}{c^2}\, m^\alpha m^\beta D_\alpha D_\beta c - c\,\mu^2. \tag{26}$$

We will eventually use this identity to determine an expression for the derivative of μ along the ray. The latter follows directly by differentiating (24), where the derivatives $\mathrm{d}\mu/\mathrm{d}s$ can be eliminated quite easily using $(21)_2$ and the fact that $\mathrm{d}(m^\alpha n_\alpha)/\mathrm{d}s = 0$. We find

$$\frac{\mathrm{d}\mu}{\mathrm{d}s} = 2\, n^\alpha m^\beta M_{\alpha\beta}\, m^\lambda\, \frac{c_{,\lambda}}{c} + m^\alpha m^\beta n^\lambda\, D_\lambda D_\beta D_\alpha \phi. \tag{27}$$

Note that the final term in (27) is similar but not identical to the expression (26). The difference arises because of the fact that in general, $D_\lambda D_\beta D_\alpha \phi \neq D_\beta D_\lambda D_\alpha \phi$, even though $D_\beta D_\alpha \phi = D_\alpha D_\beta \phi$. To be more precise, for any vector function with components v_α,

$$\left(D_\lambda D_\beta - D_\beta D_\lambda\right)\, v_\alpha = R^{\rho}{}_{.\alpha\beta\lambda}\, v_\rho, \tag{28}$$

where $R^{\rho}{}_{.\alpha\beta\lambda} = \Gamma^{\alpha}_{\beta\lambda,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\lambda} + \Gamma^{\rho}_{\beta\lambda}\Gamma^{\alpha}_{\rho\gamma} - \Gamma^{\rho}_{\beta\gamma}\Gamma^{\alpha}_{\rho\lambda}$ are the components of the surface curvature tensor [13].

Combining Eqs. (23) and (26) through (28), we obtain the desired differential equation for μ,

$$\frac{\mathrm{d}\mu}{\mathrm{d}s} = -c\,\mu^2 - m^\alpha m^\beta\, \frac{D_\alpha D_\beta c}{c^2} + \frac{1}{c}\, m^\alpha m^\beta n^\lambda n^\rho\, R_{\rho\alpha\beta\lambda}. \tag{29}$$

This is a Ricatti equation for μ, i.e. a first order nonlinear differential equation. It can be reduced to a second order linear system by introducing parameters $A(s)$ and $B(s)$ defined by

$$\mu = \frac{B}{A}, \qquad \frac{\mathrm{d}A}{\mathrm{d}s} = B\, c. \tag{30}$$

We note in passing that Eqs. (23), (24), and (30) imply the explicit form (16) for the matrix $\mathbf{M}$. The differential equation for B follows from (29) and $(30)_2$ as

$$\frac{\mathrm{d}B}{\mathrm{d}s} = \frac{1}{c}\, m^\alpha m^\beta \left[-\frac{D_\alpha D_\beta c}{c} + n^\rho n^\lambda\, R_{\rho\alpha\beta\lambda} \right] A. \tag{31}$$

Alternatively, eliminate B to obtain the linear second order equation (13) for A, where L is given by $(14)_2$ and $K = -m^\alpha m^\beta n^\rho n^\lambda\, R_{\rho\alpha\beta\lambda}$. The simplified version of K in Eq. $(14)_1$ can be derived using the equation of Gauss for the curvature tensor for a surface [13], $R_{\alpha\beta\gamma\lambda} = d_{\alpha\gamma} d_{\beta\lambda} - d_{\alpha\lambda} d_{\beta\gamma}$. The only non-zero elements are $R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$ and it can be seen quite easily that $R_{1212} = a/R_{\mathrm{I}} R_{\mathrm{II}}$ where $a = \det a_{\alpha\beta}$ and R_{I} and R_{II} are the principal curvatures.

4.4. The transport equation

The transport equation (10) for membrane waves may be rewritten, using $\mathbf{n} = c\mathbf{p}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\rho h\, c^2\, V^2\right) + \rho h\, c^2\, V^2 M^{\alpha}_{\alpha} = 0, \tag{32}$$

where M^{α}_{β} are defined in (15). Noting that the derivative along a ray is $\mathrm{d}/\mathrm{d}s = n^\alpha D_\alpha$, and using (16) and (30), Eq. (32) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}s} \log\left(\rho h\, c\, V^2\, A\right) = 0. \tag{33}$$

The amplitude therefore evolves according to Eq. (12).

4.5. Discussion and interpretation

The ray equation $(21)_2$ can be simplified if we define the total derivative of an arbitrary vector function $\mathbf{q}(\theta^\alpha)$ along the ray as

$$\frac{Dq^\alpha}{Ds} \equiv \frac{dq^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} n^\beta q^\gamma. \tag{34}$$

The equation for the geodesic that is locally parallel to the ray path is then simply $\mathbf{Dn}/\mathrm{D}s = 0$, while the ray equation $(21)_2$ becomes

$$\frac{Dn^\alpha}{Ds} = -m^\alpha m^\beta \frac{c_{,\beta}}{c}. \tag{35}$$

This is more like the standard form of the ray equation for acoustic media, normally phrased in euclidean spaces where the definition of the total derivative is straightforward. An alternative form of the ray equation on a curved surface has been derived by Collins [14] starting from Fermat's principle (see the Appendix of [14]).

It was stated above that the parameter A is the "ray tube area". We will now justify this statement, and at the same time provide a more rigorous interpretation of the term "ray tube area" in the context of surface area mappings. We first introduce the matrices

$$A^\alpha_\beta \equiv \frac{\partial\theta^\alpha}{\partial\theta^\beta_0}, \qquad B^\alpha_\beta \equiv \frac{\partial p^\alpha}{\partial\theta^\beta_0}, \tag{36}$$

where θ^α_0 are independent parameters which define the initial position of the ray. Hence, $\mathbf{A}$ and $\mathbf{B}$ describe the variation in the ray coordinates as a function of the initial position. Evolution equations for both matrices follow by taking the derivatives of the ray equations with respect to θ^α_0, yielding the so-called variational ray equations [15]. The first of these follows from $(21)_1$ by writing $n^\alpha = p^\alpha/\sqrt{a_{\lambda\gamma}p^\lambda p^\gamma}$ and using the identity $a^\alpha_\beta = n^\alpha n_\beta + m^\alpha m_\beta$. We find

$$\frac{d}{ds}A^\alpha_\beta = c\, m^\alpha m_\gamma B^\gamma_\beta - n^\alpha \Gamma^\gamma_{\lambda\rho} n^\lambda n_\gamma A^\rho_\beta. \tag{37}$$

This can be simplified by first using the general result, true for any matrix, that

$$\frac{d}{ds}\log(\det\mathbf{A}) = \mathrm{tr}\frac{d\mathbf{A}}{ds}\mathbf{A}^{-1}, \tag{38}$$

and the following identity, which is simply derived from the definitions in Eqs. (15) and (36),

$$B^\alpha_\gamma \left(A^{-1}\right)^\gamma_\beta = M^\alpha_\beta - \frac{1}{c}\Gamma^\alpha_{\lambda\beta} n^\lambda. \tag{39}$$

We obtain

$$\frac{d}{ds}\log(\det\mathbf{A}) = c\, m^\alpha m_\beta M^\beta_\alpha - \Gamma^\alpha_{\lambda\alpha} n^\lambda. \tag{40}$$

The final term can be simplified using the identity $\Gamma^\alpha_{\lambda\alpha} = (\log\sqrt{a})_{,\lambda}$, (see equation (2.82) in Eisenhart [13]), where again $a = \det a_{\alpha\beta}$ is the determinant of the metric. It then follows from (40) that

$$m^\alpha m_\beta M^\beta_\alpha = \frac{1}{c}\frac{d}{ds}\log(\sqrt{a}\,\det\mathbf{A}). \tag{41}$$

Comparison with Eqs. (24) and (30) implies that the scalar A is related to the tensor $\mathbf{A}$ by

$$A = \sqrt{a}\,\det \mathbf{A}. \tag{42}$$

We therefore identify A unambiguously as the analog of the ray tube area. More precisely it is the area mapping along the ray, because it depends upon the jacobian associated with the rays through the matrix $\mathbf{A}$, and the term $\sqrt{a}$ occurs in (42) because it defines the physical area associated with the general curvilinear coordinates θ^α.

