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WAVES IN PERIODICALLY LAYERED MEDIA: A COMPARISON OF TWO THEORIES*

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Abstract. Two distinct asymptotic theories for wave propagation in one-dimensional inhomogeneous media are compared in their common domain of validity. One theory, due to Santosa and Symes, applies to long wavelength propagation in periodic media with arbitrary contrast in material properties. The O'Doherty–Anstey theory, on the other hand, is explicitly intended to describe time-dependent wave propagation in media that are finely layered but characterized by relatively small reflectivity. The two theories are compared in detail in the doubly asymptotic limit of low-frequency wave propagation in periodic media with small contrasts. The equivalence is demonstrated by deriving the asymptotic limit of the dispersion curve of the fundamental Bloch wave according to each theory. The analysis for the O'Doherty–Anstey theory sheds some new light on its strengths and limitations, particularly in periodic media. It is shown that it correctly predicts the leading-order dispersion curve of the first branch for frequencies of $O(1)$, but fails near the first band edge.

Key words. linear waves, periodic media, layered materials, multiple scattering

AMS subject classifications. 73D25, 35B27, 73B27

1. Introduction. This paper looks at the equivalence of two apparently dissimilar theories for describing waves in nonuniform one-dimensional media, or layered materials. Both theories are asymptotic in nature and lead to considerable simplification in the description of wave propagation over many layers, for which exact numerical solutions, although feasible, are very time-consuming. Also, both theories have been compared with numerical simulations, with favorable results in each case. The first is for media in which the variation is small [1], [2] and is known as finely layered medium theory, or the O'Doherty–Anstey theory. O'Doherty and Anstey [3] first proposed their equation after noting the apparent attenuation of simulated waves traveling through many layers of nondissipative materials. The theory was firmly established by Burridge, Papanicolaou, and White [1] and Burridge and Chang [2], who explained the attenuation phenomenon in terms of cumulative double scattering over long propagation distances. A summary of the theory with extensions to elasticity problems is given by Burridge [4], and the connection with asymptotic methods of averaging has been discussed by Stanke and Burridge [5]. The attenuation phenomenon is also discussed in [6], while a current review of waves in one-dimensional layered media can be found in the article of Asch et al. [7].

The second theory considered here is specifically for media with periodic structure, but it allows for arbitrary variation in material properties within the unit cell [8], [9]. The basic idea is to expand the dispersion for the first Bloch wave of the system and to use this asymptotic form of the dispersion relation. Santosa and Symes [8] showed how this can be done for arbitrary layering within the unit cell. They also proved the important result that this level of approximation is sufficient to describe the evolution of a long wavelength initial disturbance uniformly in time.

The two theories mentioned are approximate or asymptotic in nature, each with its own range of validity depending upon material variations, the frequency content of the

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signal, and the distance of propagation in the medium. Both theories describe the evolution of a coherent pulse traveling in the forward direction. The long wavelength theory for periodic media [8], [9] is not limited in the extent of material variations within the period, but it is a long wavelength approximation based on the assumption that the shortest wavelength in the signal λ is much longer than the length of the unit period h. Let

$$\varepsilon = \frac{h}{\lambda}; \tag{1.1}$$

then, by assumption, $\varepsilon \ll 1$.

The O'Doherty–Anstey or finely layered medium theory [1]–[3] is based upon the assumption that the reflectivity is small. In the context of the one-dimensional scalar wave equation, this is tantamount to saying that the relative variation in the acoustic impedance is small. Let ζ be the acoustic impedance and $\Delta\zeta$ the change in impedance; then the relevant small parameter is

$$\delta = \frac{\Delta\zeta}{\zeta}, \tag{1.2}$$

where $\delta \ll 1$.

The theory does not explicitly involve a length; however, it is useful to define the basic length scale as the correlation length of the reflectivity. We denote this by h, not to be confused with the basic length in the periodic medium, although we will consider the special case of a periodical finely layered medium for which the lengths can be considered the same. We now describe several features of either theory, pointing out common properties and differences.

Long wavelength, periodic medium theory.

1. The distance of propagation over which dispersive effects become significant is $O(h/\varepsilon^3)$.
2. The theory is limited to describing the evolution of initial disturbances that are long wavelength.
3. Energy is conserved.

Small variations, finely layered medium theory.

1. Pulse broadening and dispersive effects become important after propagation over distances on the order of h/δ^2.
2. An explicit formula determines the dispersion and damping at all frequencies, and therefore the initial disturbance may be arbitrary.
3. Energy is not generally conserved, except, as we will see, for periodic media.

The major distinctions are in the nature of the asymptotic approximation upon which each theory is founded. For the long wavelength theory, the small parameter is ε of (1.1), whereas the small parameter in the finely layered medium theory is δ of (1.2). The main purpose of this paper is to demonstrate that the theories agree in the doubly asymptotic limit where both ε and δ are small. The physical nature of this situation is such that an initial disturbance of long wavelength ($\varepsilon \ll 1$) encounters very weak scattering ($\delta \ll 1$), and, consequently, the first dispersive effects will not be seen until the wave has propagated a very large distance on the order $h/\delta^2\varepsilon^3$. However, this scenario falls within the realm of either theory, and so we expect them to agree. The task of demonstrating equivalence is nontrivial, as we will see, but it is simplified by formulating both theories in travel-time variables. Then the dependence upon the impedance becomes apparent. The main reason for the difficulty in comparing the theories is that they are based upon different physical approximations resulting from (1.1) and (1.2), respectively.

