

Dynamic Green's functions in anisotropic piezoelectric, thermoelastic and poroelastic solids†

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A procedure is described to generate fundamental solutions or Green's functions for time harmonic point forces and sources. The linearity of the field equations permits the Green's function to be represented as an integral over the surface of a unit sphere, where the integrand is the solution of a one-dimensional impulse response problem. The method is demonstrated for the theories of piezoelectricity, thermoelasticity, and poroelasticity. Time domain analogues are discussed and compared with known expressions for anisotropic elasticity.

1. Introduction

Point force solutions for elliptic, homogeneous, linear second order partial differential equations in three spatial dimensions can be represented by either three-dimensional infinite Fourier integrals or by finite, one- or two-dimensional integrals over a closed surface. The equations of static elasticity serve as the most appropriate example, for which the basic problem is to find the displacement $\mathbf{u}(\mathbf{x})$ satisfying

$$\mathbf{Q}(\nabla)\mathbf{u}(\mathbf{x}) = \delta(\mathbf{x}). \quad (1.1)$$

Here, $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$ is the spatial Dirac delta function, and $\mathbf{Q}$ is the second order acoustical tensor, defined below in equation (4.1). Direct application of transforms yields

$$\mathbf{u}(\mathbf{x}) = (2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{Q}^{-1}(\boldsymbol{\xi}) \exp(i\boldsymbol{\xi} \cdot \mathbf{x}) d\boldsymbol{\xi}. \quad (1.2)$$

The tensor function $\mathbf{Q}^{-1}(\boldsymbol{\xi})$ is homogeneous of degree -2 in $\boldsymbol{\xi}$. Armed with this simple property alone we can use Synge's result (1957)

$$(2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{Q}^{-1}(\boldsymbol{\xi}) \exp(i\boldsymbol{\xi} \cdot \mathbf{x}) d\boldsymbol{\xi} = \frac{1}{8\pi^2|\mathbf{x}|} \oint \mathbf{Q}^{-1}(\mathbf{n}) d\theta(\mathbf{n}), \quad (1.3)$$

where the integral is around the unit circle formed by the intersection of the plane $\mathbf{n} \cdot \mathbf{x} = 0$ with the unit $\mathbf{n}$ -sphere. The one-dimensional finite integral expression for the static Green's function of elasticity is well known, and goes back to the start of the century. Thus Willis (1965) demonstrated the equivalence of the ring

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integral expression of Synge (1957) and the earlier results of Fredholm (1900), as developed by Kröner (1953). The ring integral solution was also derived by Lifshitz & Rozenzweig (1947), and has been used to develop series solutions for the Green's function (Mura & Kinoshita, 1971). There are, of course, many indirectly related works, of which we mention only Indenbom & Orlov (1968), who obtained the Green's function in terms of lower dimensional Green's functions.

This paper describes a related procedure to generate Green's functions for the time harmonic equations of theories with extra field variables. The additional variable, e.g. electric potential, temperature, leads to equations that are elliptic in the highest spatial derivatives. The starting point is the 'plane wave' representation of the Dirac delta function in R^3 (Gelfand & Shilov 1964, p. 77), see equation (2.5)₂ below. Burridge (1967) used this to express the time dependent anisotropic elastic Green's function as an integral over the unit n -sphere, his equation (4.6). Duff (1960) had previously shown that the same Green's function may be written as an integral over the slowness surface, which is the Herglotz–Petrovskii representation for elasticity (Gelfand & Shilov 1964, p. 139). The two representations are of course closely related to one another, as demonstrated by Burridge (1967). More lengthy, but equally valid derivations of the time dependent Green's function using Fourier integrals have been given by Mura (1987) (his equation (9.23) is similar to Burridge's (4.6)), and by Every & Kim (1994). The plane wave representation is essentially the same as the Radon transform, which was used recently by Wang & Achenbach (1993, 1994) to generate Green's functions for anisotropic elasticity.

The theories of poroelasticity and thermoelasticity are not hyperbolic, and hence explicit Herglotz–Petrovskii formulae cannot be found for the fundamental singular solution in the time domain. We therefore focus on the time harmonic Green's functions. Fundamental solutions for isotropic thermoelasticity and poroelasticity are known (Kupradze 1978; Bonnet 1987), but with the exception of the work by Kazi-Aoual *et al.* (1988) who consider transversely isotropic poroelasticity, there are no Green's functions available for anisotropic versions of these theories. The theory of piezoelectricity is also considered here because the formalism for all three theories is similar and each collapses to simple elasticity when the additional field variable is ignored. Also, the time dependent Green's functions can be found for each theory when the dispersion is set to zero (it is identically zero for piezoelectricity), giving formulae similar to Burridge's (1967) for elasticity.

The plane wave transform method is outlined in §2 and the general solution is derived and discussed in §3. These results are general, and do not pertain to any one field theory. Applications to the three field theories are discussed in §4. The notation is introduced as it arises, although we note that both vector and subscript notation are used, with the summation convention implied for repeated lower case Latin subscripts. Also, the terms 'matrix' and 'tensor' are sometimes used for the same object with the understanding that the two are related by the choice of a basis.

2. Problem definition

We are concerned with solutions to equations of the form

$$\mathbf{L}(\nabla, \partial_t) \mathbf{V}(\mathbf{x}, t) = -\mathbf{F}^{(0)} \delta(\mathbf{x}) H(t); \quad \mathbf{V}(\mathbf{x}, t) = 0, \quad t < 0, \quad (2.1)$$

where $\mathbf{x}$ is in R^3 , $H(t)$ is the unit step function, $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$. The field variable, $\mathbf{V}(\mathbf{x}, t)$, is a three or four dimensional vector, the first three components of which are the elastic displacement, and the fourth can be temperature or pore fluid pressure, if thermoelasticity or poroelasticity is considered. The right-hand member in equation (2.2) represents a three- or four-dimensional point force at the origin of strength $\mathbf{F}^{(0)}$. The specific form of the linear differential operator $\mathbf{L}$ for each theory will be given later. The analogous basic problem in the frequency domain is

$$\mathbf{L}(\nabla, -i\omega) \mathbf{U}(\mathbf{x}) = -\mathbf{F}^{(0)} \delta(\mathbf{x}), \quad (2.2)$$

where $\mathbf{U}(\mathbf{x}, \omega)$, or more simply just $\mathbf{U}(\mathbf{x})$, is complex valued. The two solutions are related by the Fourier transform

$$\mathbf{V}(\mathbf{x}, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathbf{U}(\mathbf{x}, \omega) (-i\omega)^{-1} e^{-i\omega t} d\omega. \quad (2.3)$$

Our emphasis is on the Green's functions of (2.2), the steady state time harmonic problem, with associated physical quantities given by $\text{Re}\{\dots e^{-i\omega t}\}$. This implicit dependence upon time is omitted henceforth except when we return to the explicit time dependent solution of (2.1). The solution to (2.2) depends upon the frequency ω through its appearance in the differential operator $\mathbf{L}(\nabla, -i\omega)$. The desired solution of (2.2) also satisfies the radiation conditions at infinity, namely, it must represent outgoing waves as $|\mathbf{x}| \rightarrow \infty$.

