Fundamental elastodynamic solutions for anisotropic media with ellipsoidal slowness surfaces

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When the slowness surface $\mathscr{S}$ of an anisotropic elastic medium consists of three concentric ellipsoids, solutions of the displacement equations of motion can be generated from functions satisfying scalar wave equations and the problem of constructing the fundamental, or Green's, tensor $\boldsymbol{G}$ for an infinite region becomes tractable. This paper has two aims: first, to find all the conditions on the linear elastic moduli under which $\mathscr{S}$ is ellipsoidal (that is the union of concentric ellipsoids), and, second, to determine $\boldsymbol{G}$ for each case in which $\mathscr{S}$ simplifies in this way. The two stages of the investigation have a key idea in common. The ellipsoidal form of $\mathscr{S}$ requires the eigenvalues of the acoustical tensor $\boldsymbol{Q}(\boldsymbol{n})$ to be quadratic forms in the unit vector argument $\boldsymbol{n}$: at least two of the associated eigenvectors are either constant or linear in $\boldsymbol{n}$ and the squared moduli of the linear eigenvectors are divisors of eigenvalue differences. These algebraic properties provide a classification of media with ellipsoidal slowness surfaces and aid in characterizing the membership of each class. The first stage culminates in four sets of conditions, labelled A, B, C(i) and C(ii): case C(i) is a restriction of transverse isotropy and the others are specializations of orthorhombic symmetry. At the second stage $\boldsymbol{n}$ is replaced by the gradient ∂ with respect to spatial position and polynomials in $\boldsymbol{n}$ become differential operators. The construction of $\boldsymbol{G}$ involves two canonical problems of classical type, an initial-value problem for a scalar wave equation and a potential problem for a pair of 'charged' ellipsoids. The divisibility property indicated above implies that the ellipsoids are confocals carrying equal and opposite charges and these characteristics render the fundamental solution causal in the sense that the entire disturbance excited by the point impulse begins with the first and ends with the last of the wavefront arrivals. The structures of the fundamental solutions in cases A, B, C(i) and C(ii) are described and the latter solution is shown to reduce to a standard result of Stokes in the degenerate case of isotropy. Mention is also made of a specialization of case B, appropriate to a transversely isotropic medium which is inextensible in the direction of the symmetry axis.

1. Introduction

It is well known that there are special forms of anisotropy for which the slowness surface $\mathscr{S}$ of an elastic material has a particularly simple shape and structure. In the

Proc. R. Soc. Lond. A (1993) **440**, 655–681
Printed in Great Britain

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case of transverse isotropy there are two relations between the elastic moduli (equations (3.13) and (6.13) below) under which $\mathscr{S}$ consists of three spheroids with a common axis of symmetry (Payton 1983, p. 96; Chadwick 1989, §5*e*). In the cubic system a connexion similar to (6.13) has been noted which reduces $\mathscr{S}$ to three identical spheroids with mutually orthogonal axes of symmetry (Chadwick & Smith 1982, §8.3). In each instance the displacement equations of motion decompose into scalar wave equations with a concurrent simplification of the linear elastodynamics.

Recently, two of the authors set out to find all the restrictions on the moduli of an elastic material of general anisotropy which resolve $\mathscr{S}$ into the union of *aligned ellipsoids*, that is three ellipsoids with common principal axes each defining a direction in which a longitudinal plane wave can propagate (Chadwick & Norris 1990). They showed that the material has to have orthorhombic symmetry and obtained five sets of conditions on the moduli, one of them necessary and each sufficient for $\mathscr{S}$ to consist of aligned ellipsoids. The five solutions reproduce, by appropriate specialization, the results for transversely isotropic and cubic materials referred to earlier.

It is shown herein that it is not necessary to assume *a priori* that the ellipsoidal sheets of $\mathscr{S}$ are aligned in the sense defined above: the restriction to orthorhombic symmetry (in one case to transverse isotropy) and the conditions on the elastic moduli found by Chadwick & Norris (1990) continue to hold when the sheets of $\mathscr{S}$ are required only to be ellipsoidal and concentric. This relaxation is achieved on the basis of algebraic properties of the eigenvalues and eigenvectors of the acoustical tensor $\boldsymbol{Q}(\boldsymbol{n})$ which are also found to have a wider significance.