The second order differential equation (13) is closely related to the Jacobi equation of differential geometry. This is obtained when the wave speed c is uniform, so that the evolution of A depends simply upon the Gaussian curvature at each point along the geodesic (the ray path coincides with the geodesic for constant c). The Jacobi equation can be quickly derived from the second fundamental form for surfaces, $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ [16], by specializing u and v to geodesic coordinates such that $E = 1$, $F = 0$, so that $ds^2 = du^2 + Gdv^2$. Gauss's general equation for the curvature in terms of the derivatives of E, F and G [16] then simplifies to $K = -(G)^{-1/2}\partial^2\sqrt{G}/\partial u^2$, which is the same as (13) for constant c if we identify $\pm A$ with $\sqrt{G}$, in accordance with the previous discussion about A. A "polar" form of the local metric, $ds^2 = c^2(du^2 + G^2dv^2)$, is the starting point for the analysis of Steel [17] for the asymptotic form of the Green's function for the reduced wave equation on a surface. Steele derived a second order evolution equation for G for smoothly varying c, equation (2.11) in [17], which is identical to (13) with the identification $G = A/c$. The scalar functions A and G are closely related to "Jacobi fields", which are vector functions that undergo parallel transport along geodesics, in the terminology of differential geometry. Also, the natural generalization of the Jacobi equation to arbitrary differentiable manifolds is Synge's equation of geodesic deviation [12].

The equations derived here are applicable to shells on which the material properties and the curvature vary continuously. The degree of ray spreading depends upon the local parameter $(K+L)$. When this vanishes, as on a uniform flat plate, the distance between neighboring rays increases linearly with initial distance from a source. In general, the increase is sub-linear or super-linear, depending as $K + L$ is positive or negative, respectively. This is the major distinction between rays on uniform, flat plates and on inhomogeneous curved shells, and leads to the possibility that initial data in the form of plane waves or point sources can generate caustics at finite distances. However, the presence of curvature and variable speed does not lead to dispersion of the waves, meaning that the spatial dependence of the waveform in the ray direction is unaltered as it propagates. In the next section we will formalize the evolution of wavefronts, and allow the possibility of complex valued wavefront curvature, i.e. gaussian beams.

5. Gaussian beam evolution

5.1. The ray matrix

The differential equations $(30)_2$ and (31) for $A(s)$ and $B(s)$ form a system of coupled first order equations,

$$\frac{dA}{ds} = c\,B, \qquad \frac{dB}{ds} = -c^{-1}(K+L)\,A. \tag{43}$$

Define the matrix

$$\mathbf{P}(s) = \begin{bmatrix} A_1(s) & A_2(s) \\ B_1(s) & B_2(s) \end{bmatrix}, \tag{44}$$

where $(A_1(s), B_1(s))$ and $(A_2(s), B_2(s))$ are two sets of linearly independent solutions of (43), such that $\mathbf{P}(0) = \mathbf{I}$. The general solutions for A and B subject to arbitrary initial conditions is

$$\mathbf{Q}(s) = \mathbf{P}(s)\,\mathbf{Q}(0), \tag{45}$$

where $\mathbf{Q}(s)$ is the two-vector formed from $A(s)$ and $B(s)$. The matrix $\mathbf{P}(s)$ propagates ray tube areas and the rate of change of the ray tube areas. We will call it the *ray matrix*, in keeping with optical terminology [18].

Note that A_1 and B_1 correspond to initial conditions like a plane wave, and A_2 and B_2 describe the ray tube for an initial point source. Also, Eqs. (43) and the initial conditions imply that the Wronskian, or Lagrange ray-invariant [18], $\det \mathbf{P}(s) = A_1(s)\,B_2(s) - A_2(s)\,B_1(s)$ is constant and equal to unity. Therefore, the two eigenvalues of $\mathbf{P}(s)$ are of the form λ, $1/\lambda$.

Now consider the element $\mu(s)$ of the $\mathbf{M}$ matrix associated with the transverse direction, $\mathbf{m}$. Its evolution follows from $(30)_1$ and the fundamental solutions for A and B, as

$$\mu(s) = \frac{B_1(s) + \mu(0)\,B_2(s)}{A_1(s) + \mu(0)\,A_2(s)}. \tag{46}$$

This is the "ABCD law" of transformation of wavefront curvature in optics [18]. If the initial conditions correspond to a gaussian beam then $\mu(0)$ is complex valued with positive imaginary part, ensuring gaussian decay in the transverse direction. The Sturm separation theorem (page 223 of Ince [19]) can be applied to the single equation (13) for $A(s)$, and implies that the zeros of A_1 and A_2 alternate. Hence the complex number $\mu(s)$ of Eq. (46) is always of finite magnitude. The Sturm comparison theorem may also be applied to the corresponding second order differential equation for B_1 and B_2, from which we conclude that $\mu(s)$ cannot vanish either. Thus, $\mu(s)$ is guaranteed to remain finite and non-zero for a gaussian beam.

Let $\mu(0) = \mu_1 + \mathrm{i}\mu_2$, where μ_1 and μ_2 are real with $\mu_2 > 0$, then using the fact that $\det \mathbf{P}(s) = 1$, (46) implies

$$\operatorname{Im} \mu(s) = \left[A_1(s) + \mu_1 A_2(s))^2 + (\mu_2 A_2(s)\right]^{-2} \mu_2 . \tag{47}$$

The imaginary part of $\mu(s)$ is therefore guaranteed to be positive if and only if the imaginary part of $\mu(0)$ is positive. We note in passing that the points where Im $\mu(s)$ is stationary are precisely those points where the real part of $\mu(s)$ vanishes, which can be verified by differentiating (47) and using (46).

The gaussian beam amplitude follows from (12) as

$$V(s) = \frac{V(0)}{\sqrt{A_1(s) + A_2(s)\,\mu(0)}} \left[\frac{(\rho h c)(0)}{(\rho h c)(s)}\right]^{1/2} \tag{48}$$

The Sturm comparison theorem also implies that $V(s)$ remains finite and has no zeroes. Hence, we have shown that *a gaussian beam remains localized and of finite but non-zero amplitude as it travels over the surface.* This is one of the fundamental properties of gaussian beams.

