

Acoustic diffraction from the junction of two flat plates

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A general solution is developed for the acoustic and structural scattered response from the junction of two flat plates under unilateral fluid loading. The plates are modelled by the classical theory of flexure, and the solution is found using the Wiener–Hopf technique for the dual integral equations for the unknown pressure on the plates. Explicit formulae are obtained for the pressure transform when the plates are in welded or clamped contact, and corresponding explicit expressions are given for the various diffraction coefficients associated with the fluid–structure interaction. The magnitudes of the reflection and transmission coefficients of structural waves display very simple analytic forms at low frequency.

1. Introduction

Subsonic flexural energy is known to be a significant part of the total vibrational energy of submerged structures, but it is only weakly coupled to the acoustic field on smooth structural segments. One of the main sources of flexural energy, and the site for acoustic radiation from flexural waves, must be at junctions where material and structural properties are discontinuous. In this paper we consider the interaction of sound with the junction of two plates which are fluid loaded on one side, as depicted in figure 1. Both plates support flexural motion in the sense of the classical theory of bending, and they may have different thickness, mass density, and bending stiffness. The presence of the material discontinuity at the junction couples all possible wave number vectors in the same way that a point attachment such as a rib on a uniform plate of infinite extent under fluid loading can generate waves in all directions. The latter problem is far simpler, however, because the effect of the structural attachment can be represented in terms of a point force and a point couple, each of which can be represented as explicit integrals suitable for a computer (Seren & Hayek 1989; Photiadis 1993, 1995). Scattering from clamped and finite impedance ribs has been widely discussed (e.g. Lyapunov 1969; Guo 1993, 1994), and the related problem of cracks and joints in otherwise uniform plates was considered by Howe (1986, 1994).

Scattering problems of the type considered here, with different boundary conditions on complementary half-lines, are invariably attacked using the Wiener–Hopf

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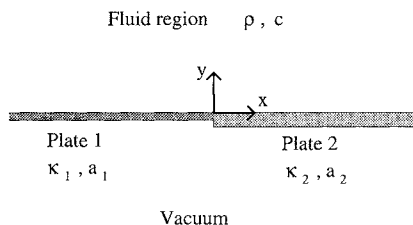


Figure 1. The plate junction and coordinate system.

technique (Noble 1958). For instance, Crighton (1970) discussed the general problem of different flexible boundaries, although he only considered the case of locally reactive boundaries in detail. The particular problem addressed here has been discussed by Kouzov (1963, 1969) and Brazier-Smith (1987). Kouzov developed a formal solution in terms of a transform which satisfies a Wiener–Hopf equation, and he outlined the different types of junction conditions possible. The essence of the problem may be reduced to two issues: (i) analytic factorization of the kernel in the Wiener–Hopf equation, and (ii) satisfaction of the edge conditions at the join. Kouzov (1963, 1969) provided only a formal solution for (i) in terms of infinite Cauchy integrals, and failed to address the difficult task of actually applying the edge conditions. The approach of Brazier-Smith (1987) was quite different, but more direct than that of Kouzov. Brazier-Smith handled the first item using what might be called a ‘brute force’ approach of performing an infinite Cauchy integral along the real axis, although the range of integration was transformed to a finite one by a clever change of variable. Brazier-Smith considered a variety of possible edge conditions (welded, hinged and free–free) and in each case they reduced to a system of eight simultaneous equations. The more complicated problem of acoustic scattering by the junction of two Timoshenko–Mindlin plates was considered by Woolley (1980). However, his assumed form for the scattered potential (equation (7) in (Woolley 1980)) is not a solution to the Helmholtz equation in the fluid, contrary to his contention. The subsequent analysis in (Woolley 1980) is therefore of questionable validity. As noted above, many authors have considered effects related to acoustic interaction with a single semi-infinite plate in a fluid. The intermediate case of a plate coplanar with an absorbing boundary was recently analysed by Asghar *et al.* (1994).

Although we consider the same problem as Kouzov (1963, 1969) and Brazier-Smith (1987), we believe our method of solution is far simpler and more physically appealing. For instance, in §5 we obtain explicit and simple formulae for the amplitudes of reflected and transmitted structural waves from welded and clamped joins under heavy fluid loading conditions. Concerning item (i) above, analytic factorization of the Wiener–Hopf kernel is performed using a new, general procedure developed by the authors (Norris & Wickham 1995), by which the desired functions are obtained as finite integrals. This form allows a simple and direct asymptotic expansion germane to a rigorous satisfaction of the edge conditions (ii), a luxury not afforded by the brute force approach of Brazier-Smith (1987). Thus the conditions at the junction of the two plates are attacked in a straightforward manner leading to far fewer equations to be solved in general. Also, the terms in these equations can be found in analytic form, and have some physical significance, in contrast to Brazier-Smith’s approach. In fact, we derive explicit solutions for the cases of welded and clamped

plates, and these solutions in turn yield new, simple and useful results for scattering under heavy fluid loading.

The outline of the paper is as follows. We begin in § 2 with the equations of motion and a statement of the scattering problem. Some related physical quantities, such as reflection coefficients, are also introduced. The formal solution of the problem is derived in § 3 in terms of an undetermined polynomial function $A(\xi)$. Some general properties of the solution and the diffraction and coupling coefficients are discussed there. A systematic procedure is described in § 4 for finding the polynomial $A(\xi)$, and the particular solutions for welded and clamped plates are obtained in § 5, where some numerical results are also presented.

2. Formulation of the scattering problem

(a) Dynamic equations

We consider time harmonic motion of frequency $\omega > 0$, the factor $\text{Re}\{e^{-i\omega t}\}$ understood but suppressed. The problem is two dimensional, with inviscid, compressible fluid in the half space $-\infty < x < \infty$, $0 < y < \infty$, and two plates lie along the x -axis. The field quantities of interest are the acoustic pressure $p(x, y)$ and the plate deflection into the fluid, $w(x)$. The pressure satisfies the Helmholtz equation in the fluid region,

$$\nabla^2 p + k^2 p = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty, \quad (2.1)$$

where $k = \omega/c$ is the acoustic wave number, and c is the fluid sound speed. The equation of kinematic continuity between the plates and the fluid is

$$\rho \omega^2 w(x) = \frac{\partial p}{\partial y}(x, 0), \quad -\infty < x < \infty, \quad (2.2)$$

where ρ is the fluid mass density.

The two plates may have different densities, elastic properties, and thicknesses, but each is uniform and its motion is modelled by the classical theory of dynamic flexure. Assuming they meet at $x = 0$, we have

$$B_j \frac{d^4 w}{dx^4}(x) - m_j \omega^2 w(x) = -p(x, 0), \quad \begin{cases} -\infty < x < 0, & j = 1, \\ 0 < x < \infty, & j = 2. \end{cases} \quad (2.3)$$

The plate parameters are the mass per unit area, $m_{1,2}$, and the bending stiffness, $B_{1,2}$, and are constant on each plate. These quantities may be related to the intrinsic plate properties; thus, $m = \rho_s h$, and $B = Eh^3/12(1 - \nu^2)$, where h , ρ_s , E and ν are the thickness, volumetric mass density, Young's modulus, and Poisson ratio, respectively. For each plate we define the flexural wave number, $\kappa_{1,2}$, and the impedance length, $a_{1,2}$, by

$$\kappa_j^4 = \omega^2 m_j / B_j, \quad a_j = m_j / \rho, \quad j = 1, 2. \quad (2.4)$$

The plate frequency defined by $ka = 1$ serves as a useful threshold distinguishing the transition from a low frequency, pressure release regime, to higher frequencies where the plate dynamics cannot be ignored. Eliminating the plate deflection between the pair of boundary conditions (2.2) and (2.3) reduces them to a single equation for the pressure,

$$\mathcal{L}_1 p(x, 0) = 0, \quad x < 0; \quad \mathcal{L}_2 p(x, 0) = 0, \quad x > 0, \quad (2.5)$$

where the boundary operators are

$$\mathcal{L}_j \equiv 1 + a_j \left(\kappa_j^{-4} \frac{\partial^4}{\partial x^4} - 1 \right) \frac{\partial}{\partial y}, \quad j = 1, 2. \quad (2.6)$$

The diffraction problem can now be formulated entirely in terms of the pressure. Thus, we need to solve the Helmholtz equation (2.1) in the fluid, subject to radiation conditions as $y \rightarrow \infty$ and the boundary conditions (2.5) on $y = 0$. Four edge conditions must be specified at the junction $x = y = 0$. The precise form of the conditions are given in §4 for the following types of junction: (i) welded, (ii) clamped-clamped, (iii) free-free, and (iv) hinged. The first two cases are discussed in detail in §5. The formulation of the problem is complete once we have specified the incident wave field. There are several possible types of incident wave to consider, both acoustic and structural, and these will be delineated in §3. First we need to discuss the dispersion relations for the infinite plates.

(b) *The dispersion function and reflection coefficients*

The fluid-loaded interaction with a plate is characterized by the functions D and $\bar{D}$, defined by

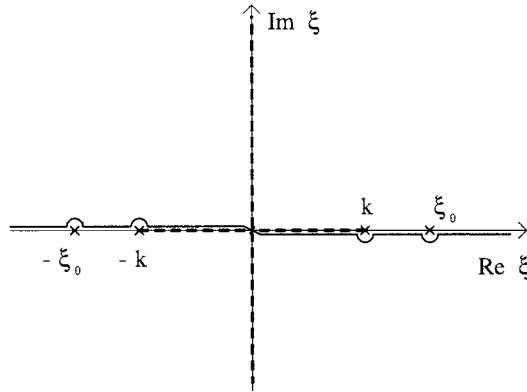
$$\mathcal{L} e^{(i\xi x - \gamma y)} = e^{(i\xi x - \gamma y)} D(\xi), \quad \mathcal{L} e^{(i\xi x + \gamma y)} = e^{(i\xi x + \gamma y)} \bar{D}(\xi), \quad (2.7)$$

where $\mathcal{L}$ and D stand for one of $\mathcal{L}_j$ and D_j , with $j = 1$ or 2 . The square root $\gamma(\xi) = (\xi^2 - k^2)^{1/2}$ is defined as an analytic function in the complex ξ -plane cut as shown in figure 2 so that its real part is non-negative. Along the real axis $\gamma(\xi) = -i\sqrt{k^2 - \xi^2}$ for $|\xi| < k$ and $\gamma(\xi) = \sqrt{\xi^2 - k^2}$ for $|\xi| > k$. We have selected this branch for γ so that certain Fourier superpositions of solutions of the form used in (2.7)₁ are outgoing at infinity. Note also that, for later convenience, we have given k a small positive imaginary part, i.e. $k = |k|e^{i\epsilon}$, $0 < \epsilon \ll 1$. We shall further assume that $D(\xi) \neq 0$, $\xi \in \mathcal{H}^+ \cap \mathcal{H}^-$, where $\mathcal{H}^\pm$ are upper and lower halves of the complex ξ -plane as shown in figure 2. Thus,

$$D(\xi) = 1 - \gamma(\xi)V(\xi), \quad \bar{D}(\xi) = 1 + \gamma(\xi)V(\xi), \quad V(\xi) = a(\kappa^{-4}\xi^4 - 1). \quad (2.8)$$

Hence, D is defined for outgoing wave solutions, and $\bar{D}$ for ingoing solutions.

