

Acoustoelasticity of solid/fluid composite systems

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SUMMARY

A heterogeneous medium composed of inviscid fluid and solid constituents is pre-stressed, resulting in relative slip of material particles at the interfaces between the solid and the fluid. The standard theory of acoustoelasticity, which is concerned with small deformation superimposed upon large initial strain, is generalized here to include the effects of the interfacial slip. Difficulties arise from the possibility that the traction, viewed as a function of either the undeformed material (Lagrangian) coordinates or of the intermediate coordinates, is not necessarily continuous across the interface. It is shown that the problem is most easily considered in the intermediate coordinates, leading to a divergence formulation of the equations of small motion from which the interface conditions arise naturally. The theory is demonstrated for the problem of a fluid-filled borehole with a pressurized fluid and pre-strained solid. An explicit expression is found for the change in the speed of the tube wave, which is the quasi-static limit of the Stoneley wave mode.

Key words: acoustoelasticity, interface, pre-stress, tube wave.

1 INTRODUCTION

Pre-stress is known to affect the speed of small-amplitude waves in crystals by effectively altering the linear moduli governing small-amplitude motion (Toupin & Bernstein 1961; Thurston 1965; Sinha 1982). The description of this non-linear process is appropriately known as ‘small-on-large’ theory or acoustoelasticity. In this paper we consider the effects of initial deformation on waves in a heterogeneous body comprising fluid and solid constituents. Under some applied static deformation particles can slip relative to one another, with the result that material particles originally side-by-side become distant relatives, and possibly strangers. The interfaces between the distinct fluid and solid regions therefore represent possible material slip surfaces, i.e. surfaces across which the material deformation is not necessarily continuous. In order to fix ideas, imagine a fluid-filled borehole (a cylindrical hole in an elastic solid of infinite extent) in which the fluid is pressurized, resulting in initial pre-stress and pre-strain in both the fluid and solid. The acoustoelastic problem is to find the change in speed of guided waves. We deal with this example later.

Acoustoelasticity of fluid/solid systems is complicated by both the possibility of slip at interfaces and the spatially inhomogeneous nature of the pre-stress/pre-strain. The latter introduces difficulties that are surmountable using the ‘standard’ theory of acoustoelasticity, but the issue of slip introduces fundamental novelties. It is therefore important

to distinguish several different, but equally valid, descriptions of the same system. In the reference configuration each position vector is associated with a material particle and the mapping from the material particles to the vector basis is continuous. Hence no slip interfaces are present in the undeformed medium because, by assumption, there is a continuous bijective mapping of every material particle to some mathematical vector space. The associated coordinates in a continuous frame will be called the reference or Lagrangian coordinates. This state of contiguity may be disrupted by an arbitrarily small deformation. For instance, an inviscid fluid in contact with a solid may slip locally relative to the solid. The existence of slip interfaces means that tractions measured in the reference frame are not necessarily continuous across such interfaces. A simple thought experiment may help in clarifying this point. Again, consider the generic situation of a fluid/solid interface, the fluid being inviscid. A static deformation is applied, resulting in slip of the fluid relative to the solid. The associated Piola–Kirchhoff (PK) stress is defined as the force per unit area acting over an elemental area defined in the reference configuration. However, the material particles are no longer adjacent across the interface, and the PK tractions are not necessarily continuous functions of the Lagrangian coordinates. Consider an extreme example in which a fluid particle becomes far removed from its original solid neighbour. The pressure at the new site of the fluid particle could be quite different from the state of affairs in

the current fluid element adjacent to its original solid particle.

The normal treatment of finite deformation of elastic solids is to consider everything in the reference frame, which is a reasonable procedure as long as the particles remain contiguous. However, once interfaces are present it is not clear that the same Lagrangian formalism is optimal, because, as pointed out above, the PK traction can be discontinuous. The associated Cauchy stress, defined in terms of current elemental area, leads to a continuous traction across any interface, bonded or otherwise. One could quite reasonably argue that the problem should be attacked in current or Eulerian coordinates completely, thereby avoiding the issue of dealing with discontinuous tractions. However, acoustoelastic problems present a fundamental difficulty when dealt with in current coordinates, namely, the positions are *a priori* unknown. Here we will describe the dynamics in terms of three alternative configurations, the reference, the intermediate and the current. The formulations are equally valid, in principle, but we will demonstrate that the intermediate coordinates are preferable for practical use.

In this paper we outline a general procedure for attacking acoustoelastic problems for fluid/solid systems. The terminology and notation of small-on-large theory is introduced in Section 2. Section 3 deals with continuity conditions across slip interfaces, with explicit formulae derived for the intermediate and reference configurations. Similar conditions have been obtained previously by Grinfel'd & Movchan (1979) for the intermediate coordinates, although the present derivation is, we believe, more physically intuitive. The interface conditions are used in Section 4, where a natural, or divergence, formulation of the acoustoelastic theory is presented. The constitutive relations appropriate to the acoustoelastic equations are described in Section 5, with particular emphasis on the moduli for an inviscid fluid. The divergence formulation of Section 4, combined with the linearized constitutive relations of Section 5, permits the standard artillery of perturbation analysis to be applied, with the details given in Section 6. Finally, a practical application of the whole theory is given in Section 7.

2 SMALL-ON-LARGE THEORY

2.1 Definitions and notation

Three configurations are distinguished: the reference, intermediate and current, denoted by $\mathcal{R}_0$, $\mathcal{R}$ and $\mathcal{R}$, respectively. The material of interest occupies the respective regions (volumes) V_0 , $\tilde{V}$ and V , each of which is a simply-connected subset of R^3 . In the undeformed or *reference* condition, the material is unstressed with position vector $\mathbf{X}$ denoting a position in V_0 , which is a continuous function of the material particles, even across fluid–solid interfaces. The

initial or *intermediate* configuration, $\mathcal{R}$, is obtained by the static deformation of each point according to $\mathbf{X} \rightarrow \xi(\mathbf{X}) \in \tilde{V}$. This mapping is not necessarily continuous, and gives rise to the possibility of interfaces across which material particles slip relative to one another. A slip interface is defined by a surface in the reference configuration, $\mathcal{L}_0: f(\mathbf{X}) = 0$, such that $\mathcal{L}_0$ forms a bounding surface between the disjoint sets of points on either side, which we label V_{01} and V_{02} for convenience. The unit normals to $\mathcal{L}_0$ are $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ directed out of V_{01} and V_{02} , respectively, with $\mathbf{N}^{(1)} + \mathbf{N}^{(2)} = \mathbf{0}$. The slip condition means that