The most basic elastodynamic problem for which an explicit solution is brought within reach by the simplification of $\mathscr{S}$ to ellipsoidal form is that of determining the fundamental, or Green's, tensor $\boldsymbol{G}$ for an unbounded body. The differential system governing $\boldsymbol{G}$ (equations (4.1) below) contains the operator $\boldsymbol{Q}(\partial)$ where ∂ is the spatial gradient. When $\mathscr{S}$ is ellipsoidal, the eigenvalues of $\boldsymbol{Q}(\partial)$ are second-order linear differential operators. The duality between algebraic and differential objects extends to the eigenvectors of $\boldsymbol{Q}(\partial)$ and leads naturally to representations of $\boldsymbol{G}$ in terms of functions satisfying scalar wave equations. The construction of $\boldsymbol{G}$ requires only the solution of a Cauchy problem for the typical wave equation and the calculation of the potential due to two 'charged' ellipsoids. These surfaces are, in essence, polar reciprocals of sheets of $\mathscr{S}$ and the 'charges' arise from the solution of the Cauchy problem. Crucially, the algebraic properties mentioned in the previous paragraph imply that the ellipsoids are confocals and the total charges equal and opposite. The causality of the fundamental solution stems from these relations.

The algebraic consequences of $\mathscr{S}$ being the union of concentric ellipsoids are elicited in §2 and are shown there to impose a threefold classification on the eigenvectors of $\boldsymbol{Q}(\boldsymbol{n})$. The three possibilities entail the results obtained, under more restrictive hypotheses, by Chadwick & Norris (1990), but only after long and elaborate manipulations. To avoid disturbing the logical development, we summarize the conclusions in §3 and defer the derivations to the Appendix. The representation of $\boldsymbol{G}$ in terms of scalar wave functions is effected in §4 and the two canonical problems underlying the actual construction of $\boldsymbol{G}$ are solved in §5. The structures of the fundamental solutions corresponding to the various ellipsoidal forms of $\mathscr{S}$ are discussed in §6 and two special cases are considered: isotropy, for which the Stokes solution is recovered, and inextensible transverse isotropy, for which a result of comparable simplicity is obtained.

2. The nature of the eigenvalues and eigenvectors of the acoustical tensor when the slowness surface consists of concentric ellipsoids

We are concerned with an elastic material that is anisotropic in relation to a natural reference configuration N. The mass density and the linear elasticity tensor of the material in N are denoted by ρ and $\boldsymbol{C}$, respectively, and the components of $\boldsymbol{C}$ relative to an arbitrary orthonormal basis $b = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ by C_{ijkl}: $\boldsymbol{C}$ is assumed to be symmetric and positive definite, so that

$$C_{jikl} = C_{klij} = C_{ijkl} \tag{2.1}$$

and

$$C_{pqrs} S_{pq} S_{rs} > 0 \quad \forall \quad \text{non-zero symmetric tensors } \boldsymbol{S}. \tag{2.2}$$

Italic subscripts take the values 1, 2, 3 throughout and summation is implied on repetitions of p, q, r, s only. In this section all vector and tensor components relate to b.

(a) Basic properties of the acoustical tensor

Given the set $\mathscr{U}$ of all unit vectors and an arbitrary member $\boldsymbol{n}$ of $\mathscr{U}$, the acoustical tensor $\boldsymbol{Q}(\boldsymbol{n})$ of the elastic material under consideration is defined component-wise by

$$Q_{ij}(\boldsymbol{n}) = C_{pirj} n_p n_r \quad \forall \quad \boldsymbol{n} \in \mathscr{U}. \tag{2.3}$$

On account of (2.1) and (2.2), $\boldsymbol{Q}(\boldsymbol{n})$ is symmetric and positive definite and thus has positive eigenvalues $\lambda_i(\boldsymbol{n})$ and mutually orthogonal real eigenvectors $\boldsymbol{q}_i(\boldsymbol{n})$ satisfying

$$\boldsymbol{Q}(\boldsymbol{n})\, \boldsymbol{q}_i(\boldsymbol{n}) = \lambda_i(\boldsymbol{n})\, \boldsymbol{q}_i(\boldsymbol{n}) \quad \forall \quad \boldsymbol{n} \in \mathscr{U}. \tag{2.4}$$

