Journal of Sound and Vibration (1994) **169**(4), 485–502

LOW FREQUENCY BENDING WAVES IN PERIODIC PLATES

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(*Received* 17 *January* 1992, *and in final form* 22 *July* 1992)

An asymptotic approximation is obtained for the dispersion relation of flexural waves propagating in an infinite, flat plate, with material properties periodic in one direction. The approximation assumes that the wavelength is long compared with the length of the unit period, but makes no assumption about the magnitude of the variation of material parameters. The leading order term corresponds to an effective plate with areal density equal to the mean and bending stiffnesses which could be predicted from purely static considerations. The first departure from the dispersion relation for an effectively uniform plate depends upon a parameter Ω_3, which is discussed in detail. An expression is found for Ω_3 for plates with arbitrary periodic variation in material properties. It turns out that Ω_3 vanishes for waves travelling normal to the layering if *either* the areal density or the bending stiffness is uniform throughout the plate. Numerical comparisons of the exact and asymptotic dispersion relations suggest that the cubic term in the dispersion relation is always small.

1. INTRODUCTION

An inhomogeneous periodic medium behaves at very long wavelengths like an effectively uniform medium with properties defined by an *equivalent effective medium* [1]. The concept of an effective medium is really only valid for static deformation, since waves at any finite frequency must exhibit dispersion if allowed to propagate far enough. However, at long wavelengths or, equivalently, low frequency, the effective medium picture is the natural starting point for understanding the dynamic response of a periodic structure. The description of waves in periodic structures is well understood [2–4] and is sufficiently developed that the finite frequency response of most structures of interest can be easily analyzed on the computer. The propagation at any given frequency depends upon the Bloch waves of the system and their dispersive behavior, as in any periodic material [5]. However, at low frequencies, only the fundamental Bloch waves are significant. These correspond to the modes which have initial behavior predicted by the effective medium.

In this paper we examine the propagation of low frequency flexural waves in a periodic plate, with a view towards deriving the static effective medium and the first correction beyond the static approximation. We show by a formal asymptotic expansion that the dispersion curve of a periodic plate is approximated to first order by an equivalent uniform plate. The first correction to this provides some deviation from the uniform effective plate dispersion. The results are obtained for obliquely propagating bending waves on a plate which is periodic in one direction only, and classical plate theory is used for simplicity. The findings here are similar to recent work on the low frequency dispersion of acoustic and elastic waves in layered media [6–9]. The major difference is that in the latter case the equivalent static effective medium is non-dispersive, whereas in the plate problem the

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effective medium supports dispersive bending waves, as on a uniform plate. In the acoustic case, the first correction to the effective medium dispersion equation is crucial as this is the first cause of signal dispersion of a pulse propagating over many layers [7, 9].

The paper proceeds as follows. The equations of motion for a plate with unidirectional inhomogeneity are presented in section 2. The main analysis is performed in section 3, where we use a regular perturbation procedure to obtain the first few terms in the low frequency dispersion relation of the fundamental Bloch waves. No assumption is made about the periodic nature of the plate, which may be smoothly varying or piecewise continuous. The only assumption is that the wavelength far exceeds the period length. The asymptotic expansion may be developed to any desired order but it becomes progressively more difficult to find the higher terms. We deal only with the first two non-zero terms in the dispersion relation. Some applications and simplifications of the general results are discussed in section 4. Specific formulae are provided for the two- and three-phase laminated materials, which are used later in section 6 for some numerical examples. In section 5, we briefly describe the propagator matrix procedure which we used to compute the "exact" dispersion curves of section 6. Finally, some comparisons between the exact and asymptotic predictions for the dispersive behavior of the fundamental waves are presented in section 6.

2. EQUATIONS OF MOTION

We consider flexural motion in an inhomogeneous, infinite thin plate, governed by the classical theory of plate flexure [11]. The undeflected surface lies in the x–y plane and it bends in the z-direction, the displacement W measuring the deflection of the middle plane of the plate. The equation of motion for the plate is

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = \rho h \frac{\partial^2 W}{\partial^2 t}, \tag{1}$$

where $\rho(x)$ is the density per unit volume. The shear forces per unit length, S_x and S_y, result from the shearing stresses τ_{xz} and τ_{yz} within the plate, and are related to the moments by

$$S_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}, \qquad S_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x}. \tag{2}$$

We assume that the plate is isotropic, with the usual constitutive relations

$$M_x = -D\left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\right), \qquad M_y = -D\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right), \qquad M_{xy} = D(1-\nu)\frac{\partial^2 W}{\partial x\, \partial y}. \tag{3}$$

The bending stiffness D is

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

and E and ν are the Young's modulus and the Poisson ratio, respectively.

The inhomogeneity is assumed to exist only in the x-direction, so that each of the fundamental parameters, $h(x)$, $\rho(x)$, $E(x)$ and $\nu(x)$, may depend upon x but not on y. We consider time harmonic motion of frequency ω. The basic equations (1)–(3) can be recast as a system of ordinary differential equations in x. The pertinent quantities that are continuous at each level of x are W, its derivative $\partial W/\partial x$, the moment M_x, and the effective shearing force $S_x - \partial M_{xy}/\partial y$. Justification for the choice of these parameters, especially the last, can be found in the book by Timoshenko [11]. The dependence of the y component of the wavenumber vector is expected to go as $\omega^{1/2}$. On the basis of these observations, we assume

$$W = w(x,\omega)\,e^{i\theta}, \qquad M_x = m(x,\omega)\,e^{i\theta}, \qquad S_x - \partial M_{xy}/\partial y = s(x,\omega)\,e^{i\theta}, \tag{4}$$

where

$$\theta = \omega^{1/2} q y - \omega t. \tag{5}$$

The parameter q is essentially a horizontal wavenumber, and is left as a free parameter to be specified. Define the four-vector $\mathbf{V}(x)$ as

$$\mathbf{V}(x) = \begin{Bmatrix} i\omega^{3/2} w \\ i\omega^{1/2} m \\ \omega\, dw/dx \\ s \end{Bmatrix}, \tag{6}$$

then the equations of motion for bending waves in the thin plate may be written in matrix form

$$d\mathbf{V}/dx = i\omega^{1/2}\mathbf{N}\mathbf{V}, \tag{7}$$

where

$$\mathbf{N}(x) = \begin{bmatrix} \mathbf{0} & \mathbf{H} \\ \mathbf{Q} & \mathbf{0} \end{bmatrix}, \tag{8}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ -2q^2 D(1-\nu) & 1 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} -\nu q^2 & 1/D \\ \rho h - D(1-\nu^2)q^4 & -\nu q^2 \end{bmatrix}. \tag{9}$$

This vector formalism is standard in unidirectional materials, such as layered media, and leads to many useful simplifications. For instance, the energy flux in the x-direction is given by

$$F = -\mathrm{Re}\,(-i\omega w\, e^{i\theta})\,\mathrm{Re}\,(s\, e^{i\theta}) + \mathrm{Re}\left(-i\omega \frac{dw}{dx}\, e^{i\theta}\right)\mathrm{Re}\,(m\, e^{i\theta}), \tag{10}$$

where θ is defined in equation (5). The time average flux over a period may be written succinctly as

$$F_{av} = \tfrac{1}{2}\omega^{-1/2}\mathbf{V}^{\mathrm{T}}\mathbf{J}\mathbf{V}^*, \tag{11}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

and the asterisk denotes the complex conjugate. Premultiplying equation (7) by $\mathbf{V}^{*\mathrm{T}}\mathbf{J}$, and making use of the fact that $\mathbf{N}$ is real and satisfies the relation

$$\mathbf{N}^{\mathrm{T}}\mathbf{J} = \mathbf{J}\mathbf{N}, \tag{13}$$

we deduce the identity

$$(\mathrm{d}/\mathrm{d}x)(\mathbf{V}^{\mathrm{T}}\mathbf{J}\mathbf{V}^*) = 0. \tag{14}$$

Combined with equation (11), this means that the average flux is conserved at any position on the plate.