$$\lim_{\epsilon \downarrow 0} \xi[\mathbf{X} - \epsilon \mathbf{N}^{(1)}] \neq \lim_{\epsilon \uparrow 0} \xi[\mathbf{X} - \epsilon \mathbf{N}^{(2)}], \quad \mathbf{X} \in \mathcal{L}_0, \quad (1)$$

with strict inequality at, possibly, all points on $\mathcal{L}_0$. The points on either side of $\mathcal{L}_0$ define material surfaces for the regions V_{01} and V_{02} in the reference configuration. We will assume that the static deformation maintains these material surfaces as bounding surfaces, i.e. that the limits in eq. (1), though distinct, are both elements of the same deformed interface $\tilde{\mathcal{L}}$. Thus, the static deformation forms *two* distinct bijective mappings between $\mathcal{L}_0(\mathbf{X})$ and $\tilde{\mathcal{L}}(\xi)$, defined by the unequal members in eq. (1). Our main interest here is in cases where the slip interface separates an inviscid compressible fluid from an elastic solid. We may then identify V_{01} and V_{02} with V_{0f} and V_{0s} , the fluid and solid regions, respectively. At the same time, we define $\mathcal{L}_{0f}$ and $\mathcal{L}_{0s}$ as the fluid and solid surfaces that coincide with $\mathcal{L}_0$. We define $\tilde{\mathcal{L}}_f$ and $\tilde{\mathcal{L}}_s$ by analogy, and note the identities $\mathcal{L}_0 = \mathcal{L}_{0f} \cup \mathcal{L}_{0s}$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_f \cup \tilde{\mathcal{L}}_s$. An important consequence of the slip is that $\tilde{\mathcal{L}}_f$ does not coincide with $\tilde{\mathcal{L}}_s$.

We will consider later the example of a pressurized borehole in which a column of fluid surrounded by solid may be compressed in a piston-like manner in the static deformation. In this case, $\mathcal{L}_{0f}$ and $\mathcal{L}_{0s}$ are defined by the cylindrical fluid/solid interface. The solid undergoes no strain in the axial direction under the static deformation, implying that the axial extents of $\tilde{\mathcal{L}}_s$ and $\mathcal{L}_{0s}$ are the same. However, the axial length of $\tilde{\mathcal{L}}_f$ is diminished compared with $\mathcal{L}_{0f}$ because of the smaller fluid surface meeting the solid after compression, so that $\tilde{\mathcal{L}}_f \subset \tilde{\mathcal{L}}_s$ in this particular case. Obviously, $\tilde{\mathcal{L}}_s \subset \tilde{\mathcal{L}}_f$ would hold if the fluid were to undergo an expansion (negative compression).

The current state of the medium results from additional small acoustic disturbances in which material particles deform according to the motion $\xi \rightarrow \mathbf{x} \in V$, where $\mathbf{x}$ denotes the current position of the material particle described by $\mathbf{X}$ in $\mathcal{R}_0$. We distinguish the static and dynamic displacement fields, $\mathbf{w} = \mathbf{w}(\mathbf{X})$ and $\mathbf{u}(\mathbf{X}, t)$, defined respectively by

$$\mathbf{w} = \xi - \mathbf{X}, \quad \mathbf{u} = \mathbf{x} - \xi.$$

The location of the slip interface in the current configuration is the set of points in R^3 defined by $\mathcal{L}$, which is the mapped version of $\tilde{\mathcal{L}}$. The different notations for the three configurations are summarized in Table 1. The surface S_0

Table 1. Parameters in the three descriptions.

Description	Position	Volume/Surface	Density	Disps./Stresses
Reference/Lagrangian $\mathcal{R}_0$	$\mathbf{X}$	$V_0, S_0, \mathbf{N}, \mathcal{L}_0$	ρ_0	$u_j, T_{Mj}, P_{Mj}^1, P_{Mj}$
Intermediate/initial $\mathcal{R}$	ξ	$\tilde{V}, \tilde{S}, \tilde{\mathbf{N}}, \tilde{\mathcal{L}}$	$\tilde{\rho}$	$\tilde{u}_j, \tilde{T}_{\alpha j}, \tilde{P}_{\alpha j}^1, \tilde{P}_{\alpha j}$
Current/Eulerian $\mathcal{R}$	$\mathbf{x}$	$V, S, \mathbf{n}, \mathcal{L}$	ρ	$u_j, \tau_{ij}, \pi_{ij}^1, \pi_{ij}$

denotes an arbitrary surface element in V_0 , which is transformed into $\bar{S}$ and S in the other configurations.

Different types of subscripts are used to distinguish the three states. Upper case Latin, lower case Greek and lower case Latin refer to the reference, intermediate and current states, respectively. Thus, for example, the components X_L , ξ_α and x_i are unambiguous, although we can equally well write ξ_i with no confusion. The summation convention on repeated subscripts is implied. Partial derivatives are defined by subscripts preceded by a comma and are specific: thus $f_{,A} = \partial f / \partial X_A$, $f_{,\alpha} = \partial f / \partial \xi_\alpha$ and $f_{,i} = \partial f / \partial x_i$. A note concerning functional dependence: vectors describing the dynamic displacements and stresses are always indicated as functions of points in R^3 . Thus $u_i(\mathbf{X})$ means the value of the displacement at the physical spot $\mathbf{X}$ in R^3 defined by the vector $\mathbf{X} \in \mathcal{R}_0$. Quantities with tildes refer to variables expressed in the intermediate coordinates. Thus $\tilde{u}_i(\xi)$ and $u_i(\mathbf{X})$ mean the same thing. In general, we will use the notation $\tilde{f}(\xi) \equiv f(\mathbf{X})$. The notation in current coordinates is a source of possible confusion: thus we use $\mathbf{u}(\mathbf{x})$ to indicate the displacement as a function of $\mathbf{x}$, while the same symbol is used to denote for the displacement as a function of $\mathbf{X}$. The reader can easily discern the meaning by context.