We are now in a position to completely describe the evolution of gaussian beams. Consider an initial direction $\mathbf{n}_0$, $n_0^\alpha n_{0\alpha} = 1$, with data prescribed on the surface curve ${n_0}_\alpha \theta^\alpha = 0$, of the form

$$\mathbf{v}(\theta^\alpha) = V_0\,\mathbf{q}_0 \exp\left\{\frac{\mathrm{i}}{2}\,\omega\,\mu_0\,m_{0\alpha} m_{0\beta}\,\theta^\alpha\theta^\beta\right\}, \qquad D_\beta v^\alpha = \mathrm{i}\,\frac{\omega}{c}\,n_{0\beta}\,v^\alpha, \tag{49}$$

where $\mathbf{m}_0 = \mathbf{a}_3 \wedge \mathbf{n}_0$, $\mathbf{q}_0 = \mathbf{n}_0$ for longitudinal waves and $\mathbf{q}_0 = \mathbf{m}_0$ for shear waves and c is the corresponding wave speed. The complex number μ_0 defines a transverse gaussian decay about the initial center, if and only if the imaginary part of μ_0 is positive definite. The gaussian beam solution is defined by solving the ray and transport equations, which are just ODEs along the ray path, for the initial conditions defined by (49). Thus, the initial ray position is at the origin of coordinates, and the initial ray direction is $\mathbf{n}(0) = \mathbf{n}_0$. The remaining initial condition is $\mu(0) = \mu_0$.

5.2. *Examples*

The ray matrix of (44) can be easily evaluated for regions that are uniform (c = constant) and of constant Gaussian curvature. Thus,

$$\mathbf{P}(s) = \begin{bmatrix} \cos\sqrt{K}\,s & \dfrac{c}{\sqrt{K}}\sin\sqrt{K}\,s \\ -\dfrac{1}{c}\sqrt{K}\sin\sqrt{K}\,s & \cos\sqrt{K}\,s \end{bmatrix}, \tag{50}$$

if the curvature is positive, while for regions of constant negative curvature

$$\mathbf{P}(s) = \begin{bmatrix} \cosh\sqrt{-K}\,s & \dfrac{c}{\sqrt{-K}}\sinh\sqrt{-K}\,s \\ \dfrac{1}{c}\sqrt{-K}\sinh\sqrt{-K}\,s & \cosh\sqrt{-K}\,s \end{bmatrix}, \tag{51}$$

The ray matrix for a uniform, flat region is simply the limit of either (50) or (51) for zero curvature, i.e.,

$$\mathbf{P}(s) = \begin{bmatrix} 1 & cs \\ 0 & 1 \end{bmatrix}. \tag{52}$$

5.3. Gaussian beams on a spheroid

Consider a spheroid with semi-major axes of length R, R, and $R_3 = \beta^{-1}R$. The beam propagates around the "waist", describing a circular path of radius R. Let θ be the angle on the circle from the starting point, then the ray matrix is given by (50) with $s = R\theta$ and $\sqrt{K} = 1/R_3$. The beam-width parameter follows from Eqs. (46) and (50) as

$$\mu(s) = \frac{\mu(0) - (cR_3)^{-1}\tan\beta\theta}{1 + \mu(0)\,cR_3\tan\beta\theta}. \tag{53}$$

The natural unit for μ is $1/cR_3$. Accordingly, let $\mu(0) = (cR_3)^{-1}(\widehat{\mu}_1 + \mathrm{i}\widehat{\mu}_2)$, where $\widehat{\mu}_2 > 0$, and define $\widehat{\mu}_0$ and θ_0 by

$$\widehat{\mu}_0 = \frac{2\widehat{\mu}_2}{\widehat{\mu}_1^2 + \widehat{\mu}_2^2 + 1 + \sqrt{(\widehat{\mu}_1^2 + \widehat{\mu}_2^2 - 1)^2 + 4\widehat{\mu}_1^2}}, \qquad \tan 2\beta\theta_0 = \frac{2\widehat{\mu}_1}{1 - \widehat{\mu}_1^2 - \widehat{\mu}_2^2}. \tag{54}$$

With no loss in generality, let the origin of θ be redefined so that $\theta_0 = 0$, then $\mu(s)$ can be written in the "standard" form, $\mu(s) = (cR_3)^{-1}\,\widehat{\mu}(s)$, where

$$\widehat{\mu}(s) = \frac{\mathrm{i}\widehat{\mu}_0\cos\beta\theta - \sin\beta\theta}{\mathrm{i}\widehat{\mu}_0\sin\beta\theta + \cos\beta\theta}. \tag{55}$$

The general solution for a gaussian beam traveling around the waist of a spheroid is therefore

$$u(s,r) = \frac{V(0)}{\sqrt{\cos\beta\theta + \mathrm{i}\widehat{\mu}_0\sin\beta\theta}}\exp\left\{\mathrm{i}k\left(R\theta + \frac{\widehat{\mu}(s)}{2}\frac{r^2}{R_3}\right)\right\}, \tag{56}$$

where $k = \omega/c$ and r is the transverse distance from the ray. The gaussian beam (56) "breathes" as it circles the spheroid because the width parameter, which depends upon $\mathrm{Im}\,\widehat{\mu}(s)$, varies between $\widehat{\mu}_0$ and $1/\widehat{\mu}_0$. The interval between maximum or minimum widths corresponds to an increment in θ of $\pi/2\beta$. Thus, for a given value of R, the "breathing" is more frequent on an oblate spheroid ($1 < \beta < \infty$) than on a prolate spheroid ($0 < \beta < 1$). Gaussian beams on a sphere are just a special case of (56) ($R/R_3 \equiv \beta = 1$).

6. Quasimodes and asymptotic modal frequencies

The gaussian beam solution associated with a periodic ray, as in the previous example, is generally not periodic. However, when it is, we call it a *quasimode* [8]. In general, one can associate a quasimode with a stable periodic ray according to the prescription described below. The basic idea is that the neighboring ray system is periodic, which requires that both the amplitude and the beam width are periodic. This implies a constraint upon the frequency, which is the semi-classical Einstein–Brillouin–Keller quantization condition [20].

The term quasimode is used here in the spirit of Arnold [8], who pointed out that although these are periodic solutions which almost satisfy the wave equation (the error is asymptotically small) they are not necessarily global approximations to modes. The frequencies are, however, good approximations to the modal frequencies. The relatively simple mechanical system composed of a pair of weakly coupled but similar pendula provides a useful analogy. There the modal frequencies are well approximated by the uncoupled frequencies, but the modes are strictly coupled. The quasimodes are the uncoupled vibrations of the separate pendula, which are good approximations over many periods, but not forever. In the same manner, a quasimode could correspond to a disturbance which is trapped locally about the central ray for a long time (many periods) but not an infinite time! Thus, a more exact description of the quasimode should discuss whether it can leak away to other parts of the structure, and if so, how long it takes. The duration would probably depend in some manner on how stable the ray system is. Without digressing too much, we note that for most practical purposes it may be safely assumed that the quasimode is a wave object that exists for many periods.

The theory as presented here is similar to that of geometrical, Hamiltonian optics, as reviewed by Arnaud [18], for example, although the applications are different. The reader should not confuse quasimodes with "beam modes" [18], which are propagating modes similar to the waveguide modes of an optical fiber. The general mathematical theory of quasimodes on surfaces and in higher dimensions has been described by Ralston [9]. There are other methods by which one can attempt to obtain geometrical optics approximations to modes, although they are sometimes restricted as to the type of system. Thus, Keller and Rubinow [21] describe a quite different procedure for approximating the modes of 2D regions, such as the interior of an ellipse, while Lazutkin [22] and Smith [23] have developed asymptotic methods specifically for "bouncing ball" types of modes. Most relevant to the present work is the paper by Babich and Lazutkin [24], in which a parabolic equation approach is used to construct modes concentrated near closed geodesics on surfaces. Whatever method one uses, the leading order approximation to the frequency must be given by the quantization condition (60).

