Nonlinear poroelasticity for a layered medium

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The equations of motion and the nonlinear constitutive theory of fluid-filled poroelastic media are derived from the fundamental equations of elasticity and fluid mechanics for the constituents. A two-scale spatial expansion and the method of homogenization are employed. Explicit equations are obtained for the special case of a medium consisting of alternating solid and fluid layers. The linearized theory is examined in depth for the particular case of isotropic solid layers. The governing effective poroelastic medium is transversely isotropic, and wave solutions are discussed and compared with previous studies. © *1995 Acoustical Society of America.*

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INTRODUCTION

Biot's theory[1,2] for linear dynamics of fluid-filled porous media is by now well established, with several experimental investigations that corroborate it very well. For example, the data of Plona *et al.*[3] display both the fast and slow wave in a layered, alternating fluid/solid system. It has recently been emphasized that sandstone and other porous granular media[4,5] display strongly nonlinear behavior, as compared with, for example, water or metals. The nonlinearity parameter B/A for water, which is the ratio of a third-order elastic modulus to the second-order bulk modulus, is approximately 5. The analogous quantity for sandstone can be on the order of 10^4.[4] What can one expect for the nonlinear behavior of a strongly nonlinear porous sandstone saturated with water? The nonlinear acoustical properties of the constituents are disparate, and it is not clear how this will affect the nonlinear acoustics of the fast and slow waves of the poroelastic medium. This paper is a first step toward an understanding of the nonlinear mechanics and dynamics of porous media, particularly the interplay between the nonlinearities of the constituents. We focus on the problem of deriving the governing equations, the strain energy functions, and describe some linear wave solutions for a layered medium.

Our objective is the governing nonlinear equations for a poroelastic medium—the nonlinear generalization of the Biot model.[2] We employ the "two-scale" technique of homogenization for heterogeneous media with disparate length scales, which has been used to obtain the linear Biot theory.[6–10] Homogenization leads directly to the macroscopic equations, and defines the microproblems which uniquely determine the coefficients in them. Burridge and Kostek[11] recently derived the nonlinear poroelasticity equations for a system with a granular solid skeleton whose elastic response is governed primarily by deformation at grain contacts. This is assumed to dominate the nonlinear effects, to the extent that the fluid can be considered as linearly elastic. Unlike Burridge and Kostek,[11] we do not assume that the solid nonlinearity overwhelms the fluid's, but keep both on an equal footing with the purpose of comparing their interaction.

Three sources of nonlinearity are traditionally distinguished in continuum mechanics. First, there is the physical nonlinearity, that is, the nonlinearity associated with the dependence of the stress tensor on the tensor of finite deformations. The second one is the nonlinearity of the universal equations of mass, momentum, etc., that is, those equations which have validity for all specific models of the continuous medium. The third one is geometrical nonlinearity, for example, the nonlinear relationship between displacements and the tensor of finite deformation. This classification of nonlinearities is not absolute, and some changes occur when one switches from, for instance, the Eulerian to the Lagrangian description (e.g., the universal momentum equation appears to be nonlinear in the Eulerian description and linear in the Lagrangian description). In order to keep this paper as simple as possible, we choose the Lagrangian description and concentrate on the physical nonlinearity, which seems to be the most significant for water saturated sandstones. The assumption of a periodic microstructure is another significant physical assumption made for the purpose of simplifying the effects of nonlinearity.

We consider in detail a layered fluid/solid medium. The general form of the nonlinear equations is derived, but we consider only linear wave solutions in this paper. Nonlinear waves and the interplay between the solid and fluid nonlinearity will be explored in a separate paper. The stratified medium is perhaps the simplest realization of a porous medium, and has been examined in several studies, both theoretical[12–17] and experimental.[3] The first study by Rytov[12] considered wave motion parallel to the layers, and showed the existence of two waves: the fast and the slow. Bedford[15] later showed that the slow wave is the long-wavelength manifestation of the second mode of the system. This is a rich system for wave motion, particularly when the waves are allowed to propagate in oblique directions. The definitive theoretical and experimental studies of Schoenberg and co-workers[3,13,14,16] describe the wave motion and slowness surfaces for fast and slow waves. We give for the first time

TABLE I. Material parameters, from Ref. 3.

Material	Compressional speed (m/s)	Shear speed (m/s)	Density (kg/m^3)
Aluminum	6450	3150	2700
Plexiglass	2700	1380	1200
Water	1490	···	1000

the complete set of Biot equations for the stratified fluid/solid system, which includes fluid viscosity and elastic anisotropy.

The paper proceeds as follows. The nonlinear equations of motion are outlined and the scaling procedure is defined in Sec. I. The main difficulty in any averaging theory of heterogeneous media is to calculate the "effective" parameters (see Table I), which require solving simpler problems on typical unit cells. Explicit solutions can be obtained for the specific case of the layered medium, and are addressed in Sec. II. All moduli and parameters can be found for this simple geometry; in particular, the time-dependent viscodynamic operator is explicit. The effective nonlinear equations of motion are summarized in Sec. III. Section IV focuses on the linear limit, and comparisons are made with Schoenberg's model[13,14] for a layered medium of fluid and isotropic solid in Sec. V.

I. GOVERNING EQUATIONS AND SCALINGS

A. The primitive field equations

The volume V comprises fluid and solid regions, V_f and V_s, with boundary ∂V between them. The displacement fields are u_i and U_i in the solid and fluid, and the stress fields are σ_{ji} and Σ_{ji}, respectively. All equations are defined in terms of the reference or Lagrangian coordinates. The governing equations of motion and constitutive relations are

$$\rho_s \partial_t^2 u_i = D_j \sigma_{ji}, \tag{1a}$$

$$\sigma_{ji} = \sigma_{ji}(F_{mn}), \tag{1b}$$

in V_s,

$$\rho_f \partial_t^2 U_i = D_j \Sigma_{ji}, \tag{2a}$$

$$\Sigma_{ji} = -p\,\delta_{ji} + 2\bar{\eta}\partial_t\{D_j U_i\}, \tag{2b}$$

$$p = \mathscr{P}(D_i U_i), \tag{2c}$$

in V_f, and the continuity conditions on ∂V are

$$U_i = u_i, \tag{3a}$$

$$n_j \Sigma_{ji} = n_j \sigma_{ji}. \tag{3b}$$

Here, $F_{ij} = D_j u_i$ is the deformation gradient tensor in the solid, $\{a_{ij}\} \equiv (a_{ij} + a_{ji})/2 - \delta_{ij} a_{kk}/3$ is the symmetric deviatoric part of a second-order tensor, and n_j is the unit normal on ∂V directed into the fluid region. The summation of repeated Latin suffices over 1, 2, and 3 is understood.