One benefit of the comparison is that it forces us to express the O'Doherty–Anstey theory in the frequency domain, and it turns out that the general form of the predicted dispersion equation has a very simple structure. In particular, we can examine it to see how well the theory predicts the band gap structure for a periodic medium with small variations in impedance.

2. Long wavelength periodic theory.

2.1. The asymptotic dispersion equation. We consider one-dimensional wave motion in $-\infty < z < \infty$ for the pressure $p(z, t)$ and velocity $w(z, t)$, subject to the equations

$$(\lambda + 2\mu) w_z + p_t = 0, \qquad p_z + \rho w_t = 0. \tag{2.1}$$

Here, the density ρ and the Lamé parameters λ and μ are periodic functions of z, with period h. The subscripts on w_z, p_t, and so forth denote partial derivatives. We are concerned with the propagation of low frequency, or long wavelength disturbances. It is well known that to a first approximation the medium acts as a homogeneous medium with effective sound speed given by

$$c_{\text{eff}} = h\left(\int_0^h \frac{1}{\lambda + 2\mu}\, dz\right)^{-1/2} \left(\int_0^h \rho\, dz\right)^{-1/2}. \tag{2.2}$$

The derivation of these results is simplified by switching from z to the dimensionless travel-time coordinate x, and from t to dimensionless time τ, where

$$x \equiv \frac{1}{T}\int_0^z \frac{1}{c(z')}\, dz', \qquad \tau \equiv \frac{t}{T}. \tag{2.3}$$

Here, $c(z)$ is the speed and T is the travel time across a unit cell,

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad T = \int_0^h \frac{1}{c(z)}\, dz. \tag{2.4}$$

Now think of p and w as functions of x and τ, so that (2.1) becomes

$$\zeta w_x + p_\tau = 0, \qquad p_x + \zeta w_\tau = 0, \tag{2.5}$$

where the single material parameter is the impedance ζ

$$\zeta = \rho c. \tag{2.6}$$

Define the average of a quantity f by

$$\langle f \rangle = \frac{1}{T}\int_0^h \frac{f(z)}{c(z)}\, dz = \int_0^1 f\, dx. \tag{2.7}$$

The effective speed of (2.2) may be written as

$$c_{\text{eff}} = \frac{h}{T} C_0, \tag{2.8}$$

where

$$C_0 = \left\langle \frac{1}{\zeta} \right\rangle^{-1/2} \langle \zeta \rangle^{-1/2}. \tag{2.9}$$

Note that $C_0 \leqq 1$ with equality if and only if ζ is constant. The effective sound speed implies that the behavior of the fundamental Bloch wave of the system can be thought of as a wave in a uniform medium that to leading order has the mechanical properties predicted by *effective medium theory*. However, the first dispersive effects of the Bloch wave are not contained in the effective medium description. To define the dispersion, consider a wave of the form

$$\{w(z,t), p(z,t)\} = \{\hat{w}(z), \hat{p}(z)\} \exp\left\{i\left(\kappa \frac{z}{h} - \Omega \frac{t}{T}\right)\right\}, \tag{2.10}$$

where $\hat{w}(z)$ and $\hat{p}(z)$ are h-periodic, and κ and Ω are a dimensionless wavenumber and frequency, both referred to the *unit* period. The starting point for the present analysis is the result that the leading-order expansion for the dispersion relation of the fundamental Bloch wave is

$$\kappa = \frac{\Omega}{C_0} + \frac{1}{2} C_0 D_3 \Omega^3 + O(\Omega^5), \tag{2.11}$$

where C_0 is the dimensionless effective speed defined in (2.9). The dispersion parameter in (2.11) is

$$\begin{aligned} D_3 = {} & \frac{1}{3}\langle \zeta \rangle^2 \left\langle \frac{1}{\zeta} \right\rangle^2 - \left\langle \frac{1}{\zeta} \right\rangle \int^1 \zeta \int \frac{1}{\zeta} \int \zeta \\ & - \langle \zeta \rangle \int^1 \frac{1}{\zeta} \int \zeta \int \frac{1}{\zeta} - \frac{1}{4}\left(\int^1 \zeta \int \frac{1}{\zeta} - \int^1 \frac{1}{\zeta} \int \zeta \right)^2. \end{aligned} \tag{2.12}$$

The double and triple integrals are shorthand for the explicit integrals of the form

$$\int^x f \int g = \int_0^x f\, dx' \int_0^{x'} g\, dx'',$$

$$\int^x f \int g \int f = \int_0^x f\, dx' \int_0^{x'} g\, dx'' \int_0^{x''} f\, dx'''.$$

The asymptotic expansion (2.11) with (2.12) is a consequence of a recent result of Santosa and Symes [8]. Their analysis was performed for the full second-order wave equation in the physical coordinates (z, t). The present form is simpler since it highlights the dependence upon the impedance. An alternative derivation was given by Norris and Santosa [9] based upon the coupled system (2.1). However, the result can be obtained in the simplest manner from the transformed equations (2.5). This is discussed in the next section, where we also present some new results for the wave fields associated with the fundamental Bloch wave in a periodic medium.