The basic measure of stress is the total Piola–Kirchhoff stress tensor $\mathbf{T}$, sometimes known as the nominal stress (Chadwick 1976). The transpose $\mathbf{T}^T$ is called the Piola–Kirchhoff stress of the first kind. However, we will refer to $\mathbf{T}$ here as simply the Piola–Kirchhoff or PK stress. The total stress can be expressed in the intermediate and current coordinates as $\tilde{\mathbf{T}}$ and $\boldsymbol{\tau}$, respectively, with

$$\tilde{T}_{\alpha j} = (\bar{\rho}/\rho_0)\xi_{\alpha,L}T_{Lj}, \quad \tau_{ij} = (\rho/\rho_0)x_{i,L}T_{Lj}, \quad (2)$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}^T$ is the Cauchy stress tensor. By definition, $\rho_0 dV_0 = \bar{\rho} d\bar{V} = \rho dV$, and hence the densities are related by $\rho_0 = \bar{\rho} \det(\xi_{\alpha,L}) = \rho \det(x_{i,L})$. The point forms of the equations of motion in the three descriptions are

$$\begin{aligned} T_{Lj,L}(\mathbf{X}) &= \rho_0 u_{j,m} & \mathbf{X} \in V_0, \\ \tilde{T}_{\alpha j,\alpha}(\xi) &= \bar{\rho} \tilde{u}_{j,m} & \xi \in \bar{V}, \\ \tau_{ij,i}(\mathbf{x}) &= \rho u_{j,m} & \mathbf{x} \in V. \end{aligned} \quad (3)$$

The tractions at any material point can be expressed using either stress tensor. The force acting across the material element of area dS_0 , with unit normal $\mathbf{N}$, is the same as the force on the current element of area dS , with unit normal $\mathbf{n}$, where the current and reference directions and areas are related by

$$dS/dS_0 = (\det \mathbf{F}) |\mathbf{N} \cdot \mathbf{F}^{-1}|, \quad \mathbf{n} = |\mathbf{N} \cdot \mathbf{F}^{-1}|^{-1} \mathbf{N} \cdot \mathbf{F}^{-1}. \quad (4)$$

Here $\mathbf{F}$ is the deformation gradient, with components $F_{iK} = x_{i,K}$, and $\mathbf{N} \cdot \mathbf{F}^{-1}$ denotes the action of the transpose of $\mathbf{F}^{-1}$ on $\mathbf{N}$. Relations similar to eqs (4) exist for the intermediate parameters $d\bar{S}$ and $\bar{\mathbf{N}}$. Thus,

$$\mathbf{n} \cdot \boldsymbol{\tau} dS = \bar{\mathbf{N}} \cdot \tilde{\mathbf{T}} d\bar{S} = \mathbf{N} \cdot \mathbf{T} dS_0, \quad (5)$$

where S_0 , $\bar{S}$ and S refer to the same element of material surface. The traction vector $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\tau}$ is continuous across any interfaces in the medium, viewed as a function of current coordinates. However, the traction is *not* necessarily a continuous function of the reference coordinates.

2.2 Acoustoelasticity assumptions

The ‘small-on-large’ theory is based on the assumptions that $|\mathbf{u}| \ll |\mathbf{w}|$,

and that the associated deformation gradients are similarly related, $|\partial \mathbf{u} / \partial \mathbf{X}| \ll |\partial \mathbf{w} / \partial \mathbf{X}|$. The standard procedure in small-on-large theory is essentially to perform a doubly asymptotic expansion where we are interested in effects of orders $O(w)$, $O(u)$ and, most importantly, $O(wu)$. It is the latter which enables us to determine the first derivative of physical parameters, such as modal frequencies, as a function of the pre-strain. The small-on-large stresses are defined by

$$T_{Mj}(\mathbf{X}) = P_{Mj}^1(\mathbf{X}) + P_{Mj}(\mathbf{X}) + \dots, \quad (6)$$

$$\tilde{T}_{\alpha j}(\xi) = \bar{P}_{\alpha j}^1(\xi) + \bar{P}_{\alpha j}(\xi) + \dots, \quad (7)$$

$$\tau_{ij}(\mathbf{x}) = \pi_{ij}^1(\mathbf{x}) + \pi_{ij}(\mathbf{x}) + \dots,$$

where $\mathbf{P}^1$ is the pre-stress in Lagrangian coordinates, and $\bar{P}_{\alpha j}^1$ and π_{ij}^1 are related to it by the standard transformations (2). The extra stresses P_{Mj} , $\bar{P}_{\alpha j}$ and π_{ij} are also related by eqs (2) and, by assumption, they are linear in the small strains. The pre-stress is in equilibrium and therefore satisfies the equilibrium equations in any of the three coordinate systems:

$$P_{Mj,M}^1 = \bar{P}_{\alpha j,\alpha}^1 = \pi_{ij,i}^1 = 0. \quad (8)$$

3 INTERFACE CONDITIONS

Consider an element of surface, with area dS_0 , $d\bar{S}$, dS in the configurations $\mathcal{R}_0$, $\bar{\mathcal{R}}$, $\mathcal{R}$, respectively. The associated outward normals are defined by the unit vectors with components N_L , $\bar{N}_\alpha$, n_i (see Table 1). We note that

$$d\bar{S}/dS = 1 - E_\perp + \dots, \quad dS_0/dS = 1 - E_\perp^1 - E_\perp + \dots, \quad (9)$$

where

$$E_\perp^1 = E_{LL}^1 - N_K N_L E_{KL}^1, \quad E_\perp = u_{k,k} - \bar{N}_k \bar{N}_l u_{l,k}, \quad (10)$$

and

$$E_{KL}^1 = \frac{1}{2}(w_{K,L} + w_{L,K}).$$

The total traction per unit current area can now be expressed in the alternative descriptions by combining eqs (5)–(7) and (9) as

$$\begin{aligned} \mathbf{n} \cdot (\boldsymbol{\pi}^1 + \boldsymbol{\pi}) &= (1 - E_\perp) \bar{\mathbf{N}} \cdot (\bar{\mathbf{P}}^1 + \bar{\mathbf{P}}) \\ &= (1 - E_\perp^1 - E_\perp) \mathbf{N} \cdot (\mathbf{P}^1 + \mathbf{P}). \end{aligned} \quad (11)$$

The traction expressions (11) are well known and their application to small-on-large motion is straightforward (see Baumhauer & Tiersten (1973) for a thorough discussion). However, it is usually assumed that the surfaces remain bonded, and hence surface elements transform in the same manner on either side of a material interface. This assumption is not valid here. There is no slippage in bonded solid materials, and therefore the area changes $d\bar{S}/dS$ and dS_0/dS in these identities are continuous across the interface. However, it is the lack of continuity of these functions in the presence of slip that is the key to our problem. The possibility of jumps in these quantities must

be taken into account. At the same time, the slip of particles at the interface makes pointwise conditions difficult to formulate in reference and intermediate coordinates, because the traction condition must be applied at the same current point on either side of the interface. These two aspects, the jump in areas and the question of identifying the points on $\tilde{\mathcal{L}}$ and $\mathcal{L}_0$, will be dealt with separately.