Zeros of these dispersion functions are of some significance. A real root of $D = 0$ exists at all frequencies, corresponding to the subsonic flexural wave (i.e. slower than acoustic). The dispersion relation $D = 0$ and its root structure has been the subject of much discussion in the literature, e.g. Crighton (1979). In general, the field at a point is represented by an integral along the real ξ -axis, where the definition of the square root function γ is unambiguous. However, there is no unique way to define the square root of the real axis, and one could choose different cuts extending to infinity. We have specifically indicated one, see figure 2. Contour deformation from the real axis, combined with a given definition of the cuts for the square root can lead to poles arising from zeros of the function $\bar{D}$. We will denote these complex roots of $\bar{D} = 0$ as ‘leaky’ wave roots. There is some ambiguity in ascribing physical significance to leaky waves, because their very existence is a sensitive function of the choice of branch cut for the analytic extension of the square root of the real axis (Crighton 1979). One may even consider incident leaky waves for the problem at hand, if, for example, a source is at a finite distance from the junction. This is not unphysical, as long as one remembers that a leaky wave cannot exist in isolation, but is simply a mathematically identifiable part of a spectrum. Therefore, for future


 Figure 2. The ξ -plane and the branch cuts for $\gamma(\xi)$.

usage, we adopt the terminology that plate waves refer to zeros of $D(\xi)$ whereas the zeros of $\bar{D}(\xi)$ are called leaky waves. In summary, the function $D(\xi)$ and its zeros are necessary for generating the solution, and it is mathematically convenient to introduce the related function $\bar{D}(\xi)$ and its zeros. Some or all of these wave numbers can be considered as incident wave numbers, whether they are leaky or not. We will consider all possibilities for incident waves in the next section, although leaky wave incidence is not addressed in the numerical examples discussed later.

Now suppose a plane wave with incident x -component of slowness ξ_0 impinges on a homogeneous boundary $y = 0$ where the acoustic pressure satisfies one of (2.5) for all x , then the total field consisting of incident plus reflected waves is

$$p^{(0)}(x, y) = e^{(i\xi_0 x + \gamma(\xi_0)y)} + R(\xi_0)e^{(i\xi_0 x - \gamma(\xi_0)y)}, \quad R(\xi) = -\bar{D}(\xi)/D(\xi). \quad (2.9)$$

The reflection coefficient for a plane acoustic wave incident at angle θ from the surface is $\mathcal{R}(\theta) = R(k \cos \theta)$, and hence $|\mathcal{R}(\theta)| = 1$ for real θ .

(c) Distinguished wave numbers for the combined plates

The reflection coefficients, $R_1(\xi)$ and $R_2(\xi)$, of the individual plates are generally distinct, but they coincide for certain values of the incident wave number. Thus,

$$R_1 - R_2 = 2\gamma P^*/D_1 D_2, \quad (2.10)$$

where $P^*(\xi)$ is the quartic polynomial

$$P^*(\xi) = V_2(\xi) - V_1(\xi) = P_0^*(\xi^4 - \zeta_1^4), \quad (2.11)$$

and

$$P_0^* = \frac{a_2}{\kappa_2^4} - \frac{a_1}{\kappa_1^4}, \quad \zeta_1^4 = (a_2 - a_1) \left(\frac{a_2}{\kappa_2^4} - \frac{a_1}{\kappa_1^4} \right)^{-1} = \kappa_1^4 \left(\frac{\alpha - 1}{\beta - 1} \right). \quad (2.12)$$

The dimensionless parameters α and β are

$$\alpha = a_2/a_1, \quad \beta = B_2/B_1. \quad (2.13)$$

Hence, if ξ is a root of $P^*(\xi) = 0$, while $D_1(\xi)D_2(\xi) \neq 0$, then $R_1(\xi) = R_2(\xi)$, and both plates reflect equally at such values. The four zeros of P^* , $\pm\zeta_1$, $\pm\zeta_2$, are necessarily outside $\mathcal{H}^+ \cap \mathcal{H}^-$, and we define them such that ζ_1 and ζ_2 are in $\mathcal{H}^+$. We will see that the zeros of P^* play a central role in the general solution.

3. Formal solution of the diffraction problem

(a) Incident and scattered fields

As a first step in the solution of the scattering problem defined in §2 we define the scattered field $p^{(s)}$ according to

$$p(x, y) = p^{(0)}(x, y) + p^{(s)}(x, y), \quad (3.1)$$

where $p^{(0)}$ is an incident wave solution with horizontal wave number ξ_0 and satisfying the boundary condition on $x < 0$. Thus, $p^{(0)}$ may be an incident plane acoustic wave with $\xi_0 = k \cos \theta_0$, where $0 < \theta_0 < \pi/2$ so that ξ_0 lies in the upper half plane, and the amplitude of $p^{(0)}$ at the origin is then $(1 + \mathcal{R}_1(\theta_0))$ times the incident pressure there. Or, it could be an incident plate wave or leaky wave, in which case ξ_0 is a zero of one of the dispersion functions $D(\xi)$ or $\bar{D}(\xi)$ with $\text{Re } \xi_0 > 0$. Finally, we include the possibility of an acoustic 'end-fire' wave (Brazier-Smith 1979). This is an acoustic wave in the fluid with horizontal wave number $\xi_0 = k$, and linear dependence upon the depth coordinate, i.e. a solution of the form $p^{(0)}(x, y) = (1 + \mu y) \exp(ikx)$. This solves the Helmholtz equation in the fluid and it satisfies the boundary conditions on the left-hand plate if μ is chosen appropriately. The end-fire wave cannot exist in an infinitely extended medium, but it may be present in real structures. The incident field is therefore assumed to be one of the following:

$$p^{(0)}(x, y) = e^{i\xi_0 x} \times \begin{cases} [e^{\gamma(\xi_0)y} + R_1(\xi_0)e^{-\gamma(\xi_0)y}], & \text{acoustic wave,} \\ e^{-\gamma(\xi_0)y}, & \text{plate wave,} \\ e^{\gamma(\xi_0)y}, & \text{leaky wave,} \\ [1 - y/V_1(k)] \quad (\xi_0 = k), & \text{end-fire wave.} \end{cases} \quad (3.2)$$

We assume, without loss of generality, that ξ_0 lies in the upper half, $\mathcal{H}^+$, of the complex ξ -plane described in figure 2.

The boundary conditions (2.5) may then be written as

$$\mathcal{L}_1 p^{(s)}(x, 0) = 0, \quad x < 0; \quad \mathcal{L}_2 p^{(s)}(x, 0) = -\mathcal{L}_2 p^{(0)}(x, 0), \quad x > 0. \quad (3.3)$$

We now introduce an outgoing Fourier superposition of plane waves for $p^{(s)}$ in the form

$$p^{(s)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{p}(\xi) e^{(i\xi x - \gamma(\xi)y)} d\xi. \quad (3.4)$$

This will satisfy (3.3) if the dual equations

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} D_1(\xi) \tilde{p}(\xi) e^{i\xi x} d\xi = 0, \quad x < 0, \quad (3.5a)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} D_2(\xi) \tilde{p}(\xi) e^{i\xi x} d\xi = -D_2(\xi_0) A_0 e^{i\xi_0 x}, \quad x > 0, \quad (3.5b)$$

hold with A_0 given by

$$R_1(\xi_0) - R_2(\xi_0), \quad 1, \quad -R_2(\xi_0), \quad -P^*(k)/V_1(k), \quad (3.6)$$

for incidence of an acoustic wave, a plate wave, a leaky wave, or an end-fire wave (for which $\xi_0 = k$), respectively. Note that A_0 vanishes for plane wave incidence if ξ_0 is a root of $P^* = 0$ (but not a root of $D_1 D_2 = 0$, cf. (2.10)). One might think that

the scattered pressure is identically zero in this case. However, the solution of the dual integral equations (3.5) must also satisfy the edge conditions. The incident field $p^{(0)}$ will not, in general, be consistent with the edge conditions, and must therefore be supplemented by a *null solution* of the dual integral equations when ξ_0 coincides with ζ_n . This particular case will not be discussed further here, and we will operate under the assumption that $A_0 \neq 0$. We note that $\mathcal{L}_1$ and $\mathcal{L}_2$ reduce to impedance operators if the bending effects are set to zero, and then no edge conditions need be considered. The solution is relatively simple and is discussed by Norris & Rebinsky (1995).

(b) *General solution*

It is evident that the first of (3.5) is satisfied by writing

$$\tilde{p}(\xi) = F^-(\xi)/D_1(\xi), \quad (3.7)$$

where F^- is *any* function analytic in $\mathcal{H}^-$ and

$$F^-(\xi) = O(\xi^{-1}), \quad \xi \rightarrow \infty, \quad \xi \in \mathcal{H}^-. \quad (3.8)$$

Substituting this ansatz into the second of (3.5) yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{D_2(\xi)}{D_1(\xi)} F^-(\xi) e^{i\xi x} d\xi = -D_2(\xi_0) A_0 e^{i\xi_0 x}, \quad x > 0. \quad (3.9)$$

Again by inspection, a particular solution of this equation is

$$F^-(\xi) = \frac{iA_0 D_2(\xi_0)}{\xi - \xi_0} \frac{K^-(\xi)}{K^+(\xi_0)}, \quad (3.10)$$

where $K^\pm(\xi)$ are particular Wiener–Hopf factors of the quotient of the two dispersion relations, such that

$$D_1(\xi)/D_2(\xi) = K(\xi) = K^-(\xi)/K^+(\xi) \quad \text{with} \quad K^-(-\xi) = 1/K^+(\xi). \quad (3.11)$$

Thus, in particular, $K^\pm(\xi)$ are analytic in the half-planes $\mathcal{H}^\pm$ of figure 2.

A little reflection on the preceding argument shows that the boundary condition on each plate is also formally satisfied by the scattered pressure field

$$\begin{aligned} p^{(s)}(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A(\xi) F^-(\xi)}{D_1(\xi)} e^{(i\xi x - \gamma(\xi)y)} d\xi \\ &= \frac{1}{2\pi} A \left(-i \frac{\partial}{\partial x} \right) \int_{-\infty}^{\infty} \frac{F^-(\xi)}{D_1(\xi)} e^{(i\xi x - \gamma(\xi)y)} d\xi, \end{aligned} \quad (3.12)$$

where $A(\xi)$ is a polynomial of degree q (say) and

$$A(\xi_0) = A_0. \quad (3.13)$$

The degree q is chosen so that $w(x)$ is bounded everywhere on the plates and $p_s(x, y)$ exists everywhere in the fluid region including the origin. It follows from the explicit expressions for $K^\pm(\xi)$ given in Appendix A that

$$K^\pm(\xi) = O(1), \quad |\xi| \rightarrow \infty, \quad (3.14)$$

and $D_j(\xi) = O(\xi^5)$ at infinity and hence the maximum value of q is 4. The general solution for the pressure transform is

$$\tilde{p}(\xi) = \frac{iA(\xi)}{\xi - \xi_0} \frac{G(\xi_0)}{G(\xi)}, \quad (3.15)$$

where G is an analytical generalization of the dispersion functions,

$$G(\xi) \equiv D_1(\xi)/K^-(\xi) = D_2(\xi)/K^+(\xi), \quad (3.16)$$

Thus we have provided a formal construction of a q -parameter family of outgoing scattered fields satisfying Helmholtz's equation and the plate boundary conditions (3.3). It remains to satisfy the conditions at the junction of the two plates. Evidently the value of q only depends on the size of the factors $K^\pm$ as $\xi \rightarrow \infty$ and we would expect that the physical constraints at the join will also number q to enable a unique construction. We will show in §4 that this is indeed the case, using an analytic construction for $K^\pm(\xi)$ developed in Appendix A. Firstly however, we discuss the general form of the solution for the physical quantities of interest.