Both the solid and fluid constitutive relations are nonlinear. We ignore, however, the "geometrical" nonlinearity in the fluid equation (2b) for the sake of simplicity (conceptually, many of our conclusions remain valid without this assumption). Also, the equation of state for the pressure, Eq. (2c), is normally expressed in terms of the current (Eulerian) density, but we take a slightly different approach, and assume that it depends upon the dilatation in the reference description. This is simpler, and leads to the same nonlinear effects that are normally present in homogeneous fluids up to second order. Also, we assume that the fluid equation of state possesses an inverse,

$$D_i U_i = \mathscr{Q}(p). \tag{4}$$

The material nonlinearity of the phases is then reflected in the nonlinear behavior of the functions $\mathscr{P}$, $\mathscr{Q}$, and $\sigma_{ji}(F_{mn})$. The stress functions σ_{ji} and the pressure are normally derived from strain energy potentials, according to

$$\sigma_{ji} = \frac{\partial E_s}{\partial(D_j u_i)}, \quad -p = \frac{\partial E_f}{\partial(D_i U_i)}, \tag{5}$$

where $E_s(D_j u_i)$ and $E_f(D_i U_i)$ are the elastic strain energy functions for the solid and fluid, respectively. The homogenization theory outlined below is not dependent upon the existence of the strain energies, and the final equations do not involve them. However, we will see that the effective poroelastic medium also possesses a strain energy function when E_s and E_f exist.

B. Asymptotic scaling

In order to simplify the system of equations (1)–(3) we employ the method of homogenization. We assume the medium is characterized by two distinct spatial lengths, $h \ll H$, associated with the micro- and macroscales, respectively. The number, $\epsilon \equiv h/H$, is a small parameter, $\epsilon \ll 1$. Furthermore, we assume that the viscosity scales as

$$\bar{\eta} = \epsilon^2 \eta. \tag{6}$$

References 7–9 and 11 provide a clear motivation for this choice. The results generated by the homogenization technique are very sensitive to the presence or absence of the multiplier ϵ^2 in the viscosity. Without it the "macro" equations turn out to be of viscoelastic type, whereas the presence of the term ϵ^2 in Eq. (6) leads to Biot-type equations, as we will demonstrate. We refer the reader to the cited papers for further details of the homogenization procedure. [See also the comment after Eq. (32).]

We assume a two-scale expansion[18] in terms of the slow or macroscopic spatial variable $\mathbf{x}$, and the fast variable $\mathbf{y} = \mathbf{x}/\epsilon$, such that the spatial differential operator is

$$D_j = \partial_{x_j} + \epsilon^{-1} \partial_{y_j}. \tag{7}$$

The fast variable $\mathbf{y}$ reflects the small-scale structure in the problem through the perturbation parameter ϵ. Our goal is to eliminate the explicit dependence on $\mathbf{y}$, leaving us with equations in $\mathbf{x}$. The governing equations (1)–(3), combined with Eq. (6), become

$$\epsilon \rho_s \partial_t^2 u_i = (\epsilon \partial_{x_j} + \partial_{y_j}) \sigma_{ji}, \tag{8a}$$

$$\sigma_{ji} = \sigma_{ji}[(\partial_{x_m} + \epsilon^{-1} \partial_{y_m}) u_n] \tag{8b}$$

in the solid phase, and

$$\epsilon \rho_f \partial_t^2 U_i = (\epsilon \partial_{x_j} + \partial_{y_j}) \Sigma_{ji}, \tag{9a}$$

$$\Sigma_{ji}=-p\,\delta_{ji}+\epsilon 2\eta\partial_t\{(\epsilon\partial_{x_j}+\partial_{y_j})U_i\}, \tag{9b}$$

$$\mathcal{Q}(p)=(\partial_{x_i}+\epsilon^{-1}\partial_{y_i})U_i \tag{9c}$$

in the fluid. We consider the *ansatz*

$$u_i(\mathbf{x},t,\epsilon)=u_i^0(\mathbf{x},t)+\sum_{n=1}^{\infty}\epsilon^n u_i^n(\mathbf{x},\mathbf{y},t),$$

$$p(\mathbf{x},t,\epsilon)=p^0(\mathbf{x},t)+\sum_{n=1}^{\infty}\epsilon^n p^n(\mathbf{x},\mathbf{y},t),$$

$$U_i(\mathbf{x},t,\epsilon)=\sum_{n=0}^{\infty}\epsilon^n U_i^n(\mathbf{x},\mathbf{y},t), \tag{10}$$

$$\sigma_{ji}(\mathbf{x},t,\epsilon)=\sum_{n=0}^{\infty}\epsilon^n\sigma_{ji}^n(\mathbf{x},\mathbf{y},t),$$

$$\Sigma_{ji}(\mathbf{x},t,\epsilon)=\sum_{n=0}^{\infty}\epsilon^n\Sigma_{ji}^n(\mathbf{x},\mathbf{y},t).$$

Substituting these expansions into the governing system and comparing like powers of ϵ yields a sequence of asymptotic equations. The leading-order equations, of order unity, are

$$\partial_{y_j}\sigma_{ji}^0=0, \tag{11a}$$

$$\sigma_{ji}^0=\sigma_{ji}(\partial_{x_m}u_n^0+\partial_{y_m}u_n^1), \tag{11b}$$

for $\mathbf{x}$ and $\mathbf{y}$ in V_s,

$$\partial_{y_j}\Sigma_{ji}^0=0, \tag{12a}$$

$$\Sigma_{ji}^0=-p^0\delta_{ji}, \tag{12b}$$

$$\partial_{y_i}U_i^0=0, \tag{12c}$$

for $\mathbf{x}$ and $\mathbf{y}$ in V_f, and on ∂V,

$$U_i^0=u_i^0, \tag{13a}$$

$$n_j\Sigma_{ji}^0=n_j\sigma_{ji}^0. \tag{13b}$$

The next set of equations, of order ϵ, is

$$\rho_s\partial_t^2u_i^0=\partial_{x_j}\sigma_{ji}^0+\partial_{y_j}\sigma_{ji}^1, \tag{14a}$$

$$\sigma_{ji}^1=\frac{\partial\sigma_{ji}(\partial_{x_m}u_n^0+\partial_{y_m}u_n^1)}{\partial F_{pq}}(\partial_{x_p}u_q^1+\partial_{y_p}u_q^2) \tag{14b}$$

in V_s,

$$\rho_f\partial_t^2U_i^0=\partial_{x_j}\Sigma_{ji}^0+\partial_{y_j}\Sigma_{ji}^1, \tag{15a}$$

$$\Sigma_{ji}^1=-p^1\delta_{ji}+2\eta\partial_t\{\partial_{y_j}U_i^0\}, \tag{15b}$$

$$\mathcal{Q}(p^0)=\partial_{x_i}U_i^0+\partial_{y_i}U_i^1, \tag{15c}$$

in V_f, and

$$U_i^1=u_i^1, \tag{16a}$$

$$n_j\Sigma_{ji}^1=n_j\sigma_{ji}^1, \tag{16b}$$

on ∂V.