The solution of elliptic equations of this type is discussed in detail by John (1955) and by Gelfand & Shilov (1964, p. 122). The method relies on a representation of $\delta(\mathbf{x})$ as an integral over a spherical surface. We look for solutions in terms of the transform,

$$g(\mathbf{x}) = \frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} \bar{g}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) d\Omega(\mathbf{n}), \quad (2.4)$$

where $g(\mathbf{x})$ can be either a scalar or vector function, and the prime denotes the derivative with respect to $\mathbf{n} \cdot \mathbf{x}$. For example, $g(\mathbf{x}) = (2\pi|\mathbf{x}|)^{-1}$ and $g = \delta(\mathbf{x})$ correspond to the 'transforms' $\bar{g} = |\mathbf{n} \cdot \mathbf{x}|$ and $\bar{g} = -\delta(\mathbf{n} \cdot \mathbf{x})$, respectively, or (Courant & Hilbert 1962; Burridge 1967)

$$\frac{1}{|\mathbf{x}|} = \frac{1}{2\pi} \int_{|\mathbf{n}|=1} \delta(\mathbf{n} \cdot \mathbf{x}) d\Omega(\mathbf{n}), \quad \delta(\mathbf{x}) = -\frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} \delta''(\mathbf{n} \cdot \mathbf{x}) d\Omega(\mathbf{n}). \quad (2.5)$$

The first of these is easily checked, while the second is a direct consequence of the first combined with the identity $\nabla^2(1/|\mathbf{x}|) = -4\pi\delta(\mathbf{x})$. Let

$$\mathbf{U}(\mathbf{x}) = \frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} \bar{\mathbf{U}}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) d\Omega(\mathbf{n}), \quad (2.6)$$

then substitution in equation (2.2) yields

$$\frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} d\Omega(\mathbf{n}) \left[\mathbf{L} \left(\mathbf{n} \frac{d}{d\mathbf{n} \cdot \mathbf{x}}, -i\omega \right) \bar{\mathbf{U}}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) - \mathbf{F}^{(0)} \delta''(\mathbf{n} \cdot \mathbf{x}) \right] = 0. \quad (2.7)$$

This is satisfied if the transform $\bar{\mathbf{U}}$ solves the ordinary differential equation

$$\mathbf{L} \left(\mathbf{n} \frac{d}{d\mathbf{n} \cdot \mathbf{x}}, -i\omega \right) \bar{\mathbf{U}}(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) = \mathbf{F}^{(0)} \delta(\mathbf{n} \cdot \mathbf{x}), \quad (2.8)$$

for each $\mathbf{n}$ on the unit sphere. We have thus reduced the spatial dimension of the problem from three to one.

3. Solutions

(a) The unidimensional Green's matrix

The general problem depends upon the matrix operator $\mathbf{L}$, which is a second order differential operator for the three field theories considered below. In each case equation (2.8) reduces to the generic form,

$$\mathbf{K}(\mathbf{n}) \overline{\mathbf{U}}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) + i\omega \mathbf{C}(\mathbf{n}) \overline{\mathbf{U}}' + \omega^2 \mathbf{M} \overline{\mathbf{U}} = \mathbf{F}^{(0)} \delta(\mathbf{n} \cdot \mathbf{x}), \quad (3.1)$$

where $\mathbf{K}$, $\mathbf{C}$, and $\mathbf{M}$ have the properties

$$\mathbf{K}(\mathbf{n}) = \mathbf{K}^T(\mathbf{n}) = \mathbf{K}(-\mathbf{n}), \quad \mathbf{C}(\mathbf{n}) = \mathbf{C}^T(\mathbf{n}) = -\mathbf{C}(-\mathbf{n}), \quad \mathbf{M} = \mathbf{M}^T. \quad (3.2)$$

All three matrices can also depend upon ω , although the explicit dependence will be omitted for brevity. The transform for direction $\mathbf{n}$ is represented as

$$\overline{\mathbf{U}}(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) = \overline{\mathbf{G}}(\mathbf{n} \cdot \mathbf{x}, \omega, \mathbf{n}) \mathbf{F}^{(0)}. \quad (3.3)$$

The 'unidimensional Green's matrix' then satisfies the matrix equation

$$\mathbf{P} \left(-i \frac{d}{d\xi}, \omega, \mathbf{n} \right) \overline{\mathbf{G}}(\xi) = \mathbf{I} \delta(\xi), \quad \xi \equiv \mathbf{n} \cdot \mathbf{x}, \quad (3.4)$$

where $\mathbf{P}(k, \omega, \mathbf{n})$, or more simply $\mathbf{P}(k)$, is the symmetric fundamental matrix governing equation (3.1),

$$\mathbf{P}(k, \omega, \mathbf{n}) \equiv \omega^2 \mathbf{M} - \omega k \mathbf{C}(\mathbf{n}) - k^2 \mathbf{K}(\mathbf{n}). \quad (3.5)$$

The fundamental eigenvalue problem associated with (3.1) is to find solutions of the form $\overline{\mathbf{U}} = \mathbf{A} e^{ik\mathbf{n} \cdot \mathbf{x}}$ for the homogeneous equation, which correspond to non-trivial solutions of

$$\mathbf{P}(k, \omega, \mathbf{n}) \mathbf{A} = \mathbf{0}, \quad \det \mathbf{P}(k, \omega, \mathbf{n}) = 0. \quad (3.6)$$