(c) *Displacement and pressure solutions*

Equations (2.2), (3.1), (3.4), and (3.15), show that the pressure and the transverse deflection may be expressed in terms of two fundamental potentials $p_0(x, y)$, and $w_0(x)$, i.e.

$$p(x, y) = p^{(0)}(x, y) - A \left(-i \frac{\partial}{\partial x} \right) p_0(x, y), \quad (3.17a)$$

$$\rho \omega^2 w(x) = \frac{\partial p^{(0)}}{\partial y}(x, 0) - A \left(-i \frac{d}{dx} \right) w_0(x), \quad (3.17b)$$

where

$$p_0(x, y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G(\xi_0)}{G(\xi)} \frac{e^{i\xi x - \gamma y}}{\xi - \xi_0} d\xi, \quad (3.18a)$$

$$w_0(x) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \gamma(\xi) \frac{G(\xi_0)}{G(\xi)} \frac{e^{i\xi x}}{\xi - \xi_0} d\xi. \quad (3.18b)$$

At first it may appear to be a simple matter to formally apply the junction continuity conditions using these expressions. However, $w_0(x)$ will in general have weak singularities at $x = 0$ and therefore it is necessary to proceed with caution. Our approach is straightforward in that we will first derive alternative expressions for $w_0(x)$ so that it is easy to find its power series as $x \rightarrow 0^\pm$ and then obtain an algebraic system of equations for the undetermined coefficients in A by substituting these expansions into (3.17b) and the junction conditions.

(d) *Diffraction coefficients*

The scattered pressure simplifies at distances far from the junction in units of the longest wavelength in the problem. A far field approximation may be obtained by the usual methods of first shifting the contour of integration from the real axis to the path of steepest descents. The saddle point contribution then yields the scattered pressure in the fluid. When the observation angle θ (see figure 1) is close to $\theta = 0$, the deformation onto the path of steepest descents will capture a pole corresponding to the subsonic flexural wave traveling to the right in $x > 0$. This pole occurs at the positive real zero of $D_2(\xi)$. Similarly when θ is close to $\theta = \pi$, the deformation captures a pole at the negative real zero of $D_1(\xi)$. When the incident field is a flexural wave advancing from the left, say, the residues at these poles are the reflected and transmitted waves respectively. Using (3.15) and the identities (3.16) with $\xi_0 = \xi_0^{(1)}$,

we thus find that the left- and right-going flexural waves are

$$p_1(x, y) = R_{\text{flex}} e^{[-i\xi_0^{(1)}x - \gamma(\xi_0^{(1)})y]}, \quad x < 0; \quad p_2(x, y) = T_{\text{flex}} e^{[i\xi_0^{(2)}x - \gamma(\xi_0^{(2)})y]}, \quad x > 0, \quad (3.19)$$

respectively, where $\xi_0^{(2)}$ is the wave number of flexural waves on the right hand plate. The reflection and transmission coefficients are

$$R_{\text{flex}} = \frac{A(-\xi_0^{(1)}, \xi_0^{(1)})G(\xi_0^{(1)})}{2\xi_0^{(1)}K^+(\xi_0^{(1)})D_1'(\xi_0^{(1)})}, \quad T_{\text{flex}} = \frac{A(\xi_0^{(2)}, \xi_0^{(1)})G(\xi_0^{(1)})K^+(\xi_0^{(2)})}{(\xi_0^{(1)} - \xi_0^{(2)})D_2'(\xi_0^{(2)})}, \quad (3.20)$$

respectively, and we have rewritten $A(\xi)$ as $A(\xi, \xi_0)$ to remind us that it is a function of both the incident and scattered directions. Note that T_{flex} is the transmitted amplitude for the surface pressure, not the displacement.

The acoustic far-field in the fluid depends upon a diffraction coefficient $C(\theta)$ defined such that

$$p^{(s)} = C(\theta) \sqrt{2/\pi kr} e^{-i\pi/4} e^{ikr}, \quad r \rightarrow \infty, \quad 0 < \theta \leq \pi. \quad (3.21)$$

The value of C may be found using the method of steepest descents applied to the integral (3.4) after making the conformal mapping $\xi = -k \cosh t$ and writing (x, y) in polar coordinates (r, θ) , see figure 1. If the incident wave is an acoustic wave with angle of incidence θ_0 such that $\xi_0 = k \cos \theta_0$, the diffraction coefficient can be considered a function of both angles, i.e. $C(\theta) = C(\theta, \theta_0)$, and $\xi = k \cos \theta$. It follows from (3.15) as

$$C(\theta, \theta_0) = \frac{1}{2} k \sin \theta \tilde{p}(k \cos \theta) = -\frac{1}{2} \gamma(\xi) \frac{G(\xi_0)}{G(\xi)} \frac{A(\xi, \xi_0)}{\xi - \xi_0}. \quad (3.22)$$

(e) Reciprocity and energy conservation

Acoustical reciprocity requires that the diffraction should be the same under the interchange of the source and receiver directions, or

$$C(\theta, \theta_0) = C(\pi - \theta_0, \pi - \theta). \quad (3.23)$$

This implies, using (3.22),

$$A(\xi, \xi_0) \gamma(\xi) G(\xi_0)/G(\xi) = A(-\xi_0, -\xi) \gamma(\xi_0) G(-\xi)/G(-\xi_0). \quad (3.24)$$

It follows from (2.10) and (3.16) that

$$G(\xi)G(-\xi) = D_1(\xi)D_2(\xi) = 2\gamma(\xi)P^*(\xi)/[R_1(\xi) - R_2(\xi)], \quad (3.25)$$

while the denominator in the last expression simplifies further for acoustic wave incidence as $R_1(\xi) - R_2(\xi) = A(\xi, \xi)$, from (3.6). Using (3.24) and (3.25) and the identity $A(\xi, \xi) = A(-\xi, -\xi)$, we see that reciprocity implies the connection

$$A(\xi, \xi) A(-\xi, \xi_0) P^*(\xi_0) = A(\xi_0, \xi_0) A(-\xi_0, \xi) P^*(\xi). \quad (3.26)$$

The polynomial A must satisfy this relation for arbitrary plane wave incidence.

The energy flux associated with an incident flexural wave has been derived by Crighton & Innes (1984) for fluid loading on both sides of a plate. The modification for unilateral loading is straightforward (although there is a typographical error in equation (5.1) of Brazier-Smith (1987)) and gives a flux of

$$\left(\frac{4B\xi_0^3\gamma^2(\xi_0)}{\rho\omega^2} + \frac{\xi_0}{\gamma(\xi_0)} \right) \frac{|p|^2}{2\rho\omega} = -D'(\xi_0)\gamma(\xi_0) \frac{|p|^2}{2\rho\omega}, \quad (3.27)$$

where $|p|$ is the surface pressure amplitude. We have assumed a value of unity in §2. The total flux of acoustic energy diffracted from the junction into the fluid follows from (3.21) as

$$\lim_{r \rightarrow \infty} \frac{1}{\rho c} \int_0^\pi |p^{(s)}(r, \theta)|^2 r \, d\theta = \frac{2}{\pi \rho \omega} \int_0^\pi |C(\theta)|^2 \, d\theta. \quad (3.28)$$

Now consider an incident flexural wave with unit flux, then (3.27) and (3.28) may be combined to provide a statement of energy conservation. We specifically assume that subsonic flexural waves on each plate provide the only means of energy transmission away from the junction, other than the acoustic diffraction loss. Thus,

$$1 = |R_{\text{flex}}|^2 + \frac{D'_2(\xi_0^{(2)})\gamma(\xi_0^{(2)})}{D'_1(\xi_0^{(1)})\gamma(\xi_0^{(1)})} |T_{\text{flex}}|^2 - \frac{1}{D'_1(\xi_0^{(1)})\gamma(\xi_0^{(1)})} \frac{4}{\pi} \int_0^\pi |C(\theta)|^2 \, d\theta. \quad (3.29)$$

The three terms in the right member are each positive and less than unity, and correspond to the fractional energy reflected on plate 1, transmitted on plate 2, and acoustically radiated into the fluid.

4. Satisfying the join conditions

We now give a general systematic procedure for evaluating $A(\xi)$ in the formal solution (3.15) so that various prescribed conditions at the junctions of the plates may be determined. Application of the junction conditions requires knowledge of the behaviour of the potential $w_0(x)$ in the neighbourhood of the join, $x = 0$. Our first order of business is to obtain an analytic expansion for this quantity, actually, as we will see, a power series in ascending powers of x . We can then apply the conditions directly in physical space.

(a) Alternative integral forms for w_0

The potential is in the form of a fourier transform which can be separated into two distinct transforms each of which vanishes for either $x > 0$ or $x < 0$. Thus,

$$w_0(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \tilde{w}_0(\xi) e^{i\xi x} \, d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \tilde{w}_0^\pm(\xi) e^{i\xi x} \, d\xi, \quad x \gtrless 0, \quad (4.1)$$

where $\tilde{w}_0^+$ and $\tilde{w}_0^-$ partition $\tilde{w}_0$ as

$$\tilde{w}_0(\xi) = \tilde{w}_0^+(\xi) + \tilde{w}_0^-(\xi), \quad (4.2)$$

and $\tilde{w}_0^\pm(\xi)$ are analytic in $\mathcal{H}^\pm$ respectively and both $\tilde{w}_0^\pm = O(\xi^{-1})$ as $|\xi| \rightarrow \infty$. The transform $\tilde{w}_0$ as defined in (3.18b) may be rewritten by noting that

$$\frac{\gamma(\xi)}{G(\xi)} = \frac{K^+(\xi) - K^-(\xi)}{V_1(\xi) - V_2(\xi)}, \quad (4.3)$$

which is easily obtained by eliminating γ between the two equations (3.16). Thus,

$$\tilde{w}_0(\xi) = G(\xi_0) \frac{(K^+(\xi) - K^-(\xi))}{(\xi - \xi_0)P^*(\xi)}, \quad (4.4)$$

where the polynomial in the denominator is defined in (2.11). The partition functions can be found quite easily from (4.4) because, apart from the split function K^+ and K^- , the only singularities are from the simple poles at $\xi = \xi_0$ and the roots of $P^* = 0$.