The traction vector of eq. (11) is continuous across a deformed interface, and it is immaterial which coordinate system is used to define the functional dependence of the vector. Continuity of the total traction vector therefore implies, using eq. (11) and expanding subject to the doubly asymptotic procedure, that the intermediate stress tensor satisfies the continuity condition

$$\{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1 + \tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}} - E_{\perp} \tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1\}_{\mathcal{L}} = 0. \quad (12)$$

The notation $\{f\}_{\mathcal{L}}$ means the difference in the quantity f evaluated at adjacent points $\mathbf{x} \pm \epsilon \mathbf{n}$, $\epsilon \rightarrow 0$, on either side of the interface $\mathcal{L}$. Similarly, $\{f\}_{\tilde{\mathcal{L}}}$ and $\{f\}_{\mathcal{L}_0}$ will be used to denote the difference evaluated at neighbouring points across $\tilde{\mathcal{L}}$ and $\mathcal{L}_0$, respectively. The jump condition (12) involves quantities at neighbouring current positions, not at neighbouring intermediate positions. The transformation from one to the other requires evaluating eq. (12) at the different points $\xi = \mathbf{x} - \mathbf{u}$ for the same $\mathbf{x}$ but different $\mathbf{u}$ on either side of $\mathcal{L}$. The only constraint upon $\mathbf{u}$ is that the normal component be continuous:

$$\{\mathbf{u} \cdot \mathbf{n}\}_{\mathcal{L}} = 0.$$

We split $\mathbf{u}$ into normal and tangential parts, according to $\mathbf{u} = \mathbf{u}_n + \mathbf{u}_{\perp}$, so that

$$\xi = (\mathbf{x} - \mathbf{u}_n) - \mathbf{u}_{\perp}. \quad (13)$$

This permits us to convert the jump condition (12) to one defined on $\tilde{\mathcal{L}}$. In so doing, the bracketed term on the right of eq. (13) is considered constant, and the quantities in eq. (12) have to be evaluated at points in $\tilde{\mathcal{L}}$ separated by $\{-\mathbf{u}_{\perp}\}_{\mathcal{L}}$. This can only affect the first term in eq. (12) because the changes in the other terms are negligible. Thus

$$\begin{aligned} \{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1\}_{\mathcal{L}} &\rightarrow \{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1 - \mathbf{u}_{\perp} \cdot \nabla \tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1\}_{\tilde{\mathcal{L}}} \\ &= \{-\mathbf{u}_{\perp} \cdot \nabla \tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1\}_{\tilde{\mathcal{L}}}, \end{aligned} \quad (14)$$

where we use the fact that the pre-stress is in equilibrium:

$$\{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1\}_{\tilde{\mathcal{L}}} = 0.$$

The desired, converted, form of eq. (11) is therefore

$$\{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}} - E_{\perp} \tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1 - \mathbf{u}_{\perp} \cdot \nabla \tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}^1\}_{\tilde{\mathcal{L}}} = 0. \quad (15)$$

The analogous form of the force balance condition in the reference configuration follows from eq. (11)₂. We first write it as a jump condition across the current interface:

$$\{\mathbf{N} \cdot \mathbf{P}^1 + \mathbf{N} \cdot \mathbf{P} - E_{\perp} \mathbf{N} \cdot \mathbf{P}^1 - E_{\perp}^1 \mathbf{N} \cdot \mathbf{P}\}_{\mathcal{L}} = 0.$$

In order to convert this to a jump condition across $\mathcal{L}_0$ we need to go one step further than before, because

$$\mathbf{X} = (\mathbf{x} - \mathbf{u}_n - \mathbf{w}_n) - \mathbf{u}_{\perp} - \mathbf{w}_{\perp},$$

with obvious notation. Therefore, in addition to the

expansion in eq. (14), we need to include

$$\{\mathbf{N} \cdot \mathbf{P}\}_{\mathcal{L}} \rightarrow \{\mathbf{N} \cdot \mathbf{P} - \mathbf{w}_{\perp} \cdot \nabla \mathbf{N} \cdot \mathbf{P}\}_{\mathcal{L}_0}.$$

Finally, we deduce that

$$\begin{aligned} \{\mathbf{N} \cdot \mathbf{P} - E_{\perp} \mathbf{N} \cdot \mathbf{P}^1 - \mathbf{u}_{\perp} \cdot \nabla \mathbf{N} \cdot \mathbf{P}^1 - E_{\perp}^1 \mathbf{N} \cdot \mathbf{P} \\ - \mathbf{w}_{\perp} \cdot \nabla \mathbf{N} \cdot \mathbf{P}\}_{\mathcal{L}_0} = 0. \end{aligned} \quad (16)$$

Equations (15) and (16) are, perhaps, the central results of the paper. The jump condition in intermediate coordinates, eq. (15), is equivalent to equation (2.16) of Grinfel'd & Movchan (1979), although the derivations are somewhat different. Grinfel'd & Movchan (1979) applied a formal expansion of the continuity conditions to deduce their result, while we have used a two-step argument, based first upon the kinematic relation (11), then identifying the 'conversion' of eq. (12) to intermediate coordinates. The general continuity conditions (15) and (16) apply to any interface conditions, whether bonded or not. They reduce to the simple, standard, forms

$$\{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}\}_{\tilde{\mathcal{L}}} = 0, \quad \{\mathbf{N} \cdot \mathbf{P}\}_{\mathcal{L}_0} = 0,$$

when the interface is bonded.

4 A DIVERGENCE FORMULATION FOR SOLID/FLUID MEDIA

The dynamical equations for the small motion follow from eqs (3) and (6)–(8) as

$$P_{Lj,L}(\mathbf{X}) = \rho_0 u_{j,w}, \quad \mathbf{X} \in V_0, \quad (17)$$

$$\tilde{P}_{\alpha j,\alpha}(\xi) = \bar{\rho} \tilde{u}_{j,w}, \quad \xi \in \tilde{V}, \quad (18)$$

$$\pi_{ij,i}(\mathbf{x}) = \rho u_{j,w}, \quad \mathbf{x} \in V. \quad (19)$$

The equations in current coordinates lead to a natural or divergence formulation, in the sense that the interface conditions can be found by integrating the 'diverged' stress. Thus, $\{\mathbf{n} \cdot \boldsymbol{\pi}\}_{\mathcal{L}} = 0$ arises directly from the equations in current coordinates. However, the position of the interface $\mathcal{L}$ is not known *a priori*, which makes this formulation rather difficult to handle. The other two formulations are not in divergence form, because the interface conditions (15) and (16) are clearly unrelated to the differential equations of motion (17) and (18).