6.1. General theory

Consider a closed periodic ray path of base length L. A quasimode associated with the ray must satisfy the periodicity condition that the neighboring rays form a congruence, i.e. the beam must have exactly the same lateral dependence after each cycle around the closed central ray. This requires (we will not consider degenerate rays [18], for which the periodicity conditions are imposed after $N > 1$ orbits)

$$\mu(L) = \mu(0). \tag{57}$$

These are two equations for the initial values of the real and imaginary parts of $\mu(0)$. We note the identity

$$|V(L)|^{-4}\,\mathrm{Im}\;\mu(L) = |V(0)|^{-4}\,\mathrm{Im}\;\mu(0), \tag{58}$$

which follows from (47), (48), and the fact that ρ, h and c are L-periodic. Hence, satisfaction of (57) implies that the absolute value of the amplitude is also periodic,

$$|V(L)| = |V(0)|. \tag{59}$$

Any possible change in phase of the amplitude is included in the quantization condition for the frequency, which is that the total phase is an integral multiple of 2π. The total phase over one cycle follows by integrating the relation $\mathrm{d}\phi/\mathrm{d}s = 1/c(s)$ and combining the phase of the amplitude V. Thus, the Einstein–Brillouin-Keller (EBK) frequency condition becomes

$$\omega \int_0^L \frac{\mathrm{d}s}{c(s)} + \int_0^L \frac{\mathrm{d}}{\mathrm{d}s}\arg V(s)\,\mathrm{d}s = n2\pi, \qquad n = 1,2,\ldots. \tag{60}$$

The closed ray is said to be stable if the eigenvalues of the ray matrix $\mathbf{P}(L)$ are of unit magnitude, and unstable otherwise. The eigenvalues of a stable ray must therefore be of the form $\lambda_1 = \mathrm{e}^{\mathrm{i}\psi}$, $\lambda_2 = \mathrm{e}^{-\mathrm{i}\psi}$, for some $0 \le \psi \le \pi$. Substituting $\mu(0) = \mu_1 + \mathrm{i}\mu_2$ into $(57)_1$ and solving for both μ_1 and μ_2, we find that a solution is possible only if the ray is stable. The condition for the ray to be stable can be shown to reduce to [18]

$$-2 \le A_1(L) + B_2(L) \le 2. \tag{61}$$

Thus, a quasimode can only exist in the neighborhood of a stable ray. Assuming that (61) holds, we find that the eigenvector rotation angle is

$$\psi = \cos^{-1}\frac{A_1(L) + B_2(L)}{2}, \qquad 0 \le \psi \le \pi. \tag{62}$$

Solving $(57)_1$ for $\mu(0)$, and demanding that the imaginary part be non-negative, we find

$$\mu(0) = \frac{\cos\psi - A_1(L)}{A_2(L)} + \mathrm{i}\,\frac{\sin\psi}{|A_2(L)|}. \tag{63}$$

There appears to be singular behavior if $A_2(L) = 0$. However, the fact that $\det\mathbf{P}(L) = 1$ implies $A_1(L)B_2(L) = 1$, and since the ray is assumed to be stable, (62) implies in turn that $A_1(L) = B_2(L) = \pm 1$, with $\psi = 0$ or π. Hence, the numerators in the right member of (63) vanish when $A_2(L) = 0$, and it is not unreasonable to assume that the limit exists, in the sense of l'Hopital's rule. This case occurs for the sphere.

The EBK frequency condition (60) reduces to

$$\omega \int_0^L \frac{\mathrm{d}s}{c(s)} = n2\pi + \frac{1}{2}\Psi(L), \qquad n = 1,2,\ldots, \tag{64}$$

where $\Psi(s)$ is a continuous function defined by, see Eqs. (48) and (63),

$$\tan\Psi(s) = \frac{\sin\psi\,|A_2(L)|^{-1}A_2(s)}{A_1(s) + A_2(L)^{-1}(\cos\psi - A_1(L))\,A_2(s)}, \tag{65}$$

with $\Psi(0) = 0$. Thus, the frequency equation becomes

$$\omega \int_0^L \frac{\mathrm{d}s}{c(s)} = 2\pi\left(n + \frac{m}{4}\right) + \frac{\psi_m}{2}, \tag{66}$$

where

$$\psi_m = \begin{cases} \psi, & m \text{ even}, \\ \pi - \psi, & m \text{ odd}, \end{cases} \tag{67}$$

and m is the largest integer less than $|\Psi(L)|/\pi$. This is precisely the Keller–Maslov index [20], which equals the number of conjugate points on the orbit, i.e., the number of times the ray intersects a caustic.

6.2. Examples: Periodic paths with constant curvature

Consider the example of the beam rotating about the waist of a spheroid, in Section 5.3. Applying the periodicity condition (57) to $\widehat{\mu}(s)$ of Eq. (55), with $L = 2\pi R$, implies that the normalized width parameter $\widehat{\mu}_0$ is unity. Therefore, $\widehat{\mu}(s)$ maintains the constant imaginary value $\widehat{\mu}(s) = i$, and (56) reduces to

$$u(s,r) = V(0)\, e^{i(kR-\frac{\beta}{2})\theta}\, e^{-kr^2\beta/2R}. \tag{68}$$

The beam propagates with a constant width and amplitude, although the phase of $V(s) = V(0)\exp\{-i\beta\theta/2\}$ changes steadily. The constant $1/e$ half-width is $\sqrt{R\lambda/\beta\pi}$, where $\lambda = 2\pi/k$ is the wavelength. The quantization condition can be deduced immediately from the general solution (68) as

$$kR = n + \beta/2, \quad n = 1, 2, 3, \ldots . \tag{69}$$

The special case of a sphere is noteworthy. The beam width parameter $\widehat{\mu}(s)$ of Eq. (55) is periodic for *any* initial value of $\widehat{\mu}_0$. The condition (57), which is necessary but not sufficient for a quasimode, is therefore automatically satisfied for rays on a great circle of a sphere. That is, the beam defined by Eqs. (55) and (56) with $\beta = 1$ is a quasimode for any $\widehat{\mu}_0$ as long as $kR = n + \frac{1}{2}$. The values of the modal frequencies and the non-uniqueness of the modes correspond to the fact that any function of the form $f(\theta,\phi) = \sum_{m=-\infty}^{\infty} a_m e^{im\phi} P_n^m(\cos\theta)$ is the solution to the Helmholtz equation on the sphere if $kR = \sqrt{n(n+1)}$, where ϕ is the azimuthal angle and P_n^m are Legendre functions of the first kind. The asymptotic eigenvalues of (69) with $\beta = 1$ are approximations to the exact eigenvalues, because $\sqrt{n(n+1)} = n + \frac{1}{2} + O(1/n)$. The modal degeneracy of the sphere is not present for spheroids.