The dispersion relation (3.6)₂ is a quartic (or cubic) equation in k^2 (see Appendix A), for which the roots are $\pm k_\alpha$, $\alpha = 1, \dots, N$, where $N = 4$ (or 3) is the problem dimension. The roots $+k_\alpha$ are assumed to be radiating, which means that they have non-negative imaginary parts, or positive real parts if $\text{Im } k_\alpha = 0$. We define for later use the N phase speeds $v_\alpha = \omega/k_\alpha$, $\alpha = 1, \dots, N$. The associated null vectors are $\mathbf{A}_\alpha$, which are normalized according to

$$\mathbf{A}_\alpha \cdot \mathbf{A}_\alpha = 1, \quad \text{for each } \alpha, \quad (3.7)$$

however, in general, they are mutually orthogonal only if $\mathbf{C} = \mathbf{0}$. The symmetry of $\mathbf{P}$ implies that $\mathbf{A}_\alpha$ is simultaneously a right and a left null vector of $\mathbf{P}(k_\alpha, \omega, \mathbf{n})$. The matrix $\mathbf{P}(k, \omega, \mathbf{n})$ is complex-valued and symmetric but not necessarily Hermitian. Matrices of this type are common in elastic wave theory, although their canonical structure has only recently been investigated (Scott 1993b). They can admit of non-trivial null vectors with the property $\mathbf{A} \cdot \mathbf{A} = 0$ under certain circumstances (Hayes 1984; Scott 1993a,b). It is therefore assumed that (3.7) holds for those values of k , ω , and $\mathbf{n}$ necessary for computing the Green's function. A simplified version of the dispersion relation appropriate for thermoelasticity and

poroelasticity is presented in Appendix A. The quartic nature of the relation is evident from equation (A 2).

The null vector of equation (3.6)₁ is not necessarily invariant under a reversal of the direction $\mathbf{n}$, or under a change of sign for k , i.e. $\mathbf{A}_\alpha(k_\alpha, -\mathbf{n}) \neq \mathbf{A}_\alpha(k_\alpha, \mathbf{n})$ and $\mathbf{A}_\alpha(-k_\alpha, \mathbf{n}) \neq \mathbf{A}_\alpha(k_\alpha, \mathbf{n})$. However, it is unchanged under a simultaneous reversal of both parameters because it depends upon k_α and $\mathbf{n}$ in the combination $k_\alpha \mathbf{n}$, or $\mathbf{A} = \mathbf{A}(k\mathbf{n})$. We will use this parity property below to simplify the Green's function. In particular, we define $\mathbf{A}(\pm 1) \equiv \mathbf{A}(\pm k\mathbf{n})$, and if the argument is absent it is taken to mean $\mathbf{A}(k\mathbf{n})$, the null vector of (3.6)₁. This distinction based upon the parity will not enter the final expression.

We now drop all explicit dependence upon ω and $\mathbf{n}$ in order to concentrate on the ξ -dependence (recall that $\xi = \mathbf{n} \cdot \mathbf{x}$). Let

$$\overline{\mathbf{G}}(\xi) = \mathbf{P}^* \left(-i \frac{d}{d\xi} \right) \psi(\xi), \quad (3.8)$$

where $\mathbf{P}^*(k)$ is the transposed matrix of cofactors of $\mathbf{P}(k)$, which satisfies

$$\mathbf{P} \mathbf{P}^* = \mathbf{P}^* \mathbf{P} = \mathbf{I} \det \mathbf{P}. \quad (3.9)$$

Note that $\mathbf{P}^*$ is symmetric, which is a consequence of the symmetry of $\mathbf{P}$. It follows from equations (3.4) and (3.8) that the scalar function ψ must satisfy the ordinary differential equation

$$\left[\det \mathbf{P} \left(-i \frac{d}{d\xi} \right) \right] \psi(\xi) = \delta(\xi). \quad (3.10)$$

This may be solved quite easily, using a Fourier transform, for example, to give

$$\psi(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{ik\xi} dk}{\det \mathbf{P}(k)} = \sum_{\alpha=1}^N \frac{i e^{ik_\alpha |\xi|}}{[\det \mathbf{P}(k)]' |_{k=k_\alpha}}, \quad (3.11)$$

where the prime denotes the derivative with respect to k . Thus, from equations (3.8) and (3.11),

$$\overline{\mathbf{G}}(\xi) = \sum_{\alpha=1}^N \frac{i \mathbf{P}^*(k_\alpha \operatorname{sgn} \xi)}{[\det \mathbf{P}(k)]' |_{k=k_\alpha}} e^{ik_\alpha |\xi|}. \quad (3.12)$$

(b) Simplification of the transform

The transform Green's matrix $\overline{\mathbf{G}}$ can be related to the null vectors by the alternative formula

$$\overline{\mathbf{G}}(\xi) = \sum_{\alpha=1}^N \frac{e^{ik_\alpha |\xi|}}{2ik_\alpha \lambda_\alpha} \mathbf{A}_\alpha(\operatorname{sgn} \xi) \otimes \mathbf{A}_\alpha(\operatorname{sgn} \xi), \quad (3.13)$$

where the parameters λ_α have dimensions of stiffness,

$$\lambda_\alpha = \frac{1}{2} \mathbf{A}_\alpha \cdot (v_\alpha^2 \mathbf{M} + \mathbf{K}) \mathbf{A}_\alpha. \quad (3.14)$$

The equivalence of equations (3.12) and (3.13) can be shown as follows. First, we note that

$$[\det \mathbf{P}(k)]' = \operatorname{tr} \mathbf{P}'(k) \mathbf{P}^*(k) = \operatorname{tr} (-2k\mathbf{K} - \omega\mathbf{C}) \mathbf{P}^*(k). \quad (3.15)$$

At the same time, equations (3.6)₂ and (3.9) imply

$$\text{tr}(-k^2 \mathbf{K} - \omega k \mathbf{C} + \omega^2 \mathbf{M}) \mathbf{P}^*(k) = 0, \quad k = k_\alpha, \quad \alpha = 1, \dots, N. \quad (3.16)$$

It is then a simple matter to see that

$$[\det \mathbf{P}(k_\alpha)]' = (-1/k) \text{tr}(\omega^2 \mathbf{M} + k^2 \mathbf{K}) \mathbf{P}^*(k), \quad k = k_\alpha, \quad \alpha = 1, \dots, N. \quad (3.17)$$

Finally, we note that the cofactor matrix is related to the dyadic formed from the null vectors by

$$\mathbf{A}_\alpha(\text{sgn } \xi) \otimes \mathbf{A}_\alpha(\text{sgn } \xi) = \mathbf{P}^*(k_\alpha \text{sgn } \xi) / \text{tr } \mathbf{P}^*(k_\alpha). \quad (3.18)$$

Equations (3.17) and (3.18) together yield the identity

$$\frac{i \mathbf{P}^*(k_\alpha \text{sgn } \xi)}{[\det \mathbf{P}(k)]'|_{k=k_\alpha}} = \frac{1}{2ik_\alpha \lambda_\alpha} \mathbf{A}_\alpha(\text{sgn } \xi) \otimes \mathbf{A}_\alpha(\text{sgn } \xi), \quad (3.19)$$

which in turn implies the equivalence of equation (3.12) and the pair (3.13) and (3.14).