It may be checked that the dispersion parameter D_3 vanishes if ζ is constant. It is also known that $D_3 \geqq 0$ for the class of two-phase layered media [9]. We show, below, that $D_3 \geqq 0$ for arbitrary layering in the limit of small contrasts in the impedance function; see (2.35). It is not obvious from (2.12) that the same inequality holds for arbitrary finite variations in ζ. The proof is not apparent to the author, although it is suspected that the inequality is general. We note that D_3 can be recast in different ways; for example,

$$D_3 = \left\langle \frac{1}{\zeta} \right\rangle \langle \zeta X_1^2 \rangle + \langle \zeta \rangle \left\langle \frac{1}{\zeta} X_2^2 \right\rangle - \frac{2}{3} \langle X_1 + X_2 \rangle^2 - \frac{1}{4} \langle X_1 - X_2 \rangle^2, \tag{2.13}$$

where

$$X_1 = \frac{1}{\zeta} \int_0^x \zeta\, dx', \qquad X_2 = \zeta \int_0^x \frac{1}{\zeta}\, dx'. \tag{2.14}$$

Both X_1 and X_2 reduce to x when the impedance is constant, in which case it follows from (2.13) that $D_3 = 0$.

2.2. Derivation of the dispersion relation. The Bloch waves of the system are solutions of the form [10], [11]

$$\begin{bmatrix} w(x,\tau) \\ p(x,\tau) \end{bmatrix} = \begin{bmatrix} \bar{w}(x) \\ \bar{p}(x) \end{bmatrix} \exp\{i(kx - \Omega\tau)\}, \tag{2.15}$$

where $\bar{w}(x)$ and $\bar{p}(x)$ are periodic in x with unit period. The vector $v(x) = [\bar{w}(x), \bar{p}(x)]^T$ satisfies, from (2.5) and (2.15),

$$\frac{dv}{dx}(x) + i\Omega \begin{bmatrix} S & -1/\zeta \\ -\zeta & S \end{bmatrix} v = 0, \tag{2.16}$$

where $S = k/\Omega$ is the *slowness*.

It can be shown [12] that S must be an even function of Ω. We therefore assume the expansions (ansatz)

$$S = S_0 + (-i\Omega)^2 S_2 + (-i\Omega)^4 S_4 \cdots, \tag{2.17}$$

$$v = v_0 + (-i\Omega)v_1 + (-i\Omega)^2 v_2 \cdots. \tag{2.18}$$

Substitution of (2.17) and (2.18) into (2.16) and comparison of terms of like power in Ω leads to a series of equations. We summarize the results here, since the analysis is similar to that of Santosa and Symes [8]. The same type of matrix methods that are used here have also been used to derive the low-frequency dispersion relation in layered anisotropic elastic composites; see [13], [14] for further details. The velocity-pressure fields are, for $0 \leqq x \leqq 1$,

$$v_0 = e^{(+)}, \tag{2.19a}$$

$$v_1 = R_1 e^{(-)} + \int_0^x M\, dx' e^{(+)}, \tag{2.19b}$$

$$v_2 = R_2^{(-)} e^{(-)} + R_2^{(+)} e^{(+)} + R_1 \int_0^x M\, dx' e^{(-)} + \int^x M \int M e^{(+)}, \tag{2.19c}$$

$$v_3 = R_3 e^{(-)} + x S_2 e^{(+)} + \int_0^x M v_2\, dx', \ldots. \tag{2.19d}$$

Here $e^{(+)}$ and $e^{(-)}$ are the vectors that have unit power flux in the positive and negative x-directions, respectively, *for the effective medium*. Thus

$$e^{(\pm)} = \begin{pmatrix} \pm\gamma \\ 1/\gamma \end{pmatrix}, \tag{2.20}$$

where

$$\gamma = \left(\frac{\langle 1/\zeta \rangle}{\langle \zeta \rangle} \right)^{1/4}. \tag{2.21}$$

We have arbitrarily chosen the leading-order solution to be a wave traveling in the positive direction. The matrix $M(x)$ is

$$M = \begin{bmatrix} S_0 & -1/\zeta \\ -\zeta & S_0 \end{bmatrix}. \tag{2.22}$$

The results in (2.19) are all consequences of integrating the ordinary differential equation (2.16) and imposing the periodicity constraint on each v_j. The latter amounts to requiring that $v_j(1) = v_j(0)$ for each j. Specifically, the form of $e^{(+)}$ follows from the requirement that v_1 is unit-periodic, which also implies that

$$S_0 = \frac{1}{C_0}. \tag{2.23}$$

Thus S_0 is the dimensionless slowness associated with the effective medium. Similarly, the periodicity of v_2 determines the first-order reflection coefficient R_1 as

$$R_1 = \frac{1}{4S_0}\left(\int^1 \frac{1}{\zeta}\int \zeta - \int^1 \zeta \int \frac{1}{\zeta}\right). \tag{2.24}$$

The condition that v_3 be periodic implies values for both S_2 and $R_2^{(-)}$. We find that

$$S_2 = -\frac{D_3}{2S_0}, \tag{2.25}$$

where D_3 is defined in (2.12). Combining (2.17), (2.23), and (2.25), we see that the asymptotic dispersion relation is indeed given by (2.11). The reflection coefficient $R_2^{(-)}$ proves to be

$$R_2^{(-)} = \frac{1}{4}\left(\langle \zeta \rangle^{-1} \int^1 \zeta \int \frac{1}{\zeta} \int \zeta - \left\langle \frac{1}{\zeta} \right\rangle^{-1} \int^1 \frac{1}{\zeta} \int \zeta \int \frac{1}{\zeta}\right). \tag{2.26}$$

The determination of the transmission coefficient $R_2^{(+)}$ and higher-order terms involves considerably more algebra. The form of the terms v_1, v_2, and so forth shows that, at each order, the field is modified by additional reflected ($e^{(-)}$) terms. The even-order fields, v_2, v_4, and so forth also include transmitted ($e^{(+)}$) terms. The remaining parts vanish at both $x = 0$ and $x = 1$. Thus the total field, considered as a power series in Ω, consists of forward and backward propagating waves *in the effective medium*, plus parts that vanish at one point or more in the unit cell.