The slowness surface $\mathscr{S}$ of the material is the three-sheeted surface in $\mathbb{R}^3$ specified by

$$\boldsymbol{s}(\boldsymbol{n}) = \{\rho^{-1} \lambda_i(\boldsymbol{n})\}^{-\frac{1}{2}} \boldsymbol{n}, \quad i = 1, 2, 3, \quad \forall \quad \boldsymbol{n} \in \mathscr{U}, \tag{2.5}$$

the slowness vector $\boldsymbol{s}(\boldsymbol{n})$ measuring position relative to the centre O. If the sheets of $\mathscr{S}$ are concentric ellipsoids, it follows from (2.5) that the eigenvalues $\lambda_i(\boldsymbol{n})$ are homogeneous quadratic forms in $\boldsymbol{n}$, and then from (2.4) and (2.3) that the eigenvectors $\boldsymbol{q}_i(\boldsymbol{n})$ are also homogeneous polynomials in $\boldsymbol{n}$. We suppose that each eigenvector has been made irreducible by the removal of scalar factors and denote by d_i the degree of $\boldsymbol{q}_i(\boldsymbol{n})$ in $\boldsymbol{n}$.

(b) Distinct eigenvalues

We consider first the situation in which no two of $\lambda_i(\boldsymbol{n})$ are equal for all $\boldsymbol{n} \in \mathscr{U}$ and label the eigenvalues and eigenvectors so that $d_1 \leqslant d_2 \leqslant d_3$. Because of the orthogonality of $\boldsymbol{q}_i(\boldsymbol{n})$ we can set

$$\boldsymbol{q}_3(\boldsymbol{n}) = \boldsymbol{q}_1(\boldsymbol{n}) \times \boldsymbol{q}_2(\boldsymbol{n}), \tag{2.6}$$

in consequence of which

$$d_3 = d_1 + d_2. \tag{2.7}$$

Let

$$m_i(\boldsymbol{n}) = \boldsymbol{q}_i(\boldsymbol{n}) \cdot \boldsymbol{q}_i(\boldsymbol{n}). \tag{2.8}$$

Then $\boldsymbol{Q}(\boldsymbol{n})$, its adjugate $\boldsymbol{Q}^{\mathrm{adj}}(\boldsymbol{n})$ and the identity tensor $\boldsymbol{I}$ have the spectral forms

$$\boldsymbol{Q}(\boldsymbol{n}) = \sum_{i=1}^{3} \lambda_i(\boldsymbol{n}) \{m_i(\boldsymbol{n})\}^{-1} \boldsymbol{q}_i(\boldsymbol{n}) \otimes \boldsymbol{q}_i(\boldsymbol{n}), \tag{2.9}$$

$$\boldsymbol{Q}^{\mathrm{adj}}(\boldsymbol{n}) = \sum_{i=1}^{3} \lambda_j(\boldsymbol{n})\, \lambda_k(\boldsymbol{n}) \{m_i(\boldsymbol{n})\}^{-1} \boldsymbol{q}_i(\boldsymbol{n}) \otimes \boldsymbol{q}_i(\boldsymbol{n}), \quad i \neq j \neq k \neq i,$$

$$\boldsymbol{I} = \sum_{i=1}^{3} \{m_i(\boldsymbol{n})\}^{-1} \boldsymbol{q}_i(\boldsymbol{n}) \otimes \boldsymbol{q}_i(\boldsymbol{n}). \tag{2.10}$$