C. Homogenization of the asymptotic equations

Introduce the relative fluid displacement vector $w_i^0(\mathbf{x},\mathbf{y},t)$, defined as

$$w_i^0(\mathbf{x},\mathbf{y},t)=\phi[U_i^0(\mathbf{x},\mathbf{y},t)-u_i^0(\mathbf{x},t)], \tag{17}$$

where $0<\phi<1$ is the porosity or volume fraction of the fluid, $\phi=V_f/V$. We can then rewrite Eq. (15c) as

$$-\mathcal{Q}(p^0)+\partial_{x_i}(u_i^0+\phi^{-1}w_i^0)=-\partial_{y_i}U_i^1. \tag{18}$$

The dependence upon the fast scale may now be eliminated by averaging over $\mathbf{y}$. A formal definition of the averaging procedure can be given for nonperiodic, statistically defined media; see, for example, Burridge and Keller.[7] Here, for the sake of simplicity, we assume periodicity on the small scale, so that the average is trivial. Thus integrating Eq. (18) over V_f, using the displacement continuity conditions (16a) and the assumed periodicity in $\mathbf{y}$, gives

$$\phi(-\mathcal{Q}(p^0)+\partial_{x_i}u_i^0)+V_f^{-1}\int_{V_f}\partial_{x_i}w_i^0\,dV(\mathbf{y})$$

$$=(1-\phi)V_s^{-1}\int_{V_s}\partial_{y_i}u_i^1\,dV(\mathbf{y}),\quad \mathbf{x}\text{ in }V_f. \tag{19}$$

Substitution of Eqs. (12b) and (15b) into Eq. (15a) yields

$$\rho_f\partial_t^2w_i^0+\partial_{y_j}(p^1\phi\delta_{ji}-2\eta\partial_t\{\partial_{y_j}w_i^0\})$$

$$=-\phi[\rho_f\partial_t^2u_i^0+\partial_{x_i}p^0],\quad \mathbf{x},\mathbf{y}\text{ in }V_f, \tag{20}$$

while Eqs. (12c) and (13a) become, respectively,

$$\partial_{y_i}w_i^0=0,\quad \mathbf{y}\text{ in }V_f, \tag{21a}$$

$$w_i^0=0,\quad \mathbf{y}\text{ on }\partial V. \tag{21b}$$

Next, integrating Eq. (14a) over V_s and Eq. (15a) over V_f, then adding the results and using the continuity conditions (16b) and periodicity, we find

$$\rho\partial_t^2u_i^0(\mathbf{x},t)+\rho_fV_f^{-1}\int_{V_f}\partial_t^2w_i^0\,dV(\mathbf{y})$$

$$=-\phi\partial_{x_i}p^0+(1-\phi)V_s^{-1}\frac{\partial}{\partial_{x_j}}\int_{V_s}\sigma_{ji}^0\,dV(\mathbf{y}), \tag{22}$$

where

$$\rho=\phi\rho_f+(1-\phi)\rho_s \tag{23}$$

is the average density. Finally, using Eq. (12b), we rewrite the boundary condition (13b) as

$$n_j\sigma_{ji}^0=-p^0n_i. \tag{24}$$

The system of equations (20) and (21) forms a closed boundary-value problem with respect to $\partial_tw_i^0(\mathbf{x},\mathbf{y},t)$, and in principle, one can express this function in terms of u_i^0 and p^0. Similarly, Eqs. (11) and (13b) give a well defined boundary-value problem with respect to u_i^1, which enables us to present u_i^1 as a function or functional of u_i^0 and p^0. Both problems are solved explicitly for the case of a periodically layered medium in the next section. Inserting the results in Eqs. (19) and (22), we arrive at a closed master system of equations with respect to u_i^0, p^0, and w_i, with the averaged relative fluid displacement defined as

$$w_i(\mathbf{x},t) \equiv V_f^{-1} \int_{V_f} w_i^0(\mathbf{x},\mathbf{y},t)\, dV(\mathbf{y}). \tag{25}$$

II. CONSTITUTIVE THEORY FOR A LAYERED MEDIUM

We consider periodically alternating fluid and solid layers. The thickness of each fluid layer is l in terms of $\mathbf{x}$ and $L=l/\epsilon$ in terms of $\mathbf{y}$. Let $y=\mathbf{y}\cdot\mathbf{n}$ be the coordinate in the direction of layering. The two microproblems defined at the end of the previous section then depend only upon y, and hence reduce to unidimensional problems. This is the great simplification that arises from the layering, and it is not present for any other configuration.

A. The permeability operator

The permeability operator results from the solution of Eqs. (20) and (21). Let

$$v_i = \partial_t w_i^0, \quad f_i(\mathbf{x},t) = -\rho_f \partial_t^2 u_i^0 - \partial_{x_i} p^0; \tag{26}$$

then Eqs. (20) and (21b) become

$$\partial_t v_i = -n_i \rho_f^{-1} \partial_y p^1 + \eta \rho_f^{-1} \partial_y^2 v_i + \phi f_i(\mathbf{x},t), \tag{27a}$$

$$n_i \partial_y v_i = 0. \tag{27b}$$

We consider y-periodic solutions of this system. The boundary conditions are those for a rigid wall at $y=0$ and $y=L$, which combined with the incompressibility condition for w_i^0, implies

$$n_i v_i = 0 \Leftrightarrow v_i = P_{ij} v_j, \tag{28}$$

where $\mathbf{P(n)}=\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$ projects on to the horizontal plane. Contracting Eq. (27a) with $\mathbf{n}$ and using Eq. (27b) allows us to eliminate the pressure gradient term. Bearing in mind Eq. (28) we deduce that v_i satisfies

$$\partial_t v_j - \nu \partial_y^2 v_i = \phi P_{ij} f_j(\mathbf{x},t), \tag{29}$$

where $\nu=\eta/\rho_f$ is the kinematic viscosity. This can be solved by standard means, i.e., using a Fourier series in y. A very similar type of problem is discussed by Sneddon.[19] Taking into account the initial data ($v=0$ for $t<0$), we find

$$v_i(\mathbf{x},y,t) = \phi P_{ij} \sum_{n=1}^{\infty} \frac{2}{n} [1-(-1)^n] \sin \frac{n\pi y}{L} \times \int_0^t d\xi\, e^{-\nu(n^2\pi^2/L^2)(t-\xi)} f_j(\mathbf{x},\xi). \tag{30}$$

Averaging $v_i(\mathbf{x},y,t)$ over the layer yields $\tilde{v}_i(\mathbf{x},t)$, which satisfies

$$\tilde{v}_i(\mathbf{x},t) = K_{ij}(t) * f_j(\mathbf{x},t), \quad K_{ij}(t) = P_{ij} K(t), \tag{31}$$

where $*$ denotes convolution, and

$$K(t) = \phi \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \exp\left(-\pi^2 (2n-1)^2 \frac{\nu t}{L^2} \right). \tag{32}$$

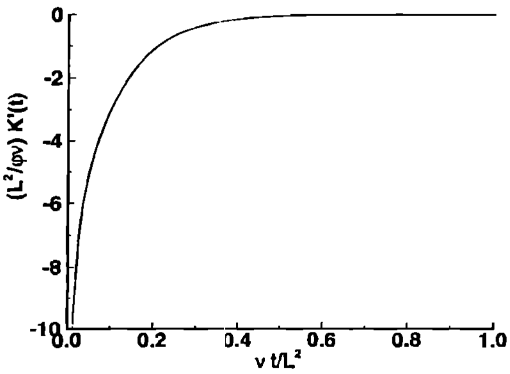

FIG. 1. The permeability function for the stratified medium in nondimensional units.