By adding and subtracting poles at the same points with suitable residues, we can arrive at explicit formulae for the partitioned functions. They therefore become

$$\tilde{w}_0^+(\xi) = \frac{G(\xi_0)}{\xi - \xi_0} \left\{ \frac{K^+(\xi)}{P^*(\xi)} - \frac{K^+(\xi_0)}{P^*(\xi_0)} - \sum_{n=1}^2 \left[\frac{(\xi - \xi_0)u_n^+}{(\xi - \zeta_n)(\zeta_n - \xi_0)} + \frac{(\xi - \xi_0)u_n^-}{(\xi + \zeta_n)(-\zeta_n - \xi_0)} \right] \right\}, \quad (4.5a)$$

$$\tilde{w}_0^-(\xi) = \frac{G(\xi_0)}{\xi - \xi_0} \left\{ \frac{K^-(\xi)}{P^*(\xi)} - \frac{K^-(\xi_0)}{P^*(\xi_0)} + \sum_{n=1}^2 \left[\frac{(\xi - \xi_0)u_n^+}{(\xi - \zeta_n)(\zeta_n - \xi_0)} + \frac{(\xi - \xi_0)u_n^-}{(\xi + \zeta_n)(-\zeta_n - \xi_0)} \right] \right\}, \quad (4.5b)$$

where

$$u_n^\pm = \text{residue of } [K^\pm(\xi)/P^*(\xi)] \text{ at } \xi = \pm\zeta_n. \quad (4.6)$$

The residues u_n^+ and u_n^- can be related to one another by noting, from (2.9)₂, (2.11), and (3.16), that $K(\pm\zeta_n) = 1$. Let ν_1 and ν_2 be the logarithms of the split function at the roots, such that

$$K^+(\zeta_n) = e^{\nu_n}. \quad (4.7)$$

Then using the fact that P^* is an even function, it is easily seen that

$$u_n^\pm = \pm e^{\pm\nu_n} / (4\zeta_n^3 P_0^*), \quad n = 1, 2. \quad (4.8)$$

(b) Behaviour of w_0 near the join

In order to satisfy the join conditions we only need the asymptotic forms for w_0 and its derivatives near $x = 0$, and these follow immediately from the power series expansion of the fourier transform about the point at infinity. Suppose that the partitions of $\tilde{w}_0(\xi)$ are of the form

$$\tilde{w}_0^\pm(\xi) = \mp \sum_{n=0}^{M-1} \lambda_n^\mp \xi^{-(n+1)} + O(\xi^{-(M+1)} \log \xi), \quad |\xi| \rightarrow \infty, \quad (4.9)$$

for some integer $M \geq 1$. Then it can be shown that

$$w_0(x) = \sum_{n=0}^{M-1} \lambda_n^\pm \frac{(ix)^n}{n!} + O(x^M \log |x|), \quad x \rightarrow 0^\pm. \quad (4.10)$$

The only terms containing logarithmic singularities in the expressions (4.5) for $\tilde{w}_0^\pm$ are those with $K^\pm(\xi)$, respectively. Referring to the asymptotic results in Appendix B, specifically (B7), and to (4.5), shows that the leading order singular term at infinity in the expansion of the fourier transforms for w_0 in (4.1) is of order $\xi^{-10} \log \xi$. On inversion, the latter yields a term of order $x^9 \log |x|$, and hence $M = 9$. The coefficients in (4.10) are listed explicitly in Appendix B. We note that $\lambda_j^+ = \lambda_j^-$ for $j = 0$ through $j = 3$, and hence $w_0(x)$ and its first three derivatives are continuous at $x = 0$.

(c) *Determination of $A(\xi)$*

The fact that there are four edge conditions to be satisfied, combined with the condition (3.13), suggests that the undetermined polynomial $A(\xi)$ is of fourth order ($q = 4$). Let

$$A(\xi) = \sum_{n=0}^4 \bar{A}_n \xi^n. \quad (4.11)$$

Then (3.13) implies the identity

$$\sum_{n=0}^4 \bar{A}_n \xi_0^n = A_0. \quad (4.12)$$

The displacement near the origin therefore follows from (3.17*b*), (4.10), and (4.11), as

$$\rho \omega^2 w(x) = \frac{\partial p^{(0)}}{\partial y}(x, 0) - \sum_{n=0}^4 \Lambda_n^{\pm} \frac{(ix)^n}{n!} + O(x^5 \log |x|), \quad x \gtrless 0, \quad (4.13)$$

where

$$\Lambda_n^{\pm} = \sum_{k=0}^4 \lambda_{k+n}^{\pm} \bar{A}_k, \quad n = 0, 1, \dots, 4. \quad (4.14)$$

The four edge conditions can now be applied for the different junctions.

(i) *Welded edges*

The four edge conditions dictate continuity of the displacement, rotation, bending moment and shear force, i.e. w , w' , $-Bw''$, $-Bw'''$, respectively, where the prime denotes the x -derivative. These can now be expressed, using (4.13), as

$$\begin{aligned} \Lambda_0^+ &= \Lambda_0^-, & \Lambda_1^+ &= \Lambda_1^-, \\ \beta \Lambda_2^+ - \Lambda_2^- &= \xi_0^2 (\beta - 1) p_{,y}^{(0)}(0, 0), & \beta \Lambda_3^+ - \Lambda_3^- &= \xi_0^3 (\beta - 1) p_{,y}^{(0)}(0, 0). \end{aligned} \quad (4.15)$$

(ii) *Clamped edges*

If the ends of the plates are clamped at $x = 0$ then both w and w' vanish at either plate termination. The four conditions follow from (4.13) as

$$\Lambda_0^+ = \Lambda_0^-, \quad \Lambda_1^+ = \Lambda_1^-, \quad \Lambda_0^+ = p_{,y}^{(0)}(0, 0), \quad \Lambda_1^+ = \xi_0 p_{,y}^{(0)}(0, 0). \quad (4.16)$$

(iii) *Free edges*

Alternatively, the plates may be free-free, in which case w'' and w''' vanish on either side of $x = 0$. The four conditions become

$$\Lambda_2^+ = \Lambda_2^-, \quad \Lambda_3^+ = \Lambda_3^-, \quad \Lambda_2^+ = \xi_0^2 p_{,y}^{(0)}(0, 0), \quad \Lambda_3^+ = \xi_0^3 p_{,y}^{(0)}(0, 0). \quad (4.17)$$

(iv) *Hinged edges*

If the plates are hinged at $x = 0$ then w and Bw''' are continuous across the joint, while w'' vanishes on both plates there, implying the conditions

$$\Lambda_0^+ = \Lambda_0^-, \quad \Lambda_2^+ = \Lambda_2^-,$$

$$A_2^+ = \xi_0^2 p_{,y}^{(0)}(0,0), \quad \beta A_3^+ - A_3^- = \xi_0^3(\beta-1)p_{,y}^{(0)}(0,0). \quad (4.18)$$

Equations (4.12) and the relevant one of (4.15)–(4.18) constitute a set of five equations for the five unknowns $\bar{A}_n$, $n = 0, 1, \dots, 4$. An explicit linear system of equations for these unknowns follows from (4.14). Explicit solutions for the first two cases will be developed in the next section.

5. Welded or clamped plates

(a) The general form of A for welded plates

It turns out that the two kinematic conditions (4.15)₁ and (4.15)₂ are trivially satisfied. Thus, as mentioned previously, $\lambda_j^+ = \lambda_j^-$ for $j = 0$ through $j = 3$, and therefore, (4.15)₁ and (4.15)₂ imply respectively that $\bar{A}_4 = 0$ and $\bar{A}_3 = 0$. Hence, A is of second order ($q \rightarrow 2$) and there are only *three* equations to be satisfied for welded plates in contact: (4.12), (4.15)₃, and (4.15)₄. The three equations are

$$\begin{bmatrix} 1 & \xi_0 & \xi_0^2 \\ \beta\lambda_2^+ - \lambda_2^- & \beta\lambda_3^+ - \lambda_3^- & \beta\lambda_4^+ - \lambda_4^- \\ \beta\lambda_3^+ - \lambda_3^- & \beta\lambda_4^+ - \lambda_4^- & \beta\lambda_5^+ - \lambda_5^- \end{bmatrix} \begin{bmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \bar{A}_2 \end{bmatrix} = \begin{bmatrix} A_0 \\ \xi_0^2(\beta-1)p_{,y}^{(0)}(0,0) \\ \xi_0^3(\beta-1)p_{,y}^{(0)}(0,0) \end{bmatrix}. \quad (5.1)$$

This system can be simplified by first redefining the A polynomial as

$$A(\xi) = A_0 + (\tilde{A}_1 + \tilde{A}_2\xi)(\xi - \xi_0), \quad (5.2)$$

so that

$$\begin{aligned} \frac{1}{\beta-1} \begin{bmatrix} (\beta-1)(\lambda_3^\pm - \xi_0\lambda_2^\pm) & \beta(\lambda_4^+ - \xi_0\lambda_3^\pm) - (\lambda_4^- - \xi_0\lambda_3^\pm) \\ \beta(\lambda_4^+ - \xi_0\lambda_3^\pm) - (\lambda_4^- - \xi_0\lambda_3^\pm) & \beta(\lambda_5^+ - \xi_0\lambda_4^\pm) - (\lambda_5^- - \xi_0\lambda_4^\pm) \end{bmatrix} \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \\ = \begin{bmatrix} \xi_0^2 p_{,y}^{(0)}(0,0) - A_0\lambda_2^\pm \\ \xi_0^3 p_{,y}^{(0)}(0,0) - A_0\lambda_3^\pm \end{bmatrix}. \end{aligned} \quad (5.3)$$

The matrix elements can be reduced using the results of Appendix B, specifically the final three identities in (B 12). These results, combined with (B 10) and (B 11) for $p_{,y}^{(0)}(0,0)$ and $\lambda_2^\pm, \lambda_3^\pm$, imply that (5.3) multiplied by the factor $2P_0^*/G(\xi_0)$ becomes exactly

$$(\mathbf{M}_1 + \mathbf{M}_2) \begin{bmatrix} \tilde{A}_2 \\ \tilde{A}_1 \end{bmatrix} = -A_0 \left(\frac{\mathbf{M}_1}{\zeta_1^2 - \xi_0^2} + \frac{\mathbf{M}_2}{\zeta_2^2 - \xi_0^2} \right) \begin{bmatrix} 1 \\ \xi_0 \end{bmatrix}, \quad (5.4)$$

where

$$\mathbf{M}_n = \begin{bmatrix} \cosh \nu_n & \zeta_n^{-1} \sinh \nu_n \\ \zeta_n \sinh \nu_n & \cosh \nu_n \end{bmatrix}, \quad n = 1, 2. \quad (5.5)$$

Both $\mathbf{M}_1$ and $\mathbf{M}_2$ have determinant of unity, and are easily inverted. Using the identity $\zeta_1^2 + \zeta_2^2 = 0$, the system (5.4) implies that

$$\begin{bmatrix} \tilde{A}_2 \\ \tilde{A}_1 \end{bmatrix} = \frac{A_0}{\xi_0^4 - \zeta_1^4} \left\{ \xi_0^2 \mathbf{I} + \frac{\zeta_1^2 (\mathbf{M}_2^{-1} \mathbf{M}_1 - \mathbf{M}_1^{-1} \mathbf{M}_2)}{2(\cosh \nu_1 \cosh \nu_2 + 1)} \right\} \begin{bmatrix} 1 \\ \xi_0 \end{bmatrix}, \quad (5.6)$$

where $\mathbf{I} = \text{diag}[1, 1]$. This is easily evaluated and combined with (5.2) to give, finally,

$$A(\xi) = \frac{A_0}{\xi_0^4 - \xi_1^4} \left\{ \xi^2 \xi_0^2 - \xi_1^4 + \frac{(\xi - \xi_0)}{(\cosh \nu_1 \cosh \nu_2 + 1)} [(\xi - \xi_0) \zeta_1 \zeta_2 \sinh \nu_1 \sinh \nu_2 + (\xi \xi_0 + \zeta_1^2) \zeta_1 \sinh \nu_1 \cosh \nu_2 + (\xi \xi_0 + \zeta_2^2) \zeta_2 \sinh \nu_2 \cosh \nu_1] \right\}. \quad (5.7)$$

This clearly satisfies the reciprocity condition (3.26).