These representations combine to give

$$\left.\begin{aligned} \boldsymbol{Q}^{\mathrm{adj}}(\boldsymbol{n}) + \lambda_1(\boldsymbol{n})\,\boldsymbol{Q}(\boldsymbol{n}) &= \lambda_1(\boldsymbol{n})\{\lambda_2(\boldsymbol{n})+\lambda_3(\boldsymbol{n})\}\boldsymbol{I} \\ &\quad -\{m_1(\boldsymbol{n})\}^{-1}\{\lambda_1(\boldsymbol{n})-\lambda_2(\boldsymbol{n})\}\{\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})\}\boldsymbol{q}_1(\boldsymbol{n})\otimes\boldsymbol{q}_1(\boldsymbol{n}), \\ m_2(\boldsymbol{n})\,\boldsymbol{Q}(\boldsymbol{n}) &= \lambda_3(\boldsymbol{n})\,m_2(\boldsymbol{n})\,\boldsymbol{I} + \{\lambda_2(\boldsymbol{n})-\lambda_3(\boldsymbol{n})\}\boldsymbol{q}_2(\boldsymbol{n})\otimes\boldsymbol{q}_2(\boldsymbol{n}) \\ &\quad - m_2(\boldsymbol{n})\{m_1(\boldsymbol{n})\}^{-1}\{\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})\}\boldsymbol{q}_1(\boldsymbol{n})\otimes\boldsymbol{q}_1(\boldsymbol{n}), \end{aligned}\right\} \tag{2.11}$$

$$\left.\begin{aligned} \boldsymbol{Q}^{\mathrm{adj}}(\boldsymbol{n}) + \lambda_2(\boldsymbol{n})\,\boldsymbol{Q}(\boldsymbol{n}) &= \lambda_2(\boldsymbol{n})\{\lambda_3(\boldsymbol{n})+\lambda_1(\boldsymbol{n})\}\boldsymbol{I} \\ &\quad -\{m_2(\boldsymbol{n})\}^{-1}\{\lambda_1(\boldsymbol{n})-\lambda_2(\boldsymbol{n})\}\{\lambda_2(\boldsymbol{n})-\lambda_3(\boldsymbol{n})\}\boldsymbol{q}_2(\boldsymbol{n})\otimes\boldsymbol{q}_2(\boldsymbol{n}), \\ m_1(\boldsymbol{n})\,\boldsymbol{Q}(\boldsymbol{n}) &= \lambda_3(\boldsymbol{n})\,m_1(\boldsymbol{n})\,\boldsymbol{I} - \{\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})\}\boldsymbol{q}_1(\boldsymbol{n})\otimes\boldsymbol{q}_1(\boldsymbol{n}) \\ &\quad + m_1(\boldsymbol{n})\{m_2(\boldsymbol{n})\}^{-1}\{\lambda_2(\boldsymbol{n})-\lambda_3(\boldsymbol{n})\}\boldsymbol{q}_2(\boldsymbol{n})\otimes\boldsymbol{q}_2(\boldsymbol{n}). \end{aligned}\right\} \tag{2.12}$$

All the terms in equations (2.11) and (2.12) except those involving $\{m_1(\boldsymbol{n})\}^{-1}$ or $\{m_2(\boldsymbol{n})\}^{-1}$ are polynomials in $\boldsymbol{n}$. We infer from (2.11) that $m_1(\boldsymbol{n})$ divides $\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})$ and from (2.12) that $m_2(\boldsymbol{n})$ divides $\lambda_2(\boldsymbol{n})-\lambda_3(\boldsymbol{n})$. In view of (2.8), this means that $d_1 \leqslant 1, d_2 \leqslant 1$, and, bearing in mind (2.7), we arrive at the following possibilities.

A. $d_1 = 0$, $d_2 = 0$: $\boldsymbol{q}_1, \boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are all constant.

B. $d_1 = 0$, $d_2 = 1$: $\boldsymbol{q}_1$ is constant; $\boldsymbol{q}_2(\boldsymbol{n})$ and $\boldsymbol{q}_3(\boldsymbol{n})$ are homogeneous linear forms in $\boldsymbol{n}$.

C. $d_1 = 1$, $d_2 = 1$: $\boldsymbol{q}_1(\boldsymbol{n})$ and $\boldsymbol{q}_2(\boldsymbol{n})$ are homogeneous linear forms and $\boldsymbol{q}_3(\boldsymbol{n})$ a homogeneous quadratic form in $\boldsymbol{n}$.