We now show how eq. (18) in intermediate coordinates can be rewritten in divergence form for a particular type of heterogeneity. We consider a fluid/solid composite medium in which the fluid phase is connected. A fluid-saturated porous medium serves as a good example. The connected property is required to ensure that the initial pressure in the fluid phase, caused by the pre-stress, is homogeneous throughout the fluid, i.e.

$$\tilde{\mathbf{P}}^1 = -p^1 \mathbf{I} \quad \text{in the fluid,} \quad (20)$$

where p^1 is constant. Furthermore, the inviscid nature of the fluid means that the initial traction on the interface is everywhere a normal stress of $-p^1$. Consider the new stress

$$\tilde{\tilde{\mathbf{P}}} = \tilde{\mathbf{P}} + p^1 \{\mathbf{I} \operatorname{div} \mathbf{u} - (\nabla \mathbf{u})^T\}, \quad (21)$$

where $(\nabla \mathbf{u})_{\alpha i} = u_{i,\alpha}$. This addition to $\tilde{\mathbf{P}}$ is divergence-free, and hence the equations of motion in intermediate

coordinates, eq. (18), may be equally well written

$$\tilde{P}_{\alpha j, \alpha}(\xi) = \tilde{\rho} \tilde{u}_{j, \alpha}, \quad \xi \in \tilde{V}. \quad (22)$$

We will now demonstrate that the interface condition (15) can be expressed as

$$\{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}\}_{\tilde{\mathcal{S}}} = 0. \quad (23)$$

The jump condition (15) may be rewritten, using eqs (20) and (21), as

$$\begin{aligned} 0 = & \{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}\}_{\tilde{\mathcal{S}}} + p^1 \{(\mathbf{u}_\perp \cdot \nabla) \tilde{\mathbf{N}}\}_{\tilde{\mathcal{S}}} \\ & + p^1 \{E_\perp \tilde{\mathbf{N}} - \tilde{\mathbf{N}} \operatorname{div} \mathbf{u} + \tilde{\mathbf{N}} \cdot (\nabla \mathbf{u})^T\}_{\tilde{\mathcal{S}}}. \end{aligned} \quad (24)$$

The differential operator $\mathbf{u}_\perp \cdot \nabla$ in the middle term of the right member is the same as $\mathbf{u} \cdot \nabla_\perp$ where $\nabla_\perp$ represents the in-surface derivatives. The final term in eq. (24) can be simplified, using this operator and eq. (10)₂, to give

$$\begin{aligned} 0 = & \{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}\}_{\tilde{\mathcal{S}}} + p^1 \{(\mathbf{u} \cdot \nabla_\perp) \tilde{\mathbf{N}}\}_{\tilde{\mathcal{S}}} + p^1 \{\tilde{\mathbf{N}} \cdot (\nabla_\perp \mathbf{u})^T\}_{\tilde{\mathcal{S}}} \\ = & \{\tilde{\mathbf{N}} \cdot \tilde{\mathbf{P}}\}_{\tilde{\mathcal{S}}} + p^1 \nabla_\perp \{\tilde{\mathbf{N}} \cdot \mathbf{u}\}_{\tilde{\mathcal{S}}}. \end{aligned}$$

But the final term is zero because of the continuity of $\tilde{\mathbf{N}} \cdot \mathbf{u}$ across the interface. Hence we have proved the equivalence of eqs (15) and (23).

5 SMALL-ON-LARGE CONSTITUTIVE RELATIONS

5.1 General theory

It remains to define the linearized constitutive relations for the small-on-large motion. We will demonstrate that these are as follows (Sinha 1982):

$$P_{Mj} = G_{MjQk} u_{k,Q}, \quad (25)$$

where

$$\begin{aligned} G_{MjQk} = & C_{MjQk} + P_{MQ}^1 \delta_{jk} + w_{j,P} C_{MPkQ} \\ & + C_{MjPQ} w_{k,P} + C_{MjQkPl} w_{l,P}, \end{aligned} \quad (26)$$

and the second- and third-order elastic moduli are defined below. Note that the extra moduli G_{MjQk} are linear in the applied fields. The linearized constitutive equations for the perturbed stresses in the intermediate description are

$$\tilde{P}_{\alpha j} = \tilde{C}_{\alpha j \beta \gamma} \tilde{u}_{\gamma, \beta}, \quad (27)$$

where

$$\begin{aligned} \tilde{C}_{\alpha j \beta \gamma} = & C_{\alpha j \beta \gamma} + P_{\alpha \beta}^1 \delta_{j\gamma} - C_{\alpha j \beta \gamma} w_{\delta, \delta} + C_{\alpha j \beta \gamma Pl} w_{l, P} \\ & + w_{\alpha, P} C_{Pj \beta \gamma} + w_{j, P} C_{\alpha P \beta \gamma} + w_{\beta, P} C_{\alpha j P \gamma} \\ & + w_{\gamma, P} C_{\alpha j \beta P}. \end{aligned} \quad (28)$$