Hence the permeability of the medium is a projection operator onto the transverse plane. Also, $K(t)$ is independent of ϵ, because $\nu/L^2 = \bar{\eta}/\rho_f l^2$.

Note that Eq. (31) implies

$$\partial_t \tilde{v}_i(\mathbf{x},t) = K_{ij}(0) f_j(\mathbf{x},t) + K'_{ij}(t) * f_j(\mathbf{x},t), \tag{33}$$

where $K(0)=\phi$ for the layered medium, and $K'(t)$ is plotted in Fig. 1. It follows from Eq. (32) as

$$K'(t) = -\phi \frac{8\nu}{L^2} \sum_{n=1}^{\infty} \exp\left(-\pi^2 (2n-1)^2 \frac{\nu t}{L^2} \right). \tag{34}$$

This is a well behaved and convergent series, except for values of t approaching zero, where it is evidently singular. The behavior near zero can be seen by transforming the sum using the Poisson summation formula: for a given function $g(x)$,

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-i2\pi m x}\, dx. \tag{35}$$

Applying this to the sum in Eq. (34) and performing the integration yields

$$K'(t) = -\phi \frac{4\nu}{L\sqrt{\pi \nu t}} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{n^2 L^2}{4\nu t} \right) \right\}. \tag{36}$$

The function $K'(t)$ therefore has an integrable $t^{-1/2}$ singularity at zero.

B. The stress and strain in the solid layers

We now turn to the system defining the displacement u_i^1 in terms of the macroscopic parameters u_i^0 and p^0, Eqs. (11) and (24). Proceeding as we did for the previous microproblem, we look for the affine solution depending on the vertical coordinate y only. Hence

$$\partial_{y_j} u_i^1 = n_j G_i, \tag{37}$$

where the vector G_i does not depend upon y. Then u_i^1 automatically satisfies the equilibrium equations within the solid region, Eq. (11). The traction boundary conditions of Eq. (24) become, using Eq. (11b),

$$n_j\sigma_{ji}(\partial_{x_m}u_n^0+n_mG_n)=-p^0n_i. \tag{38}$$

This is an algebraic system of three equations in three unknowns, with solution

$$G_n=G_n(\partial_{x_j}u_i^0,p^0). \tag{39}$$

Explicit versions of this will be discussed later, but for the moment the implicit solution of Eq. (39) is sufficient. The leading-order stress in the solid then follows from Eqs. (11b) and (37) as

$$\sigma_{ji}^0=\sigma_{ji}(\partial_{x_m}u_n^0+n_mG_n)\equiv T_{ji}(\partial_{x_m}u_n^0,p^0). \tag{40}$$

III. MACROSCOPIC EQUATIONS

A. The general nonlinear equations

We are now in a position to state the leading-order macroscopic equations of motion. The fundamental field variables are the solid displacement $u_i^0(\mathbf{x},t)$ and the averaged fluid displacement $w_i(\mathbf{x},t)$, and their governing dynamic equations follow from Eqs. (22), (26), and (31). In order to simplify the notation, we make the replacements $u_i^0(\mathbf{x},t)\to u_i(\mathbf{x},t)$ and $p^0(\mathbf{x},t)\to p(\mathbf{x},t)$. The dynamic equations are

$$\rho\partial_t^2u_i(\mathbf{x},t)+\rho_f\partial_t^2w_i(\mathbf{x},t)=\tau_{ji,j}(\mathbf{x},t), \tag{41a}$$

$$\partial_tw_i(\mathbf{x},t)=-K_{ij}(t)*[\partial_t^2u_j(\mathbf{x},t)+\rho_f^{-1}p_{,j}(\mathbf{x},t)], \tag{41b}$$

where $K_{ij}(t)$ is given explicitly in Eq. (32). The stress and pressure are related to the displacements by the right-hand side of Eq. (22) and Eq. (40), and by Eqs. (19) and (37), respectively, or

$$\tau_{ji}(\mathbf{x},t)=(1-\phi)T_{ji}(\mathbf{F},p)-\phi p(\mathbf{x},t)\delta_{ij}, \tag{42a}$$

$$\zeta(\mathbf{x},t)=\phi e(\mathbf{x},t)-\phi\mathcal{Q}(p)-(1-\phi)n_iG_i(\mathbf{F},p), \tag{42b}$$

where $F_{ij}=\partial_{x_j}u_i\equiv u_{i,j}$ is the macroscopic deformation gradient tensor, $e\equiv u_{i,i}$, and $\zeta\equiv-w_{i,i}$ is the relative fluid dilatation.[2]

Equations (42) provide the bulk stress τ_{ji} and the relative fluid dilatation ζ in terms of the solid deformations F_{ij} and the fluid pressure p. When the pressure is zero, the bulk stress is

$$\tau_{ji}=(1-\phi)T_{ji}(\mathbf{F},0),\quad p\equiv0. \tag{43}$$

Hence we can identify $(1-\phi)T_{ji}(\mathbf{F},0)$ as the stress function for the dry frame, or the "open" system.[2] If the pores are sealed, or "closed,"[2] then the associated stress function follows from Eqs. (42) with $\zeta\equiv0$.

B. Energy potentials

The macroscopic elastic constitutive relation for the stress in Eq. (42a) can be related to the effective strain energy. Comparison of Eqs. (42) and (A13) implies that the former can be replaced by the simpler relations

$$\tau_{ji}=\frac{\partial W}{\partial F_{ji}},\quad p=\frac{\partial W}{\partial\zeta}. \tag{44}$$

The effective strain energy follows from Eq. (A1), which becomes in the present notation

$$W(\mathbf{F},\zeta)=(1-\phi)E_s[F_{ji}+n_iG_j(\mathbf{F},p)]+\phi E_f(\mathcal{Q}(p)). \tag{45}$$

Hence the effective medium is elastic in the sense that it possesses a stored energy potential W. Note that W is a function of the kinematic strains $\mathbf{F}$ and ζ, but its functional form is defined by $\mathbf{F}$ and p. The remaining identity (42b) is an implicit relation for p in terms of $\mathbf{F}$ and ζ, i.e., $p=p(\mathbf{F},\zeta)$.