3. THE LOW FREQUENCY DISPERSION RELATION FOR PERIODIC PLATES

3.1. ASYMPTOTIC ANALYSIS

We now consider plates which are periodic in the x-direction with period L. All material parameters, including the matrix $\mathbf{N}$, are periodic functions of x, such that $f(x + L) = f(x)$ for any parameter f. The system (7) therefore admits of Bloch wave solutions, which are analogous to free waves in a uniform medium. The associated Floquet or periodicity condition is

$$\mathbf{V}(x + L) = \mathbf{V}(x)\,\mathrm{e}^{\mathrm{i}pL}. \tag{15}$$

Solutions of equations (7) and (15) are known as Bloch waves, and the relation between the frequency ω and the x-wavenumber p, for given q, defines the dispersion relation. We will be particularly interested in the fundamental, or lowest, Bloch wave dispersion equation.

We follow the method developed by Norris [8] and Norris and Santosa [9] to derive an asymptotic approximation to the dispersion relation at low frequencies. We non-dimensionalize the problem by letting

$$x = \bar{x}L, \qquad \omega = \bar{\omega}\omega_0, \tag{16}$$

where ω_0 is a constant frequency which will be defined later. The equations of motion (7) may be rewritten as

$$\mathrm{d}\bar{\mathbf{V}}/\mathrm{d}\bar{x} = \mathrm{i}\bar{\omega}^{1/2}\bar{\mathbf{N}}\bar{\mathbf{V}}, \tag{17}$$

where

$$\bar{\mathbf{N}}(\bar{x}) = \omega_0^{1/2}L\mathbf{N}(x), \qquad \bar{\mathbf{V}}(\bar{x}) = \mathbf{V}(x). \tag{18}$$

The solution to equation (17) for $\bar{\omega} \ll 1$ may be formally expressed as a Neumann, or Peano [10], expansion

$$\bar{\mathbf{V}}(\bar{x}) = \left[\mathbf{I} + \mathrm{i}\bar{\omega}^{1/2}\int^{\bar{x}}\bar{\mathbf{N}} + (\mathrm{i}\bar{\omega}^{1/2})^2\int^{\bar{x}}\bar{\mathbf{N}}\int\bar{\mathbf{N}} + (\mathrm{i}\bar{\omega}^{1/2})^3\int^{\bar{x}}\bar{\mathbf{N}}\int\bar{\mathbf{N}}\int\bar{\mathbf{N}} + \cdots\right]\bar{\mathbf{V}}(0), \tag{19}$$

where $\mathbf{I}$ is the unit matrix and, for brevity, we adopt the notation

$$\int^{\bar{x}}\bar{\mathbf{N}} \equiv \int_0^{\bar{x}}\bar{\mathbf{N}}(x_1)\,\mathrm{d}x_1, \qquad \int^{\bar{x}}\bar{\mathbf{N}}\int\bar{\mathbf{N}} \equiv \int_0^{\bar{x}}\bar{\mathbf{N}}(x_1)\,\mathrm{d}x_1\int_0^{x_1}\bar{\mathbf{N}}(x_2)\,\mathrm{d}x_2,$$

$$\int^{\bar{x}}\bar{\mathbf{N}}\int\bar{\mathbf{N}}\int\bar{\mathbf{N}} \equiv \int_0^{\bar{x}}\bar{\mathbf{N}}(x_1)\,\mathrm{d}x_1\int_0^{x_1}\bar{\mathbf{N}}(x_2)\,\mathrm{d}x_2\int_0^{x_2}\bar{\mathbf{N}}(x_3)\,\mathrm{d}x_3.$$

Let $\bar{p} = pL$ and $\bar{x} = 0$, then the Floquet condition (15) becomes

$$\bar{\mathbf{V}}(1) = \bar{\mathbf{V}}(0)\,\mathrm{e}^{\mathrm{i}\bar{p}}. \tag{20}$$

Substituting from equation (19) into equation (20), we derive the dispersion relation in the form

$$\det\left\{\left[\mathbf{I}+\mathrm{i}\bar{\omega}^{1/2}\int^1\tilde{\mathbf{N}}+(\mathrm{i}\bar{\omega}^{1/2})^2\int^1\tilde{\mathbf{N}}\int\tilde{\mathbf{N}}+(\mathrm{i}\bar{\omega}^{1/2})^3\int^1\tilde{\mathbf{N}}\int\tilde{\mathbf{N}}\int\tilde{\mathbf{N}}+\cdots\right]\right.$$
$$\left.-\mathbf{I}[1+\mathrm{i}\bar{p}+(\mathrm{i}\bar{p})^2/2+(\mathrm{i}\bar{p})^3/6+\cdots]\right\}=0. \tag{21}$$

Although equation (21) is theoretically valid and convergent at all frequencies [10], it is not of much practical use except at low frequency, $\bar{\omega} \ll 1$. We will now develop an asymptotic expansion for the dispersion relation in this asymptotic limit, assuming also that $\bar{p} \ll 1$. Specifically, we will find solutions for which the two small parameters scale according to $\bar{\omega} = O(\bar{p}^2)$. The starting point is the Ansatz

$$\bar{\omega}^{1/2}=\bar{p}[\bar{\Omega}_1+\bar{\Omega}_2\mathrm{i}\bar{p}+\bar{\Omega}_3(\mathrm{i}\bar{p})^2+\cdots]. \tag{22}$$

The motivation for the initial term comes from the dispersion relation in a uniform plate or beam. It can be shown that the exact dispersion relation for the Bloch modes is such that ω^2 is an even function of p:

$$\omega^2(p)=\omega^2(-p). \tag{23}$$

The proof of this is very similar to the same proof for Bloch waves in periodic elastic solids; see Behrens [1] for details. One consequence of equation (23) is that the coefficient Ω_2 in equation (22) is identically zero. We will see below that this comes out of the asymptotic analysis, although in a more indirect manner. The term Ω_3 is not zero, in general, and this coefficient will be our primary objective. Higher order coefficients are too complicated to consider.

Using equation (22), equation (21) may be written as

$$\det \mathbf{S}(\mathrm{i}\bar{p})=0, \tag{24}$$

where the 4×4 matrix $\mathbf{S}$ can be expressed as a power series in $(\mathrm{i}\bar{p})$,

$$\mathbf{S}(\mathrm{i}\bar{p})=\mathbf{S}_0+\mathrm{i}\bar{p}\mathbf{S}_1+(\mathrm{i}\bar{p})^2\mathbf{S}_2+\cdots. \tag{25}$$

The coefficient matrices are increasingly complicated:

$$\mathbf{S}_0=\Omega_1\int^1\mathbf{N}-\mathbf{I}, \tag{26a}$$

$$\mathbf{S}_1=\Omega_1^2\int^1\mathbf{N}\int\mathbf{N}-\tfrac{1}{2}\mathbf{I}+\Omega_2\int^1\mathbf{N}, \tag{26b}$$

$$\mathbf{S}_2=\Omega_1^3\int^1\mathbf{N}\int\mathbf{N}\int\mathbf{N}-\tfrac{1}{6}\mathbf{I}+\Omega_3\int^1\mathbf{N}+2\Omega_1\Omega_2\int^1\mathbf{N}\int\mathbf{N}, \tag{26c}$$

where

$$\Omega_i=\bar{\Omega}_i\omega_0^{1/2}L, \qquad i=1,2,3. \tag{27}$$

Notice that in order to ensure the scaling $\bar{p}\sim\bar{\omega}^{1/2}$, we need to have $\bar{\Omega}_1=O(1)$. Expanding equation (24) as a function of $\mathrm{i}\bar{p}$, using standard identities for determinants, we obtain the dispersion relation explicitly as a power series in the asymptotic parameter $\bar{p}$,

$$|\mathbf{S}_0|+\mathrm{i}\bar{p}\,\mathrm{tr}\,\{\hat{\mathbf{S}}_0\mathbf{S}_1\}+(\mathrm{i}\bar{p})^2[\mathrm{tr}\,\{\hat{\mathbf{S}}_0\mathbf{S}_2\}+\tfrac{1}{2}\,\mathrm{tr}\,\{\hat{\mathbf{S}}_0'\mathbf{S}_1\}]+\cdots=0. \tag{28}$$

In this equation, $\hat{\mathbf{S}}_0$ is the cofactor matrix of $\mathbf{S}_0$, $\hat{\mathbf{S}}_0'$ is the derivative with respect to $\mathrm{i}\bar{p}$ of the cofactor matrix of $\mathbf{S}$ at $\mathrm{i}\bar{p}=0$ and $|\ |$ denotes determinant. Each of the coefficients of the different powers in $\bar{p}$ must vanish identically, implying identities for the coefficients in the dispersion relation (22). We will derive the first three of these, beginning with Ω_1.