One can generalize these results to closed orbits on any path with constant positive Gaussian curvature. Then the beam width of the quasimode is constant, $\mu(s) = i c^{-1}\sqrt{K}$, and the phase angle is $\Psi(s) = \sqrt{K}\, s$. The quasimode is a generalization of Eq. (68),

$$u(s,r) = V(0)\, e^{is\left(k-\frac{1}{2}\sqrt{K}\right)}\, e^{-\frac{k}{2}\sqrt{K}\, r^2}, \tag{70}$$

and the modal frequencies are therefore

$$kL = 2\pi n + \sqrt{K}\, L/2, \quad n = 1, 2, 3, \ldots . \tag{71}$$

An example of such a path, apart from those mentioned above, is the exterior circumference of a doughnut. In general, if the surface is subject to inextensible deformation, or bending, it is known that the Gaussian curvature at all points on the surface is unchanged. This is Gauss' famous "Theorema egregium" to the effect that the Gaussian curvature is a bending invariant [25]. Hence, the quasimode described by Eqs. (70) and (71) is also an invariant of the surface under bending.

7. Periodic rays and stability on a complex shell structure: An example

The structure depicted in Fig. 1 comprises a cylinder of radius R_1 and length L_1, with identical assemblages attached at either end, composed of a spherical section, a conical section, and a spherical cap. The spherical section, S_1, has the same radius as the cylinder, and is of angular extent θ_1, $0 < \theta_1 < \pi/2$. The conical section, *Co*, has semi-angle θ_1 and is attached to S_1 at one end and to the spherical cap, S_2, of radius $R_2 < R_1$, at the other. The end assemblage, consisting of S_1, *Co* and S_2, is parameterized by the radii R_1 and R_2 and the cone angle θ_1. The semi-polar angle subtended by the cap S_2 is $\pi/2 - \theta_1$, and the length of the conical section is $L_0 = (R_1 - R_2)\cot\theta_1$.

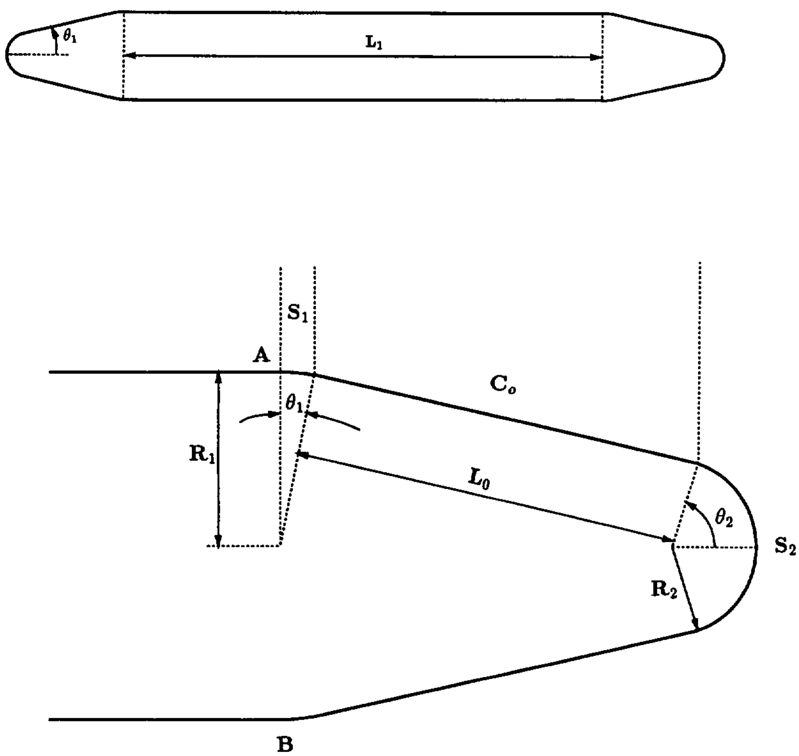

Fig. 1. A schematic of the shell structure composed of a cylinder with identical endcaps. A more detailed view of one endcap is shown.

7.1. The polar ray

There is a unique ray which goes through both "poles", with a period of length $L = 2L_1 + 2\pi R_2 + 4L_0(1 + \theta_1 \tan\theta_1)$. As it traverses each section the Gaussian curvature is constant and equal to zero on the cylinder and cone, and $1/R_1^2$ or $1/R_2^2$ on the spherical regionss. The ray matrix across each section therefore follows from Eqs. (50) and (52). For instance, the ray matrix across one end assemblage is (see Fig. 1)

$$\mathbf{P}_{A\to B} = \begin{bmatrix} -1 & -2c\, L_0 R_1/R_2 \\ 0 & -1 \end{bmatrix}. \tag{72}$$

In order to find the stability of the ray and the parameters of the quasimode, if it exists, we choose one point on the ray path and construct the total round-trip ray matrix. For example, referring to Fig. 1,

$$\mathbf{P}_{A\to A} = \begin{bmatrix} 1 & 2cL_1 + 4c\, L_0 R_1/R_2 \\ 0 & 1 \end{bmatrix}. \tag{73}$$

We find that $\psi = 0$ and $\mu(0) = 0$ for this periodic ray, from Eqs. (62) and (63). The fact that $\psi = 0$, or equivalently, $A_1(L) + B_2(L) = 2$, means the ray is neutrally stable; while $\mu(0) = 0$ implies that the quasimode is a plane wave on the cylindrical sections. By examining the ray matrix through each section, it can be easily shown that the plane wave focuses to a point at the poles, and the Keller–Maslov index is $m = 2$. The latter can be seen by considering the structure as a modification of the sphere (increase θ_1 and L_1 from zero to their final values), and noting that as the structure changes the K-M index is constant because $\psi = 0$ for any θ_1 and L_1.

7.2. Oblique rays

We next consider ray paths on the same structure which do not cross the poles. The ray is defined by the angle it makes with a plane slicing the cylindrical section, γ. Alternatively, the ray subtends an angle $\pi/2 - \gamma$ with the the generator axis as it traverses the cylinder. The previously considered polar ray path corresponds to $\gamma = \pi/2$, so we restrict attention to values less than this. In general, the path of the ray on the cylinder is spiral, but the ray path on the ends depends upon the value of γ. We distinguish two cases:

(1) $\gamma < \theta_1$. The spiral on the cylinder connects to great semi-circles on the spheres S_1 at either end, but the ray does not reach the conical section. The ray path and ray congruence are isometric to those of a ray which traverses the poles on a cylinder with hemi-spherical endcaps. Based upon the previous findings we conclude that the ray is neutrally stable ($\psi = 0$) and the quasimodes are plane waves ($\mu(0) = 0$).

(2) $\gamma > \theta_1$. The ray path reaches the conical section. Let γ_1 be the spherical angle subtended by the section of the ray path crossing S_1, and γ_2 the angle the ray makes with the interface between S_1 and the cone. It follows from some straightforward geometry that

$$\sin\gamma_1 = \frac{\sin\theta_1}{\sin\gamma}, \qquad \cos\gamma_2 = \frac{\cos\gamma}{\cos\theta_1}. \tag{74}$$

There exists two possibilities, depending upon whether or not the ray reaches the sphere S_2. We note that the conical section is isometric to a section of a circular annulus of outer radius $R_1 \csc\theta_1$, inner radius $R_2 \csc\theta_1$ and angular extent $2\pi\sin\theta_1$. The ray makes an angle γ_2 with the tangent to the outer circumference, and therefore it intersects the inner circle only if $R_1 \csc\theta_1 \cos\gamma_2 < R_2 \csc\theta_1$. Using $(74)_2$, the two cases become:

(2.1) The rays do not cross the sphere S_2:

$$\theta_1 < \gamma < \cos^{-1}\left(\frac{R_2}{R_1}\cos\theta_1\right),$$

(2.2) The rays cross S_2:

$$\cos^{-1}\left(\frac{R_2}{R_1}\cos\theta_1\right) < \gamma < \pi/2.$$

Case (2.2) is obviously the more complicated, but can be easily handled by the methods outlined here. However, case (2.1) is sufficiently rich that it describes most of the ray physics one can expect to find on this structure. Hence, we will not consider case (2.2) further. Regarding (2.1), we note that the length of the ray path across Co is $2L_2$, where $L_2 = R_1 \csc\theta_1 \sin\gamma_2$. The ray matrix for propagation across an end assemblage follows from (50) for the two paths across S_1 of length $R_1\gamma_1$ each, and from (52) for the path of length $2L_2$ across the cone. This can be simplified by noting, from (74), that $\cos\theta_1 \sin\gamma_2 = \cos\gamma_1 \sin\gamma$, which implies $L_2 = R_1 \sec\theta_1 \cot\gamma_1$. Using this to eliminate L_2, we find

$$\mathbf{P}_{A\to B} = \begin{bmatrix} -1 - 2\cos^2\gamma_1 (\sec\theta_1 - 1) & cR_1 \sin 2\gamma_1 (\sec\theta_1 \cot^2\gamma_1 + 1) \\ (cR_1)^{-1} \sin 2\gamma_1 (\sec\theta_1 - 1) & -1 - 2\cos^2\gamma_1 (\sec\theta_1 - 1) \end{bmatrix}. \tag{75}$$

If the ray is periodic then the ray matrix for the full round trip is the square of the matrix for half of the orbit, since the two halves are identical. Also, because the matrices are unimodular (the determinant is unity), the full orbit is stable or unstable if and only if the same is true of the half orbit. The ray matrix for the half cycle follows by multiplying the matrix of (75) with that of (52) for $s = L_3$, where $L_3 = L_1 \csc\gamma$ is the length of the spiral path on the cylinder. We find that

$$\frac{1}{2}\left[A_1(L/2) + B_2(L/2)\right] = -1 + (\sec\theta_1 - 1)\, 2\cos\gamma_1 \left[\frac{L_3}{R_1}\sin\gamma_1 - \cos\gamma_1\right]. \tag{76}$$

Therefore, according to (61) and (76) a periodic ray of angle γ will be stable only if

$$0 < \left(\frac{L_1}{R_1}\frac{\sin\gamma_1}{\sin\gamma} - \cos\gamma_1\right)\cos\gamma_1 < \frac{\cos\theta_1}{1-\cos\theta_1}. \tag{77}$$

A periodic ray must also satisfy the condition that the change in azimuthal angle, with respect to the axis of the structure, is a multiple of 2π over one orbit. For rays of type (2.1) this condition becomes a constraint on γ, for given values of θ_1 and L_1/R_1,

$$\frac{L_1}{R_1}\cot\gamma + 2\sin^{-1}(\sin\theta_1\cot\gamma) + 2\csc\theta_1\cos^{-1}(\sec\theta_1\cos\gamma) = j\pi, \quad j = 1,2,3,\ldots. \tag{78}$$

The left hand side represents the azimuthal angle for half of the total orbit. The first term is the angle due to the spiral path on the cylinder. The second comes from the two crossings of S_1, each contributing $\sin^{-1}(\sin\gamma_1\cos\gamma)$, and the final term is the increment in azimuthal angle as the ray crosses the conical section, $2\gamma_2/\sin\theta_1$. Eliminating the angles γ_1 and γ_2 using (74) yields the expression (78). The quasimode frequency therefore has two indices, n and j, from Eqs. (66) and (78).

7.3. An example

Let $\theta_1 = 30°$ and $L_1/R_1 = 10$. Then (77) is satisfied for all angles γ larger than 30° except for those in the range $32.9° < \gamma < 45.8°$. These angles correspond to unstable periodic orbits, if they exist. The orbital condition (78) has four roots for periodic rays of type (2.1): at $\gamma = 31.7°$, 38.5°, 48.5°, and 64.7°. The corresponding values of the index j are 6, 5, 4, and 3, respectively. Therefore, the $j = 3$, 4 and 6 rays are stable, but the $j = 5$ ray is unstable.

8. Flexural waves

Flexural waves on thin shells are characterized by displacements which are primarily in the normal direction. The associated tangential components are small and vanish on flat plates (within the limits of the thin shell theory used here, which ignores the Poisson effect). The form of the polarization field is discussed in detail by Norris and Rebinsky [6]. There is, however, another important distinction between membrane and flexural wave solutions, which concerns the scaling of frequency versus wavenumber, k. In the former case we have $k = \sqrt{k^\alpha k_\alpha}$, where $\mathbf{k} = \omega\mathbf{p}$. This is what is commonly understood as the wavenumber, and we note in particular that $\omega^2 = O(k^2)$ for large wavenumber, as one would expect for nondispersive waves. However, for flexural waves, the scaling is quite different, $\omega^2 = O(h^2k^4)$ (the precise definition of k is given below). The appearance of the non-dimensional factor (h^2k^2) means that for flexural waves ω can be $O(1)$ even though k is large (recall that $kh \ll 1$ is a condition of thin shell theory, *sine qua non*). Hence, ray theory for flexural waves does not require that the frequency is large, only that the wavenumber be large. This is the essential difference between membrane and flexural ray solutions. It leads to the expected dispersion of flexural waves, and also to the possibility of significant anisotropy which depends upon the surface curvature.

8.1. The dispersion relation

Instead of (4) the ansatz is now

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}^{(\epsilon)}(\mathbf{x}) \exp\left\{\mathrm{i}\left(\epsilon^{-1}\Phi(\mathbf{x}) - \omega t\right)\right\}, \tag{79}$$

with amplitude functions scaled as

$$w^{(\epsilon)}(\mathbf{x}) = W(\mathbf{x}) + \epsilon\,\widehat{W}(\mathbf{x}) + \dots, \qquad v^{(\epsilon)\alpha}(\mathbf{x}) = \epsilon\,V^{\alpha}(\mathbf{x}) + \epsilon^2\,\widehat{V}^{\alpha}(\mathbf{x}) + \dots. \tag{80}$$

Because of the inherent dispersion in the flexural wave solution, it is more natural to work with the wave number vector $\mathbf{k}$ rather than the wave slowness $\mathbf{p}$, where $\mathbf{k}$ is defined as

$$k_\alpha = D_\alpha \Phi. \tag{81}$$

The relation between ϕ and Φ, and $\mathbf{p}$ and $\mathbf{k}$ follows by comparing Eqs. (4), (7), (79), and (81), which imply $\Phi = \omega\,\phi$ and $\mathbf{k} = \omega\,\mathbf{p}$. At this stage we are finished with the scaling arguments and it is more convenient to renormalize by setting $\epsilon = 1$. The leading order terms in the equations of motion imply the following dispersion relation,

$$\frac{\omega^2}{c_\mathrm{p}^2} - \frac{h^2}{12}k^4 - (1-\nu^2)\left(m_\alpha m_\beta d^{\alpha\beta}\right)^2 = 0, \tag{82}$$

Here, $\mathbf{m}$ is the unit vector perpendicular to the wave normal unit vector $\mathbf{n} = \mathbf{k}/k$, i.e., $\mathbf{m} = \mathbf{a}_3 \wedge \mathbf{n}$. The derivation of (82) is described in the Appendix. We note that it is identical to flexural dispersion relations discussed by Germogenova [7], Steele [1], Pierce [3], and Norris and Rebinsky [6]. In principal coordinates, $m_\alpha m_\beta d^{\alpha\beta} = n_{\mathrm{II}}^2/R_{\mathrm{I}} + n_{\mathrm{I}}^2/R_{\mathrm{II}}$, and hence the final term in (82) introduces anisotropy only if $R_{\mathrm{I}} \neq R_{\mathrm{II}}$. In general, (82) determines the magnitude of the wave number k as a function of ω and the direction of the wave normal $\mathbf{n}$, *i.e.*, $k = k(\omega,\mathbf{n})$.