The related potential Π is defined in the Appendix. It follows from Eq. (A7) that the two constitutive relations of Eq. (42) are equivalent to

$$\tau_{ji}=\frac{\partial\Pi}{\partial F_{ji}},\quad -\zeta=\frac{\partial\Pi}{\partial p}. \tag{46}$$

The functional definition of $\Pi=\Pi(\mathbf{F},p)$ is given in Eq. (A6), which becomes in the present notation

$$\Pi(\mathbf{F},p)=(1-\phi)E_s(F_{ji}+n_iG_j(\mathbf{F},p))+(1-\phi)pn_mG_m(\mathbf{F},p)-\phi pe+\phi\int_{p_{\mathrm{amb}}}^{p}\mathcal{Q}(\eta)d\eta. \tag{47}$$

This is an explicit equation for the potential, apart from the fact that the functions $G_i(\mathbf{F},p)$ are given by the implicit relation (38). A simpler form for Π can be obtained by noting that it is a partial Legendre transform of the energy function $W(\mathbf{F},\zeta)$. The differential relations (44) and (46) imply the connection

$$W=\Pi+\zeta p. \tag{48}$$

IV. LINEAR THEORY FOR THE STRATIFIED MEDIUM

It is instructive to consider the limiting case for small amplitude waves, for which the linear constitutive equations in the solid and fluid are sufficient. The individual energy potentials are now

$$E_s(e_{ij})=\tfrac{1}{2}C_{ijkl}e_{ij}e_{kl},\quad E_f(e)=\tfrac{1}{2}\kappa_fe^2, \tag{49}$$

where κ_f is the fluid bulk modulus and the elastic moduli possess the symmetries $C_{ijkl}=C_{klij}$ and $C_{ijkl}=C_{jikl}$. The linear stress in the solid is

$$\sigma_{ji}(u_{m,n})=C_{jikl}u_{k,l}. \tag{50}$$

The deformation gradient tensor $\mathbf{F}$ can be replaced in all expressions by the macroscopic linear strain tensor $\mathbf{e}=(\mathbf{F}+\mathbf{F}^T)/2$, or $e_{ij}=(u_{i,j}+u_{j,i})/2$. Equation (38) becomes a linear system for G_i which can be solved easily, yielding

$$G_i(\mathbf{e},p)=-b_{ikl}e_{kl}-Q_{ik}^{-1}n_kp, \tag{51}$$

where

$$Q_{ik}=C_{ijkl}n_jn_l,\quad b_{ikl}=Q_{ip}^{-1}C_{pjkl}n_j. \tag{52}$$

The functions T_{ji} of Eq. (40) then follow as

$$T_{ji}(\mathbf{e},p)=(C_{ijkl}-b_{mij}b_{nkl}Q_{mn})e_{kl}-b_{kij}n_kp. \tag{53}$$

Using the fact that $\mathcal{Q}(p)=-p/\kappa_f$, which follows from Eqs. (5) and (49), the constitutive relations of Eq. (42) become

$$\tau_{ji}(\mathbf{x},t)=C^0_{ijkl}e_{kl}-M^{-1}M_{ij}p, \tag{54a}$$

$$\zeta(\mathbf{x},t)=M^{-1}M_{ij}e_{ij}+M^{-1}p, \tag{54b}$$

where

$$C^0_{ijkl}=(1-\phi)(C_{ijkl}-b_{mij}b_{nkl}Q_{mn}), \tag{55a}$$

$$M=[\phi/\kappa_f+(1-\phi)n_in_jQ^{-1}_{ij}]^{-1}, \tag{55b}$$

$$M_{ij}=[\phi\delta_{ij}+(1-\phi)n_kb_{kij}]M. \tag{55c}$$

Note that C^0_{ijkl} are the dry frame or open pore moduli; see Eq. (43). Equations (54) are equivalent to

$$\tau_{ji}=C^c_{ijkl}e_{kl}-M_{ij}\zeta, \tag{56a}$$

$$p=M\zeta-M_{ij}e_{ij}, \tag{56b}$$

where

$$C^c_{ijkl}=C^0_{ijkl}+M^{-1}M_{ij}M_{kl}, \tag{57}$$

are the "closed pore" moduli ($\zeta\equiv0$). Also, $(1-\phi)^{-1}C^0_{ijkl}$ can be identified as the moduli for a state of *plane stress* in the elastic solid. Thus Eq. (54a) indicates that macroscopic plane stress prevails when the pressure is zero, as expected.

The potential Π follows by integrating τ_{ji} and ζ, using Eqs. (46). The energy density then follows from Eq. (48). We find, after some simplification, that

$$W=\frac{1}{2}C^0_{ijkl}e_{ij}e_{kl}+\frac{p^2}{2M},\quad \Pi=\frac{1}{2}C^c_{ijkl}e_{ij}e_{kl}-\frac{M}{2}\zeta^2. \tag{58}$$

The former is a remarkably simple and physically appealing form. It clearly separates the strain energy into solid and fluid parts, where the latter depends upon the effective bulk modulus M.

V. ISOTROPIC ELASTIC LAYERS AND WAVES

A. The general equations

Let λ and μ be the Lamé moduli of the solid; then Eqs. (52) become

$$\mathbf{Q}^{\pm1}=(\lambda+2\mu)^{\pm1}\mathbf{n}\otimes\mathbf{n}+\mu^{\pm1}\mathbf{P}(\mathbf{n}), \tag{59a}$$

$$b_{ikl}=\frac{\lambda}{\lambda+2\mu}n_iP_{kl}+n_kP_{il}+n_lP_{ik}+n_in_kn_l. \tag{59b}$$

It is a simple matter of algebra to show that the parameters in Eqs. (55) are now

$$C^0_{ijkl}=(1-\phi)(\lambda_0P_{ij}P_{kl}+2\mu\bar{I}_{ijkl}), \tag{60a}$$

$$M=\left(\frac{\phi}{\kappa_f}+\frac{1-\phi}{\lambda+2\mu}\right)^{-1}, \tag{60b}$$

$$M_{ij}=\left[n_in_j+\left(\frac{\lambda+2\mu\phi}{\lambda+2\mu}\right)P_{ij}\right]M, \tag{60c}$$

where $\bar{I}_{ijkl}\equiv(P_{ik}P_{jl}+P_{il}P_{jk})/2$ is the "in-plane" fourth-order identity tensor and $\lambda_0=2\mu\lambda/(\lambda+2\mu)$ is the "plane stress" Lamé modulus. The two constitutive equations are (54a) and (54b), the former simplifying to

$$\tau_{ij}=(1-\phi)(\lambda_0\bar{e}P_{ij}+2\mu\bar{e}_{ij})-M^{-1}M_{ij}p, \tag{61}$$

where $\bar{e}_{ij}=\bar{I}_{ijkl}e_{kl}$ is the in-plane strain, and $\bar{e}=\bar{e}_{jj}$. The constitutive relations (54b) and (61) are written with the solid strain e_{ij} and the fluid pressure p as the fundamental variables for the poroelastic medium. Alternatively, if we choose the strains e_{ij} and ζ as the primitive variables, then the constitutive relations are Eqs. (56). The energy potential W of Eq. (58) reduces to

$$W=\frac{1}{2}(1-\phi)(\lambda_0\bar{e}^2+2\mu\bar{e}_{ij}\bar{e}_{ij})+\frac{p^2}{2M}. \tag{62}$$

The Biot equations are those of a transversely isotropic poroelastic medium, as expected. The general Biot equations for a material with this symmetry[20] have eight independent moduli: five for the solid, one for the fluid, and two for the fluid/solid interaction. It is interesting that the present equations only have five independent moduli, two each for C^0_{ijkl} and M_{ij}, and one for M. Three of the eight possible constants are identically zero, which can be ascribed to the plane-stress configuration of the solid.