We have seen that the dispersion relation (2.17) agrees with the original form (2.11), which was stated but not proved. There is a subtle distinction between these slightly different forms of the dispersion relation. The derived relation (2.17) pertains to the Bloch wave description (2.15) for which the wavenumber k relates to x, whereas (2.11) was proposed for the waveform (2.10) with wavenumber κ associated with z/h. The correspondence between them follows from the fact that an increase of one in the value of x is also an increase of one for the value of z/h. Also, the analysis of Santosa and Symes [8] (see also [9]) begins with an ansatz of the form (2.10) and ends with a dispersion relation of the form (2.11). It can be shown that the present form of the dispersion relation agrees with that of [8] after the appropriate transformations are made. Therefore the wavenumber parameters κ and k of (2.11) and (2.17) with $k = \Omega S$ are interchangeable.

2.3. The small variations limit. The theory just described is asymptotic in the frequency but is not restricted in the range of variations in material properties. We now consider the particular limit of a periodic medium with impedance close to constant, although the speed c can still vary arbitrarily. This is therefore a doubly asymptotic theory—long wavelength and small reflectivity. Let

$$F(x) = \frac{1}{2}\log\left(\frac{\zeta}{\zeta_0}\right), \tag{2.27}$$

where ζ_0 is chosen for convenience to be such that

$$\langle F \rangle = 0 \Leftrightarrow \log \zeta_0 = \langle \log \zeta \rangle. \tag{2.28}$$

Thus $F = 0$ when the impedance is constant, in which case $D_3 = 0$. We consider the approximation to the dispersion relation (2.11) under the assumption that $|F| \ll 1$.

Expanding (2.23), using (2.9), we find that

$$S_0 = 1 + 2\langle F^2 \rangle + \cdots, \tag{2.29}$$

and therefore the effective medium slowness is greater than the average slowness T/h, in agreement with (2.8).

The expansion of D_3 is a bit more tedious, and its simplification requires using (2.28), (2.29), and the identity

$$\int^1 F^2 \int 1 \int 1 + \int^1 1 \int F^2 \int 1 + \int^1 1 \int 1 \int F^2 = \frac{1}{2}\langle F^2 \rangle, \tag{2.30}$$

but eventually yields, to leading order in the small variation,

$$\begin{aligned} D_3 = 8\Bigg[&\int^1 F \int F \int 1 + \int^1 1 \int F \int F \\ &- \int^1 F \int 1 \int F - 2\left(\int^1 1 \int F\right)^2\Bigg] + O(F^3). \end{aligned} \tag{2.31}$$

Then, using the identities

$$2\int^1 F \int F \int 1 = -\int^1 F \int 1 \int F - \langle F \rangle \int^1 F \int 1,$$

$$2\int^1 1 \int F \int F = -\int^1 F \int 1 \int F + \langle F \rangle \int^1 1 \int F,$$

and $\langle F \rangle = 0$, we deduce

$$D_3 = -16\left[\int^1 F \int 1 \int F + \left(\int^1 1 \int F\right)^2\right] + O(F^3). \tag{2.32}$$

To simplify (2.32), further define the sequence of functions $F^{(n)}$, $n = 1, 2, \ldots$ such that

$$\frac{dF^{(n+1)}}{dx} = F^{(n)}, \tag{2.33a}$$

$$\langle F^{(n+1)} \rangle = 0, \tag{2.33b}$$

$$F^{(1)} \equiv F(x). \tag{2.33c}$$

We note that $F^{(n)}$ inherits the periodicity of F. For now, we need only $F^{(2)}$,

$$F^{(2)} = \int_0^x F(x')\,dx' - \int_0^1 dx \int_0^x F(x')\,dx'. \tag{2.34}$$

Integrating (2.32) by parts then yields the relatively simple result that, to leading order,

$$D_3 = 16\langle F^{(2)^2} \rangle + O(F^3). \tag{2.35}$$

Note that $D_3 \geqq 0$ in the limit of small contrasts, with equality only for a constant impedance profile.

The dispersion equation (2.11) becomes, using (2.29) and (2.35),

$$\kappa = (1 + 2\langle F^2\rangle)\Omega + 8\left\langle F^{(2)^2}\right\rangle\Omega^3 + \cdots . \tag{2.36}$$

We defer discussion of (2.36) until after we have obtained the same limit within the context of the finely layered medium theory.