3.2. THE COEFFICIENT Ω_1 AND THE EFFECTIVE PLATE

The leading order term in equation (28) implies

$$|\mathbf{S}_0| = 0, \tag{29}$$

from which we deduce that Ω_1 satisfies

$$D_x \Omega_1^{-4} + 2(D_1 + 2D_{xy}) q^2 \Omega_1^{-2} + D_y q^4 - \langle \rho h \rangle = 0, \tag{30}$$

where

$$D_x = \langle D^{-1} \rangle^{-1}, \qquad D_y = \langle D \rangle + \langle \nu \rangle^2 \langle D^{-1} \rangle^{-1} - \langle \nu^2 D \rangle, \tag{31a, b}$$

$$2D_{xy} = \langle D \rangle - \langle D\nu \rangle, \qquad D_1 = \langle \nu \rangle D_x \tag{31c, d}$$

and $\langle \cdot \rangle$ denotes the average of a quantity over the unit cell. The dispersion relation (30) corresponds to an equivalent plate which is uniform but anisotropic, with the equation of motion for the effective normal displacement given by

$$\left(D_x \frac{\partial^4}{\partial x^4} + 2(D_1 + 2D_{xy}) \frac{\partial^4}{\partial x^2 \partial y^2} + D_y \frac{\partial^4}{\partial y^4} \right) W + \langle \rho h \rangle \frac{\partial^2 W}{\partial t^2} = 0. \tag{32}$$

Another way of deriving this *static effective medium* is to simply replace the inhomogeneous matrix $\mathbf{N}$ by its average, $\mathbf{N} \to \langle \mathbf{N} \rangle$. Disentangling this effective matrix yields the following relations for the moments in the effective medium

$$M_x = -D_x \frac{\partial^2 W}{\partial x^2} - D_1 \frac{\partial^2 W}{\partial y^2}, \tag{33a}$$

$$M_y = -D_1 \frac{\partial^2 W}{\partial x^2} - D_y \frac{\partial^2 W}{\partial y^2}, \qquad M_{xy} = 2D_{xy} \frac{\partial^2 W}{\partial x \partial y}. \tag{33b, c}$$

Substitution of these relations into equations (2) for the shear forces and then into equation (1), with the replacement $\rho h \to \langle \rho h \rangle$, yields equation (32).

The constitutive relations (33) are identical to those of a uniform orthotropic plate [11], which is not surprising considering the microstructure. The same type of effective equations for static deformation of inhomogeneous plates have been derived by Kohn and Vogelius [12], Lewiński [13] and others using mathematical techniques from the theory of homogenization. These studies do not, however, yield the first corrections to the static behavior. We note that the effective areal density is simply the average over the plate, and the principal bending stiffnesses are D_x and D_y. If D is constant then $D_x \geqslant D_y$, whereas $D_y \geqslant D_x$ if ν is constant. Using the inequality

$$\langle fg \rangle \langle f/g \rangle - \langle f \rangle^2 \geqslant 0,$$

it follows that both D_x and D_y are smaller than the averaged bending stiffness $\langle D \rangle$. This is quite different from the analogous situation in acoustics or elasticity [8, 9]. For instance, a periodically layered elastic medium with each layer isotropic produces an effective medium which is transversely isotropic. The effective shear modulus associated with SH-wave propagation normal to the layers is the harmonic average of the shear moduli, analogous to the bending stiffness D_x, which is the harmonic average of D. However, the modulus associated with SH waves travelling parallel to the layers is simply the average shear modulus, whereas in the plate the analogous stiffness D_y is less than or equal to the average stiffness, with equality only if the Poisson ratio is uniformly zero throughout the plate. The difference between the present results and those for acoustic waves can, therefore, be ascribed to a Poisson effect.

In general, equation (30) implies that Ω_1 has four roots for a given q. Define

$$\Omega_1^{(\pm)} = \{(1/D_x)[(D_x\langle\rho h\rangle + q^4[-D_xD_y + (D_1 + 2D_{xy})^2])^{1/2} \pm q^2(D_1 + 2D_{xy})]\}^{-1/2}. \quad (34)$$

Two roots are always purely imaginary and of equal magnitude, $\Omega_1 = \pm i\Omega_1^{(+)}$, corresponding to equally evanescent waves in either x-direction. The other two roots, $\Omega_1 = \pm\Omega_1^{(-)}$, are real and of opposite sign if q is small. However, if the horizontal slowness exceeds the critical value defined by the horizontal stiffness, i.e., if

$$q^4 > \langle\rho h\rangle/D_y, \quad (35)$$

then all four roots are purely imaginary, implying that no wave can propagate.

In closing this subsection, we note that the previously undefined constant frequency ω_0 of equation (16) may be determined by setting $\bar{\Omega}_1 = 1$, implying that

$$\omega_0 = L^{-2}\Omega_1^2. \quad (36)$$

The present asymptotic theory is valid for frequencies which are small relative to ω_0. At this frequency, the wavelength of a flexural wave in the "effective" plate is commensurate with the length of the unit period. The asymptotic theory is therefore valid for wavelengths which are much longer than the periodic length. Finally, we note that in uniform materials, $\omega_0 = hC_L/L^2\sqrt{12}$, where $C_L = \sqrt{E/\rho(1-\nu^2)}$ is the longitudinal wave velocity in thin plates.

3.3. THE COEFFICIENT Ω_2

Although we know from equation (23) that the second coefficient in the asymptotic expansion of the low frequency dispersion relation vanishes, the explicit derivation of this identity provides some results which are essential for finding the next term Ω_3. We begin with the identity

$$\int^1 \mathbf{N}\int\mathbf{N} = \tfrac{1}{2}\left[\int^1\mathbf{N}\right]^2 + \tfrac{1}{2}\left[\int^1\mathbf{N}\int\mathbf{N} - \left(\int^1\mathbf{N}^{\mathrm{T}}\int\mathbf{N}^{\mathrm{T}}\right)^{\mathrm{T}}\right],$$

which allows us to rewrite $\mathbf{S}_1$ as

$$\mathbf{S}_1 = \tfrac{1}{2}\mathbf{S}_0^2 + (1 + \Omega_2/\Omega_1)\mathbf{S}_0 + (\Omega_2/\Omega_1)\mathbf{I} + \tfrac{1}{2}\Omega_1^2\mathbf{T}, \quad (37)$$

where

$$\mathbf{T} = \begin{bmatrix} \int^1\mathbf{H}\int\mathbf{Q} - \left(\int^1\mathbf{Q}^{\mathrm{T}}\int\mathbf{H}^{\mathrm{T}}\right)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \int^1\mathbf{Q}\int\mathbf{H} - \left(\int^1\mathbf{H}^{\mathrm{T}}\int\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}} \end{bmatrix}.$$

Then, since $\hat{\mathbf{S}}_0\mathbf{S}_0 = |\mathbf{S}_0|\mathbf{I}$, substitution of equation (37) into the second order term in equation (28) gives

$$|\mathbf{S}_0|(\tfrac{1}{2}\operatorname{tr}\{\mathbf{S}_0\} + 4(1 + \Omega_2/\Omega_1)) + (\Omega_2/\Omega_1)\operatorname{tr}\{\hat{\mathbf{S}}_0\} + \tfrac{1}{2}\Omega_1^2\operatorname{tr}\{\hat{\mathbf{S}}_0\mathbf{T}\} = 0,$$

and with the help of equation (29), we obtain explicitly

$$\Omega_2 = -\Omega_1^3\frac{\operatorname{tr}\{\hat{\mathbf{S}}_0\mathbf{T}\}}{2\operatorname{tr}\{\hat{\mathbf{S}}_0\}}. \quad (38)$$