The equations of motion can be further simplified. Substituting τ_{ij} from Eq. (61) into Eq. (41a) gives a decoupled system of equations. We now let $\mathbf{n}=\mathbf{e}_3$, where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ form an orthonormal triad. Then, using Eq. (33), we find

$$\rho\partial_t^2u_3+p_{,3}=0, \tag{63a}$$

$$\rho_s\partial_t^2u_\alpha-\lambda_0\bar{e}_{,\alpha}-2\mu\bar{e}_{\alpha\beta,\beta}+(\lambda_0/2\mu)p_{,\alpha}$$
$$=(1-\phi)^{-1}K'(t)*(\rho_f\partial_t^2u_\alpha+p_{,\alpha}), \tag{63b}$$

where the Greek suffices α and β are restricted to 1 and 2, and the summation convention is implicit. Finally, eliminating ζ between Eqs. (41b) and (56b) gives

$$\partial_t(p+M_{ij}e_{ij})=MK(t)*(\partial_t^2\bar{e}+\rho_f^{-1}p_{,\alpha\alpha}), \tag{64}$$

where K is given in Eq. (32). The four equations in Eqs. (63) and (64) form a closed set of equations of motion for $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$. Next, we consider four examples of wave solutions in the layered medium.

B. Examples of wave motion

1. Propagation in the vertical direction

As a first example consider motion in the vertical direction only: $\mathbf{u}=u(x_3,t)\mathbf{e}_3$ and $\mathbf{w}=w(x_3,t)\mathbf{e}_3$. Then Eq. (41b) implies $w=0$ while Eq. (42) gives $\tau_{ji}=e_{33}M_{ji}$ and $p=-e_{33}M$. The equation of motion (63a) becomes simply

$$\rho\partial_t^2u(x,t)-M\partial_x^2u(x,t)=0. \tag{65}$$

This is a scalar wave equation with wave speed $\sqrt{M/\rho}$.

2. A damped shear wave

Consider shear motion in the horizontal plane: $\mathbf{u}=u(x_1,t)\mathbf{e}_2$ and $\mathbf{w}=w(x_1,t)\mathbf{e}_2$. The dynamic equations again reduce to a single equation for u:

$$\rho_s\partial_t^2u(x,t)-\mu\partial_x^2u(x,t)=(1-\phi)^{-1}K'(t)*\rho_f\partial_t^2u(x,t). \tag{66}$$

The shear wave propagates with a time delayed memory function for the inertial term. The appearance of K' reflects the influence of the fluid viscosity on the shear wave in the solid.

3. Wave motion when the fluid is inviscid: Schoenberg's problem

The limiting case of an inviscid fluid is relevant to experiments performed by Plona *et al.*[3] The fast and slow waves in the system are both nondispersive and nonattenuating in this limit. But the anisotropy of the configuration means that both wave types exhibit directional dependence, which can be best understood by considering the slowness surfaces. Schoenberg[13,14] derived the equations for the slowness surface from the dispersion relation for the long-wavelength modes of a layered system of fluid and isotropic solid. We will demonstrate that exactly the same slowness surface follows from the governing Biot equations outlined above.

The inviscid nature of the fluid means that the operator K is instantaneous, or $K' \equiv 0$. Differentiating Eq. (64) with respect to time then implies, using Eq. (33), that

$$\partial_t^2(p+M_{ij}e_{ij})=MK(0)(\partial_t^2\bar{e}+\rho_f^{-1}p_{,\alpha\alpha}), \tag{67}$$

where $K(0)=\phi$. Consider motion in the x_1-x_3 plane with solutions of the form

$$u_1(t-s_1x_1-s_3x_3), \quad u_2=0, \quad u_3(t-s_1x_1-s_3x_3),$$
$$p=p(t-s_1x_1-s_3x_3), \tag{68}$$

where s_1 and s_3 are the horizontal and vertical components of the slowness. Let $\mathbf{v} \equiv \partial_t \mathbf{u}$; then Eqs. (63) imply

$$\rho v_3 - s_3 p = 0, \tag{69a}$$

$$(1-c_{pl}^2 s_1^2)v_1 - \sqrt{1-c_{pl}^2/c_p^2}\, s_1 p = 0, \tag{69b}$$

where c_p and c_{pl} are the wave speeds for longitudinal waves in the solid in bulk and as a thin plate, respectively. That is,

$$c_p^2=(\lambda+2\mu)/\rho_s, \quad c_{pl}^2=(\lambda_0+2\mu)/\rho_s, \tag{70}$$

and therefore $\lambda_0/2\mu = \sqrt{1-c_{pl}^2/c_p^2}$. Eliminating v_3 between Eqs. (67) and (69a) yields the third equation,

$$(1-\phi)\sqrt{1-c_{pl}^2/c_p^2}\, s_1 v_1 - \left(\frac{1}{M} - \phi\frac{s_1^2}{\rho_f} - \frac{s_3^2}{\rho}\right)p = 0. \tag{71}$$

Equations (69b) and (71) are satisfied if s_1 and s_3 are related according to

$$\frac{s_3^2}{\rho} - \left[\frac{\phi}{\rho_f}(c_f^{-2}-s_1^2) + \frac{(1-\phi)}{\rho_s(1-c_{pl}^2 s_1^2)}(c_p^{-2}-s_1^2)\right] = 0, \tag{72}$$

where $c_f = \sqrt{\kappa_f/\rho_f}$ is the speed of sound in the fluid. Equation (72) defines the slowness surface for the fast and slow waves, and it agrees with Schoenberg's[13] equation, derived by taking the low-frequency limit of the exact dispersion relation for a finely laminated medium of alternating solid and fluid layers (see also Refs. 3 and 14). We may rewrite Eq. (72) as

$$s_3^2 = \frac{\phi\rho}{\rho_f(c_{pl}^{-2}-s_1^2)}(s_{\text{fast}}^2 - s_1^2)(s_{\text{slow}}^2 - s_1^2), \tag{73}$$