2.4. Example: A two-phase composite. Consider a laminate made of two alternating uniform constituents, with parameters ζ_1, ζ_2, c_1, and c_2. The volume fractions are n_1 and n_2, and the average of a quantity f becomes

$$\langle f\rangle = n_1' f_1 + n_2' f_2, \tag{2.37}$$

where

$$n_1' = \frac{n_1}{c_1}\bigg/\left(\frac{n_1}{c_1} + \frac{n_2}{c_2}\right), \qquad n_2' = \frac{n_2}{c_2}\bigg/\left(\frac{n_1}{c_1} + \frac{n_2}{c_2}\right). \tag{2.38}$$

We can then readily compute the reference value for impedance ζ_0 and the elements in (2.36). We find that

$$\langle F^2\rangle = 16 n_1' n_2' \Delta^2, \qquad \left\langle F^{(2)^2}\right\rangle = \tfrac{4}{3}(n_1' n_2')^2\Delta^2, \tag{2.39}$$

where Δ is the measure of the reflectivity

$$\Delta = \frac{1}{2}\log\left(\frac{\zeta_2}{\zeta_1}\right). \tag{2.40}$$

The first of these allows us to calculate the approximate dimensionless slowness $S_0 \approx 1 + 2\langle F^2\rangle$, while the approximate formula for the dispersion parameter D_3 follows from (2.35) and (2.39). In summary, the small contrast approximations are

$$S_0 = 1 + 2n_1' n_2' \Delta^2, \tag{2.41a}$$

$$D_3 = \tfrac{4}{3}(n_1' n_2')^2\Delta^2. \tag{2.41b}$$

These should be compared with the exact forms for S_0, which follows from (2.9) and (2.23), and for D_3, which may be obtained from (2.12) or, more simply, from known results for the dispersion parameter in bilaminates [15], [16], [9]. The exact results are

$$S_0 = (1 + 4n_1' n_2' \sinh^2\Delta)^{1/2}, \tag{2.42a}$$

$$D_3 = \tfrac{1}{3}(n_1' n_2')^2 \sinh^2 2\Delta. \tag{2.42b}$$

3. Finely layered medium theory. We now convert from the dimensionless variables (x, τ) to dimensionless characteristic variables (ξ, η),

$$\xi = \tfrac{1}{2}(\tau + x), \qquad \eta = \tfrac{1}{2}(\tau - x). \tag{3.1}$$

Define the continuous version of the reflection coefficient

$$r(\xi - \eta) = r(x) \equiv F'(x), \tag{3.2}$$

where F is defined in (2.27). The local reflection coefficient r is undefined at interfaces where the impedance is discontinuous. We assume that the medium is everywhere continuous for simplicity. The presence of discontinuities does not change the general results here, although the reader is referred to [17] for a discussion of how to include both discrete and continuous variations simultaneously. Define the down- and up-going waves

$$D(\xi, \eta) = \tfrac{1}{2}[\zeta^{-1/2}p + \zeta^{1/2}w], \tag{3.3a}$$

$$U(\xi, \eta) = \tfrac{1}{2}[\zeta^{-1/2}p - \zeta^{1/2}w]. \tag{3.3b}$$

Then it follows from (2.5) and (3.1)–(3.3) that [1]

$$D_\xi(\xi, \eta) + r(\xi - \eta)U(\xi, \eta) = 0, \tag{3.4a}$$

$$U_\eta(\xi, \eta) - r(\xi - \eta)D(\xi, \eta) = 0. \tag{3.4b}$$

Burridge et al. [17] showed that the coupled equations (3.4) can be reduced to a single equation for an approximation to the down-going component D. The form of the equation for D had been proposed earlier by O'Doherty and Anstey [3]. Let $\bar{D}$ be the approximate solution and define its one-sided transform as

$$\hat{\bar{D}}(\xi, \Lambda) = \int_0^\infty \bar{D}(\xi, \eta)e^{i\Lambda\eta}\, d\eta. \tag{3.5}$$

Then the O'Doherty–Anstey equation is

$$\frac{\partial}{\partial \xi}\hat{\bar{D}}(\xi, \Lambda) + \hat{a}(\Lambda)\hat{\bar{D}}(\xi, \Lambda) = 0. \tag{3.6}$$

The parameter $\hat{a}$ is the one-sided transform of the autocorrelation function

$$a(\eta - \eta') = \lim_{\xi \to \infty} \frac{1}{\xi}\int_0^\xi r(\xi' - \eta)r(\xi' - \eta')\, d\xi'. \tag{3.7}$$

3.1. A periodic medium. The crucial physical parameter in the finely layered medium theory is the autocorrelation function of (3.7), which is a spatial, or deterministic, average, as opposed to a stochastic quantity. The theory as originally proposed by O'Doherty and Anstey [3] and rigorously derived by Burridge, Papanicolaou, and White [1] can deal with arbitrary fine structure. The only requirement is that the impedance variations are small, so that scattering is *weak* in the sense that the cumulative effects of irregularity are significant over large distances of propagation. We will now demonstrate that the dispersion predicted by the O'Doherty–Anstey equation agrees with the low-frequency periodic theory in the limit of small contrasts. Burridge, Papanicolaou, and White [1] reported some numerical results for periodic media but gave no analytical results for this particular limit of the general theory. To compare the theories, we must first obtain the general form of $\hat{a}$ for periodic media and then take its low-frequency limit.

The autocorrelation function becomes, for periodic media,

$$a(\eta) = \int_0^1 r(\xi' - \eta)r(\xi')\, d\xi'. \tag{3.8}$$

Note that $a(\eta)$ inherits the periodicity of the medium, i.e., $a(\eta + 1) = a(\eta)$, and is an even function, $a(-\eta) = a(\eta)$. The former property presents some formal difficulties arising from the fact that the autocorrelation function does not decay at infinity, and hence the Fourier transform is ill-defined in the usual sense. However, by reducing the infinite integral to an infinite sum of integrals over the unit period, then summing the phase terms for each of these integrals, in the sense of

$$1 + e^{i\theta} + e^{i2\theta} + \cdots = \frac{1}{1 - e^{i\theta}}, \tag{3.9}$$

the one-sided Fourier transform can be easily shown to be

$$\hat{a}(\Lambda) = \frac{i}{2\sin\Lambda/2}\int_0^1 a(\eta)\cos\Lambda\left(\eta - \frac{1}{2}\right) d\eta. \tag{3.10}$$

Note that $\hat{a}$ is purely imaginary, and hence the effective down-going wave experiences no dissipation. Generally, when the medium is not periodic, the real part of $\hat{a}$ is strictly positive, implying decay in the down-going, positive x (or z) direction.