It can be shown without much difficulty that

$$-\hat{\mathbf{S}}_0=\begin{bmatrix} \mathbf{I}+\Omega_1^2\langle\hat{\mathbf{Q}}\rangle(\langle\mathbf{H}\rangle-2\mathbf{I}) & \Omega_1(\langle\mathbf{H}\rangle-\Omega_1^2\langle\hat{\mathbf{Q}}\rangle) \\ \Omega_1\langle\mathbf{Q}\rangle(\mathbf{I}-\Omega_1^2\langle\hat{\mathbf{Q}}\rangle\langle\hat{\mathbf{H}}\rangle) & \mathbf{I}+\Omega_1^2(\langle\mathbf{H}\rangle-2\mathbf{I})\langle\hat{\mathbf{Q}}\rangle \end{bmatrix}, \tag{39}$$

where $\hat{\mathbf{Q}}$ and $\hat{\mathbf{H}}$ are the cofactor matrices of $\mathbf{Q}$ and $\mathbf{H}$ of equation (9), and

$$\begin{gathered} T_{11}=\iint Q_{11}-\int Q_{11}\int, \qquad T_{12}=\iint Q_{12}-\int Q_{12}\int, \\ T_{21}=\int H_{21}\int Q_{11}-\int Q_{11}\int H_{21}+\iint Q_{21}-\int Q_{21}\int, \\ T_{22}=\int H_{21}\int Q_{12}-\int Q_{12}\int H_{21}+\iint Q_{11}-\int Q_{11}\int, \\ T_{33}=-T_{22}, \qquad T_{34}=-T_{12}, \qquad T_{43}=-T_{21}, \qquad T_{44}=-T_{11}. \end{gathered} \tag{40}$$

Hence, the denominator and numerator in equation (38) are, respectively,

$$-\operatorname{tr}\{\hat{\mathbf{S}}_0\}=4+4q^2\Omega_1^2(D_1+2D_{xy})D_x^{-1}, \qquad \operatorname{tr}\{\hat{\mathbf{S}}_0\mathbf{T}\}=0, \tag{41}$$

which implies, as expected, that

$$\Omega_2=0. \tag{42}$$

Hence, the presence of the periodic structure does not introduce any apparent damping in the dispersion relation for the Bloch waves of the system.

3.4. THE COEFFICIENT Ω_3

Because Ω_2 is identically zero, the first dispersive effects above and beyond those associated with a uniform plate are defined by the magnitude of the coefficient Ω_3 in the dispersion relation (22). It follows from the requirement that the third order term in equation (28) must vanish, implying that

$$\Omega_3=\frac{1}{\operatorname{tr}\{\hat{\mathbf{S}}_0\langle\mathbf{N}\rangle\}}\left[-\Omega_1^3\operatorname{tr}\left\{\hat{\mathbf{S}}_0\int^1\mathbf{N}\int\mathbf{N}\int\mathbf{N}\right\}+\tfrac{1}{6}\operatorname{tr}\{\hat{\mathbf{S}}_0\}-\tfrac{1}{2}\operatorname{tr}\{\hat{\mathbf{S}}_0'\mathbf{S}_1\}\right]. \tag{43}$$

In order to simplify the evaluation of the right member, we note that it may be shown, using equations (8), (27) and (39), that

$$\operatorname{tr}\{\hat{\mathbf{S}}_0\langle\mathbf{N}\rangle\}=-4\Omega_1^{-1}[1+q^2\Omega_1^2D_x^{-1}(D_1+2D_{xy})], \tag{44}$$

$$\begin{aligned} \operatorname{tr}\left\{\hat{\mathbf{S}}_0\int^1\mathbf{N}\int\mathbf{N}\int\mathbf{N}\right\} &= -\Omega_1\operatorname{tr}\left\{(\langle\mathbf{H}\rangle-\Omega_1^2\langle\hat{\mathbf{Q}}\rangle)\int^1\mathbf{Q}\int\mathbf{H}\int\mathbf{Q}\right. \\ &\quad \left.+\langle\mathbf{Q}\rangle(\mathbf{I}-\Omega_1^2\langle\hat{\mathbf{Q}}\rangle)\int^1\mathbf{H}\int\mathbf{Q}\int\mathbf{H}\right\} \\ &= (\hat{\mathbf{S}}_0)_{13}\left(\int^1 Q_{1k}\int H_{kl}\int Q_{l1}+\int^1 Q_{2k}\int H_{kl}\int Q_{l2}\right) \\ &\quad +(\hat{\mathbf{S}}_0)_{14}\int^1 Q_{2k}\int H_{kl}\int Q_{l1}+(\hat{\mathbf{S}}_0)_{23}\int^1 Q_{1k}\int H_{kl}\int Q_{l2} \\ &\quad +(\hat{\mathbf{S}}_0)_{31}\left(\int^1 H_{1k}\int Q_{kl}\int H_{l1}+\int^1 H_{2k}\int Q_{kl}\int H_{l2}\right) \\ &\quad +(\hat{\mathbf{S}}_0)_{32}\int^1 H_{2k}\int Q_{kl}\int H_{l1}+(\hat{\mathbf{S}}_0)_{41}\int^1 H_{1k}\int Q_{kl}\int H_{l2}. \end{aligned} \tag{45}$$

The summation convention on repeated suffices is assumed in equation (45). The one remaining term in equation (43),

$$\Lambda = \tfrac{1}{2}\operatorname{tr}\{\mathbf{S}_0'\mathbf{S}_1\}, \tag{46}$$

is a little more difficult to evaluate. However, using some identities from Appendix A, it is possible to express it in the easily computed form

$$\Lambda = |\mathbf{S}_0 + \mathbf{T}| - |\mathbf{T}|. \tag{47}$$

The coefficient Ω_3 can therefore be evaluated, in general, from equations (43)–(47).

3.5. NORMAL INCIDENCE

Before discussing some of the implications of the asymptotic approximation to the dispersion relation, we first demonstrate that the coefficients Ω_1 and Ω_3 simplify considerably for waves propagating normal to the layer, for which $q = 0$. We refer to this particular case as normal incidence. The explicit form of Ω_1 follows from equation (34) as

$$\Omega_1^4 = \langle D^{-1}\rangle^{-1}\langle \rho h\rangle^{-1}. \tag{48}$$

The coefficient Ω_3 is still a little involved, even for $q = 0$. However, the identities (45) and (47) simplify to

$$\operatorname{tr}\left\{\mathbf{S}_0\int^1\mathbf{N}\int\mathbf{N}\int\mathbf{N}\right\}$$
$$= -\Omega_1\left[\int^1(1/D)\iint\rho h + \int^1\rho h\iint(1/D) + \langle 1/D\rangle\int^1\int\rho h x + \langle\rho h\rangle\int^1\int(x/D)\right], \tag{49}$$

and

$$\Lambda = -\tfrac{1}{4}\Omega_1^4\left|\int^1\int\mathbf{Q} - \int^1\mathbf{Q}x\right| = -(\Omega_1^4/2)\left[\int^1\int(1/D) - \int^1(x/D)\right]\left[\int^1\int\rho h - \int^1\rho h x\right]. \tag{50}$$

The remaining terms in equation (43) follow from equations (41) and (44), which, when combined with equations (49) and (50), yield a fairly explicit expression

$$\Omega_3 = -(\Omega_1^5/4)\left\{\int^1(1/D)\iint\rho h + \int^1\rho h\iint(1/D) + \langle 1/D\rangle\int^1\int\rho h x + \langle\rho h\rangle\int^1\int(x/D)\right.$$
$$\left. + \tfrac{1}{2}\left[\int^1\int(1/D) - \int^1(x/D)\right]\left[\int^1\int\rho h - \int^1\rho h x\right] - \tfrac{2}{3}\Omega_1^{-4}\right\}. \tag{51}$$

Note that Ω_3 vanishes identically if *either* D or ρh is constant, which does *not* imply that the medium is non-dispersive. It means that the first possible term in the low frequency expansion of the dispersion curve is zero. This contrasts with the case of acoustic waves travelling normal to a periodic set of layers, for which the first dispersive term, analogous to Ω_3, vanishes if and only if the acoustic impedance is constant [7, 9].