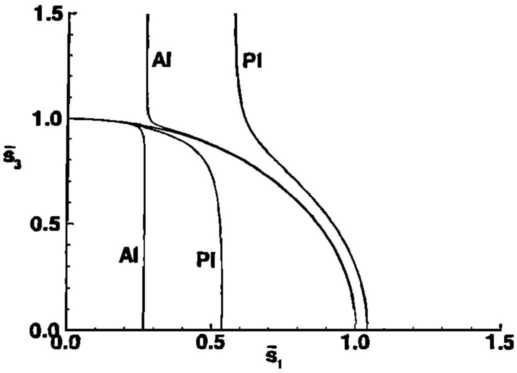

FIG. 2. The slowness surfaces for two different stratified media, consisting of water/aluminum (Al) and water/plexiglas (Pl). The material parameters are in Table I and the porosity is ϕ=0.5 in each case. The ordinate and abscissa are the dimensionless horizontal and vertical slownesses $\bar{s}_1 = s_1\sqrt{\phi M/\rho_f}$ and $\bar{s}_3 = s_3\sqrt{M/\rho}$, respectively. The ellipse of Eq. (76), which becomes a circle in these units, is also shown. The plate slowness c_{pl}^{-1} corresponds to $\bar{s}_1=0.268$ for aluminum and $\bar{s}_1=0.561$ for plexiglas.

where $s_{\text{fast}} < s_{\text{slow}}$ are the horizontal slowness for the fast and slow waves propagating horizontally. They satisfy

$$\frac{\phi}{\rho_f}(s_{\text{fast}}^2+s_{\text{slow}}^2) = \frac{\phi}{\rho_f c_f^2} + \left(\frac{\phi}{\rho_f} + \frac{1-\phi}{\rho_s}\right)\frac{1}{c_{pl}^2}, \tag{74a}$$

$$\frac{\phi}{\rho_f}s_{\text{fast}}^2 s_{\text{slow}}^2 = \left(\frac{\phi}{\rho_f c_f^2} + \frac{1-\phi}{\rho_s c_p^2}\right)\frac{1}{c_{pl}^2}. \tag{74b}$$

These slowness values were also obtained by Rytov[12] who analyzed waves traveling horizontally in a fluid/solid layered medium.

Schoenberg[13] considered a Plexiglas/water system with ϕ=0.6, while Plona *et al.*[3] considered the same system with ϕ=0.5, and also aluminum/water with ϕ=0.5. In all cases it was found that $s_{\text{fast}} < c_{pl}^{-1} < s_{\text{slow}}$, and therefore the slowness surface of Eq. (73) comprises an inner closed sheet for $0 \leq s_1 \leq s_{\text{fast}}$, corresponding to the fast mode, and an outer sheet for $c_{pl}^{-1} < s_1 \leq s_{\text{slow}}$ which gives the slow mode. A stop band exists for $s_{\text{fast}} < s_1 < c_{pl}^{-1}$. The value of s_{fast} is very close to c_{pl} in all these cases, and very accurate approximations can be obtained by iterating from this starting value; thus

$$s_{\text{slow}}^2 \approx \frac{1}{c_f^2} + \frac{(1-\phi)\rho_f}{\phi\rho_s c_p^2} = \frac{\rho_f}{\phi M}, \tag{75a}$$

$$s_{\text{fast}}^2 \approx \frac{1}{c_{pl}^2} - \frac{(1-\phi)\rho_f}{\phi\rho_s c_{pl}^2}\left(\frac{c_{pl}^{-2}-c_p^{-2}}{c_f^{-2}-c_p^{-2}}\right). \tag{75b}$$

Thus the slowness surface is essentially the ellipse

$$\frac{\phi s_1^2}{\rho_f} + \frac{s_3^2}{\rho} = \frac{1}{M}, \tag{76}$$

punctuated by a small stop band to the left of $s_1 = c_{pl}^{-1}$. The slowness surfaces for two material combinations are plotted in Fig. 2, which also shows the ellipse of Eq. (76).

The displacement polarization follows from Eqs. (69a). For a given value of s_1, we have

$$\frac{v_1}{p} = \frac{s_1}{\rho_s}\frac{\sqrt{1-c_{pl}^2/c_p^2}}{(1-c_{pl}^2 s_1^2)}, \quad \frac{v_3}{p} = \frac{s_3}{\rho}, \tag{77}$$

where s_3 is defined by Eq. (73). Assume that $s_1>0$ and $s_3>0$, as in Fig. 2. The ratios v_1/p and v_3/p are both positive for the fast wave (the branch $s_1 \leq s_{\text{fast}}$). The same ratios are negative and positive, respectively, for the slow wave. Thus the horizontal velocity is in phase with the acoustic fluid pressure for the fast wave, but out of phase for the slow wave.

4. *Dynamic compatibility*

The phenomenon of dynamic compatibility was noted by Biot in his first paper on the dynamics of isotropic porous media.[1] If the material parameters satisfy a certain constraint, then a wave solution exists which has no viscous attenuation and its relative fluid motion is zero, or $\mathbf{w}=0$. Analogous compatibility conditions exist for the layered medium, as we now demonstrate for horizontal wave motion. Consider longitudinal motion in the horizontal plane, $\mathbf{u}=u(x_1,t)\mathbf{e}_1$ and $\mathbf{w}=w(x_1,t)\mathbf{e}_1$. This leads to a pair of coupled equations for $u(x,t)$ and $w(x,t)$,

$$\rho\partial_t^2 u-[(1-\phi)(\lambda_0+2\mu)+Mq^2]\partial_x^2 u+\rho_f\partial_t^2 w -Mq\partial_x^2 w=0, \tag{78a}$$

$$\partial_t w+K(t)*\left[\partial_t^2 u-\frac{M}{\rho_f}\,\partial_x^2(qu+w)\right]=0, \tag{78b}$$

where $q=(\lambda+2\mu\phi)/(\lambda+2\mu)$. The system reduces to

$$\rho_f\partial_t^2 u-Mq\partial_x^2 u=0, \quad w=0, \tag{79}$$

if the physical parameters are related by

$$Mq/\rho_f=[(1-\phi)(\lambda_0+2\mu)+Mq^2]/\rho, \tag{80}$$

or equivalently, if the densities are in the ratio

$$\frac{\rho_s}{\rho_f}=\frac{\lambda_0}{2\mu}+\left[\frac{\lambda_0+2\mu}{\lambda_0(1-\phi)+2\mu\phi}\right]\frac{2\mu}{M}. \tag{81}$$

This is the condition of *dynamic compatibility*, and when it is satisfied waves can propagate unattenuated by viscous drag with wave speed $\sqrt{Mq/\rho_f}$. Alternatively, the right-hand side of Eq. (81) may be written in terms of the bulk compressional sound speeds and Poisson's ratio of the solid, $\nu_s=\lambda/2(\lambda+\mu)$, yielding

$$\frac{\rho_s}{\rho_f}=\frac{1-\nu_s-\phi(1-2\nu_s)}{\nu_s-\phi(1-2\nu_s)[(1-\nu_s)^{-1}(c_p^2/c_f^2)-1]}. \tag{82}$$

The right-hand member is an increasing function of ϕ for most material combinations, i.e., $c_p^2/c_f^2(1-\nu_s)-1>1$ normally. Therefore, for a given pair of materials, there is a unique value of the porosity at which compatibility is attained as long as $\rho_s/\rho_f>(1-\nu_s)/\nu_s$.