We can now expand $\hat{a}$ as a Taylor series in Λ. Noting that

$$\int_0^1 a(\eta)\, d\eta = 0, \tag{3.11}$$

on account of the periodicity of the medium, it is clear that the first term in the expansion of $\hat{a}$ is $O(\Lambda)$ and that it and subsequent terms may be found by expanding the trigonometric functions in (3.10). A simpler and more appealing approach is to introduce new functions $a^{(2n)}(\eta)$, $n = 1, 2, 3, \ldots$ such that

$$(-1)^n \frac{d^{2n}}{d\eta^{2n}} a^{(2n)}(\eta) = a(\eta), \tag{3.12a}$$

$$\int_0^1 a^{(2n)}(\eta)\, d\eta = 0. \tag{3.12b}$$

These functions are also defined to be periodic and even in η (note that, if we were to represent $a(\eta)$ as a Fourier series in $[0, 1]$, then the periodicity and evenness properties follow immediately). Integration of (3.10) by parts yields

$$\begin{aligned} \hat{a}(\Lambda) &= \frac{-i}{2 \sin \Lambda/2} \int_0^1 a^{(2)''}(\eta) \cos \Lambda\left(\eta - \frac{1}{2}\right) d\eta \\ &= \frac{-i\Lambda}{2 \sin \Lambda/2} \int_0^1 a^{(2)'}(\eta) \sin \Lambda\left(\eta - \frac{1}{2}\right) d\eta \\ &= -i\Lambda a^{(2)}(0) + \frac{i\Lambda^2}{2 \sin \Lambda/2} \int_0^1 a^{(2)}(\eta) \cos \Lambda\left(\eta - \frac{1}{2}\right) d\eta. \end{aligned} \tag{3.13}$$

Proceeding in the same manner, we find by induction that

$$\hat{a}(\Lambda) = -i[a^{(2)}(0)\Lambda + a^{(4)}(0)\Lambda^3 + a^{(6)}(0)\Lambda^5 + \cdots]. \tag{3.14}$$

This Taylor series has a finite radius of convergence since it is clear, for example, that, according to (3.10), $\hat{a}(\Lambda)$ is singular at $\Lambda = 2n\pi$, $n = 1, 2, \ldots$.

The terms in (3.14) can be interpreted in another manner. For any unit-periodic function f, we have

$$-\frac{d^2}{d\eta^2} \int_0^1 f(\xi - \eta) f(\xi)\, d\xi = \int_0^1 f'(\xi - \eta) f'(\xi)\, d\xi. \tag{3.15}$$

It then follows from (2.34), (3.2), (3.8), (3.12), and (3.15) that

$$a^{(2)}(\eta) = \int_0^1 F(\xi - \eta) F(\xi)\, d\xi, \tag{3.16a}$$

$$a^{(4)}(\eta) = \int_0^1 F^{(2)}(\xi - \eta) F^{(2)}(\xi)\, d\xi, \tag{3.16b}$$

and hence

$$a^{(2)}(0) = \langle F^2 \rangle, \qquad a^{(4)}(0) = \langle F^{(2)^2} \rangle. \tag{3.17}$$

It may be shown, although we omit the details, that similar formulae hold for $a^{(2n)}$, $n > 2$, in particular,

$$a^{(2n)}(0) = \langle F^{(n)^2} \rangle, \tag{3.18}$$

where $F^{(n)}$ are defined by (2.33). Thus the expansion of the real wavenumber function $i\hat{a}(\Lambda)$ of (3.14) becomes simply

$$i\hat{a}(\Lambda) = \langle F^2 \rangle \Lambda + \langle F^{(2)^2} \rangle \Lambda^3 + \langle F^{(3)^2} \rangle \Lambda^5 + \cdots. \tag{3.19}$$

Note that *each* of the coefficients in this power series is positive.

We are now in a position to compare these results with the doubly asymptotic limit discussed in the previous section. Consider solutions to (3.6) of the form

$$\bar{D}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\bar{D}}(\xi, \Lambda) e^{i(i\hat{a}\xi - \Lambda\eta)} \, d\Lambda. \tag{3.20}$$

Expanding the phase, using (3.1), (3.17), and the first two terms in (3.14), and also converting the frequency according to

$$\Lambda = 2\Omega, \tag{3.21}$$

we obtain

$$i\hat{a}(\Lambda)\xi - \Lambda\eta = \left[x - \tau + 2\langle F^2 \rangle \left(\frac{x + \tau}{2} \right) \right] \Omega + 8 \langle F^{(2)^2} \rangle \left(\frac{x + \tau}{2} \right) \Omega^3. \tag{3.22}$$

This is to be compared with the phase $(\kappa x - \Omega\tau)$, which follows from (2.36) as

$$\kappa x - \Omega\tau = (x - \tau + 2\langle F^2 \rangle x)\Omega + 8\langle F^{(2)^2} \rangle x \Omega^3. \tag{3.23}$$