4. DISCUSSION AND APPLICATIONS

4.1. SPEEDS AND MODAL FREQUENCIES

The first three terms of the dispersion relation (22) may be used to define an asymptotic approximation to the dispersion relation, valid for low frequency and long wavelength.

The dimensional form of the approximation can be expressed as $\omega = \omega(p)$, or $p = p(\omega)$, where

$$\omega^{1/2} = p[\Omega_1 - p^2L^2\Omega_3 + \cdots], \qquad p = \omega^{1/2}[1/\Omega_1 + \omega L^2(\Omega_3/\Omega_1^4) + \cdots]. \tag{52}$$

Once Ω_1 and Ω_3 are obtained, the associated phase and group speeds in the direction normal to the layering, C_p and C_g, respectively, can be calculated from equation (52) using $C_p = \omega/p$ and $C_g = \mathrm{d}\omega/\mathrm{d}p$. Thus, as functions of frequency,

$$C_p = \omega^{1/2}\Omega_1[1 - \omega L^2(\Omega_3/\Omega_1^3) + \cdots], \qquad C_g = 2\omega^{1/2}\Omega_1[1 - 3\omega L^2(\Omega_3/\Omega_1^3) + \cdots], \tag{53a, b}$$

or

$$C_g = 2C_p - 4\omega^{3/2}L^2(\Omega_3/\Omega_1^2) + O(\omega^{5/2}). \tag{54}$$

The final relation is interesting because it clearly shows how these dispersion relations differ from the case of a uniform anisotropic plate. Hence, $C_g = 2C_p$ at all frequencies for normal incidence ($q = 0$) according to simple plate theory for a uniform plate, but $C_g = 2C_p + O(\omega^{3/2})$ according to equation (54), with the difference depending upon the parameter Ω_3.

The asymptotic dispersion relation can also be used to estimate the resonance frequencies of a finite length of a periodic plate or beam. Consider, for example, a beam of length L_1 in the x-direction, with given support conditions at either end. Then the modal condition is that $pL_1 = r_n$, where r_n, $n = 1, 2, 3, \ldots$, are known numbers; e.g., $r_n = n\pi$ for a simply supported beam. A first approximation to the modal frequencies uses the dispersion relation for the effective beam, implying that

$$\omega_n^{(0)} = r_n^2(\Omega_1^2/L_1^2). \tag{55}$$

A better estimate follows from equation (52) as

$$\omega_n^{(1)} = \omega_n^{(0)}[1 - 2r_n^2(\Omega_3/\Omega_1)(L^2/L_1^2)]. \tag{56}$$

This approximation is only valid if the length L_1 includes many periods, or $L \ll L_1$, so that the correction to the effective medium prediction will be relatively small.

4.2. LAMINATED MATERIALS

The general expressions that we have obtained for Ω_1 and Ω_3 are valid for arbitrary periodic variation in the material properties. If the material is piecewise constant, or *laminated*, then the integrals in these expressions may be reduced to finite sums. For simplicity, we assume that the periodic medium is composed of no more than three uniform phases over the unit cell, and labelled consecutively in the same order in which they are arranged. Let the volume fraction for phase i be n_i, such that $n_1 + n_2 + n_3 = 1$; then the spatial average of any material parameter g becomes simply $\langle g\rangle = \sum_{i=1}^3 n_i g_i$. The value of Ω_1 may then be obtained directly from equation (34). In order to obtain Ω_3, we note the identities

$$\int^1 a\int b\int a = \tfrac{1}{6}\sum_{k=1}^{3} a_k^2 b_k n_k^3 - \tfrac{1}{2}\sum_{k=1}^{3} a_k b_k n_k^2 \sum_{i=1}^{k} a_i n_i - \tfrac{1}{2}\sum_{k=1}^{3} a_k n_k \sum_{i=1}^{k} a_i b_i n_i^2 + \sum_{k=1}^{3} a_k n_k \sum_{j=1}^{k} b_j n_j \sum_{i=1}^{j} a_i n_i,$$

$$\int^1 b\iint a = \tfrac{1}{6}\sum_{i=1}^{3} a_i b_i n_i^3 + a_1 n_1 n_2 b_3 n_3 + \tfrac{1}{2}[a_1 b_3 n_1 n_3(n_1 + n_3) + a_2 b_3 n_2 n_3(n_2 + n_3) + a_1 b_2 n_1 n_2(n_1 + n_2)],$$

$$\int^1 \int ax = \tfrac{1}{3}\sum_{i=1}^{3} a_i n_i^3 + \tfrac{1}{2}\sum_{i=1}^{3} a_i n_i^2 + a_2 n_1 n_2 n_3,$$

$$\int^1 b \int a - \int^1 a \int b = \sum_{j=1}^{3}\sum_{i=1}^{j} (b_j a_i - a_j b_i) n_i n_j.$$

We restrict our attention to the case of normal incidence ($q = 0$), for which Ω_3 follows from the previous identities and equation (51) as

$$\Omega_3 = (\Omega_1^5/24)\{n_1^2 n_2^2 F_{12} G_{12} + n_2^2 n_3^2 F_{23} G_{23} + n_3^2 n_1^2 F_{31} G_{31}$$
$$- n_1 n_2 n_3 [n_1(F_{12}G_{31} + F_{31}G_{12}) + n_2(F_{12}G_{23} + F_{23}G_{12}) + n_3(F_{23}G_{31} + F_{31}G_{23})]\}, \tag{57}$$

where

$$F_{ij} = 1/D_i - 1/D_j, \qquad G_{ij} = (\rho h)_i - (\rho h)_j.$$

Thus, as mentioned previously, Ω_3 vanishes if either D or ρh are independent of x. Finally, we note that the dispersion parameter simplifies considerably for the practically significant case of a two-phase laminate ($n_3 = 0$),

$$\Omega_3 = (\Omega_1^5/24) n_1^2 n_2^2 (1/D_1 - 1/D_2)((\rho h)_1 - (\rho h)_2). \tag{58}$$

Note that the dispersive effect of Ω_3 may either slow the wave ($\Omega_3 > 0$), or make it faster ($\Omega_3 < 0$), in contrast to the acoustic case in which the first dispersive term always reduces the wave speed [9].

4.3. EVANESCENT WAVES

The dispersion relation (52) implicitly assumes that both Ω_1 and Ω_3 are real. However, it is possible to have Ω_1 purely imaginary, and since Ω_3 depends upon Ω_1, it will also be imaginary, corresponding to evanescent waves by analogy with the evanescent waves which exist in a uniform plate. However, unlike the uniform case, the magnitudes of the real and evanescent branches can differ for the inhomogeneous plate. Noting that Ω_3 is an odd function of Ω_1, it follows from equations (30) and (52) that the complete set of four asymptotic dispersion curves is described by

$$p = \pm\mu, \qquad p = \pm i\eta, \tag{59}$$

where, assuming for the moment that $q^4 < \langle \rho h \rangle / D_y$, both μ and η are real and positive,

$$\mu = \frac{\omega^{1/2}}{\Omega_1^{(-)}}[1 + \omega L^2 \Gamma_3(\Omega_1^{(-)})], \qquad \eta = \frac{\omega^{1/2}}{\Omega_1^{(+)}}[1 + \omega L^2 \Gamma_3(i\Omega_1^{(+)})], \tag{60}$$

with $\Omega_1^{(\pm)}$ defined in equation (34), and

$$\Gamma_3(\Omega_1) = \Omega_1^{-3}\Omega_3(\Omega_1). \tag{61}$$

Consider, for instance, the case of normal incidence ($q = 0$), for which $\Omega_1^{(+)} = \Omega_1^{(-)}$ but $\Gamma_3(i\Omega_1^{(+)}) = -\Gamma_3(\Omega_1^{(-)})$; see equations (51) and (61). Hence, referring to equations (59) and (60), we see that $\mu \neq \eta$, with the difference occurring in the higher order terms. Finally, we note that if $q^4 > \langle \rho h \rangle / D_y$, then, as mentioned in section 3.2, all four branches are evanescent.