(b) *The general solution for clamped plates*

The deflection and rotation are continuous across the junction, as for the welded plates, i.e. the first two conditions in (4.15) and (4.16) are the same. Therefore, by the same argument as before, it follows that $A(\xi)$ is at most a quadratic polynomial ($\bar{A}_3 = \bar{A}_4 = 0$). The three conditions to be satisfied are then (4.12), (4.16)₃, and (4.16)₄. The first of these is automatically met by writing $A(\xi)$ in the form of (5.2), and the remaining pair can be simplified using (B 10) and (B 11) and the first three identities in (B 12), yielding

$$(\mathbf{M}_1 - \mathbf{M}_2) \begin{bmatrix} \tilde{A}_2 \\ \tilde{A}_1 \end{bmatrix} = -A_0 \left(\frac{\mathbf{M}_1}{\zeta_1^2 - \xi_0^2} - \frac{\mathbf{M}_2}{\zeta_2^2 - \xi_0^2} \right) \begin{bmatrix} 1 \\ \xi_0 \end{bmatrix}, \quad (5.8)$$

where the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are defined in (5.5). This system may be easily solved and combined with (5.2) to eventually yield

$$A(\xi) = \frac{A_0}{\xi_0^4 - \xi_1^4} \left\{ \xi^2 \xi_0^2 - \xi_1^4 + \frac{(\xi - \xi_0)}{(\cosh \nu_1 \cosh \nu_2 - 1)} [(\xi - \xi_0) \zeta_1 \zeta_2 \sinh \nu_1 \sinh \nu_2 + (\xi \xi_0 + \zeta_1^2) \zeta_1 \sinh \nu_1 \cosh \nu_2 + (\xi \xi_0 + \zeta_2^2) \zeta_2 \sinh \nu_2 \cosh \nu_1] \right\}. \quad (5.9)$$

Again, this clearly satisfies the reciprocity condition (3.26). We note the remarkable similarity between the solutions for the welded and clamped plates. Equations (5.7) and (5.9) differ only in the terms $(\cosh \nu_1 \cosh \nu_2 \pm 1)$. The full solutions are otherwise identical apart from this apparently small change in the analytic result.

(c) *Light fluid loading limit*

Light fluid loading is defined here as the limit in which the lengths $a_{1,2}$ are much greater than all others, so that appropriate approximations can be used. Thus, referring to (2.8)₃, and (3.11), we have $K \rightarrow V_1/V_2$, with the explicit analytic factors

$$K^\pm(\xi) = \beta^{\pm 1/2} ((\kappa_2 \pm \xi)(i\kappa_2 \pm \xi))^{\pm 1} ((\kappa_1 \pm \xi)(i\kappa_1 \pm \xi))^{\mp 1}. \quad (5.10)$$

The scattered pressure transform is

$$\tilde{p}(\xi) = \frac{iA(\xi)}{(\xi - \xi_0)} \frac{(\xi_0 + \kappa_1)(\xi_0 + i\kappa_1)(\xi_0 - \kappa_2)(\xi_0 - i\kappa_2)}{(\xi + \kappa_1)(\xi + i\kappa_1)(\xi - \kappa_2)(\xi - i\kappa_2)} \frac{\gamma(\xi_0)}{\gamma(\xi)}. \quad (5.11)$$

The function A can be simplified using the identities, which result from (5.10),

$$\cosh \nu_n = \sqrt{\beta} [X_n^4 - \eta^2 + iX_n^2(\eta - 1)^2] / (X_n^4 - 1), \quad (5.12a)$$

$$\sinh \nu_n = \sqrt{\beta} X_n(\eta - 1) [X_n^2 + \eta + i(X_n^2 - \eta)] / (X_n^4 - 1), \quad (5.12b)$$

where $X_n \equiv \zeta_n/\kappa_1$ for $n = 1$ or 2 , and $\eta \equiv \kappa_2/\kappa_1 = (\alpha/\beta)^{1/4}$.

A flexural wave incident from the left has $\xi_0^{(1)} \approx \kappa_1$ and $A_0 = 1$. Its reflection and transmission coefficients are given by (3.20), or alternatively, follow from (5.11) using $R_{\text{flex}} = (-i) \operatorname{res} \tilde{p}(-\kappa_1)$ and $T_{\text{flex}} = i \operatorname{res} \tilde{p}(\kappa_2)$. The latter give

$$R_{\text{flex}} = iA(-\kappa_1, \kappa_1) \left(\frac{1-\eta}{1+\eta} \right) \left(\frac{1-i\eta}{1+i\eta} \right), \quad T_{\text{flex}} = \frac{2A(\kappa_2, \kappa_1)}{\eta(1+\eta)} \frac{\gamma(\kappa_1)}{\gamma(\kappa_2)}. \quad (5.13)$$

Combining (5.12) and (5.13)₁ with the expressions for A in (5.7) and (5.9), and eliminating X_1 and X_2 , gives R_{flex} as i and

$$((1-\eta^2) - (1-i\eta)^2 r^2) / ((1+\eta)^2 - (1+\eta^2) r^2), \quad (5.14)$$

for clamped and welded conditions, respectively, where $r \equiv (\beta\eta^2 - 1) / (\beta\eta^2 + 1)$. These values agree with known expressions for R_{flex} , e.g. as given by Mace (1984), and hence provide a reassuring check on the validity of the general solutions for welded and clamped plates.

The term $\gamma(\kappa_1)/\gamma(\kappa_2)$ in the transmission coefficient of (5.13) appears because T_{flex} defines the pressure on the right-hand plate. It is more convenient to work with the displacement transmission coefficient in the light fluid loading limit because it is independent of the fluid properties. Thus, proceeding as before, we find that (5.13)₂ gives $T_{\text{flex}}^{\text{disp}} \equiv T_{\text{flex}} \gamma(\kappa_2) / \gamma(\kappa_1)$ as zero for clamped edges, and

$$2(1+\eta)(1-r) / ((1+\eta)^2 - (1+\eta^2) r^2), \quad (5.15)$$

for welded conditions, as expected (Mace 1984).

(d) Heavy fluid loading

The heavy fluid loading limit occurs at low frequency, or more precisely, when $\lambda \gg \kappa$, where

$$\lambda \equiv (\rho\omega^2/B)^{1/5} = (\kappa^4/a)^{1/5}, \quad (5.16)$$

although Crighton (1972) showed that this can be relaxed to the requirement $\kappa \gg k$. The explicit solutions for the pressure transform for welded and clamped contact can be simplified in this limit by using the appropriate asymptotic form of K^+ . This can be derived from Appendix A, although for the sake of brevity we take advantage of the previous work of Crighton & Innes (1984) for the factorization of $D(\xi) = D^+(\xi)D^-(\xi)$, where $D^+(\xi) = D^-(-\xi)$, $D^\pm(0) = 1$, and $D(\xi) = 1 - \lambda^{-5}\xi^5 \operatorname{sgn} \operatorname{Re} \xi$ in this limit (Crighton & Innes 1984). It follows from equation (A13) of Crighton & Innes (1984) that

$$D^+(\xi) = \prod_{n=1}^5 \left(1 + \frac{\xi}{\lambda z_n} \right)^{1/2} \exp \left\{ \frac{i}{\pi} \sum_{n=1}^5 (-1)^{n-1} \int_0^{\xi/\lambda z_n} \frac{\ln z \, dz}{z^2 - 1} \right\}, \quad (5.17)$$

where $z_n = e^{i(n-1)\pi/5}$. The asymptotic expansion for $\xi = O(\kappa)$ is

$$D^+(\xi) = e^{\xi d_1/\lambda} \left(1 + d_3 \frac{\xi^3}{3\lambda^3} + \dots \right), \quad (5.18)$$

where $d_1 = 1 - i \cot \pi/5$ and $d_3 = 1 + i \tan \pi/10$. Terms up to quadratic in equation (5.18) are given explicitly in equation (A33) of Crighton & Innes (1984), while the cubic term follows from equations (A29)–(A32) of the same paper after a bit of manipulation. The limiting form of the split function $K^+(\xi)$ can be determined using

these results, from which the functions $A(\xi, \xi_0)$ may be simplified. For instance, we find for welded contact that $A(\xi, \lambda_1) = \xi^2/\lambda_1^2$ for $\xi = O(\lambda_1)$.

