

LETTERS TO THE EDITOR

FLEXURAL EDGE WAVES

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1. INTRODUCTION

The possibility of flexural edge waves propagating parallel to a free edge of an isotropic plate was first noted by McKenna *et al.* [1], and discussed in specific detail by Thurston and McKenna [2]. The purpose of this letter is to show that such waves are always possible in orthotropic thin plates under the most general anisotropy.

2. GUIDED EDGE WAVES

We consider flexural motion in a homogeneous, anisotropic, infinite thin plate, governed by the classical theory of plate flexure [3]. The deflection $W(x, y, t)$ satisfies the equation of motion

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = \rho h \frac{\partial^2 W}{\partial t^2}, \quad (1)$$

where ρ is the density per unit volume and h is the thickness. The shear forces per unit length, S_x and S_y , are related to the moments by

$$S_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}, \quad S_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x}. \quad (2)$$

These relations are independent of the material symmetry, or anisotropy, of the plate, which, because of the 2-D nature of the problem, is orthotropic in the most general case. The constitutive relations for an orthotropic plate are [3]

$$M_x = -D_x \frac{\partial^2 W}{\partial x^2} - D_1 \frac{\partial^2 W}{\partial y^2}, \quad M_y = -D_1 \frac{\partial^2 W}{\partial x^2} - D_y \frac{\partial^2 W}{\partial y^2}, \quad (3a, b)$$

$$M_{xy} = 2D_{xy} \frac{\partial^2 W}{\partial x \partial y}. \quad (3c)$$

Substitution of these relations into equations (1) and (2) yields

$$\left(D_x \frac{\partial^4}{\partial x^4} + 2H \frac{\partial^4}{\partial x^2 \partial y^2} + D_y \frac{\partial^4}{\partial y^4} \right) W + \rho h \frac{\partial^2 W}{\partial t^2}, \quad (4)$$

where $H = D_1 + 2D_{xy}$. We note that positive definiteness of the strain energy density, $U = M_{xy} W_{,xy} - (M_x W_{,xx} + M_y W_{,yy})/2$, requires that the bending stiffnesses must satisfy the constraints

$$D_{xy} > 0, \quad D_x + D_y > 0, \quad D_x D_y - D_1^2 > 0. \quad (5)$$

Now consider a plate that is infinite in the x -direction and semi-infinite in the y -direction, with the infinite edge $y = 0$ free of tractions. Assuming that the plate lies in $y > 0$, we look for an inhomogeneous travelling wave solution of the form

$$W = (A_1 e^{-\gamma_1 k_0 y} + A_2 e^{-\gamma_2 k_0 y}) e^{i(\xi k_0 x - \omega t)}, \quad (6)$$

with $\gamma_1 > 0$ and $\gamma_2 > 0$. Here, $\omega > 0$ is the frequency and k_0 is the wavenumber for a plane wave travelling in the x -direction in an infinite plate; i.e., $k_0 = \omega^{1/2}(\rho h/D_x)^{1/4}$. Substitution of equation (6) into equation (4) implies that γ_1 and γ_2 are roots of

$$D_y(\gamma_1^2 + \gamma_2^2) = (D_1 + 2D_{xy})2\xi^2, \quad D_y\gamma_1^2\gamma_2^2 = D_x(\xi^4 - 1). \quad (7)$$

The boundary conditions require that both M_y and V_y vanish at the plate edge, where $V_y = S_y - \partial M_{xy}/\partial x$ is the Kirchhoff shear force [3]. Applying these, using equations (2), (3a) and (6), yields

$$(D_1\xi^2 - D_y\gamma_1^2)A_1 + (D_1\xi^2 - D_y\gamma_2^2)A_2 = 0, \quad (8a)$$

$$[(D_1 + 4D_{xy})\xi^2 - D_y\gamma_1^2]\gamma_1 A_1 + [(D_1 + 4D_{xy})\xi^2 - D_y\gamma_2^2]\gamma_2 A_2 = 0. \quad (8b)$$

Setting the determinant to zero, we have

$$\gamma_1(D_1\xi^2 - D_y\gamma_2^2)[(D_1 + 4D_{xy})\xi^2 - D_y\gamma_1^2] = \gamma_2(D_1\xi^2 - D_y\gamma_1^2)[(D_1 + 4D_{xy})\xi^2 - D_y\gamma_2^2]. \quad (9)$$

We then square both sides, divide by $(\gamma_1^2 - \gamma_2^2)$, and use the relations (7) to eliminate γ_1^2 and γ_2^2 : this gives a quadratic equation for ξ^4 . The phase speed of the edge wave relative to the plane wave in the infinite plate is $c = 1/\xi$, which solves the related equation

$$D_x^2 D_y^2 c^8 - 2c^4 D_x D_y (D_x D_y - D_1^2 - 8D_{xy}^2) + (D_x D_y - D_1^2)^2 - 16D_x D_y D_{xy}^2 = 0. \quad (10)$$

The roots are

$$D_x D_y c^4 = D_x D_y - D_1^2 - 8D_{xy}^2 \pm 4D_{xy}(D_1^2 + 4D_{xy}^2)^{1/2}, \quad (11)$$

where only positive roots are of interest. Taking the + sign gives

$$c^4 = 1 - \frac{(\sqrt{4D_{xy}^2 + D_1^2} - 2D_{xy})^2}{D_x D_y}. \quad (12)$$

This is the main result of this letter.