4.4. COMPARISON WITH SHEAR AND ROTARY INERTIA EFFECTS

It is of some interest to compare the effects of periodic inhomogeneity with those of more sophisticated approximations to the exact dispersion equation for flexural wave propagation in a uniform thin plate. Both lead to deviations from the classical dispersion of Kirchhoff plate theory. Here, we consider the effects of shear and rotary inertia of the thin plate, which are not taken into account in the classical theory. The systematic procedure for including these effects is contained in the Mindlin theory for plates, e.g., reference [14], section 8.3.1. However, if we restrict our attention to waves travelling normal to the layering, i.e., $q = 0$, then Mindlin's theory reduces essentially to Timoshenko's theory of beam flexure [14, 15], which may be written in a form similar to equation (7),

$$\mathrm{d}\mathbf{V}/\mathrm{d}x = \mathrm{i}\omega^{1/2}[\mathbf{N}(x) + \omega\mathbf{B}(x)]\mathbf{V}(x), \tag{62}$$

where $\mathbf{V}(x)$ and $\mathbf{N}(x)$ are as before, and

$$\mathbf{B}(x) = \begin{bmatrix} 0 & 0 & 0 & 1/\kappa \\ 0 & 0 & \Theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{63}$$

Here, $\kappa = Gh/\chi$ and $\Theta = \rho I$, where $G = E/2(1+\nu)$ is the shear modulus, χ is a dimensionless factor indicative of the transfer of shear force across a section and is approximately unity, and I is the moment of inertia for the plate, $I = h^3/12$.

The dispersion equation for wave solutions of the form $\mathbf{V}(x) = \mathbf{V}(0)\,\mathrm{e}^{\mathrm{i}\omega^{1/2}\lambda x}$ follows from equation (62) as

$$\lambda^4 - \omega(\Theta/D + \rho h/\kappa)\lambda^2 - (1 - \omega^2(\Theta/\kappa))\rho h/D = 0. \tag{64}$$

The root of interest is

$$\lambda = (1/\Lambda_1)[1 + (\omega/4)\Lambda_1^2(\Theta/D + \rho h/\kappa) + O(\omega^2)], \tag{65}$$

where

$$\Lambda_1 = (D/\rho h)^{1/4}. \tag{66}$$

Comparing equation (66) with equation (52), it is clear that Λ_1 can be identified with Ω_1, and $\frac{1}{4}\Lambda_1^2(\Theta/D + \rho h/\kappa)$ with $\Omega_1^{-3}\Omega_3 L^2$. Hence, the deviations from classical theory depend upon $4\Omega_1^{-5}\Omega_3 L^2$ for the inhomogeneous plate, and $(\Theta/D + \rho h/\kappa)$ for effects due to shear and rotary inertia. Considering, for simplicity, the case of a two-phase periodic plate, then the former follows from equation (58) as

$$\tfrac{1}{6}n_1^2 n_2^2 L^2(1/D_1 - 1/D_2)((\rho h)_1 - (\rho h)_2), \tag{67}$$

while the latter is

$$(\rho h^3/12D)[1 + 2\chi/(1-\nu)]. \tag{68}$$

The maximum value of $n_1^2 n_2^2$ is 1/16, and if we simplify matters further by taking $\chi = 1 - \nu$, then the ratio of the two dispersive effects may be expressed as

$$\frac{\text{periodic effect}}{\text{shear and rotary}} = \frac{1}{24}\frac{L^2}{h^2}\frac{\Delta\rho h}{\rho h}\frac{\Delta(1/D)}{(1/D)}, \tag{69}$$

with obvious notation. In general, the term L^2/h^2 in equation (69) is necessarily large, and the remaining terms are small. The utility of equation (69) is in estimating which effect is more significant for long wavelength propagation in an inhomogeneous plate. Of course, at high frequencies both of these refinements to the classical plate theory become equally inadequate.

5. THE EXACT DISPERSION RELATION

In this section we summarize the theory for the exact dispersion relation in a piecewise uniform plate, according to the classical plate theory. No attempt is made to include effects such as those discussed in the previous section. The results derived here will be used for numerical comparisons with the asymptotic theory in the next section. The starting point is the well known result for laminated materials that

$$\mathbf{V}(x+L) = \mathbf{P}\mathbf{V}(x), \tag{70}$$

where $\mathbf{P}$ is the propagation matrix for one period,

$$\mathbf{P} = \prod_{i=1}^{n} \mathbf{p}_{n-i+1}, \tag{71}$$

$\mathbf{p}_i$ is the propagation matrix for the ith layer, given in Appendix B, and n is the number of layers in the period. The Floquet condition (15) can therefore be written as

$$\mathbf{P}\mathbf{V}(x) = \mathrm{e}^{\mathrm{i}pL}\mathbf{V}(x). \tag{72}$$

Then, using the fact that $\det \mathbf{P} = 1$, the condition that there is a non-trivial solution for $\mathbf{V}$ becomes

$$(\mathrm{e}^{\mathrm{i}pL})^4 - I_1(\mathrm{e}^{\mathrm{i}pL})^3 + I_2(\mathrm{e}^{\mathrm{i}pL})^2 - I_3\,\mathrm{e}^{\mathrm{i}pL} + 1 = 0, \tag{73}$$

with

$$I_1 = \mathrm{tr}\,(\mathbf{P}), \qquad I_2 = \tfrac{1}{2}[(\mathrm{tr}\,(\mathbf{P}))^2 - \mathrm{tr}\,(\mathbf{P}^2)], \qquad I_3 = \mathrm{tr}\,(\mathbf{P}^{-1}). \tag{74}$$

Equation (73) is the exact form of the dispersion relation, the four roots of which may be written as $\mathrm{e}^{\mathrm{i}p_{(j)}L}$, $j = 1, 2, 3, 4$, where $p_{(1)}$, $p_{(2)}$, $p_{(3)}$ and $p_{(4)}$ are complex. It follows from equation (23) that the eigenvalues must have the properties

$$p_{(2)} = -p_{(1)}, \qquad p_{(4)} = -p_{(3)}. \tag{75}$$

One immediate consequence of equation (75) is that $I_1 = I_3$, and hence we have, from equation (73), that

$$\cosh(\mathrm{i}pL) = \tfrac{1}{4}\left(I_1 \pm \sqrt{I_1^2 - 4I_2 + 8}\right), \tag{76}$$

which is consistent with equation (286) of Cremer *et al.* [2]. Here, we give an explicit expression for the propagation matrix which was not given by reference [2].

Let

$$\mathrm{i}p = \mathrm{i}\gamma + \delta, \tag{77}$$

where γ and δ are real, then equation (76) may be written as

$$\cosh(\delta L)\cos(\gamma L) + \mathrm{i}\sinh(\delta L)\sin(\gamma L) = \left(I_1 \pm \sqrt{I_1^2 - 4I_2 + 8}\right)/4. \tag{78}$$

When $(I_1 \pm \sqrt{I_1^2 - 4I_2 + 8})/4$ is real, the explicit forms of the dispersion relation can be obtained, as shown in Appendix C, which are used to check our asymptotic dispersion relations in section 6.

6. NUMERICAL TESTS AND CLOSURE

The general theory outlined in sections 3 and 4 is demonstrated here for a very simple periodic plate. Our main point in this example is to illustrate the order of magnitude of

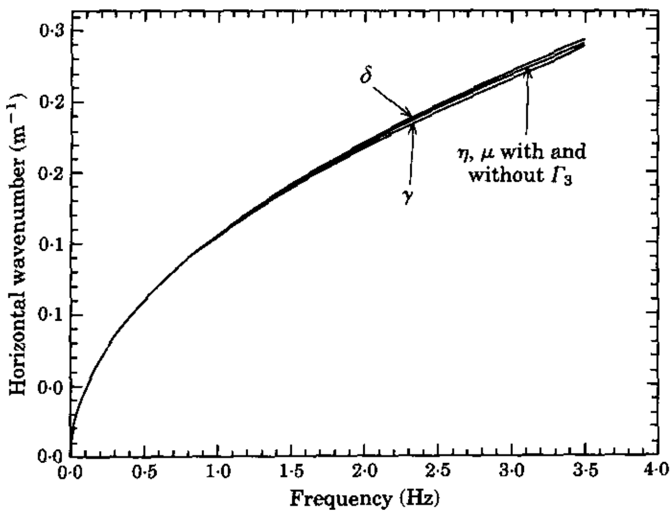

Figure 1. Dispersion curves for normally incident waves. The parameters plotted are given in equations (60) and (77). Note that the μ-curves and the η-curves, with and without Γ_3, overlap each other.

the dispersive effects described by the asymptotic theory. The plate parameters used in the following numerical tests are listed in Table 1.