(c) *Coincident eigenvalues*

When $\lambda_2(\boldsymbol{n}) = \lambda_3(\boldsymbol{n})$ for all $\boldsymbol{n} \in \mathscr{U}$, equations (2.9) and (2.10) yield

$$\boldsymbol{Q}(\boldsymbol{n}) = \lambda_3(\boldsymbol{n})\,\boldsymbol{I} - \{m_1(\boldsymbol{n})\}^{-1}\{\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})\}\boldsymbol{q}_1(\boldsymbol{n})\otimes\boldsymbol{q}_1(\boldsymbol{n}). \tag{2.13}$$

We deduce that $m_1(\boldsymbol{n})$ divides $\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})$, and hence that $d_1 \leqslant 1$. In this case $\boldsymbol{q}_2(\boldsymbol{n})$ and $\boldsymbol{q}_3(\boldsymbol{n})$ are any pair of vectors mutually orthogonal with $\boldsymbol{q}_1(\boldsymbol{n})$ and we can take

$$\boldsymbol{q}_2(\boldsymbol{n}) = \boldsymbol{k} \times \boldsymbol{q}_1(\boldsymbol{n}),$$

together with (2.6), $\boldsymbol{k}$ being a constant vector. Then $d_1 = d_2$, and either $d_1 = d_2 = 0$ or $d_1 = d_2 = 1$. We thus return to possibility A or C.

If the three eigenvalues are identically equal, we have

$$\boldsymbol{Q}(\boldsymbol{m}) = \lambda(\boldsymbol{m})\,\boldsymbol{I}, \quad \boldsymbol{Q}(\boldsymbol{n}) = \lambda(\boldsymbol{n})\,\boldsymbol{I} \quad \forall \quad \boldsymbol{m}, \boldsymbol{n} \in \mathscr{U},$$

and hence

$$\lambda(\boldsymbol{m}) = \boldsymbol{n}\cdot\{\boldsymbol{Q}(\boldsymbol{m})\,\boldsymbol{n}\}, \quad \lambda(\boldsymbol{n}) = \boldsymbol{m}\cdot\{\boldsymbol{Q}(\boldsymbol{n})\,\boldsymbol{m}\}. \tag{2.14}$$

The right-hand sides of equations (2.14) are equal, by (2.3) and (2.1). The repeated eigenvalue is therefore a constant, λ, and

$$\boldsymbol{Q}(\boldsymbol{n}) = \lambda \boldsymbol{I} \quad \forall \quad \boldsymbol{n} \in \mathscr{U}. \tag{2.15}$$

As proved in part (*a*) of the Appendix, (2.15) is incompatible with the positive definiteness condition (2.2), so at most two of the eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ can coincide.

3. Classification of elastic materials with ellipsoidal slowness surfaces

The elastic moduli in N relative to the basis b are defined by

$$c_{\alpha\beta} = \operatorname{tr}\{\boldsymbol{B}_\alpha\,\boldsymbol{C}[\boldsymbol{B}_\beta]\}, \tag{3.1}$$

where

$$\left.\begin{aligned} B_1 = B_{11}, \quad B_2 = B_{22}, \quad B_3 = B_{33}, \\ B_4 = \tfrac{1}{2}(B_{23}+B_{32}), \quad B_5 = \tfrac{1}{2}(B_{31}+B_{13}), \quad B_6 = \tfrac{1}{2}(B_{12}+B_{21}), \end{aligned}\right\} \tag{3.2}$$

and

$$B_{ij} = b_i \otimes b_j. \tag{3.3}$$

In (3.1), $C[A]$ is the second-order tensor with components $C_{ijrs}A_{rs}$ and tr denotes the trace: when no indication is made to the contrary, Greek subscripts take the values $1, \ldots, 6$. The notations $c_{\alpha\beta}$ and n_i are used in the sequel for all elastic moduli and all components of $\boldsymbol{n} \in \mathscr{U}$, care being taken to specify the basis to which reference is made. The useful inequalities

$$c_{ii} > 0, \quad c_{ii}+c_{jj}+2c_{ij} > 0, \quad i \neq j, \tag{3.4}$$

result from the choices $\boldsymbol{S} = \boldsymbol{B}_{ii}$ and $\boldsymbol{S} = \boldsymbol{B}_{ii}+\boldsymbol{B}_{jj}$ in (2.2).