Equations (3.22) and (3.23) are equivalent under the correspondence $(x + \tau)/2 \to x$, which is reasonable in the limit that $|\tau - x| \ll x$, which is certainly true if the pulse has propagated through many wavelengths. The dispersion equation associated with the O'Doherty–Anstey equation follows from (3.19), (3.22), and (3.23),

$$\kappa = (1 + 2\langle F^2 \rangle)\Omega + \langle F^{(2)^2} \rangle (2\Omega)^3 + \cdots + \langle F^{(n+1)^2} \rangle (2\Omega)^{2n+1} + \cdots. \tag{3.24}$$

Some comments are in order on this equation. First, we have shown by explicit calculation that the two theories agree in the doubly asymptotic limit of small impedance contrast and long wavelength. Second, in arriving at the comparison, we have found that the Taylor series expansion of the O'Doherty–Anstey dispersion relation (3.10) has coefficients that are all positive. Hence the wavenumber dispersion predicted by the O'Doherty–Anstey equation is strictly monotone as a function of frequency.

4. Discussion. We have seen that the two theories agree in their common domain of validity. The long wavelength theory provides the first dispersive term for arbitrary variation in material properties. The O'Doherty–Anstey formula gives the dispersion for small reflectivity but is valid for finite frequencies. It is not clear, however, just how high in frequency the O'Doherty–Anstey formula goes. The theory of Burridge, Papanicolaou, and White [1] makes no restrictions upon the frequency content; in fact, their analysis considered the impulse response. In this section, we examine the range of validity of the O'Doherty–Anstey formula (3.10) and show that, *within the first passing band* $0 < k < \pi$, *it is only valid for values of* k *that may be* $\Omega = O(1)$ *but must lie away from the band edge at* $k = \pi$. Hence the frequency is not confined to small values exclusively, but it is restricted to $\pi - k \neq o(1)$.

First, let us write the equation of motion in a different form. Let

$$p(x, \tau) = q(x)F(x)\, e^{-i\Omega\tau}, \tag{4.1}$$

where F is defined in (2.27). The two equations in (2.5) can be combined into a single Hill equation [10] for $q(x)$

$$q''(x) - [\Omega^2 + F''(x) - (F'(x))^2]q(x) = 0. \tag{4.2}$$

We look for a Bloch wave solution of the form

$$q(x) = Q(x)\, e^{ikx}, \tag{4.3}$$

where $Q(x)$ is unit periodic. Its equation is

$$Q''(x) + 2ikQ'(x) - [\Omega^2 - k^2 + F''(x) - (F'(x))^2]Q(x) = 0. \tag{4.4}$$

Let

$$Q(x) = 1 + \sum_{n\neq 0} d_n\, e^{i2\pi nx}. \tag{4.5}$$

If we assume that the Bloch wave is dominated by the effective medium wave propagating in the positive x-direction, i.e.,

$$d_n = O(F), \qquad \Omega^2 - k^2 = O(F^2), \tag{4.6}$$

then a simple perturbation analysis yields

$$d_n = \frac{-\pi n}{\pi n + k} \int_0^1 F(x) e^{-i2\pi nx}\, dx. \tag{4.7}$$

These coefficients can in turn be used to find the leading-order dispersion,

$$\Omega^2 - k^2 = \int_0^1 (F')^2\, dx + \sum_n \frac{\pi n}{\pi n + k} \int_0^1 F''(x)\, e^{i2\pi nx}\, dx \int_0^1 F(x')\, e^{-i2\pi nx'}\, dx'. \tag{4.8}$$

Integrating by parts and substituting for $r(x) = F'(x)$ gives

$$\Omega^2 - k^2 = \int_0^1 \int^2 r(x)r(x')\, dx\, dx'[\delta(x - x') - G(x - x')], \tag{4.9}$$

where

$$G(x) = \sum_n \frac{\pi n}{\pi n + k}\, e^{i2\pi nx}. \tag{4.10}$$

The function G can be found by standard means, so that (4.9) becomes finally

$$\Omega^2 = k^2 + \frac{k}{\sin k} \int_0^1 a(x) \cos k(2x - 1)\, dx, \tag{4.11}$$

where a is defined in (3.8). Furthermore, since a is $O(F^2)$, which is small by assumption, we may write this as

$$\Omega = k + \frac{1}{2 \sin k} \int_0^1 a(x) \cos k(2x - 1)\, dx + \cdots. \tag{4.12}$$

This is very similar to the O'Doherty–Anstey result (3.10) for $\hat{a}$. The equivalence is $i\hat{a}(2\Omega) \leftrightarrow (k - \Omega)$. Therefore we see that (4.12) is slightly different from (3.10), the difference being that (4.12) predicts $\Omega \to \infty$ as k approaches π, whereas the O'Doherty–

Anstey formula gives a finite limit for Ω. The singularity in (4.11) and (4.12) is not realistic since the frequency should reach a finite value less than π as $k \to \pi$. This is the frequency at which the first stopping band begins, and the frequencies below are the first passing band. The O'Doherty–Anstey formula can be obtained from (4.11) by substituting $k \to \Omega$ on the right-hand side. The resulting formula

$$\Omega^2 = k^2 + \frac{\Omega}{\sin \Omega} \int_0^1 a(x) \cos \Omega(2x - 1)\, dx \tag{4.13}$$

is entirely consistent with (3.10) and gives a finite value for the frequency that marks the beginning of the first stop band.