The general solution for the point forces is given by equations (2.6), (3.8), and either (3.12) or (3.13). Using the latter, and the result

$$\frac{d^2}{d\xi^2} e^{ik|\xi|} = 2ik \delta(\xi) - k^2 e^{ik|\xi|}, \quad (3.20)$$

we deduce

$$\begin{aligned} \mathbf{U}(\mathbf{x}) = & \frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} \sum_{\alpha=1}^N [\delta(\mathbf{n} \cdot \mathbf{x}) + \frac{1}{2} i k_\alpha e^{ik_\alpha |\mathbf{n} \cdot \mathbf{x}|}] \\ & \times \lambda_\alpha^{-1} \mathbf{A}_\alpha(\text{sgn } \xi) \otimes \mathbf{A}_\alpha(\text{sgn } \xi) \mathbf{F}^{(0)} d\Omega(\mathbf{n}). \end{aligned} \quad (3.21)$$

The $\delta(\mathbf{n} \cdot \mathbf{x})$ picks out the ring integral around the circle $\{\mathbf{n} \cdot \mathbf{x} = 0\} \cap \{|\mathbf{n}| = 1\}$, and the remaining integral can be reduced to one over the hemisphere $\{\mathbf{n} \cdot \mathbf{x} > 0\} \cap \{|\mathbf{n}| = 1\}$ by using the parity property of the null vector. Thus

$$\begin{aligned} \mathbf{U}(\mathbf{x}, \omega) = & \frac{1}{8\pi^2 |\mathbf{x}|} \oint \sum_{\alpha=1}^N \frac{\mathbf{A}_\alpha \otimes \mathbf{A}_\alpha}{\lambda_\alpha} \mathbf{F}^{(0)} d\theta(\mathbf{n}) \\ & + \frac{1}{8\pi^2} \int_{\cap} \sum_{\alpha=1}^N \frac{ik_\alpha}{\lambda_\alpha} \mathbf{A}_\alpha \otimes \mathbf{A}_\alpha \mathbf{F}^{(0)} e^{ik_\alpha \mathbf{n} \cdot \mathbf{x}} d\Omega(\mathbf{n}), \end{aligned} \quad (3.22)$$

where the θ -integral is around the same unit circle as in equation (1.3), and the second integral is over the hemisphere. This is the central result of the paper. We include the 'force' $\mathbf{F}^{(0)}$ in the integrals because it may also depend upon the direction $\mathbf{n}$, as occurs in the piezoelectric equations discussed in §4, where it has the parity $\mathbf{F}^{(0)}(-\mathbf{n}) = \mathbf{F}^{(0)}(\mathbf{n})$, consistent with the reduction to the hemisphere integral. When $\mathbf{F}^{(0)}$ is independent of $\mathbf{n}$ it can be removed from the final result, and $\mathbf{U}$ is then the fundamental matrix solution.

(c) Solutions in the time domain

Analytic evaluation of the integral in (2.3) is not generally feasible because the frequency occurs in the integrand in a very nonlinear manner. Progress can be made if the equations are nondispersive in the sense that $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$ are

each independent of ω . They are consequently real-valued matrices (note that $i = \sqrt{-1}$ can only occur in the combination $-i\omega$) and all of the phase speeds v_α , $\alpha = 1, \dots, N$, are real and constant as functions of frequency. That is, they are nondispersive. Also, it can be seen that the integrand in the ring integral of (3.22) does not involve frequency, while the final integral depends upon ω through the combination $\{i\omega \exp(i\omega \mathbf{n} \cdot \mathbf{x}/v_\alpha)\}$. The inverse transforms are then simple, with the outcome

$$\begin{aligned} \mathbf{V}(\mathbf{x}, t) = & \frac{1}{8\pi^2|\mathbf{x}|} \oint \sum_{\alpha=1}^N \frac{\mathbf{A}_\alpha \otimes \mathbf{A}_\alpha}{\lambda_\alpha} \mathbf{F}^{(0)} d\theta(\mathbf{n}) \\ & - \frac{1}{8\pi^2} \int_{\cap} \sum_{\alpha=1}^N \delta(v_\alpha t - \mathbf{n} \cdot \mathbf{x}) \frac{\mathbf{A}_\alpha \otimes \mathbf{A}_\alpha}{\lambda_\alpha} \mathbf{F}^{(0)} d\Omega(\mathbf{n}), \quad t > 0. \end{aligned} \quad (3.23)$$

The second term gives the transient wavefronts that spread from the source point, ultimately leaving only the first term, which is the static response. The latter can also be identified as the zero frequency component of the frequency domain solution in equation (3.22). The second integral is the same whether it is over the hemisphere or the whole sphere, because $t > 0$ by assumption.

The impulse response solution for the related problem

$$\mathbf{L}(\nabla, \partial_t) \mathbf{W}(\mathbf{x}, t) = -\mathbf{F}^{(0)} \delta(\mathbf{x}) \delta(t); \quad \mathbf{W}(\mathbf{x}, t) = 0, \quad t < 0, \quad (3.24)$$

is simply $\mathbf{W} = \partial \mathbf{V} / \partial t$. The explicit form for $\mathbf{W}$ when the material is non-dispersive follows from equation (3.23) as

$$\mathbf{W}(\mathbf{x}, t) = -\frac{1}{8\pi^2} \int_{\cap} \sum_{\alpha=1}^N \delta'(v_\alpha t - \mathbf{n} \cdot \mathbf{x}) \frac{v_\alpha}{\lambda_\alpha} \mathbf{A}_\alpha \otimes \mathbf{A}_\alpha \mathbf{F}^{(0)} d\Omega(\mathbf{n}), \quad t > 0. \quad (3.25)$$

This generalizes equation (4.6) of Burridge (1967) for anisotropic elasticity. In fact his equation is the limiting case of all three theories described in the next section when the additional field variable is 'turned off'. The step function response of equation (3.23) has also been derived by Every & Kim (1994) for the elastic case.