VI. CONCLUSION

The theory of homogenization has been used to derive the nonlinear equations for a fluid infiltrated solid skeleton starting from the fundamental equations governing the motion on the microscale of the pores and grains. The canonical microgeometry of a layered system of alternating solid and fluid constituents has been analyzed in detail. The main results are summarized in the "averaged" or macroscopic equations of motion (41), and the nonlinear constitutive relations of Eqs. (42). The viscous effects of the pore fluid are contained in the permeability convolution operator defined by $K_{ij}(t)$ of Eq. (32). The stress–strain relations in Eqs. (42) allow for arbitrary nonlinear behavior in the elastic solid and in the fluid. The key quantity is the vector function G_i defined by Eqs. (38) and (39). We have also shown how the stress and strain are related to a macroscopic potential energy, in the spirit of Biot's later work on nonlinear mechanics of porous media.[21]

The linear limit of these equations is of interest, and we have derived for the first time in Secs. IV and V the full set of linear Biot equations for a layered medium. The predicted form of the slowness surface for isotropic elastic layers agrees with that obtained by Schoenberg[13,14] from the exact dispersion relation for the system. This gives us confidence in the general validity of the nonlinear Biot theory as derived here.

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APPENDIX: ENERGY CONSIDERATIONS

The leading-order term for the macroscopic energy density, W, in the porous medium is a linear combination of the energy in the solid and the energy in the fluid, each averaged over its domain. The fluid energy follows from Eq. (15c) as $E_f[\mathcal{Q}(p^0)]$, which is independent of $\mathbf{y}$, so that the sum of the average partial energy is

$$W=V^{-1}\int_{V_s} E_s(\partial_{x_m}u_n^0+\partial_{y_m}u_n^1)dV(\mathbf{y})+\phi E_f[\mathcal{Q}(p^0)]. \tag{A1}$$

The purpose of this appendix is to convince the reader that W of Eq. (A1), considered as a function of the independent macroscopic strain variables, does indeed yield the stress and pressure as partial derivatives according to Eq. (44). We follow Sanchez-Palencia[9] and first consider the function

$$\Lambda(\partial_{x_m}u_n^0,p^0)=V^{-1}\int_{V_s} E_s(\partial_{x_m}u_n^0+\partial_{y_m}u_n^1)dV(\mathbf{y}) +p^0V^{-1}\int_{\partial V_s} u_i^1 n_i\, dS(\mathbf{y}). \tag{A2}$$

The point is that Λ is clearly a function of the arguments shown, because the field variables u_i^1 are implicit functions of both $\partial_{x_m}u_n^0$ and p^0, as discussed above [see Eqs. (11) and (13b)]. The first variation is

$$\delta\Lambda=V^{-1}\int_{V_s}\frac{\partial E_s(\partial_{x_m}u_n^0+\partial_{y_m}u_n^1)}{\partial\Gamma_{ij}} \times(\partial_{x_i}\delta u_j^0+\partial_{y_i}\delta u_j^1)dV(\mathbf{y})$$

$$+\delta p^0 V^{-1}\int_{\partial V_s} u_i^1 n_i\, dS(\mathbf{y})+p^0V^{-1}\int_{\partial V_s}\delta u_i^1 n_i\, dS(\mathbf{y}). \tag{A3}$$

The terms involving δu_j^1 annihilate one another on account of Eqs. (12b), (13b), and the first of (5). We are then left with

$$\delta\Lambda=\delta(\partial_{x_m}u_n^0)V^{-1}\int_{V_s}\sigma_{mn}^0\, dV(\mathbf{y})$$
$$+\delta p^0 V^{-1}\int_{V_s}\partial_{y_i}u_i^1\, dV(\mathbf{y}). \tag{A4}$$

Thus, referring to Eqs. (37) and (40), we have

$$(1-\phi)n_iG_i(\partial_{x_m}u_n^0,p^0)=\frac{\partial\Lambda}{\partial p^0},$$
$$(1-\phi)T_{ji}(\partial_{x_m}u_n^0,p^0)=\frac{\partial\Lambda}{\partial(\partial_{x_j}u_i^0)}, \tag{A5}$$

respectively.

Consider the related function

$$\Pi(\partial_{x_m}u_n^0,p^0)=\Lambda(\partial_{x_m}u_n^0,p^0)-\phi p^0\partial_{x_i}u_i^0$$
$$+\phi\int_{p_{\text{amb}}}^{p^0}\mathcal{Q}(\eta)d\eta, \tag{A6}$$

where p_{amb} is the ambient pressure. Π, like Λ, is a function of the independent variables $\partial_{x_m}u_n^0$ and p^0. Taking its partial derivatives with respect to these, and using the identities of Eqs. (A5) and (42), we obtain the identities of Eq. (46). Thus

$$\delta\Pi(\mathbf{F},p)=\tau_{ji}\delta F_{ji}-\zeta\delta p \tag{A7}$$

or

$$\delta\tilde{\Pi}(\mathbf{F},\zeta)=\tau_{ji}\delta F_{ji}+p\,\delta\zeta, \tag{A8}$$

where

$$\tilde{\Pi}(\mathbf{F},\zeta)=\Pi(\mathbf{F},p)+\zeta p. \tag{A9}$$

We will now demonstrate that $\tilde{\Pi}$ and W are one and the same. The latter can be rewritten using Eq. (15c) as

$$W=V^{-1}\int_{V_s}E_s(\partial_{x_m}u_n^0+\partial_{y_m}u_n^1)dV(\mathbf{y})$$
$$+V^{-1}\int_{V_f}E_f(\partial_{x_m}U_m^0+\partial_{y_m}U_m^1)dV(\mathbf{y}). \tag{A10}$$

The first variation is

$$\delta W=V^{-1}\int_{V_s}(\partial_{x_i}\delta u_j^0+\partial_{y_i}\delta u_j^1)\sigma_{ij}^0\, dV(\mathbf{y})$$
$$-V^{-1}\int_{V_f}p^0\delta\mathcal{Q}(p^0)dV(\mathbf{y}), \tag{A11}$$

where Eqs. (15c) and the second of (5) have been used. The integral over the fluid region can be simplified further using Eqs. (16a), (18), and integration by parts, giving

$$\delta W=V^{-1}\int_{V_s}(\partial_{x_i}\delta u_j^0+\partial_{y_i}\delta u_j^1)\sigma_{ij}^0\, dV(\mathbf{y})$$
$$-\phi p^0\delta(\partial_{x_i}u_i^0)-p^0\delta(\partial_{x_i}w_i)$$
$$+p^0V^{-1}\int_{\partial V_s}\delta u_i^1 n_i\, dS(\mathbf{y}). \tag{A12}$$

The terms with δu_j^1 again disappear because of Eqs. (12b), (13b), and the first of (5), leaving

$$\delta W=\delta(\partial_{x_m}u_n^0)\left(V^{-1}\int_{V_s}\sigma_{mn}^0\, dV(\mathbf{y})-\phi p^0\delta_{mn}\right)$$
$$-p^0\delta(\partial_{x_i}w_i). \tag{A13}$$

This implies the identities of Eq. (44), and the equivalence

$$\tilde{\Pi}=W. \tag{A14}$$

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