Similarly, the linearized Cauchy stresses in the current configuration are

$$\pi_{ij} = \tilde{C}_{ijkl} u_{l, k}. \quad (29)$$

Also, the pre-stress satisfies the linearized stress-strain relation

$$P_{Lj}^1 = C_{LjQk} w_{k, Q}.$$

We need only deduce eqs (25) and (26), because the

others, (27)–(29), follow from these using the general relations (2) and appropriate linearizations. Expanding eq. (6) for small-on-large deformations yields

$$T_{Mi} = P_{Mi}^1 + G_{MiQk} u_{k, Q} + \dots,$$

where

$$G_{MiQk} = \left. \frac{\partial T_{Mi}}{\partial u_{k, Q}} \right|_1, \quad (30)$$

and the subscript '1' denotes quantities in the intermediate state, i.e. with $\mathbf{u} = 0$. We assume the existence of a stored energy function $U = U(\mathbf{E}, \mathbf{X})$ per unit mass in $\mathcal{R}_0$, where the finite strain, $\mathbf{E}$, is defined in the usual manner as $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})$. Then the total Piola–Kirchhoff stress follows as

$$T_{Lj} = \rho_0 \frac{\partial U}{\partial E_{KL}} F_{jK}. \quad (31)$$

Eq. (30) then becomes

$$\begin{aligned} G_{MiQk} = & \rho_0 \left. \frac{\partial U}{\partial E_{MQ}} \right|_1 \delta_{ik} + \rho_0 \left. \frac{\partial^2 U}{\partial E_{MN} \partial E_{QP}} \right|_1 \\ & \times (\delta_{iN} + w_{i, N})(\delta_{kP} + w_{k, P}) \\ = & \rho_0 \left. \frac{\partial U}{\partial E_{MQ}} \right|_1 \delta_{ik} + \rho_0 \left. \frac{\partial^2 U}{\partial E_{Mi} \partial E_{Qk}} \right|_1 + \dots \\ & + \rho_0 \left. \frac{\partial^2 U}{\partial E_{Mi} \partial E_{QP}} \right|_1 w_{k, P} \\ & + \rho_0 \left. \frac{\partial^2 U}{\partial E_{MP} \partial E_{Qk}} \right|_1 w_{i, P} + \dots, \end{aligned} \quad (32)$$

where we have ignored terms of quadratic or higher order in $w_{i, P}$. We now assume that the strain energy possesses a Taylor expansion of the form

$$\begin{aligned} \rho_0(\mathbf{X}) U(\mathbf{E}, \mathbf{X}) = & \frac{1}{2} C_{ABCD} E_{AB} E_{CD} \\ & + \frac{1}{6} C_{ABCDEF} E_{AB} E_{CD} E_{EF} + \dots, \end{aligned} \quad (33)$$

where the second- and third-order elastic moduli possess the symmetries $C_{ABCD} = C_{BACD} = C_{CDAB}$, $C_{ABCDEF} = C_{BACDEF} = C_{CDABEF} = C_{EFCADB}$. The heterogeneity of the medium is reflected in the dependence of ρ_0 and the energy function on the Lagrangian coordinates. In practice we are concerned with piecewise constant material properties. Combining eqs (32) and (33) implies that G_{MiQk} are as defined in eq. (26), thus verifying the stress-strain relation (25). The remaining stress-strain relations can then be found by using eqs (7) and (31).

5.2 Moduli and equations for an inviscid fluid

The general equations simplify considerably for the special case of an inviscid fluid. The second- and third-order moduli are (Kostek, Sinha & Norris 1993)

$$\begin{aligned} C_{ijkl} = & A \delta_{ij} \delta_{kl}, \\ C_{ijklmn} = & (A - B) \delta_{ij} \delta_{kl} \delta_{mn} - 2A (\delta_{ij} I_{klmn} \\ & + \delta_{kl} I_{ijmn} + \delta_{mn} I_{ijkl}), \end{aligned} \quad (34)$$

where A and B are the usual linear and non-linear moduli of

a fluid (Pierce 1981), with $A = \rho_0 c^2$ and c the acoustic sound speed. Also, $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ are the elements of the fourth-order identity tensor. The static pre-strain in the fluid must be both uniform (homogeneous) and symmetric, $w_{i,M} = E_{iM}^1$ and $P_{Lj}^1 = -p^1 \delta_{Lj}$ where $p^1 = -AE_{LL}^1$ is the initial hydrostatic pressure. Therefore, eqs (26) and (34) imply that

$$G_{MiQk} = A\delta_{Mi}\delta_{Qk} + [(A^{-1}B - 1)\delta_{Mi}\delta_{Qk} + \delta_{Mk}\delta_{Qi}]p^1 - A(\delta_{Mi}E_{Qk}^1 + \delta_{Qk}E_{Mi}^1). \quad (35)$$

The PK stress in the reference coordinates follows from eqs (25) and (35) as

$$P_{Lj}^1 = [(A + A^{-1}Bp^1)u_{K,K} - AE_{MK}^1 u_{K,M}] \delta_{Lj} - Au_{K,K} E_{Lj}^1 + (u_{L,M} \delta_{Mj} - u_{K,K} \delta_{Lj}) p^1. \quad (36)$$

This stress is not symmetric in general since the displacement can have a small but non-zero rotational part, as we will demonstrate. We first note that the final part of eq. (36) is self-equilibrating, because $(u_{L,K} \delta_{Kj} - u_{K,K} \delta_{Lj})_{,L} = 0$, regardless of the precise form of $\mathbf{u}$. The equations of motion in reference coordinates are, from eqs (17) and (36),

$$v^2(p^1)u_{K,Kj} - c^2 E_{KL}^1 u_{K,Lj} - c^2 E_{Lj}^1 u_{K,KL} - u_{j,u} = 0,$$

where

$$v^2(p^1) = (1 + A^{-2}Bp^1)c^2. \quad (37)$$

Now consider a wave solution of the form

$$u_j(\mathbf{X}, t) = U_j f(N_L^0 X_L - v_0 t),$$

where $\mathbf{N}^0$ is a unit vector. Standard perturbation analysis shows that

$$v_0 = v(p^1) - cN_K^0 N_L^0 E_{KL}^1 + O[(p^1)^2],$$

and hence the speed depends not only upon the pressure p^1 but also the strain. The difference $v(p^1) - v_0$ is the standard Lagrangian-to-Eulerian correction, reflecting the change in length between the two configurations in the direction of propagation. Perturbation analysis also shows that the amplitude vector $\mathbf{U}$ is not parallel to the propagation direction, but the angle of deviation is of second order in the pre-strain. This small departure from longitudinal wave motion indicates that the motion is not irrotational in the reference coordinates.

The equation of motion in the current coordinates, eq. (19), can be shown to reduce to

$$v^2(p^1)u_{k,ki} - u_{i,u} = 0, \quad (38)$$

where the speed v is defined in (37). Taking the divergence of eq. (38) implies a scalar wave equation for the dilatation, with wave speed in current coordinates equal to $v(p^1)$, in agreement with standard non-linear acoustics (Pierce 1981). The equations of motion imply that $(u_{i,j} - u_{j,i})_{,u} = 0$, and hence the small-on-large motion is irrotational in the current coordinates.