We consider a subsonic flexural wave incident from plate 1, with $\xi_0^{(1)} = \lambda_1$ and $A_0 = 1$. After some algebra, we find that the reflection and transmission coefficients for welded or clamped conditions at the junction are

$$R_{\text{weld}} = \frac{e^{-i3\pi/4}}{(D_2^+(\lambda_1))^2} \frac{\Delta B}{B_1}, \quad (5.19a)$$

$$T_{\text{weld}} = \frac{e^{-i3\pi/4}}{D_2^+(\lambda_1)D_1^+(\lambda_2)} \frac{2}{(B_1B_2)^{2/5}} \frac{\Delta B}{\Delta B^{1/5}}, \quad (5.19b)$$

$$R_{\text{clam}} = \left[1 + \frac{2}{\beta^{1/5} - 1} \left(\frac{1}{d_1} + \frac{1}{d_3} \right) - \frac{3(\beta^{1/5} + 1)[2 + (\beta^{1/5} - 1)d_1]}{(\beta^{3/5} - 1)d_3 - \frac{1}{4}(\beta^{1/5} - 1)^3 d_1^3} \right] R_{\text{weld}}, \quad (5.19c)$$

$$T_{\text{clam}} = \{-e^{i\pi/5} \sin \frac{1}{10}\pi\} T_{\text{weld}}, \quad (5.19d)$$

where $\Delta f \equiv f_2 - f_1$. The magnitudes simplify even further by using the identity

$$|D_2^+(\lambda_1)| = (\beta^{1/5} + 1)^{1/2} ((\beta - 1)/(\beta^{1/5} - 1))^{1/2}, \quad (5.20)$$

which follows from (5.17). Thus, we obtain the remarkably simple expressions,

$$|R_{\text{weld}}| = \frac{|\beta^{1/5} - 1|}{\beta^{1/5} + 1}, \quad |T_{\text{weld}}| = \frac{2\beta^{1/10}}{\beta^{1/5} + 1}, \quad (5.21a)$$

$$|R_{\text{clam}}| = \frac{\sqrt{\beta^{2/5} + 2\beta^{1/5} \cos \frac{1}{5}\pi + 1}}{\beta^{1/5} + 1}, \quad |T_{\text{clam}}| = \frac{2\beta^{1/10}}{\beta^{1/5} + 1} \sin \frac{1}{10}\pi. \quad (5.21b)$$

Note that these clearly satisfy the energy conservation identity $|R|^2 + |T|^2 = 1$, which is the appropriate form of (3.29) in this limit; i.e. no energy radiates into the fluid. The general results of (5.19) also simplify when the material contrast is small. Using the identity $D^+(\lambda) = \sqrt{10}e^{-i3\pi/8}$ (equation (A22) of Crighton & Innes (1984)), to give

$$R_{\text{weld}} = \frac{\Delta B}{10B}, \quad T_{\text{weld}} = 1, \quad R_{\text{clam}} = ie^{i\pi/5} \cos \frac{1}{10}\pi, \quad T_{\text{clam}} = -e^{i\pi/5} \sin \frac{1}{10}\pi, \quad (5.22)$$

to leading order as $\Delta B/B \rightarrow 0$. These limits for R_{clam} and T_{clam} can be derived independently using the heavy fluid loading approximation to the line drive admittance of a uniform plate (Crighton 1972), and agree with (5.22).

(e) Numerical results

We concentrate on the case first considered by Brazier-Smith (1987), namely that where the contrast at the junction is provided by a discontinuity in the plate thickness. Thus we suppose that both plates are made of the same material so that the ratio of thicknesses is simply α of (2.13), and $\kappa_2 = \kappa_1/\sqrt{\alpha}$. The dimensionless material parameter $\epsilon = \sqrt{B\rho^2/m^3c^2}$ is therefore the same for both plates. Brazier-Smith (1987) plotted the reflection and transmission coefficients as a function of the dimensionless frequency parameter $\Omega_- = k^2/\kappa_1^2$. Here we denote the same parameter by Ω . Figures 3–10 show results for a subsonic flexural wave incident from the left onto a thickness change of 100%, or $\alpha = 2$. The junction is either welded or clamped, and the material combination is either steel and water ($\epsilon = 0.134$) or aluminium

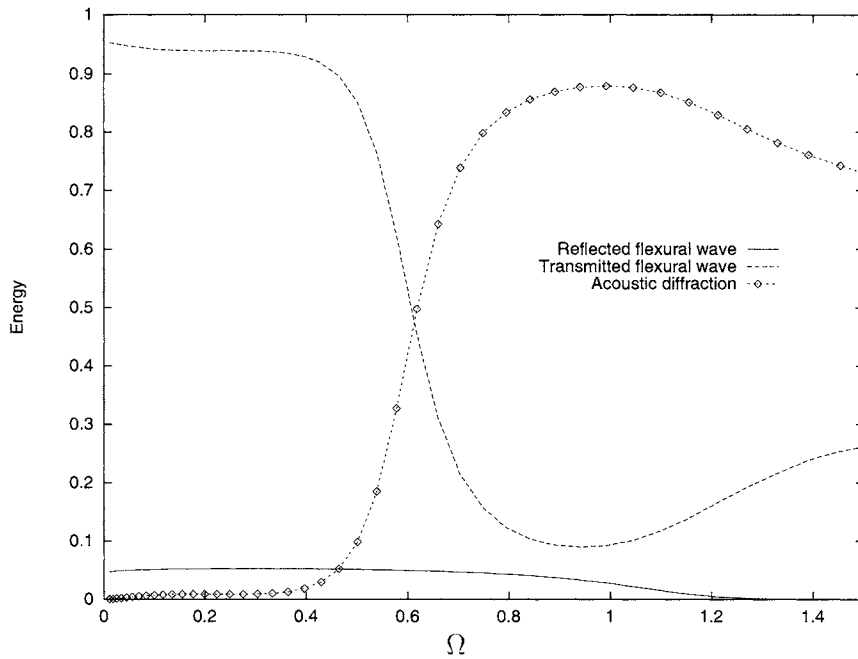


Figure 3. The reflected, transmitted and diffracted energy at a welded junction of steel plates loaded by water.

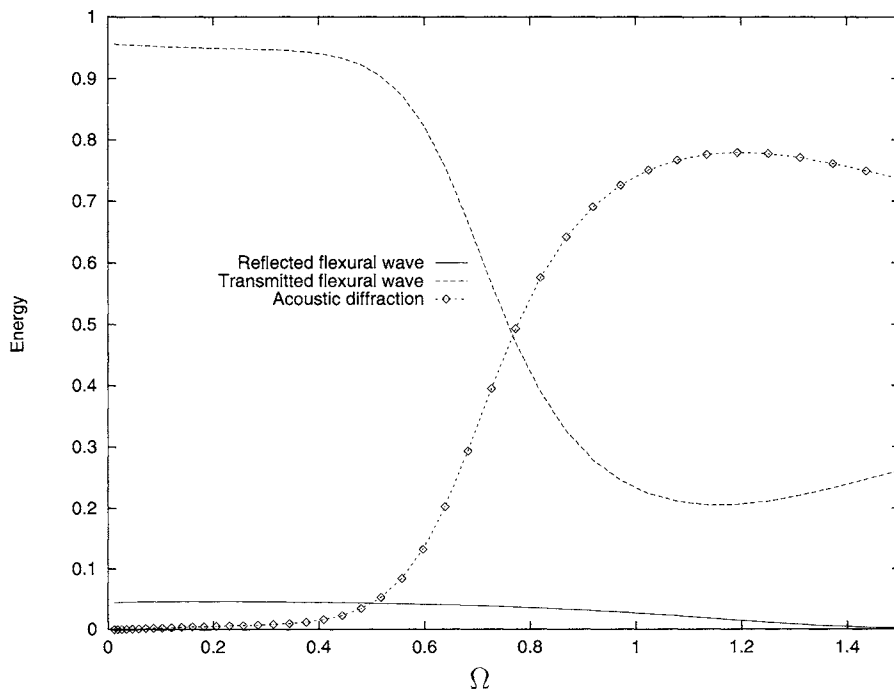


Figure 4. The energy redistribution at a welded junction of two aluminium plates in water.

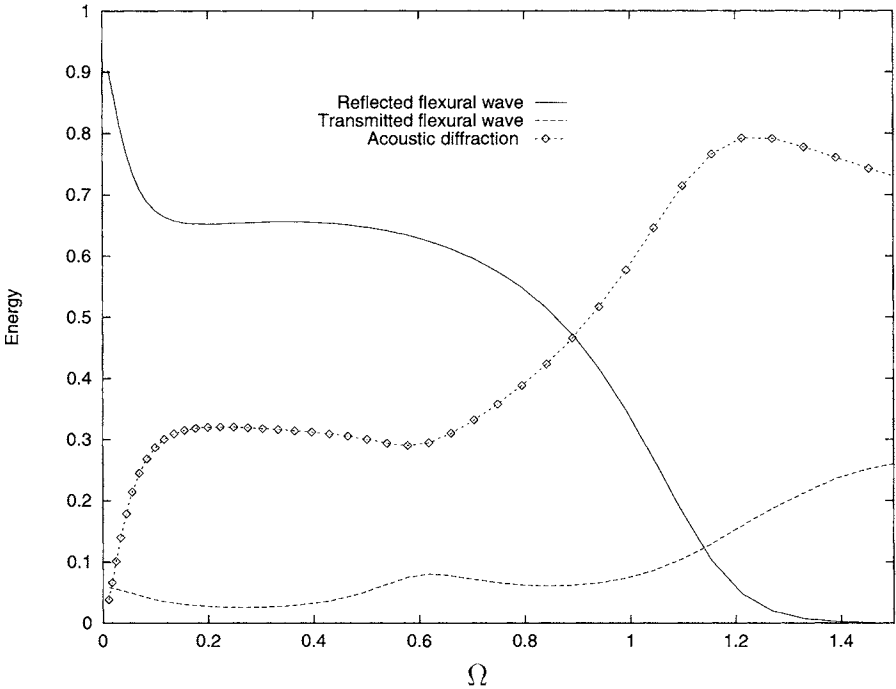


Figure 5. Clamped steel plates in water.

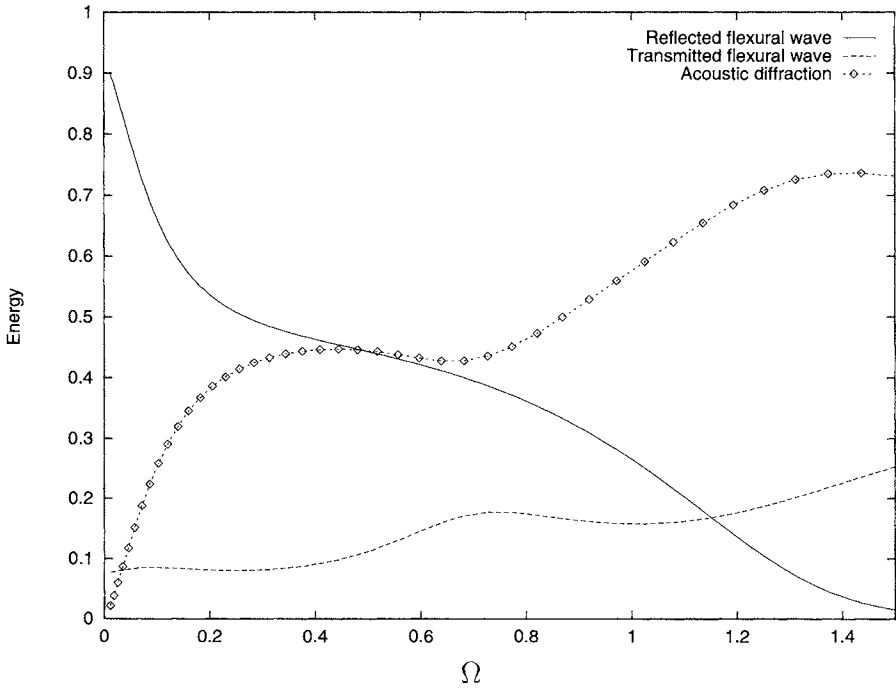


Figure 6. Clamped aluminium plates in water.