Finally, we note that the time harmonic Green's function of equation (3.22) can be computed directly from the time dependent version of equation (3.25). The two are related by the forward Fourier transform

$$\mathbf{U}(\mathbf{x}, \omega) = \int_0^\infty \mathbf{W}(\mathbf{x}, t) e^{i\omega t} dt. \quad (3.26)$$

The equivalence follows from

$$-\int_0^\infty \delta'(v_\alpha t - \mathbf{n} \cdot \mathbf{x}) v_\alpha e^{i\omega t} dt = \delta(\mathbf{n} \cdot \mathbf{x}) + ik_\alpha e^{ik_\alpha \mathbf{n} \cdot \mathbf{x}} H(\mathbf{n} \cdot \mathbf{x}), \quad (3.27)$$

where $k_\alpha = \omega/v_\alpha$. The identity is easily proved by integrating by parts, with $v_\alpha > 0$ by assumption. It is then straightforward to obtain (3.22) by combining equations (3.25) through (3.27). Thus the time harmonic elastic Green's function follows simply from Burridge's formula. It can also be obtained by Fourier integral methods (Every & Kim 1994) or by the Radon transform (Wang & Achenbach 1994).

4. Three different theories

Subscript notation will be used where i, j, k and l take the values 1 through 3, and refer to components relative to an orthonormal basis. In each theory u_i are the elastic displacements, and σ_{ij} and $e_{kl} = (u_{k,l} + u_{l,k})/2$ are the stresses and strains. Subscripts preceded by a comma denote the partial derivative. The mass density is represented by ρ , and the elastic stiffnesses, which satisfy the usual symmetries, are $C_{ijkl} = C_{klij} = C_{ijlk}$. We define the second order tensor $\mathbf{Q}$, with components

$$Q_{ik}(\mathbf{n}) = C_{ijkl}n_jn_l. \quad (4.1)$$

The source strength $\mathbf{F}^{(0)}$ is a four-dimensional vector for the theories considered, comprising a three-vector, $\mathbf{f}^{(0)}$, and a scalar, $R^{(0)}$, both constants. We also define

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}^{(0)} \delta(\mathbf{x}), \quad R(\mathbf{x}) = R^{(0)} \delta(\mathbf{x}), \quad (4.2)$$

where $\mathbf{f}$ represents the applied force and the meaning of R depends upon the theory. Each theory may be reduced to the classical elastic limit, in which case $N = 3$, the stiffness and mass tensors are simply $\mathbf{K} = \mathbf{Q}$ and $\mathbf{M} = \rho \mathbf{I}$, respectively, and $\mathbf{C} = \mathbf{0}$. Then $\mathbf{A}_\alpha$ are the eigenvectors of the acoustical tensor $\mathbf{Q}$, with eigenvalues $\lambda_\alpha = \rho v_\alpha^2$. Finally, we note that the Green's functions all reduce to explicit formulae for isotropic materials. This particular limit is discussed in Appendix B.

(a) Piezoelectricity

The governing dynamic equations are (Auld 1973)

$$\rho u_{i,tt} = \sigma_{ij,j} + \rho f_i, \quad D_{j,j} = \rho R, \quad (4.3)$$

and the linear constitutive relations are

$$\sigma_{ij} = C_{ijkl}e_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}e_{kl} + \epsilon_{ik}E_k. \quad (4.4)$$

Here, $\mathbf{D}$ and $\mathbf{E}$ are the electric displacement and electric field, and $\mathbf{f}$ and R are the applied force and charge densities per unit mass. The piezoelectric stress constants and the permittivity constants satisfy the usual symmetries, $e_{kij} = e_{kji}$, and $\epsilon_{ij} = \epsilon_{ji}$, respectively. The piezoelectric effect cannot exist for isotropic materials because the only isotropic third order tensor with the required symmetries is $e_{ijk} = 0$. Define the vector $\mathbf{p}$, and the scalar μ , by

$$p_i(\mathbf{n}) = e_{kij}n_kn_j, \quad \mu(\mathbf{n}) = \epsilon_{ij}n_i n_j, \quad (4.5)$$

and introduce the electric potential ϕ , such that $\mathbf{E} \equiv -\nabla\phi$, then equations (4.3) become

$$\mathbf{Q}(\nabla)\mathbf{u} - \rho\mathbf{u}_{,tt} + \mathbf{p}(\nabla)\phi = -\rho\mathbf{f}, \quad \mathbf{p}(\nabla) \cdot \mathbf{u} - \mu(\nabla)\phi = \rho R. \quad (4.6)$$

We assume the solutions can be represented by transforms $\bar{\mathbf{u}}$ and $\bar{\phi}$. Substitution into (4.6) gives

$$\mathbf{Q}(\mathbf{n})\bar{\mathbf{u}}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) + \rho\omega^2\bar{\mathbf{u}} + \mathbf{p}(\mathbf{n})\bar{\phi}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) = \rho\mathbf{f}^{(0)}\delta(\mathbf{n} \cdot \mathbf{x}), \quad (4.7a)$$

$$\mathbf{p}(\mathbf{n}) \cdot \bar{\mathbf{u}}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) - \mu(\mathbf{n})\bar{\phi}''(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) = -\rho R^{(0)}\delta(\mathbf{n} \cdot \mathbf{x}). \quad (4.7b)$$

Eliminating $\bar{\phi}''$ from (4.7a) using (4.7b) yields a single equation for $\bar{\mathbf{u}}$, of the

form of (3.1) with

$$\left. \begin{aligned} \overline{\mathbf{U}} &= \overline{\mathbf{u}}, & \mathbf{F}^{(0)} &= \rho [\mathbf{f}^{(0)} - R^{(0)} \mu^{-1} \mathbf{p}], \\ \mathbf{K}(\mathbf{n}) &= \mathbf{Q} + \mu^{-1} \mathbf{p} \otimes \mathbf{p}, & \mathbf{M} &= \rho \mathbf{I}, & \mathbf{C} &= 0. \end{aligned} \right\} \quad (4.8)$$