Formula (4.13) is rather interesting in its own right. The dispersion that it predicts is of $O(a)$, which is $O(F^2)$, except as the band edge is approached. There we have

$$\Omega^2 = \pi^2 - \frac{\pi}{\sin \Omega} \int_0^1 a(x) \cos 2\pi x\, dx. \tag{4.14}$$

The integral may be simplified using (3.8), and the implicit equation for Ω can be solved easily to give

$$\Omega = \pi - \frac{1}{2} \left| \int_0^1 r(x)\, e^{-i2\pi x}\, dx \right|^2. \tag{4.15}$$

The difference $\Omega - \pi$ is $O(F)$, as expected [10]. However, the magnitude of the difference is wrong. A related calculation in Brillouin's book [10] shows that the leading-order deviation of the frequency at the end of the first band gap ($k = \pi$) is actually

$$\Omega = \pi - \left| \int_0^1 r(x)\, e^{-i2\pi x}\, dx \right|^2 + \cdots. \tag{4.16}$$

Hence, the O'Doherty–Anstey theory gives the leading-order dispersion within the first band except near the band edge. As the frequency approaches the band edge, the dispersion effects change from $O(F^2)$ to $O(F)$. The O'Doherty–Anstey theory predicts the same transition but is only valid in the former regime, i.e., where the dispersion effects are $O(F^2)$. In this regime, the ansatz (4.6) holds, but it breaks down near the band edge at $k = \pi$, corresponding to the fact that d_{-1} becomes of order unity there [10].

4.1. Example. We conclude with an illustration of the preceding general analysis for a two-phase composite, as defined previously in § 2.4. The exact dispersion relation can be shown to be [18]

$$\cos k = \cos \Omega - 2 \sin n_1' \Omega \sin n_2' \Omega \sinh^2 \Delta. \tag{4.17}$$

The reflection coefficient and correlation function are, from (2.27), (3.2), and (3.8),

$$r(x) = [\delta(x - 0) - \delta(x - n_1')]\Delta, \tag{4.18a}$$

$$a(x) = [\delta(x - 0) - \delta(x - n_1')]\Delta^2. \tag{4.18b}$$

The modified O'Doherty–Anstey formula (4.13) now becomes

$$k^2 = \Omega^2 + \frac{\Omega \Delta^2}{\sin \Omega} [\cos (n_1' - n_2')\Omega - \cos \Omega]. \tag{4.19}$$

The systematic discrepancy of the modified O'Doherty–Anstey prediction at the band edge is apparent in Fig. 1. Note that it always underestimates the width of the first band gap by 50%.

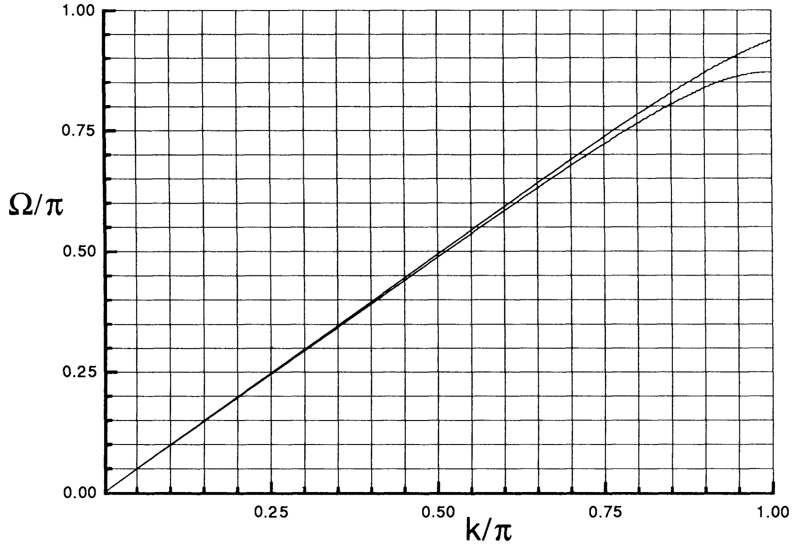

FIG. 1. *The exact (lower curve) and approximate dispersion curves in the first pass band according to* (4.17) *and* (4.19), *respectively. The parameters are* $n'_1 = n'_2 = 0.5$, $\zeta_2/\zeta_1 = 1.5$. *Note how the O'Doherty–Anstey prediction (upper curve) gives a band gap at* $k = \pi$, *which is one-half the correct value.*

5. Conclusions. The main purpose of this paper has been to show how two different asymptotic theories for waves in nonuniform media agree in their common domain of validity. The long wavelength theory for periodic media predicts the asymptotic dispersion relation (2.11), where the parameter D_3 of (2.12) describes the first dispersive effects for arbitrary variations in impedance. The finely layered medium theory, based upon the O'Doherty–Anstey equation (3.6), gives a quite different representation for the wave number (3.10). This is restricted to small contrasts in material properties but is not explicitly restricted in frequency. We have seen that the two theories agree in the simultaneous limit of low frequencies and small variations. Although this result is not surprising, it serves as a useful corroboration of both theories. It should be remarked that these are among the few rigorous asymptotic theories for nonuniform media. The O'Doherty–Anstey theory is capable of handling both periodic and nonperiodic media, although it is primarily used in applications to nonperiodic media, for which there are very few precise results available.

In deriving the equivalence, we have found that the O'Doherty–Anstey dispersion relation for a periodic medium has a Taylor series expansion in frequency with coefficients that are all positive; see (3.24). It has also been demonstrated that the O'Doherty–Anstey theory does not give the correct band gap in a periodic medium. Hence its range of validity in the frequency domain is limited to the first pass band with wavenumber of order unity but bounded away from π.

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