For each of the possibilities A, B and C distinguished in §2*b*, the elastic material turns out to be orthorhombic or transversely isotropic and each of the ellipsoidal sheets of $\mathscr{S}$ has, as its principal axes, mutually orthogonal axes of symmetry. In relation to the orthonormal basis $a = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ aligned with these axes, the elastic moduli therefore satisfy the conditions

$$c_{14} = c_{15} = c_{16} = c_{24} = c_{25} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = c_{46} = c_{56} = 0, \tag{3.5}$$

and the equations of the sheets of $\mathscr{S}$ are of the form

$$A_i s_1^2 + B_i s_2^2 + C_i s_3^2 = \rho,$$

where $s_i = \boldsymbol{a}_i \cdot \boldsymbol{s}(\boldsymbol{n})$ are the components of the slowness. From (2.5), the eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ are

$$\lambda_i(\boldsymbol{n}) = A_i n_1^2 + B_i n_2^2 + C_i n_3^2,$$

with $n_i = \boldsymbol{a}_i \cdot \boldsymbol{n}$.

The results stated in the following subsections are proved in the Appendix. They refer exclusively to the crystallographic basis a.

(*a*) *Possibility* A

The material is orthorhombic and the non-zero elastic moduli are related by

$$c_{44} = -c_{23}, \quad c_{55} = -c_{13}, \quad c_{66} = -c_{12}. \tag{3.6}$$

The eigenvalues and associated eigenvectors of $\boldsymbol{Q}(\boldsymbol{n})$ are

$$\left.\begin{aligned} \lambda_1(\boldsymbol{n}) &= c_{11} n_1^2 + c_{66} n_2^2 + c_{55} n_3^2, \\ \lambda_2(\boldsymbol{n}) &= c_{66} n_1^2 + c_{22} n_2^2 + c_{44} n_3^2, \\ \lambda_3(\boldsymbol{n}) &= c_{55} n_1^2 + c_{44} n_2^2 + c_{33} n_3^2, \end{aligned}\right\} \tag{3.7}$$

and

$$\boldsymbol{q}_i = \boldsymbol{a}_i. \tag{3.8}$$

Equations (3.6) to (3.8) reproduce case 5 of Chadwick & Norris (1990).

(*b*) *Possibility* B

The symmetry is again orthorhombic and the non-zero elastic moduli satisfy

$$c_{44} = c_{55} = -c_{13} = -c_{23}, \quad c_{66} = (c_{11}+c_{22}+2c_{12})^{-1}(c_{11}c_{22}-c_{12}^2). \tag{3.9}$$

The eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ are

$$\left.\begin{aligned}\lambda_1(\boldsymbol{n}) &= c_{44}(n_1^2+n_2^2)+c_{33}\,n_3^2,\\ \lambda_2(\boldsymbol{n}) &= c_{11}\,n_1^2+c_{22}\,n_2^2+c_{44}\,n_3^2,\\ \lambda_3(\boldsymbol{n}) &= c_{66}(n_1^2+n_2^2)+c_{44}\,n_3^2,\end{aligned}\right\} \tag{3.10}$$

and corresponding eigenvectors are

$$\left.\begin{aligned}\boldsymbol{q}_1 &= \boldsymbol{a}_3,\\ \boldsymbol{q}_2(\boldsymbol{n}) &= (c_{11}+c_{12})\,n_1\,\boldsymbol{a}_1+(c_{12}+c_{22})\,n_2\,\boldsymbol{a}_2,\\ \boldsymbol{q}_3(\boldsymbol{n}) &= -(c_{12}+c_{22})\,n_2\,\boldsymbol{a}_1+(c_{11}+c_{12})\,n_1\,\boldsymbol{a}_2.\end{aligned}\right\} \tag{3.11}$$

Equations (3.9) to (3.11) duplicate case 4 of Chadwick & Norris (1990). Two of the sheets of $\mathscr{S}$ are spheroids with a common rotational axis.

If, in addition to (3.9), $c_{44} = c_{66}$, we recover equations $(\mathrm{C2})_3$, $(\mathrm{S2})_3$ and $(\mathrm{P2})_3$ of Chadwick & Norris (1990) and hence their case 2.