Several comments are in order. First, real-valued solutions with $0 < c \leq 1$ always exist. Thus the first inequality of equation (5) implies that $(\sqrt{4D_{xy}^2 + D_1^2} - 2D_{xy})^2 < D_1^2$ which, combined with the third of equation (5), guarantees that the right member of equation (12) is positive and bounded above by unity. The speed c is a material parameter independent of frequency, and hence the dispersion characteristics of the edge wave are similar to those for plane waves on an infinite plate; i.e., the wavenumber is proportional to the square root of the frequency [2]. It is also clear from equation (12) that the relative speed c is the same when the free edge is parallel to the y -axis, in which case the reference wavenumber is $k_0 = \omega^{1/2}(\rho h/D_y)^{1/4}$.

For an isotropic plate, $D_x = D_y = D$, $D_1 = \nu D$ and $2D_{xy} = (1 - \nu)D$, where D is the bending stiffness and ν is the Poisson ratio. The relative speed then satisfies, from equation (12),

$$c^4 = -(1 - \nu)(1 - 3\nu) + 2(1 - \nu)(1 - 2\nu + 2\nu^2)^{1/2}, \quad (13)$$

TABLE 1

The edge wave parameters for three highly anisotropic materials (data from reference [4], moduli units are GPa) and an isotropic one with $\nu = 1/3$: the root γ_1 is associated with the plane wave parallel to the edge, and γ_2 is strongly evanescent; the ratio A_2/A_1 provides the magnitude of the latter relative to the former

Material	E_1	E_2	ν_{12}	G_{12}	$1 - c$	γ_1	γ_2	A_2/A_1
Isotropic	8/3	8/3	1/3	1	1.552×10^{-3}	5.58×10^{-2}	1.415	0.1985
Glass/epoxy	54.2	18.1	0.25	8.96	8.225×10^{-5}	2.00×10^{-2}	1.568	0.1130
Boron/epoxy	208	20.8	0.30	6.95	1.054×10^{-4}	4.69×10^{-2}	1.385	0.1839
Graphite/epoxy	208	5.21	0.25	2.59	5.960×10^{-6}	1.95×10^{-2}	1.580	0.1111

in agreement with Thurston and McKenna [2]. This result is more revealing written as a Taylor series in ν ,

$$c = 1 - \frac{\nu^4}{16} (1 + 2\nu + \frac{5}{2}\nu^2 + 2\nu^3 + \dots). \quad (14)$$

This provides an excellent approximation for the range of practical interest, $0 \leq \nu < 1/2$, with the greatest relative error less than 0.2%. More importantly, it shows that the speed is very close to unity for isotropic plates. The associated decay terms are $\gamma_1 = (1 - c^2)^{1/2} \approx \nu^2/\sqrt{8}$ and $\gamma_2 = (1 + c^2)^{1/2} \approx \sqrt{2}$. The smallness of γ_1 indicates the edge wave is a *weakly inhomogeneous plane wave*, in that the decay or variation in the direction normal to the free edge occurs over many wavelengths.

This aspect is even more pronounced for anisotropic plates, as the following examples demonstrate. The in-plane mechanical properties of a single lamina of fiber reinforced material are E_1 , E_2 , G_{12} and ν_{12} , in terms of which $D_x = (h^3/12)E_1/(1 - \nu_{12}\nu_{21})$, $D_y = (\nu_{21}/\nu_{12})D_x$, $D_1 = \nu_{21}D_x$ and $D_{xy} = (h^3/12)G_{12}$ [4], where $\nu_{12}E_2 = \nu_{21}E_1$. The edge wave parameters for four materials, one of which is isotropic for comparison, are listed in Table 1.

The difference $(1 - c)$, while small for isotropic plates, is even smaller in the presence of anisotropy (by at least an order of magnitude for the examples in Table 1). Thurston and McKenna [2] noted that the edge wave exists because the transverse stresses of the plane wave are relieved at the edge, hence slowing the edge wave. Based upon these considerations, the data for $(1 - c)$ in Table 1 suggest that the transverse stresses accompanying plane bending waves on orthotropic plates are small. We note that the transverse stresses vanish in an isotropic plate when $\nu = 0$, in which case the edge wave is just the plane wave ($c = 1$). The natural analogue of the isotropic ν is the *effective* Poisson ratio for flexural waves in orthotropic plates, $(\nu_{12}\nu_{21})^{1/2}$. Substituting this for ν in equation (14), and retaining only the leading order term, gives $1 - c \approx \frac{1}{16}(\nu_{12}\nu_{21})^2 = \frac{1}{16}\nu_{12}^4(E_2/E_1)^2$, which is diminished by the factor $(E_2/E_1)^2$. Therefore, looked at in another way, the smallness of $(1 - c)$ can be understood in terms of a small effective Poisson ratio governing flexural effects.

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