The time harmonic displacement transform for arbitrary point sources at the origin then follows from equation (3.22). The phase speeds $v_\alpha = \omega/k_\alpha$ are real and non-dispersive, so that the time domain results of equations (3.23) and (3.25) apply. Also, $\lambda_\alpha(\mathbf{n})$ are the eigenvalues of $\mathbf{K}(\mathbf{n})$ with $\lambda_\alpha = \rho v_\alpha^2$, and $\mathbf{A}_\alpha$ are the corresponding eigenvectors. The electric potential then follows from (4.7b) and the solution for $\mathbf{u}$. For example, the time dependent solution analogous to (3.23) is

$$\begin{aligned} \phi(\mathbf{x}, t) &= \frac{1}{8\pi^2 |\mathbf{x}|} \oint \frac{1}{\mu} \left[\rho R^{(0)} + \mathbf{p} \cdot \sum_{\alpha=1}^N \frac{\mathbf{A}_\alpha \otimes \mathbf{A}_\alpha}{\lambda_\alpha} \mathbf{F}^{(0)} \right] d\theta(\mathbf{n}) \\ &\quad - \frac{1}{8\pi^2} \int_{\cap} \sum_{\alpha=1}^N \delta(v_\alpha t - \mathbf{n} \cdot \mathbf{x}) \mathbf{p} \cdot \frac{\mathbf{A}_\alpha \otimes \mathbf{A}_\alpha}{\mu \lambda_\alpha} \mathbf{F}^{(0)} d\Omega(\mathbf{n}), \quad t > 0. \end{aligned} \quad (4.9)$$

The eigenvectors need not be evaluated explicitly, but can be related to the matrices via the identity (3.18), and $\mathbf{P}^*(k)$ may be taken as the cofactor matrix of $\lambda_\alpha \mathbf{I} - \mathbf{K}$. Note that the 'force vector' $\mathbf{F}^{(0)}$ of (4.8)₂ is anisotropic in the sense that it depends upon $\mathbf{n}$. This feature distinguishes the piezoelectric equations from those considered below. In fact, the reader will note that the piezoelectric formulation does not follow the prescription outlined in §2, because we cannot begin with an equation like (2.2).

(b) Thermoelasticity

We follow the notation of Chandrasekharaiah (1986) for the theory of thermoelasticity with thermal relaxation. The equations are

$$\rho u_{i,tt} = \sigma_{ij,j} + \rho f_i, \quad q_{j,j} + \theta_0 \rho S_{,t} = \rho R, \quad (4.10)$$

and the linear constitutive relations are

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} \theta, \quad \rho S = (\rho c / \theta_0) \theta + \beta_{ji} e_{ij}, \quad q_i + \tau q_{i,t} = -k_{ij} \theta_{,j}. \quad (4.11)$$

The new field variables, $\mathbf{q}$, S and θ , are heat flux, entropy deviation, and the temperature elevation above the ambient temperature θ_0 , respectively. The thermoelasticity coefficients $\beta_{ij} = \beta_{ji}$ can be related to the coefficients of thermal expansion, c is the specific heat per unit mass in the isothermal state, τ is the thermal relaxation time, k_{ij} are the elements of the thermal conductivity tensor, and R is the internal heat source strength per unit mass.

Define

$$b_i(\mathbf{n}) = \beta_{ij} n_j, \quad \kappa(\mathbf{n}) = [\theta_0 (\tau + (-i\omega)^{-1})]^{-1} k_{ij} n_i n_j, \quad (4.12)$$

then the equations (4.11) become, after transforming to the frequency domain and eliminating σ_{ij} , S and $\mathbf{q}$,

$$\mathbf{Q}(\nabla) \mathbf{u} + \rho \omega^2 \mathbf{u} + i\omega \mathbf{b}(\nabla) \frac{\theta}{(-i\omega)} = -\rho \mathbf{f}, \quad (4.13a)$$

$$i\omega \mathbf{b}(\nabla) \cdot \mathbf{u} + \kappa(\nabla) \frac{\theta}{(-i\omega)} + \omega^2 \frac{\rho c}{\theta_0} \frac{\theta}{(-i\omega)} = -\frac{\rho R}{\theta_0}. \quad (4.13b)$$

The singular solution, or Green's function, is defined by the point force and source of equation (4.2). The problem is therefore explicitly four dimensional, and can be reduced to the generic form. Referring to equation (3.1), we have

$$\overline{\mathbf{U}} = \{\overline{\mathbf{u}}, (-i\omega)^{-1} \overline{\theta}\}^T, \quad \mathbf{F}^{(0)} = \{\rho \mathbf{f}^{(0)}, \rho R^{(0)}/\theta_0\}^T, \quad (4.14a)$$

$$\mathbf{K}(\mathbf{n}) = \begin{bmatrix} \mathbf{Q}(\mathbf{n}) & 0 \\ 0 & \kappa(\mathbf{n}) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \rho \mathbf{I} & 0 \\ 0 & \rho c/\theta_0 \end{bmatrix}, \quad \mathbf{C}(\mathbf{n}) = \begin{bmatrix} 0 & \mathbf{b}(\mathbf{n}) \\ \mathbf{b}^T(\mathbf{n}) & 0 \end{bmatrix}. \quad (4.14b)$$

The dispersion relation (3.6)₂ can be simplified using a method of Chadwick (1979), with the details in Appendix A. The matrices $\widehat{\mathbf{K}}$ and $\widetilde{\mathbf{K}}$ defined there correspond to the isentropic and isothermal acoustical tensors, respectively. Dispersion enters the solution through the frequency dependence of κ , see equation (4.12)₂. The dependence vanishes as $\omega \rightarrow \infty$, and the time domain results are therefore relevant to the propagation of wavefronts, or more specifically, to the propagation of signals with frequency content $\omega \gg \tau^{-1}$.