(c) *Possibility* C

There are two alternatives.

(i) The material is transversely isotropic with the axis of symmetry in the direction of $\boldsymbol{a}_3$. Equations (3.5) are accordingly supplemented by

$$c_{11} = c_{22}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55}, \quad c_{66} = \tfrac{1}{2}(c_{11}-c_{12}), \tag{3.12}$$

and the relation

$$c_{44} = (c_{11}+c_{33}+2c_{13})^{-1}\,(c_{11}\,c_{33}-c_{13}^2) \tag{3.13}$$

also holds. The eigenvalues and accompanying eigenvectors of $\boldsymbol{Q}(\boldsymbol{n})$ are

$$\lambda_1(\boldsymbol{n}) = c_{11}(n_1^2+n_2^2)+c_{33}\,n_3^2, \quad \lambda_2(\boldsymbol{n}) = c_{66}(n_1^2+n_2^2)+c_{44}\,n_3^2, \quad \lambda_3 = c_{44}, \tag{3.14}$$

and

$$\left.\begin{aligned}\boldsymbol{q}_1(\boldsymbol{n}) &= (c_{11}+c_{13})\,(n_1\,\boldsymbol{a}_1+n_2\,\boldsymbol{a}_2)+(c_{13}+c_{33})\,n_3\,\boldsymbol{a}_3,\\ \boldsymbol{q}_2(\boldsymbol{n}) &= -n_2\,\boldsymbol{a}_1+n_1\,\boldsymbol{a}_2,\\ \boldsymbol{q}_3(\boldsymbol{n}) &= -(c_{13}+c_{33})\,n_3(n_1\,\boldsymbol{a}_1+n_2\,\boldsymbol{a}_2)+(c_{11}+c_{13})\,(n_1^2+n_2^2)\,\boldsymbol{a}_3.\end{aligned}\right\} \tag{3.15}$$

Equations (3.12) to (3.15) represent case 3 of Chadwick & Norris (1990), and (3.13) defines one of the specializations of transverse isotropy mentioned in the opening paragraph of §1. Two sheets of $\mathscr{S}$ are spheroids, with $\boldsymbol{a}_3$ along the axis of rotation, and the third sheet is a sphere.

(ii) Orthorhombic symmetry once more prevails with the non-zero elastic moduli connected by

$$\begin{aligned}c_{44} = c_{55} = c_{66} &= (c_{22}+c_{33}+2c_{23})^{-1}\,(c_{22}\,c_{33}-c_{23}^2)\\ &= (c_{11}+c_{33}+2c_{13})^{-1}\,(c_{11}\,c_{33}-c_{13}^2) = (c_{11}+c_{22}+2c_{12})^{-1}\,(c_{11}\,c_{22}-c_{12}^2).\end{aligned} \tag{3.16}$$

The eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ are

$$\lambda_1(\boldsymbol{n}) = c_{11}\,n_1^2+c_{22}\,n_2^2+c_{33}\,n_3^2, \quad \lambda_2 = \lambda_3 = c_{44}, \tag{3.17}$$

indicating that $\mathscr{S}$ consists of an ellipsoid and two coincident spheres. Associated eigenvectors are

$$\boldsymbol{q}_1(\boldsymbol{n}) = (c_{13}+c_{44})\,n_1\,\boldsymbol{a}_1+(c_{23}+c_{44})\,n_2\,\boldsymbol{a}_2+(c_{33}-c_{44})\,n_3\,\boldsymbol{a}_3 \tag{3.18}$$

and any pair forming with $\boldsymbol{q}_1(\boldsymbol{n})$ a mutually orthogonal set.

This alternative replicates case 1 of Chadwick & Norris (1990) and demonstrates that the coincidence of two eigenvalues, considered in §2*c*, can arise only under possibility C.

(*d*) *Summary*

The four ellipsoidal slowness surfaces to which possibilities A, B and C give rise are henceforth referred to as cases A, B, C(i) and C(ii). As noted above, they are equivalent, in turn, to cases 5, 4, 3 and 1 in the classification of Chadwick & Norris (1990). Case 2 of Chadwick & Norris is a particularization of case B and does not require individual consideration.