The ratio defined in equation (69) is about $1{\cdot}70 \times 10^3$ for this plate and a uniform plate with the same E, ρ, ν and $h = 0{\cdot}075$ m, meaning that the effects of periodic inhomogeneity are much more significant than those of shear and rotary inertia.

In Figures 1 and 2 are shown dispersion curves for the plate, using both the exact dispersion relation and the asymptotic relation (60), with and without the Γ_3 term in the latter. The case of normal incidence ($q = 0$) is considered. It turns out that the inclusion of the correction to the static effective medium predictions has almost no impact on the asymptotic curves, μ and η, of equation (60). The deviation of the exact parameters δ and γ (see equation (77)) from one another and from the asymptotic curves is due to higher order effects. Note that the dimensionless wavenumber pL ranges from 0 to 3·0 in Figure 1, which is far beyond the limits where one would expect the asymptotic theory to hold. However, the agreement between the exact and asymptotic theory is reasonable even at $pL = 1$. These results suggest that the first correction to the static effective medium theory

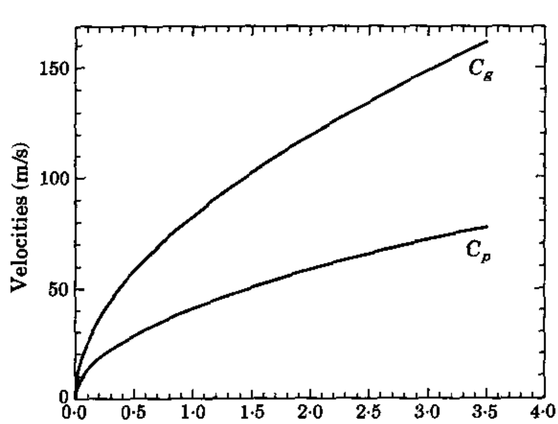

Figure 2. The phase velocity C_p and group velocity C_g, corresponding to $\Omega_1^{(+)}$ (see equations (53a, b)).

TABLE 1

Plate parameters

E (N/m^2)	ρ (kg/m^3)	ν	h_1 (m)	h_2 (m)	n_1	n_2	L(m)
$1{\cdot}98 \times 10^{11}$	$7{\cdot}85 \times 10^3$	0·30	0·05	0·10	0·25	0·50	10·00

is generally small, corresponding to the fact that Ω_3 itself is small. For this example, the pertinent dimensionless quantity Ω_3/Ω_1 is approximately $2{\cdot}70 \times 10^{-3}$. In general, this quantity will probably be smaller, and will never be of order unity. On this basis we can safely conclude that *the first correction to the static effective medium is insignificant.*

The modal frequencies can also be computed using equations (55) or (56). The first correction to the effective medium for the first few modal frequencies will be insignificant, since both L/L_1 and Ω_3/Ω_1 are small.

ACKNOWLEDGMENT

This work was supported by the United States Office of Naval Research.

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APPENDIX A: DERIVATION OF EQUATION (47)

We begin with an identity for the determinant of the sum of two matrices, which is very useful in the following developments:

$$|\mathbf{C} + z\,\mathbf{D}| = |\mathbf{C}| + z \operatorname{tr}\{\hat{\mathbf{C}}\mathbf{D}\} + (z^2/2) \operatorname{tr}\{\hat{\mathbf{F}}_0'\mathbf{D}\} + \cdots + (z^{d-1}/2) \operatorname{tr}\{\mathbf{C}\hat{\mathbf{D}}\} + z^d|\mathbf{D}|. \quad \text{(A1)}$$

Here, $\mathbf{C}$ and $\mathbf{D}$ are matrices of order d, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ are the cofactor matrices of $\mathbf{C}$ and $\mathbf{D}$, respectively, and $\hat{\mathbf{F}}_0'$ is the derivative with respect to z of the cofactor of $\mathbf{F} = \mathbf{C} + z\mathbf{D}$ at $z = 0$. According to the definition of the cofactor, we have

$$\hat{S}'_{0im} = \frac{1}{(d-2)!} e_{ij\cdots kl} e_{mn\cdots pq} S_{1lq} S_{0jn} \cdots S_{0kp}, \tag{A2}$$

where $e_{ij\cdots kl}$ is the permutation tensor of order d on the integers $1, 2, \ldots, d$, e.g., $e_{12\cdots d} = 1$, etc., and the summation convention is assumed in (A2) and the following. It follows from the definition in equation (46), that

$$\Lambda = \frac{1}{2(d-2)!} S_{1im} S_{1lq} e_{ij\cdots kl} e_{mn\cdots pq} S_{0jn} \cdots S_{0kp}. \tag{A3}$$

Substituting for $\mathbf{S}_1$ from equation (37), we find

$$\begin{aligned}\Lambda &= \tfrac{1}{8}\operatorname{tr}(\hat{\mathbf{A}}_0'\mathbf{S}_0^2) + ((d-1)/2)\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{S}_0) + ((d-1)/2)\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{S}_0^2) + (d-1)\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{T}) \\ &\quad + \operatorname{tr}(\hat{\mathbf{A}}_0'\mathbf{T}) + (1/2(d-2)!) T_{im} T_{lq} e_{ij\cdots kl} e_{mn\cdots pq} S_{0jn} \cdots S_{0kp},\end{aligned} \tag{A4}$$

where

$$\mathbf{A}(z) = \mathbf{S}_0 + z\mathbf{S}_0^2. \tag{A5}$$

Applying equations (A1)–(A5), we find that the first term in equation (A4) is the coefficient of z^2 in the expansion of the determinant of $\mathbf{A}$, which vanishes on account of equation (29). The second and third terms also vanish, since $\hat{S}_0 S_0 = |S_0|\mathbf{I}$. Finally, we are left with

$$\Lambda = (d-1)\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{T}) + \operatorname{tr}(\hat{\mathbf{A}}_0'\mathbf{T}) + \frac{1}{2(d-2)!} T_{im} T_{lq} e_{ij\cdots kl} e_{mn\cdots pq} S_{0jn} \cdots S_{0kp}. \tag{A6}$$

Now let

$$\mathbf{B}(z) = \mathbf{S}_0 + z\mathbf{T}. \tag{A7}$$

Again making use of equation (A1), we obtain

$$|\mathbf{S}_0 + z\mathbf{T}| = |\mathbf{S}_0| + z\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{T}) + (z^2/2)\operatorname{tr}\{\hat{\mathbf{B}}_0'\mathbf{T}\} + (z^3/2)\operatorname{tr}\{\mathbf{S}_0\hat{\mathbf{T}}\} + z^4|\mathbf{T}|. \tag{A8}$$

The first term vanishes on account of equation (29). It is easy to show that

$$\hat{B}'_{0im} = \frac{1}{(d-2)!} e_{ij\cdots kl} e_{mn\cdots pq} T_{lq} S_{0jn} \cdots S_{0kp}, \tag{A9}$$

and therefore

$$\Lambda = 3\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{T}) + \operatorname{tr}(\hat{\mathbf{A}}_0'\mathbf{T}) + \tfrac{1}{2}\operatorname{tr}\{\hat{\mathbf{B}}_0'\mathbf{T}\}. \tag{A10}$$

Combining equations (A8) and (A10), we have

$$\Lambda = |\mathbf{S}_0 + z\mathbf{T}| + (3 - z)\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{T}) + \operatorname{tr}(\hat{\mathbf{A}}_0'\mathbf{T}) - (z^3/2)\operatorname{tr}(\mathbf{S}_0\hat{\mathbf{T}}) - z^4|\mathbf{T}|. \tag{A11}$$

The term $\operatorname{tr}(\hat{\mathbf{S}}_0\mathbf{T})$ vanishes, using equation (41), and it may be proved without much difficulty that

$$\operatorname{tr}(\hat{\mathbf{A}}_0'\mathbf{T}) = \tfrac{1}{2}\operatorname{tr}(\mathbf{S}_0\hat{\mathbf{T}}) = 0. \tag{A12}$$

Hence, we deduce the following general expression for Λ for arbitrary values of z:

$$\Lambda = |\mathbf{S}_0 + z\mathbf{T}| - z^4|\mathbf{T}|. \tag{A13}$$

Putting $z = 1$ gives the identity (47).