8.2. Ray and transport equations

The eiconal equation (82) can be expressed as $H = \omega/2$, where the Hamiltonian $H = H(\theta^\alpha, \Phi_{,\beta})$ is now

$$H \equiv \frac{1}{2\omega}\left\{c_\mathrm{p}^2\frac{h^2}{12}\left(a^{\alpha\beta}D_\alpha\Phi\,D_\beta\Phi\right)^2 + c_\mathrm{p}^2(1-\nu^2)\frac{\left(\bar{d}^{\alpha\beta}D_\alpha\Phi\,D_\beta\Phi\right)^2}{\left(a^{\lambda\gamma}D_\lambda\Phi\,D_\gamma\Phi\right)^2}\right\}. \tag{83}$$

The modified curvature tensor $\bar{\mathbf{d}}$ is defined to have the same principal directions as $\mathbf{d}$ but the values of the principal curvatures are reversed. Thus, for example, $n_\alpha n_\beta \bar{d}^{\alpha\beta} = m_\alpha m_\beta d^{\alpha\beta}$. As we will see, the speed along a ray is not a simple parameter, such as the local wave speed c for membrane waves, and it is therefore more convenient to use the travel time as the ray parameter. The ray equations then become,

$$\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}t} = \frac{\partial H}{\partial k_\alpha}, \qquad \frac{\mathrm{d}k_\alpha}{\mathrm{d}t} = -\frac{\partial H}{\partial \theta^\alpha}. \tag{84}$$

The first of these becomes $\mathrm{d}\theta^\alpha/\mathrm{d}t = c_\mathrm{g}^\alpha$, where $\mathbf{c}_\mathrm{g}$ is the local group velocity vector, which may be determined from (83) and (84) as

$$\mathbf{c}_\mathrm{g} = \frac{c_\mathrm{p}^2}{\omega}\left[2\frac{h^2}{12}k^3\,\mathbf{n} - \frac{2}{k}(1-\nu^2)\left(m^\alpha m^\beta d_{\alpha\beta}\right)\left(m^\lambda n^\gamma d_{\lambda\gamma}\right)\mathbf{m}\right]. \tag{85}$$

This equation has been discussed in detail by Norris and Rebinsky [6], so we will not dwell on it here, except to note that the anisotropy depends upon the term $m^\alpha n^\beta d_{\alpha\beta} = 1/R_{\mathrm{II}} - 1/R_{\mathrm{I}}$, and therefore vanishes on locally spherical regions. The general form of $(84)_2$ for the evolution of $\mathbf{k}$, is very complex and is not worth pursuing except possibly by numerical means, see Norris and Rebinsky [6] for an application to conical shells.

Despite the complexity of the ray equations, it is still possible to simplify the transport equation for flexural waves of the type given in (79) and (75). We only cite the result here,

$$D_\alpha \left(\rho h\, W^2 c_g^\alpha\right) = 0. \tag{86}$$

Details can be found in the Appendix.

There is a subtle point concerning the connection between (86) and energy conservation. It can be shown that the vector $c_g^\alpha\, \rho h\, W^2$ is closely related to the time-averaged energy flux vector, $\langle \mathcal{F}^\alpha \rangle$, as defined by Pierce [26] and discussed in the present context by Norris and Rebinsky [6]. The latter is a real-valued vector, whereas the former involves W^2 which can in general be complex valued. However, the time average of W^2 is asymptotically the same as W^2, since the rapid phase variation has been accounted for in the exponential term in (79). Therefore, one way of understanding the appearance of W^2 rather than $|W|^2$ in (86) is that if $W = |W| e^{i\psi}$, then the complex phase angle ψ is essentially constant according to ray theory. This explanation is perfectly adequate for classical wavefront analysis, where the phase of the amplitude only changes at discrete events, such as reflection, transmission, or passage through a caustic. It does not apply to gaussian beam solutions, for which the phase of the amplitude, ψ, changes smoothly as the beam propagates. Energy flux is not conserved in a point-wise sense along the ray, rather, the spatially integrated energy associated with the beam is conserved. These comments apply equally well to Eq. (10) for membrane waves.

8.3. *High frequency simplifications*

The ray equations for the dispersive, anisotropic and inhomogeneous Hamiltonian of (83) are exceedingly complicated, mainly because of the presence of the final term, which introduces the anisotropy. However, this term is only significant at frequencies near the generalized ring frequency [3,6], defined as $\omega_{\text{ring}} = c_p \sqrt{K}\,(2\nu + R_I/R_{II} + R_{II}/R_I)^{1/2}$. At medium to high frequencies relative to ω_{ring} the anisotropy in the flexural dispersion relation (82) can be safely overlooked, resulting in significant simplification. The dispersion relation then reduces to $k = \omega/c$, where the phase velocity $c = c(\omega)$ is defined as $c = (12)^{-1/4}\sqrt{\omega\, h c_p}$, and the ray velocity (group velocity) of Eq. (85) becomes $\mathbf{c}_g = 2c\,\mathbf{n}$. Omitting the details, which are similar to those of Section 4, the ray equations reduce to exactly the same form as those found previously for membrane waves, i.e. Eqs. (11) or (21), where c is the phase speed and the arc-length is now $\mathrm{d}s = c_g\,\mathrm{d}t$.

The transport equation (A.8) simplifies in that we can ignore the contributions from the tangential displacement and keep only the terms involving W^2. In order to simplify it further, let $\Phi = \omega\phi$, and define $M_{\alpha\beta}$ as in Eq. (15). Then (A.8) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(B\, W^2\right) + cB\, W^2 \left(M_\alpha^\alpha + 2\, n^\alpha n^\beta M_{\alpha\beta}\right) = 0. \tag{87}$$

At the same time, it can be shown that Eq. (16) again holds. Using this to simplify the transport equation we find that W satisfies an equation just like (33) for the membrane waves, with $V(s)$ replaced by $W(s)$, and where $A(s)$ satisfies the same evolution equation, (13), as before. In summary, the flexural ray equations are identical to those of the membrane waves, with $c(s)$ the flexural phase speed. The only difference is that the group velocity $c_g = 2c$ defines the arc length of the ray, which indicates the wave is dispersive, unlike membrane waves. Because we are only considering time harmonic waves here, the effects of this dispersion are not significant. When translated into the time domain, it causes wave spreading, or diffusion, in the direction of propagation, in addition to the geometrical spreading in the transverse direction.

9. Conclusions

The propagation of membrane and flexural waves on smooth thin shells has been analyzed using the methods of geometrical, optics. The ray paths are identical to the usual geodesics if the wave speeds are constant, but can depart from these if the speed varies with position. Hence, if the shell thickness is variable, the shear and longitudinal speeds are constant but the flexural wave speed will change, implying that flexural ray paths could differ from those of membrane waves. The evolution of wavefronts is governed by the ordinary differential equation (13) for the ray tube area A. The rate of change of A depends critically upon the local Gaussian curvature, with distinct behavior on regions of positive or negative K.