(c) Poroelasticity

The notation of Biot (1962) will be adopted. The equations of motion are

$$\rho u_{i,tt} + \rho_f w_{i,tt} = \sigma_{ij,j} + \rho f_i, \quad \rho_f u_{i,tt} + \widehat{m}_{ij} * w_{j,tt} = -p_{,i}, \quad (4.15)$$

where $*$ denotes the convolution operator. The linear constitutive relations are

$$\sigma_{ij} = C_{ijkl} e_{kl} + M_{ij} \zeta, \quad p = M_{ij} e_{ij} + M \zeta, \quad \zeta_{,t} = -\text{div } \mathbf{w}_{,t} + R. \quad (4.16)$$

The field variables ζ , $\mathbf{w}$ and p , are related to the fluid. Thus, p is the pressure deviation from the ambient pore pressure, $\mathbf{w}$ is the relative fluid displacement, and ζ is a fluid dilatation measure. The coefficients $M_{ij} = M_{ji}$ are elastic moduli coupling the solid and fluid deformation, and $\widehat{m}_{ij}(t) = \widehat{m}_{ji}(t)$ are linear viscodynamic operators in the time domain (Biot 1962). They are more easily dealt with in the frequency domain, and we define for later use the transforms $m_{ij}(\omega) = m_{ji}(\omega)$. The convolution operator reduces to simple multiplication in the frequency domain. The elements $m_{ij}(\omega)$ are frequency dependent complex densities associated with the pore fluid mobility and permeability. Also, ρ_f is the fluid density, and R represents the source in the fluid phase. Equation (4.16)₃ defines R as a rate of volume injection, which can be related to the conventional notion of a volumetric mass source (Cleary 1977; Rudnicki 1986).

Define

$$\widetilde{Q}_{ik}(\mathbf{n}) = (C_{ijkl} - M^{-1} M_{ij} M_{kl}) n_j n_l, \quad (4.17a)$$

$$b_i(\mathbf{n}) = -[M^{-1} M_{ij} + \rho_f (m^{-1})_{ij}] n_j, \quad \kappa(\mathbf{n}) = (m^{-1})_{ij} n_i n_j, \quad (4.17b)$$

then the frequency dependent equations of poroelasticity reduce to

$$\widetilde{\mathbf{Q}}(\nabla) \mathbf{u} + \omega^2 (\rho \mathbf{I} - \rho_f^2 \mathbf{m}^{-1}) \mathbf{u} + i\omega \mathbf{b}(\nabla) \frac{p}{(-i\omega)} = -\rho \mathbf{f}, \quad (4.18a)$$

$$i\omega \mathbf{b}(\nabla) \cdot \mathbf{u} + \kappa(\nabla) \frac{p}{(-i\omega)} + \frac{\omega^2}{M} \frac{p}{(-i\omega)} = -R. \quad (4.18b)$$

The first of these is obtained by using equations (4.16)₁ and (4.16)₂ to express σ_{ij} in terms of e_{ij} and p . This is used in (4.15)₁, while $\mathbf{w}$ is eliminated from the same equation by using (4.15)₂ to write it in terms of $\mathbf{u}$ and ∇p . Equation (4.18b) is found by equating the two different expressions for ζ obtained first from equation (4.16)₂, and secondly from (4.15)₂ by using it as before to express $\mathbf{w}$, then taking the divergence, and finally invoking the definition of ζ in (4.16)₃. The fundamental solution is generated by the point sources of equation (4.2), so that the problem is again four dimensional, with

$$\bar{\mathbf{U}} = \{\bar{\mathbf{u}}, (-i\omega)^{-1} \bar{p}\}^T, \quad \mathbf{F}^{(0)} = \{\rho \mathbf{f}^{(0)}, R^{(0)}\}^T, \quad (4.19a)$$

$$\mathbf{K}(\mathbf{n}) = \begin{bmatrix} \tilde{\mathbf{Q}}(\mathbf{n}) & 0 \\ 0 & \kappa(\mathbf{n}) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \rho \mathbf{I} - \rho_f^2 \mathbf{m}^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}, \quad (4.19b)$$

and $\mathbf{C}$ is given by equation (4.14b).

The dispersion depends upon the behaviour of the 'mass' tensor $\mathbf{m}(\omega)$ as a function of frequency. Its low frequency content is determined by the permeability of the pore space, and at high frequencies it assumes the properties of inertia, in that $\mathbf{m}^{(\infty)} \equiv \lim_{\omega \rightarrow \infty} \mathbf{m}(\omega)$ is positive definite. The existence of this limit is physically clear. Thus, $\mathbf{m}^{(\infty)}$ is determined by the inertial resistance of the pore space to an inviscid fluid. The crossover from low to high frequencies is a sensitive function of the pore space geometry, and beyond the scope of the present paper. However, the time domain results of §3 are applicable to signals with frequency content in the high frequency regime, and they follow simply by the replacement $\mathbf{m}(\omega) \rightarrow \mathbf{m}^{(\infty)}$.

Part of this work was completed while the author was a visitor at Schlumberger-Doll Research Laboratories, Ridgefield, CT, U.S.A.

Appendix A. The dispersion relation

The dispersion relations for thermoelasticity and poroelasticity are similar because in both cases the matrix $\mathbf{P}$ is of the form

$$\mathbf{P}(k) = \begin{bmatrix} \omega^2 \tilde{\mathbf{M}} - k^2 \tilde{\mathbf{K}} & -\omega k \mathbf{b} \\ -\omega k \mathbf{b}^T & \omega^2 m - k^2 \kappa \end{bmatrix}, \quad (A1)$$

where $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$ are symmetric, 3×3 matrices. For the thermoelastic medium we have, from equation (4.14), $\tilde{\mathbf{M}} = \rho \mathbf{I}$, $m = \rho c/c_0$, $\tilde{\mathbf{K}} = \mathbf{Q}$, with $\mathbf{b}$ and κ given by (4.12). In the poroelastic case $\tilde{\mathbf{M}} = \rho \mathbf{I} - \rho_f^2 \mathbf{m}^{-1}$, $m = 1/M$, $\tilde{\mathbf{K}} = \tilde{\mathbf{Q}}$, with the other parameters as defined in (4.18). The similarities between the two theories has been noted by many, including M. A. Biot, who repeatedly drew upon the analogy in his works. More recently, Bonnet (1987) pointed out that the Green's function for isotropic poroelasticity can be written down directly using Kupradze's (1978) formula for the Green's function of isotropic thermoelasticity. The analogy between the theories is also useful for considering the static

behaviour of heterogeneous media (Norris 1992). The dynamic theories differ for anisotropic materials in that the mass matrix $\widetilde{\mathbf{M}}$ is isotropic for thermoelasticity but it can be anisotropic for porous media. This distinction does not affect the present analysis.