APPENDIX B: THE PROPAGATOR MATRIX

The propagator matrix for the ith layer follows from equation (1) and the fact that $\mathbf{V}$ is continuous:

$$\mathbf{p}_i = \mathbf{U}_i\mathbf{R}\mathbf{O}_i\mathbf{S}_i(x_i - x_{i-1})\mathbf{O}_i^{-1}\mathbf{R}^{-1}\mathbf{U}_i^{-1}, \tag{B1}$$

where

$$\mathbf{U}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ (Dv)_i q^2 & 0 & (D\alpha^2)_i & 0 \\ 0 & -\mathrm{i}\alpha_i & 0 & 0 \\ 0 & -\mathrm{i}(D(2-v)\alpha^2)_i q^2 & 0 & -\mathrm{i}(D\alpha^3)_i \end{bmatrix}, \tag{B2}$$

$$\mathbf{U}_i^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}/\alpha_i & 0 \\ -(v/\alpha^2)_i q^2 & 1/(D\alpha^2)_i & 0 & 0 \\ 0 & 0 & -\mathrm{i}(2-v_i)/\alpha_i^3 q^2 & \mathrm{i}/(D\alpha^3)_i \end{bmatrix}, \tag{B3}$$

$$\mathbf{O}_i = \begin{bmatrix} \mathbf{I} & \mathbf{E}_i \\ \mathbf{0} & \mathbf{F}_i \end{bmatrix}, \qquad \mathbf{O}_i^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{E}_i\mathbf{F}_i^{-1} \\ \mathbf{0} & \mathbf{F}_i^{-1} \end{bmatrix}, \tag{B4}$$

$$\mathbf{F}_i = \frac{1+\beta_i^2}{4}\begin{bmatrix} 1+\beta_i & 1-\beta_i \\ 1-\beta_i & 1+\beta_i \end{bmatrix}, \qquad \mathbf{F}_i^{-1} = \frac{1}{\beta_i(1+\beta_i^2)}\begin{bmatrix} 1+\beta_i & -1+\beta_i \\ -1+\beta_i & 1+\beta_i \end{bmatrix}, \tag{B5}$$

$$\mathbf{E}_i = \frac{1-\beta_i^2}{4}\begin{bmatrix} 1-\mathrm{i}\beta_i & 1+\mathrm{i}\beta_i \\ 1+\mathrm{i}\beta_i & 1-\mathrm{i}\beta_i \end{bmatrix}, \qquad \mathbf{E}_i\mathbf{F}_i^{-1} = \frac{1-\beta_i}{2}\begin{bmatrix} 1-\mathrm{i} & 1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{bmatrix}, \tag{B6}$$

$$\alpha_i = \left[\left(\frac{\rho_i h_i}{D_i}\right)^{1/2} - q^2\right]^{1/2}, \qquad \beta_i = \left[\frac{(\rho_i h_i/D_i)^{1/2} + q^2}{(\rho_i h_i/D_i)^{1/2} - q^2}\right]^{1/2}, \tag{B7}$$

$$\mathbf{S}_i(x) = \mathrm{diag}\,(\mathrm{e}^{\mathrm{i}\omega^{1/2}\alpha_i x}, \mathrm{e}^{-\mathrm{i}\omega^{1/2}\alpha_i x}, \mathrm{e}^{\omega^{1/2}\alpha_i \beta_i x}, \mathrm{e}^{-\omega^{1/2}\alpha_i \beta_i x}), \tag{B8}$$

and

$$2\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mathrm{i} & -\mathrm{i} & 1 & -1 \\ 1 & 1 & -1 & -1 \\ \mathrm{i} & -\mathrm{i} & -1 & 1 \end{bmatrix}, \qquad \mathbf{R}^{-1} = (\mathbf{R}^*)^{\mathrm{T}}, \tag{B9}$$

where the asterisk denotes the complex conjugate. These results are motivated by the fact that the displacement in the ith uniform layer has the form

$$W = (a_1\,\mathrm{e}^{\mathrm{i}\omega^{1/2}\alpha_i x} + a_2\,\mathrm{e}^{-\mathrm{i}\omega^{1/2}\alpha_i x} + a_3\,\mathrm{e}^{\omega^{1/2}\alpha_i \beta_i x} + a_4\,\mathrm{e}^{-\omega^{1/2}\alpha_i \beta_i x})\,\mathrm{e}^{\mathrm{i}(\omega^{1/2}qy - \omega t)}, \tag{B10}$$

where a_j, $j = 1, 2, 3, 4$, are constants.

We note that the propagator matrix in equation (46) of Cremer *et al.* [2] is for the special case of normal incidence and for the four-vector

$$(\partial W/\partial t,\ \partial^2 W/\partial x \partial t,\ M_x,\ S_x)^{\mathrm{T}}, \tag{B11}$$

rather than the vector $\mathbf{V}$ of equation (6). Therefore, the propagator of Cremer *et al.* [2] is equivalent to $\mathbf{C}\mathbf{p}_i\mathbf{C}^{-1}$, with $q = 0$ in $\mathbf{p}_i$ and

$$\mathbf{C}_i = \begin{bmatrix} -\omega^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & -\mathrm{i}\omega^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B12}$$

APPENDIX C: EXPLICIT DISPERSION RELATIONS

It may be shown, using equation (59), that

$$B^{(\pm)} \equiv \tfrac{1}{4}(I_1 \pm \sqrt{I_1^2 - 4I_2 + 8}) \tag{C1}$$

are always real for low frequencies. Let us assume that this is true for all frequencies. Actually, this assumption can be checked once I_1 and I_2 are obtained. If this is the case, then it may be shown from equation (C1) that

$$|B^-| < 1, \quad |B^+| > 1, \qquad \text{if } |I_1^2 - 2I_2 - 4| < \sqrt{I_1^2 - 4I_2 + 8}, \tag{C2}$$

and

$$|B^-| < 1, \quad |B^+| < 1, \qquad \text{if } I_1^2 - 2I_2 - 4 < -\sqrt{I_1^2 - 4I_2 + 8}, \tag{C3}$$

while

$$|B^-| > 1, \quad |B^+| > 1, \qquad \text{if } I_1^2 - 2I_2 - 4 > \sqrt{I_1^2 - 4I_2 + 8}. \tag{C4}$$

In summary, when equation (C2) is valid, the roots are

$$\gamma L = \cos^{-1}[(I_1 - \sqrt{I_1^2 - 4I_2 + 8})/4], \qquad \delta = 0,$$

and

$$\delta L = \cosh^{-1}[(I_1 + \sqrt{I_1^2 - 4I_2 + 8})/4], \qquad \gamma L = n\pi, \tag{C5}$$

where $n = 0, 2, 4, \ldots$ if $B^{(+)} > 0$, and $n = 1, 3, 5, \ldots$ if $B^{(+)} < 0$. When equation (C3) is valid, the roots are

$$\gamma L = \cos^{-1}[(I_1 \pm \sqrt{I_1^2 - 4I_2 + 8})/4], \qquad \delta = 0. \tag{C6}$$

When equation (C4) holds, we have

$$\delta L = \cosh^{-1}[(I_1 \pm \sqrt{I_1^2 - 4I_2 + 8})/4], \qquad \gamma L = n\pi, \tag{C7}$$

where $n = 0, 2, 4, \ldots$ if $B^{(\pm)} > 0$, and $n = 1, 3, 5, \ldots$ if $B^{(\pm)} < 0$.