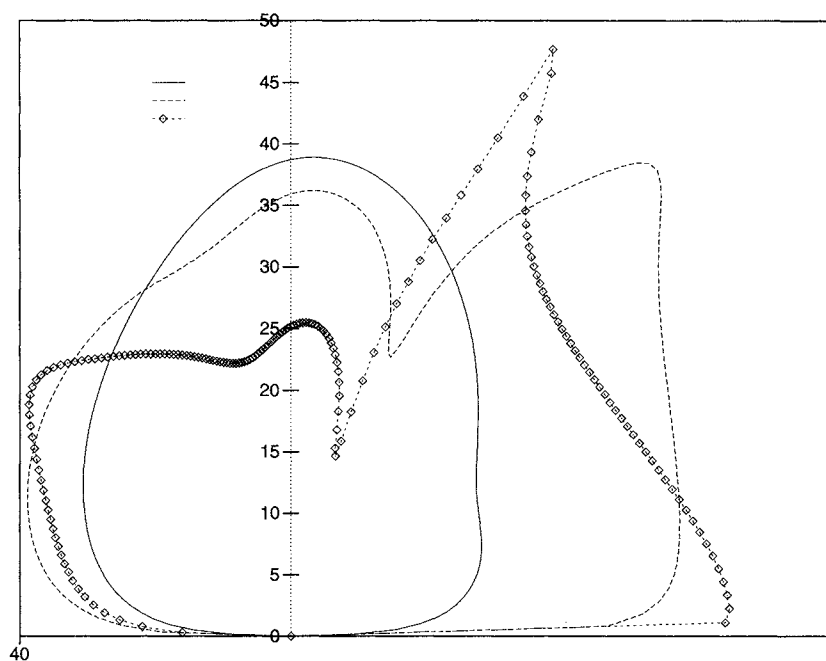


Figure 7. Polar plot of the scattered acoustic pressure amplitude from a welded junction of steel plates in water.

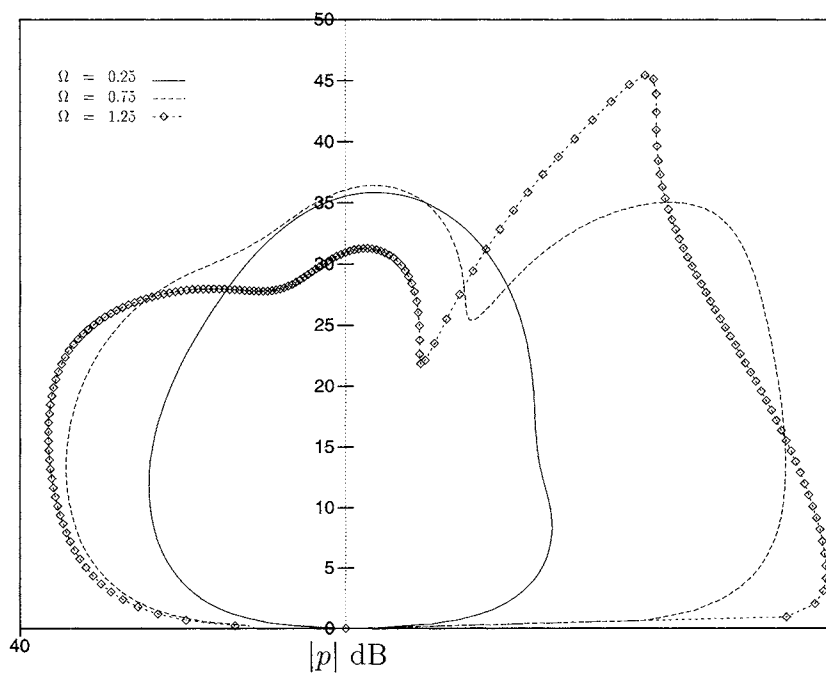


Figure 8. The acoustic diffraction from a welded junction of aluminium plates loaded by water.

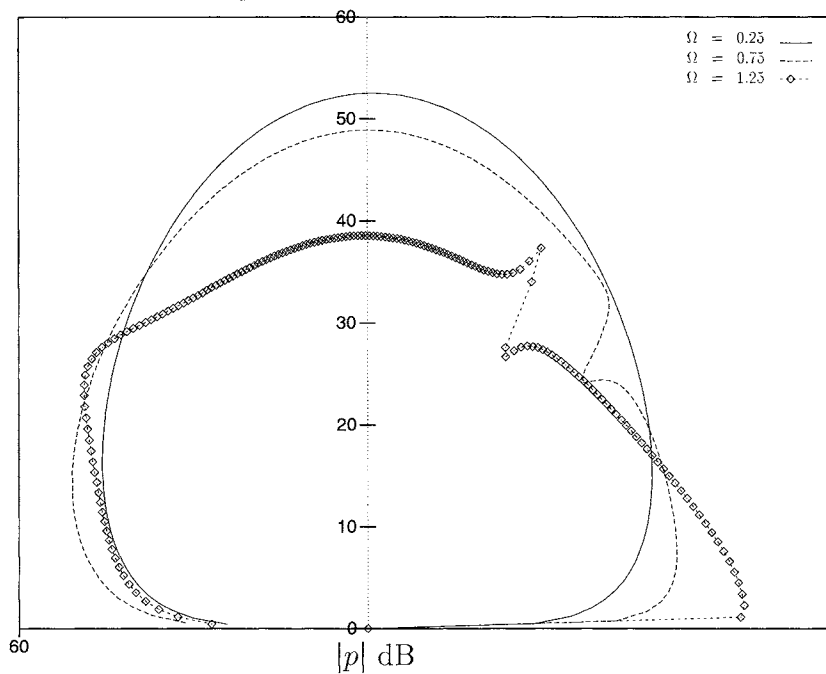


Figure 9. The acoustic response from two clamped steel plates loaded by water.

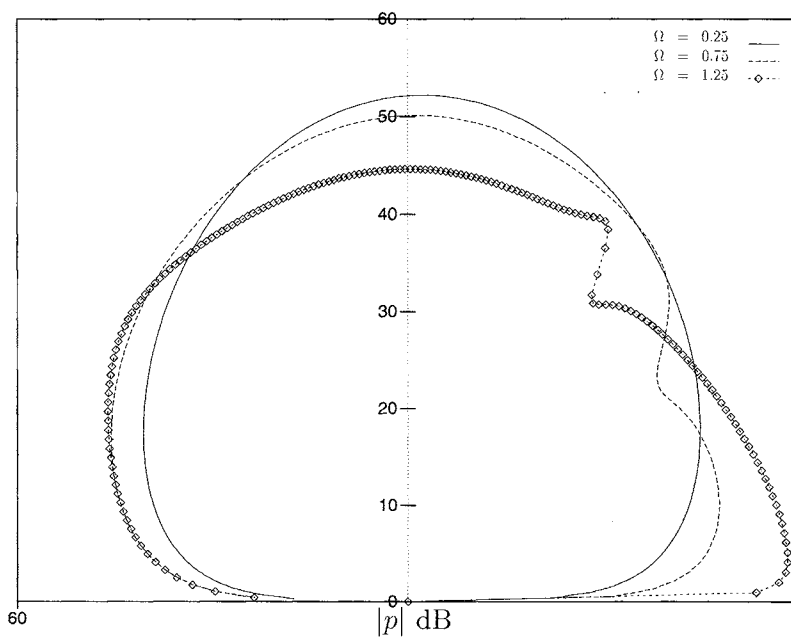


Figure 10. The diffracted acoustic pressure from two clamped aluminium plates in water.

and water ($\epsilon = 0.4$). In figures 3–6 we have plotted the proportion of energy reflected and transmitted in coupled subsonic flexural waves together with the total relative power radiated to infinity in the acoustic field diffracted from the junction. According to equation (3.29) these three quantities should sum to unity. Our numerical calculations have satisfied this equality to a very high degree of accuracy (in all cases shown the error was less than 10^{-6}). Our results for the transmission and reflection are in good agreement with Brazier-Smith in the case of the welded junction. The clamped junction is a little more difficult to compute, even though there is a remarkable similarity in the formulae (5.7) and (5.9). The reason for this is that the product $\cosh \nu_1 \cosh \nu_2$ is close to unity for small values of Ω . Thus it is very important to be able to compute it accurately. Our analytic expressions for $K^+(\xi)$ in Appendices A and B provide numerically well conditioned formulae for both small and large complex values of the argument.

The low frequency, or heavy fluid loading results of (5.21) indicate that the limiting values of $|R|$ and $|T|$ depends only on the ratio of the bending stiffnesses, β of (2.13)₂. When the plates differ only in thickness we have that $\beta = (h_2/h_1)^3$, and hence the limiting values of $|R|$ and $|T|$ are independent of the material properties. All the examples considered here are for $h_2/h_1 = 2$ and hence $|R_{\text{weld}}|^2 = 0.042$, $|T_{\text{weld}}|^2 = 0.958$, $|R_{\text{clam}}|^2 = 0.909$, and $|T_{\text{clam}}|^2 = 0.091$. The numerical calculations in figures 3–6 are consistent with these numbers, and suggest that the low frequency approximations for $|R|$ and $|T|$ should be valid over a wide and practically useful frequency range for welded plates. When the plates are clamped, on the other hand, $|R|$ decreases rapidly with frequency from its low frequency asymptote, with the bulk of the lost structural energy apparently converted to acoustic radiation.

It is interesting to study the directivity of the acoustic diffraction pattern for various frequency regimes and different edge conditions. Like Brazier-Smith, we find that it is dramatically dependent on both. Figures 7–10 show the radiation patterns at $\Omega = 0.25$, 0.75 and $\Omega = 1.25$ for welded and clamped steel and aluminium plates bathed in water. At low frequencies the total radiated power for both clamped and welded plates is small. Its directivity is similar to that of a dipole placed at the junction with its axis perpendicular to the alignment of the plates. At higher frequencies, the field develops relatively intense beams the direction of which appears to be well correlated with the mechanical constraints.

6. Conclusion

Our main results are summarized by (3.1) and (3.2) for the total response, and (3.4), (3.15), and (3.16) for the scattered field, where the split function K^+ is given in Appendices A and B in forms that are easily computed. The scattered response depends upon the polynomial function $A(\xi, \xi_0)$, which has been derived explicitly for welded and clamped plate edges, in (5.7) and (5.9) respectively. The incident wave is defined completely by the two parameters: its wave number ξ_0 , and its amplitude, A_0 of (3.6). These formulae provide an irreducible analytic solution which is useful for practical calculations of the acoustic–flexural interactions at junctions between flat plate segments. The explicit formulae also lead to new and simple results for the structural scattering coefficients in the low frequency, or heavy fluid loading limit. These provide useful approximations over a wide frequency range for welded junctions.