Finally, we consider the intermediate configuration. The PK stress in the intermediate coordinates is

$$\tilde{P}_{\alpha j} = (A + A^{-1}Bp^1)\tilde{u}_{\gamma,\gamma}\delta_{\alpha j} + p^1\tilde{u}_{\alpha,\gamma}\delta_{\gamma j},$$

which is not symmetric. However, the 'modified' PK stress

of eq. (21) is symmetric:

$$\tilde{\tilde{P}}_{\alpha j} = [A + (A^{-1}B + 1)p^1]\tilde{u}_{\gamma,\gamma}\delta_{\alpha j}. \quad (39)$$

The equations of motion, (18) or (22), reduce to isotropic form:

$$v^2(p^1)\tilde{u}_{\gamma,\gamma\alpha} - \tilde{u}_{\alpha,u} = 0,$$

where $v(p^1)$ is defined above. The motion is again irrotational and the wave speed is the same as in the current coordinates. This is not surprising since the difference between the two configurations depends upon the small motion, not on the pre-stress.

6 APPLICATIONS

6.1 Problem definition and representation

We are interested in how the initial deformation changes a mode of a composite solid/fluid system. The undeformed mode ($\mathbf{w} = \mathbf{0}$) is defined by the time-dependent displacement field $\mathbf{u}^m$ and the deformed mode by $\mathbf{u}$. The original mode is time-harmonic with radial frequency ω^m and the perturbed mode has frequency ω . The equations of motion for both modes can be formulated in Lagrangian, intermediate or current coordinates. However, the current coordinates have the unavoidable difficulty that the interface position depends upon the solution $\mathbf{u}$. The Lagrangian coordinates do not have this problem, but the interface conditions (16) are not the natural ones for the equilibrium equations (17). Furthermore, there is a conceptual difficulty in dealing with Lagrangian coordinates for heterogeneous media. For instance, the 'speed' of a small-on-large wave is not a useful concept in Lagrangian coordinates, as the case of the pressurized borehole demonstrates. Recall, in that case, that the axial deformation of the solid and fluid can be different, so that a wave speed defined relative to the original length of a fluid column is not the same as a wave speed referred to the solid.

We therefore propose that the divergence formulation of the acoustoelastic equations, (21)–(23), is optimal for dealing with practical situations. It does not suffer from the problems mentioned above, and its divergence form permits us to use many of the techniques normally applied to acoustoelastic problems. We focus here on an integral formulation analogous to the perturbation procedure of Sinha (1982). The analysis from now on is entirely in intermediate coordinates, and to simplify the expressions we adopt a simplified notation. To be specific, we use the notation for *current* coordinates, so that position is denoted by $\mathbf{x}$ rather than ξ , stress is denoted by σ_{ij} rather than $\tilde{P}_{\alpha j}$, etc. The point of this change is that we want to emphasize that the acoustoelastic problem is now in the 'standard' form. The critical parameters are then the changes in the effective elastic moduli and density, both of which are linear in the pre-stress.

6.2 The perturbation integral

Consider the perturbed and unperturbed equations of motion as, respectively,

$$\sigma_{ji,j}(\mathbf{x}) + \rho\omega^2 u_i(\mathbf{x}) = 0, \quad \sigma_{ji,j}^m(\mathbf{x}) + \rho_0(\omega^m)^2 u_i^m(\mathbf{x}) = 0.$$

Contract the first with $\mathbf{u}^m$ and the second with $\mathbf{u}$, and integrate the difference over an arbitrary volume V , yielding

$$\int_V [\rho\omega^2 - \rho_0(\omega^m)^2] u_j u_j^m dV = \int_V (\sigma_{ji,j}^m u_i - \sigma_{ji,j} u_i^m) dV.$$

Then integrate by parts, using the fact that the equations are in divergence form, allowing us to integrate 'through' interfaces. This gives

$$\int_V [\rho\omega^2 - \rho_0(\omega^m)^2] u_j u_j^m dV = - \int_V (\sigma_{ji}^m u_{i,j} - \sigma_{ji} u_{i,j}^m) dV. \quad (40)$$

We can now estimate the shift in modal frequency, using

$$\rho = \rho_0 + \Delta\rho, \quad \mathbf{C} = \mathbf{C}_0 + \Delta\mathbf{C}, \quad \omega = \omega^m + \Delta\omega,$$

where $\Delta\rho = -w_{k,k}\rho_0$ is the change in density associated with the pre-strain, and the incremental elastic moduli follow from eqs (21), (27) and (28) as

$$\begin{aligned} \Delta C_{ijkl} &= \bar{C}_{ijkl} + p^1(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) - C_{ijkl} \\ &= p^1(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) - C_{ijkl}w_{m,m} \\ &\quad + C_{ikmn}w_{m,n}\delta_{jl} + C_{ijklmn}w_{m,n} \\ &\quad + C_{pjkl}w_{i,p} + C_{ipkl}w_{j,p} \\ &\quad + C_{ijpl}w_{k,p} + C_{pjkl}w_{l,p}. \end{aligned} \quad (41)$$

Substituting into eq. (40), making the approximation $\mathbf{u} \approx \mathbf{u}^m$ and retaining only the terms linear in the deviations gives

$$2\omega^m \Delta\omega = \frac{\int_V \Delta C_{ijkl} u_{i,j}^m u_{k,l}^m dV}{\int_V \rho_0 u_q^m u_q^m dV} - (\omega^m)^2 \frac{\int_V \Delta\rho u_i^m u_i^m dV}{\int_V \rho_0 u_q^m u_q^m dV}. \quad (42)$$

The change in phase speed associated with the material change is

$$\Delta v/v = \Delta\omega/\omega^m. \quad (43)$$

In applying eqs (42) and (43) to fluid-solid media we need only consider the integrals over the solid portion V_s , because the effects of the fluid perturbations enter through the dependence of v on ρ_f and A . The change in the fluid density is simply $\Delta\rho_f = (p^1/A)\rho_f$, and the perturbation in A follows from eq. (39) as $\Delta A = (1 + B/A)p^1$. Hence

$$\begin{aligned} \frac{\Delta v}{v} &= \frac{p^1}{A} \frac{\rho_f}{v} \frac{\partial v}{\partial \rho_f} + \frac{p^1}{v} \left(1 + \frac{B}{A}\right) \frac{\partial v}{\partial A} + \frac{\int_{V_s} \Delta C_{ijkl} u_{i,j}^m u_{k,l}^m dV}{2(\omega^m)^2 \int_V \rho_0 u_q^m u_q^m dV} \\ &\quad - \frac{\int_{V_s} \Delta\rho u_i^m u_i^m dV}{2 \int_V \rho_0 u_q^m u_q^m dV}. \end{aligned} \quad (44)$$

7 EXAMPLE: THE TUBE WAVE

We consider a circular borehole of radius a in an isotropic elastic formation. The bore is filled with fluid and the axis is

aligned with the x_3 - or z -direction. The initial stress is caused by an applied pressure in the borehole fluid, which induces an inhomogeneous deformation in the solid. The general result (44) can be used to determine the change in phase speed of any given waveguide mode. However, we will focus here on the change in the tube-wave speed due to the pressure.