The basic elements developed in Section 4 allow us to derive all the necessary parameters for describing the propagation of gaussian beams, with the details in Section 5. For every closed periodic ray, which is also stable in the sense of Eq. (61), we can construct explicit geometrical optics approximations which have the property of being periodic and locally confined about the ray. These quasimodes are not structural modes, but if excited by suitable external forcing could exist for a time sufficiently long that they become indistinguishable from pure modes. The procedure for constructing quasimodes is described in detail in Section 6. In general, quasimodes will not provide a full description of the spectrum of a complex structure, just as closed orbits do not provide the full picture in systems which exhibit stochastic or chaotic rays [20]. The issue of how to incorporate such non-semiclassical effects remains an interesting question and the topic of current research.

Acknowledgment

This work was supported by the Office of Naval Research.

Appendix A. . The flexural ray equations

Some of the details leading to the flexural ray and transport equations are presented here. By substitution of the ansatz (79) and (75) into the equations of motion (2), and identification of the leading order term in powers of ϵ, we find

$$\frac{1}{2}\left[(1-\nu)k^2 a^\alpha_\beta + (1+\nu)k^\alpha k_\beta\right] V^\beta - \mathrm{i}G^\alpha_\beta k^\beta W = 0, \tag{A.1a}$$

$$-\mathrm{i}\, G_{\alpha\beta}\, k^\beta V^\alpha + \left(\frac{\omega^2}{c_\mathrm{p}^2} - \frac{h^2}{12}k^4 - G^\alpha_\beta d^\beta_\alpha\right) W = 0, \tag{A.1b}$$

where $\mathbf{k}$ is defined in (81), and

$$G^\alpha_\beta = (1-\nu)\, d^\alpha_\beta + \nu\, d^\gamma_\gamma\, a^\alpha_\beta. \tag{A.2}$$

Solving for V^α gives $V^\alpha = \mathrm{i}WS^\alpha$, where

$$S^\alpha = \frac{2}{(1-\nu)\,k^4}\left[k^2\, G^{\alpha\beta} k_\beta - \frac{1}{2}(1+\nu)\, G^{\beta\gamma} k_\beta k_\gamma\, k^\alpha\right]. \tag{A.3}$$

Then eliminating V^α from the W-equation implies the flexural wave dispersion relation (82).

The next to leading order terms in the asymptotic series that result from inserting (79) and (80) into the equations of motion (2) give a set of equations from which we can deduce a transport equation for the wave amplitude. Specifically, terms of order ϵ in comparison with the leading order terms must be retained. For example, the strains become

$$e_{\alpha\beta} = W\, d_{\alpha\beta} + \frac{\mathrm{i}}{2}(k_\alpha V_\beta + k_\beta V_\alpha) + \epsilon\, \widehat{W}\, d_{\alpha\beta} + \epsilon \frac{\mathrm{i}}{2}(k_\alpha \widehat{V}_\beta + k_\beta \widehat{V}_\alpha) + \frac{\epsilon}{2}(D_\alpha V_\beta + D_\beta V_\alpha) + \dots, \tag{A.4}$$

where only those terms omitted may be neglected. We also note that the first and second order terms which involve W in the asymptotic expansion of $D_\alpha D^\alpha D_\beta D^\beta w$ are

$$D_\alpha D^\alpha D_\beta D^\beta \left(W \mathrm{e}^{\mathrm{i}\Phi/\epsilon}\right) = \frac{1}{\epsilon^3}\left(\frac{k^4}{\epsilon} W - 4\mathrm{i}k^2 k^\alpha D_\alpha W - 2ik^2 N^\beta_\beta W - 4\mathrm{i}k_\alpha k^\beta N^\alpha_\beta W + \dots\right) \mathrm{e}^{i\Phi/\epsilon}, \tag{A.5}$$

where N^α_β are defined by analogy with (15) as $N_{\alpha\beta} = D_\alpha D_\beta \Phi$. Using (B.1) to eliminate the leading order terms, we obtain

$$\begin{aligned} 0 = {} & C\left(\mathrm{i}G^\alpha_\beta k^\beta \widehat{W} - F^\alpha_\beta \widehat{V}^\beta\right) + \frac{\mathrm{i}}{2} D_\beta \left[C(1-\nu)\left(k^\alpha V^\beta + k^\beta V^\alpha\right)\right] + \mathrm{i}D^\alpha \left(C\nu\, k_\beta V^\beta\right) \\ & + \frac{\mathrm{i}}{2} C(1-\nu) k_\beta \left(D^\alpha V^\beta + D^\beta V^\alpha\right) + \mathrm{i}\, C\nu\, k^\alpha D_\beta V^\beta + D_\beta\left(C G^{\alpha\beta} W\right), \end{aligned} \tag{A.6}$$

for the in-surface equations, and for the normal displacement

$$\begin{aligned} 0 = {} & C\left[\mathrm{i}\, G^\beta_\alpha k_\beta \widehat{V}^\alpha - \left(\Omega^2 - \frac{h^2}{12}k^4 - G^\alpha_\beta d^\beta_\alpha\right) \widehat{W}\right] + C G^\alpha_\beta D_\alpha V^\beta \\ & -2\mathrm{i}\left[B\, k^2 k^\alpha D_\alpha W + k^2 k^\alpha D_\alpha (BW) + B\left(k^2 N^\alpha_\alpha + 2k_\alpha k^\beta N^\alpha_\beta\right) W\right]. \end{aligned} \tag{A.7}$$

Multiply the first pair of equations by $\mathrm{i}V_\alpha$ and the final equation by $\mathrm{i}W$, and add the results to get, using (B.1), a single equation for the leading order amplitude,

$$\begin{aligned} 0 = {} & 2W\left[2B\, k^2 k^\alpha D_\alpha W + W k^2\, k^\alpha D_\alpha B + B\left(k^2 N^\alpha_\alpha + 2k_\alpha k^\beta N^\alpha_\beta\right) W\right] + \mathrm{i}WCG^\alpha_\beta D_\alpha V^\beta + \mathrm{i}V_\alpha D_\beta \left(CG^{\alpha\beta}W\right) \\ & - V_\alpha \left\{\frac{1}{2} D_\beta \left[C(1-\nu)\left(k^\alpha V^\beta + k^\beta V^\alpha\right)\right] + D^\alpha\left(C\nu\, k_\beta V^\beta\right)\right. \\ & \left. + \frac{1}{2} C(1-\nu) k_\beta \left(D^\alpha V^\beta + D^\beta V^\alpha\right) + C\nu\, k^\alpha D_\beta V^\beta \right\}. \end{aligned} \tag{A.8}$$

Expanding and combining terms gives

$$D_\alpha \left[2Bk^2 k^\alpha W^2 + iCG^\alpha_\beta V^\beta W - \frac{C}{2}(1-\nu) V^\beta V_\beta k^\alpha - \frac{C}{2}(1+\nu) V^\alpha V_\beta k^\beta\right] = 0. \tag{A.9}$$

The tangential components V^α can be eliminated using $V^\alpha = \mathrm{i}WS^\alpha$, where S^α is defined in (A.3). It then follows, after much algebra, that the transport equation (A.9) simplifies to (86), where $\mathbf{c}_g$ is the group velocity vector of (85).

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