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Appendix A. Factorization of the kernel

The analytic factorization of kernels similar to D_1 and D_2 was considered by Cannell (1975, 1976), and by Crighton & Innes (1984). The authors have recently developed a simpler procedure for dealing with more general quotient kernels of this type, with $K(\xi)$ being a particular case. The full derivation of the analytic factorization is rather involved, and the details can be found in a related paper (Norris & Wickham 1995). We summarize the final results as they apply here. Starting from the identity

$$(K^+/K^-)^2 = (D_2/D_1)^2 = P_2 R_1 / (P_1 R_2), \quad (\text{A } 1)$$

we have

$$(K^+)^2 = P_2^+ R_2^+ / (P_1^+ R_1^+), \quad (\text{A } 2)$$

where $R_{1,2}$ are the fluid/plate reflection coefficients, with factors satisfying $R(\xi) = R^-(\xi)/R^+(\xi)$, and $R^-(-\xi) = 1/R^+(\xi)$, while the polynomials $P_{1,2}^+$ are ‘+’ functions for the product factorizations of

$$P(\xi) = D(\xi)\bar{D}(\xi) = 1 - a^2 \left(\frac{\xi^{10}}{\kappa^8} - \frac{k^2 \xi^8}{\kappa^8} - \frac{2\xi^6}{\kappa^4} + 2\frac{k^2}{\kappa^4} \xi^4 + \xi^2 - k^2 \right), \quad (\text{A } 3)$$

such that $P(\xi) = P^+(\xi)P^-(\xi)$, and $P^+(\xi) = P^-(-\xi)$. Let $\xi = \pm\xi_n$, $n = 1, 2, \dots, 5$, be the zeros of $P(\xi)$ such that $\text{Im } \xi_n > 0$, with no loss in generality, then

$$P^\pm(\xi) = \frac{a}{\kappa^4} \prod_{n=1}^5 (\xi_n \pm \xi). \quad (\text{A } 4)$$

The factorization of R is a bit more involved, but can be achieved by noting that

$$R'/R = 1/\gamma Q, \quad (\text{A } 5)$$

where $Q(\xi)$ is the ratio of two polynomials the numerator one of which is $P(\xi)$, and Q therefore shares the same zeros as P . It is also an odd function of ξ , with

$$\text{res}(1/Q)|_{\xi=\pm\xi_n} = -s_n \gamma(\xi_n), \quad (\text{A } 6)$$

where

$$s_n = 1 \text{ if } D(\xi_n) = 0, \quad s_n = -1 \text{ if } \bar{D}(\xi_n) = 0. \quad (\text{A } 7)$$

We use the basic identities (Noble 1958) $\gamma(\xi) = \gamma^+(\xi) + \gamma^-(-\xi)$, $\gamma^-(-\xi) = \gamma^+(\xi)$, with $\gamma^+(\xi) = [\gamma(\xi)/\pi] \arccos(\xi/k)$, where the branch of the inverse cosine is

$$\arccos(\xi/k) = i \log(\xi/k + \gamma(\xi)/k). \quad (\text{A } 8)$$

and the logarithm is defined so that its argument lies in the interval $(-\pi, \pi)$. The sum split of R'/R can then be effected by adding and subtracting pole terms, with the details in Norris & Wickham (1995). The final result is

$$K^+(\xi) = (P_2^+(\xi)/P_1^+(\xi))^{1/2} (R_1(0)/R_2(0))^{1/4} e^{[\Phi_1(\xi) - \Phi_2(\xi)]/2}, \quad (\text{A } 9)$$

where

$$\Phi(\xi) \equiv \frac{1}{\pi} \int_0^\xi \left[\sum_{n=1}^5 [\log R(z)]' \arccos(z/k) + \frac{2s_n \xi_n \theta_n}{z^2 - \xi_n^2} \right] dz, \quad (\text{A } 10)$$

and

$$\theta_n = \arccos(\xi_n/k). \quad (\text{A } 11)$$

Expressing the first term in the integrand in partial fractions we obtain

$$\Phi(\xi) = \frac{2}{\pi} \int_{\pi/2}^{\arccos(\xi/k)} \sum_{n=1}^5 \frac{\theta \cos \theta \sin \theta_n - \theta_n \cos \theta_n \sin \theta}{\cos^2 \theta - \cos^2 \theta_n} s_n d\theta, \quad (\text{A } 12)$$

Equation (A 9) expresses K^+ in a form which is easily evaluated for small values of ξ relative to k . Evidently the integrand in $\Phi_{1,2}$ has only removable singularities in the upper half plane, $\mathcal{H}^+$, and so K^+ is demonstrably of the correct analyticity. Further it is readily shown that it has a branch point at $\xi = -k$ as well as simple poles at the zeros of $D_1(\xi)$ lying in the lower half-plane. Physically, these correspond to the left-going flexural and leaky waves.

An alternative form for K^+ can also be obtained which is reminiscent of the light fluid loading limit, but is generally valid,

$$K^+(\xi) = (D_2(0)/D_1(0))^{1/2} \prod' ((1 + \xi/\xi_n^{(2)})/(1 + \xi/\xi_n^{(1)})) e^{[\phi_1(\xi) - \phi_2(\xi)]/2}, \quad (\text{A } 13)$$

where the products $\prod'$ are taken only over the three roots for which $s_n = 1$, and

$$\phi(\xi) = \frac{1}{\pi} \int_{\pi/2}^{\arccos(\xi/k)} \sum_{n=1}^5 \left[\frac{\theta \sin \theta_n - \theta_n \sin \theta}{\cos \theta - \cos \theta_n} + \frac{\theta \sin \theta_n - (\pi - \theta_n) \sin \theta}{\cos \theta + \cos \theta_n} \right] s_n d\theta. \quad (\text{A } 14)$$

Evidently, $\phi_1 - \phi_2 \rightarrow 0$ in the limit of light fluid loading. Equation (A 13) is preferred for numerical computations because it does not contain any square roots in the pre-exponent.

Appendix B. Expansion coefficients

The expansions of the functions $\tilde{w}_0^\pm$ for large ξ are straightforward except for the terms involving $K^\pm(\xi)$, see (4.5). Hence, we first need the asymptotic form for K^+ at large argument. Referring to Appendix A, and the definition of K^+ , it can be shown that

$$\lim_{\xi \rightarrow \infty} K^+(\xi) = \lim_{\xi \rightarrow \infty} (P_2^+(\xi)/P_1^+(\xi))^{1/2} = \sqrt{\beta}. \quad (\text{B } 1)$$

Then it is easily seen from (A 9) that

$$K^+(\xi) = (P_2^+(\xi)/P_1^+(\xi))^{1/2} \exp \{(\Phi_1^\infty(\xi) - \Phi_2^\infty(\xi))/2\}, \quad (\text{B } 2)$$

where

$$\Phi^\infty(z) = -\frac{1}{\pi} \int_\xi^\infty \left[[\log R(z)]' \arccos(z/k) + \sum_{n=1}^5 \frac{2s_n \xi_n \theta_n}{z^2 - \xi_n^2} \right] dz. \quad (\text{B } 3)$$

A routine calculation yields

$$\frac{d}{d\xi} \log R(\xi) = -\frac{10\kappa^4}{a\xi^6} - \frac{7k^2\kappa_1^4}{a\xi^8} + O(\xi^{-10}), \quad (\text{B } 4)$$

and so expanding the integrand in (B 3) and then integrating term by term we find that

$$K^+(\xi) = \sqrt{\beta} \exp \left[\sum_{n=1}^4 \xi^{-n} (\mu_n^{(2)} - \mu_n^{(1)}) + O(\xi^{-5} \log \xi) \right], \quad (\text{B } 5)$$

where, for the sake of convenience we have promoted the pre-factor in (B 2) into the exponent before expanding. The coefficients are all explicitly expressed in terms of the zeros of $P_1(\xi)$ and $P_2(\xi)$ as

$$\mu_j^{(k)} = \frac{1}{2j} \sum_{n=1}^5 \left\{ \frac{1}{\pi} s_n^{(k)} \theta_n^{(k)} [(\xi_n^{(k)})^j - (-\xi_n^{(k)})^j] - (-\xi_n^{(k)})^j \right\}. \quad (\text{B } 6)$$

Returning to the difficult terms in (4.5), we have from (B 5) and (2.11), that

$$\begin{aligned} \frac{K^\pm(\xi)}{(\xi - \xi_0)P^*(\xi)} &= \frac{\beta^{\pm 1/2}}{P_0^*} \xi^{-5} \exp \left[\sum_{n=1}^4 \frac{\mu_n^\pm}{\xi^n} + O(\xi^{-5} \log \xi) \right] \\ &= \frac{\beta^{\pm 1/2}}{P_0^*} \sum_{n=0}^4 \frac{\delta_n^\pm}{\xi^{(5+n)}} + O(\xi^{-10} \log \xi), \end{aligned} \quad (\text{B } 7)$$

where all terms have been combined in the exponent, and

$$\delta_0^\pm = 1, \quad \delta_1^\pm = \mu_1^\pm, \quad \delta_2^\pm = \frac{1}{2}(\mu_1^\pm)^2 + \mu_2^\pm, \quad \delta_3^\pm = \frac{1}{6}(\mu_1^\pm)^3 + \mu_1^\pm \mu_2^\pm + \mu_3^\pm, \quad (\text{B } 8a)$$

$$\delta_4^\pm = \frac{1}{24}(\mu_1^\pm)^4 + \frac{1}{2}(\mu_1^\pm)^2 \mu_2^\pm + \frac{1}{2}(\mu_2^\pm)^2 + \mu_1^\pm \mu_3^\pm + \mu_4^\pm, \quad (\text{B } 8b)$$

and

$$\mu_j^\pm = (\pm 1)^{j-1} (\mu_j^{(2)} - \mu_j^{(1)}) + \frac{1}{j} (\xi_0)^j + \zeta_1^4 \delta_{j4}. \quad (\text{B } 9)$$

Hence, the expansions for $\tilde{w}_0^\pm$ of (4.5), imply using (4.9), (4.10), and the identity

$$G(\xi_0)K^+(\xi_0)/P^*(\xi_0) = p_{,y}^{(0)}(0,0)/A_0, \quad (\text{B } 10)$$

that

$$\lambda_n^\pm = \frac{(\xi_0)^n}{A_0} p_{,y}^{(0)}(0,0) + G(\xi_0) \left\{ \sum_{m=1}^2 \left[\frac{u_m^+(\zeta_m)^n}{\zeta_m - \xi_0} - \frac{u_m^-(-\zeta_m)^n}{\zeta_m + \xi_0} \right] - \frac{\beta^{\mp 1/2}}{P_0^*} \delta_{n-4}^\mp \right\}, \quad (\text{B } 11)$$

for $n = 0, \dots, 8$ and where $\delta_k^\pm = 0$ for $k < 0$. The definition in (4.7) and the expressions in (B 11) imply the following identities, which are used in § 5,

$$\left\{ \begin{array}{l} \lambda_1^\pm - \xi_0 \lambda_0^\pm \\ \lambda_2^\pm - \xi_0 \lambda_1^\pm \\ \lambda_3^\pm - \xi_0 \lambda_2^\pm \\ \lambda_4^\pm - \xi_0 \lambda_3^\pm \\ \lambda_5^\pm - \xi_0 \lambda_4^\pm \end{array} \right\} = \frac{G(\xi_0)}{2P_0^*} \sum_{m=1}^2 \left\{ \begin{array}{l} \zeta_m^{-3} \sinh \nu_m, \\ \zeta_m^{-2} \cosh \nu_m, \\ \zeta_m^{-1} \sinh \nu_m, \\ (\cosh \nu_m - \beta^{\mp 1/2}), \\ (\zeta_m \sinh \nu_m - \mu_1^\mp \beta^{\mp 1/2}). \end{array} \right. \quad (\text{B } 12)$$

We note, from (B 9), that $\mu_1^+ = \mu_1^-$.

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