The dispersion relation given by equations (3.6)₂ and (A 1), can be expressed in terms of the determinants of symmetric, 3×3 matrices. Thus

$$\begin{aligned} \det \mathbf{P} &= \omega^2 m \det(\omega^2 \widetilde{\mathbf{M}} - k^2 \widehat{\mathbf{K}}) - k^2 \kappa \det(\omega^2 \widetilde{\mathbf{M}} - k^2 \widetilde{\mathbf{K}}) \\ &= \omega^2 m \det(\omega^2 \widetilde{\mathbf{M}} - k^2 \widetilde{\mathbf{K}}) - k^2 \kappa \det(\omega^2 \widehat{\mathbf{M}} - k^2 \widetilde{\mathbf{K}}), \end{aligned} \quad (\text{A } 2)$$

where

$$\widehat{\mathbf{K}} = \widetilde{\mathbf{K}} + \frac{1}{m} \mathbf{b} \otimes \mathbf{b}, \quad \widehat{\mathbf{M}} = \widetilde{\mathbf{M}} + \frac{1}{\kappa} \mathbf{b} \otimes \mathbf{b}. \quad (\text{A } 3)$$

Equation (A 2) may be obtained by expanding the determinant of $\mathbf{P}$ in (A 1). Focusing on the 44 element, we have

$$\det \mathbf{P}(k) = -k^2 \kappa \det(\omega^2 \widetilde{\mathbf{M}} - k^2 \widetilde{\mathbf{K}}) + \omega^2 \det \begin{bmatrix} \omega^2 \widetilde{\mathbf{M}} - k^2 \widetilde{\mathbf{K}} & -k \mathbf{b} \\ -k \mathbf{b}^T & m \end{bmatrix}. \quad (\text{A } 4)$$

The second determinant can be simplified as

$$\det \begin{bmatrix} \omega^2 \widetilde{\mathbf{M}} - k^2 \widehat{\mathbf{K}} + (k^2/m) \mathbf{b} \otimes \mathbf{b} & -k \mathbf{b} \\ -k \mathbf{b}^T & m \end{bmatrix} = m \det(\omega^2 \widetilde{\mathbf{M}} - k^2 \widehat{\mathbf{K}}), \quad (\text{A } 5)$$

proving the identity (A 2)₁. The other can be derived in the same way.

A result of Scott (1989) concerning the ordering of the eigenvalues of the isentropic and isothermal acoustical tensors can be generalized to the matrices $\widehat{\mathbf{K}}$ and $\widetilde{\mathbf{K}}$ (or $\widehat{\mathbf{M}}$ and $\widetilde{\mathbf{M}}$).

If we define their real eigenvalues as $\widehat{\lambda}_1 \leq \widehat{\lambda}_2 \leq \widehat{\lambda}_3$ and $\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \widetilde{\lambda}_3$, respectively, then it clear that under appropriate conditions of positivity on the moduli they are all positive, with $\widetilde{\lambda}_j \leq \widehat{\lambda}_j$ for $j = 1, 2$, and 3. A simple extension of Scott's (1989) analysis shows that the two sets satisfy the additional inequalities $\widehat{\lambda}_1 \leq \widetilde{\lambda}_2$ and $\widehat{\lambda}_2 \leq \widetilde{\lambda}_3$. The six eigenvalues are therefore interlaced,

$$\widetilde{\lambda}_1 \leq \widehat{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \widehat{\lambda}_2 \leq \widetilde{\lambda}_3 \leq \widehat{\lambda}_3. \quad (\text{A } 6)$$

Further useful results concerning the null vectors for thermoelasticity can be found in Scott (1993a).

Appendix B. The isotropic limit

In this limit we have, referring to equation (A 1), $\widetilde{\mathbf{M}} = \widetilde{\rho} \mathbf{I}$, $\mathbf{b}(\mathbf{n}) = b \mathbf{n}$, and $\widetilde{\mathbf{K}} = A \mathbf{n} \otimes \mathbf{n} + \mu(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$, with $\widetilde{\rho}$, b , κ and A all independent of $\mathbf{n}$. The roots of the dispersion relation are then given by $k_\alpha = \omega/v_\alpha$, where v_α are independent of $\mathbf{n}$ and

$$v_1^2 + v_2^2 = \frac{A}{\widetilde{\rho}} + \frac{\kappa}{m} + \frac{b^2}{\widetilde{\rho}m}, \quad v_1^2 v_2^2 = \frac{A}{\widetilde{\rho}} \frac{\kappa}{m}, \quad v_3^2 = v_4^2 = \frac{\mu}{\widetilde{\rho}}. \quad (\text{B } 1)$$

The cofactor matrices associated with $\mathbf{A}_\alpha \otimes \mathbf{A}_\alpha$ can then be easily obtained. The integrals in equation (3.22) can be evaluated using the identity

$$\frac{1}{8\pi^2|\mathbf{x}|} \oint d\theta(\mathbf{n}) + \frac{1}{8\pi^2} \int_{\Omega} ik_\alpha e^{ik_\alpha \mathbf{n} \cdot \mathbf{x}} d\Omega(\mathbf{n}) = \frac{1}{4\pi|\mathbf{x}|} e^{ik_\alpha |\mathbf{x}|}, \quad (\text{B } 2)$$

with the final result being

$$\begin{aligned} \mathbf{U}(\mathbf{x}, \omega) = & \left\{ \begin{bmatrix} v_3^{-2} \mathbf{I} + \omega^{-2} \nabla \otimes \nabla & 0 \\ 0 & 0 \end{bmatrix} \frac{e^{ik_3 |\mathbf{x}|}}{4\pi|\mathbf{x}|} \right. \\ & + \frac{1}{m(v_2^2 - v_1^2)} \sum_{\alpha=1}^2 (-1)^\alpha \begin{bmatrix} \omega^{-2}(\kappa - mv_\alpha^2) \nabla \otimes \nabla & (i\omega)^{-1} b \nabla \\ (i\omega)^{-1} b \nabla & \tilde{\rho} - Av_\alpha^{-2} \end{bmatrix} \frac{e^{ik_\alpha |\mathbf{x}|}}{4\pi|\mathbf{x}|} \left. \right\} \\ & \times \tilde{\rho}^{-1} \mathbf{F}^{(0)}. \end{aligned} \quad (\text{B } 3)$$

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