The unperturbed wave has axial dependence of the form $\cos(kz - \omega^m t)$, where k is the axial wave number. The low-frequency limit of the fluid motion is defined by the low-frequency behaviour of the Stoneley wave mode, for which we use the appropriate quasi-static approximations (White 1983). The pressure in the fluid, $r < a$, and the strain in the solid, $r > a$, are

$$p^m = p_0 \cos[\omega^m(t - v^{-1}z)] \text{ in } V_f,$$

$$\mathbf{E}^m = \frac{1}{2} p_0 N_1 a^2 r^{-2} \mathbf{e} \cos[\omega^m(t - v^{-1}z)] \text{ in } V_s,$$

where

$$e_{\alpha\beta} = \delta_{\alpha\beta} - 2r^{-2}x_\alpha x_\beta, \quad \alpha, \beta = 1, 2, \quad e_{3k} = e_{k3} = 0. \quad (45)$$

Here, p_0 is the tube-wave pressure, N_1 the linear compliance of the borehole wall and $r = (x_1^2 + x_2^2)^{1/2}$. Thus, $N_1 = 1/\mu$ for an isotropic formation of shear modulus μ (White 1983). The unperturbed speed is v , where

$$v^{-2} = \rho_f(A^{-1} + N_1), \quad (46)$$

and ρ_f is the reference fluid density. The problem depends only upon the modal behaviour in the (x_1, x_2) -plane, and hence we drop the explicit dependence upon z and t in subsequent equations.

The applied pressure, p^1 , like the tube-wave pressure, p_0 , induces plane strain of the form

$$\mathbf{E}^1 = \frac{1}{2} p^1 N_1 a^2 r^{-2} \mathbf{e} \text{ in } V_s,$$

where $\mathbf{e}$ is defined in eq. (45). Thus $E_{kk}^1 = w_{m,m} = 0$, implying that the mass density of the formation is unchanged under the pre-stress. The dependence of the tube-wave speed with pressure then follows from eqs (44) and (46) as

$$\frac{\Delta v}{v} = -\frac{p^1}{2A} + \left(\frac{1 + A^{-1}B}{1 + AN_1}\right) \frac{p^1}{2A} + \frac{\int_{V_s} \Delta C_{ijkl} u_{i,j}^m u_{k,l}^m dV}{2(\omega^m)^2 \int_V \rho_0 u_q^m u_q^m dV}. \quad (47)$$

The upper integrand can be simplified, using eq. (41) and the symmetries of the elastic moduli, to give

$$\begin{aligned} \Delta C_{ijkl} u_{i,j}^m u_{k,l}^m &= \frac{1}{4} p_0^2 N_1^2 a^4 r^{-4} (5C_{\alpha\alpha kk} e_{kl} \\ &\quad + C_{ijklmn} e_{ij} e_{kl} e_{mn} - 2p^1) \text{ in } V_s, \end{aligned} \quad (48)$$

where the repeated Greek suffixes indicate summation only over $\alpha = 1$ and 2 . Substituting the general form for isotropic elasticity into eq. (48), which involves two second-order and three third-order elastic moduli, it turns out that the terms involving C_{ijkl} and C_{ijklmn} are individually zero, independently of the five moduli. The volumetric integrals in eq. (47) should be understood as time averaged over one cycle, or alternatively, as averages over one axial wavelength, and

they can be replaced by integrals in any cross-plane. Thus,

$$\int_{V_i} \Delta C_{ijk} u_{i,j}^m u_{k,i}^m dV \rightarrow -p^1 p_0^2 N_1^2 a^2 2\pi \times \int_a^\infty \frac{r dr}{2r^4} = -\frac{1}{2} p^1 p_0^2 N_1^2 \pi a^2. \quad (49)$$

The integral in the denominator of eq. (47) is dominated, in the quasi-static limit, by the integral of $(u_3^m)^2$ over the fluid volume, because all other displacements are $O(\omega)$ in magnitude relative to the axial displacement in the bore fluid. Furthermore, the axial speed is approximated by the relation $u_{3,i}^m = (\rho_f v)^{-1} p^m$, appropriate to a medium with acoustic impedance $\rho_f v$. Hence

$$(\omega^m)^2 \int_V \rho_0 u_q^m u_q^m dV \rightarrow \int_{V_f} \rho_f (u_{3,i}^m)^2 dV = p_0^2 (\rho_f v^2)^{-1} \pi a^2. \quad (50)$$

Combining eqs (46), (47), (49) and (50), we find that

$$\frac{\Delta v}{v} = \frac{p^1}{2A(1 + AN_1)} (A^{-1}B - AN_1 - \frac{1}{2}A^2N_1^2), \quad (51)$$

or

$$\rho_f dv^2/dp|_{p=0} = (A^{-1}B + \frac{1}{2})(v/c)^4 - \frac{1}{2}. \quad (52)$$

Note that the speed in eq. (51) depends upon the applied pressure p^1 but not on the specific fluid strain field. This is physically reasonable. One could visualize different initial strain fields in the fluid, all with the same pressure but different values of the axial strain E_{33}^1 . For instance, a piston-like loading in the borehole induces a fluid strain proportional to the tube-wave strain, whereas an axial or line load along the axis would give $E_{33}^1 = 0$. However, all yield the same tube-wave speed. We also note that eq. (52) agrees with Johnson, Koster & Norris (1993). They considered the possibility of variable elastic properties within the formation, but their general formula collapses to eq. (52) for a uniform formation. When the formation

compliance becomes infinite (rigid walls), then $v = c$ and the result (52) reduces to eq. (37), as expected. Eq. (52) indicates that the fluid non-linearity parameter B/A , which is positive, is diminished by the presence of the formation. However, the non-linearity parameter is magnified when the formation properties are not uniform, as discussed by Johnson *et al.* (1994).

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