4. Representations of the fundamental elastodynamic tensor in terms of scalar wave functions

The first step in the determination of fundamental elastodynamic tensors for anisotropic media with ellipsoidal slowness surfaces is to decompose the initial-value problem for $\boldsymbol{G}$ into uncoupled Cauchy problems for scalar wave functions from which $\boldsymbol{G}$ can subsequently be constructed. In this section we formulate the scalar Cauchy problems by a formal symbolic argument the results of which may be confirmed by direct calculation for each of cases A, B, C(i) and C(ii).

The initial-value problem for the fundamental tensor can be stated as

$$\left.\begin{array}{l} \rho^{-1}\boldsymbol{Q}(\partial)\,\boldsymbol{G} = \partial^2\boldsymbol{G}/\partial t^2, \quad t>0, \\ \boldsymbol{G}=\boldsymbol{0}, \quad \partial\boldsymbol{G}/\partial t = \delta(\boldsymbol{x})\,\boldsymbol{I}, \quad t=0 \end{array}\right\} \tag{4.1}$$

(Burridge 1967, §1), where ∂ is the gradient with respect to position $\boldsymbol{x}$ from the point at which the impulsive force acts and t is the time elapsed from the instant of application. If $x_i = \boldsymbol{b}_i\cdot\boldsymbol{x}$ are cartesian coordinates in b, ∂ has components $\partial/\partial x_i$ in this basis and $\delta(\boldsymbol{x})$ is the product $\delta(x_1)\,\delta(x_2)\,\delta(x_3)$ of delta functions.

(*a*) *The spectral form of* $\rho^{-1}\boldsymbol{Q}(\partial)$

We denote by $L_i(\partial)$ the eigenvalues and by $\boldsymbol{D}_i(\partial)$ associated eigenvectors of the symmetric tensor-valued differential operator $\rho^{-1}\boldsymbol{Q}(\partial)$ and define

$$M_i(\partial) = \boldsymbol{D}_i(\partial)\cdot\boldsymbol{D}_i(\partial), \quad \boldsymbol{E}_i(\partial) = M_i^{-1}(\partial)\,\boldsymbol{D}_i(\partial)\otimes\boldsymbol{D}_i(\partial). \tag{4.2}$$

When $\mathscr{S}$ is the union of concentric ellipsoids, $L_i(\partial)$ are second-order differential operators with constant coefficients, derived from $\rho^{-1}\lambda_i(\boldsymbol{n})$ by substituting ∂ for $\boldsymbol{n}$. The eigenvectors, assumed to be irreducible as in §2*a*, are either constants or differential operators formed from $\boldsymbol{q}_i(\boldsymbol{n})$ by the same replacement. The tensors $\boldsymbol{E}_i(\partial)$ are projection operators, satisfying

$$\boldsymbol{E}_i(\partial)\,\boldsymbol{E}_j(\partial) = \delta_{ij}\,\boldsymbol{E}_i(\partial). \tag{4.3}$$

The spectral forms analogous to (2.9) and (2.10) are

$$\rho^{-1}\boldsymbol{Q}(\partial) = \sum_{i=1}^{3} L_i(\partial)\,\boldsymbol{E}_i(\partial), \quad \boldsymbol{I} = \sum_{i=1}^{3}\boldsymbol{E}_i(\partial). \tag{4.4}$$

Taking first the case in which $L_i(\partial)$ are distinct, we eliminate $\boldsymbol{E}_3(\partial)$ between equations (4.4) to obtain

$$\rho^{-1}\boldsymbol{Q}(\partial) = L_3(\partial)\,\boldsymbol{I} - \{L_3(\partial)-L_1(\partial)\}\,\boldsymbol{E}_1(\partial) + \{L_2(\partial)-L_3(\partial)\}\,\boldsymbol{E}_2(\partial), \tag{4.5}$$

or, using $(4.2)_2$,

$$\rho^{-1}\boldsymbol{Q}(\partial) = L_3(\partial)\boldsymbol{I}-\{L_3(\partial)-L_1(\partial)\}M_1^{-1}(\partial)\boldsymbol{D}_1(\partial)\otimes\boldsymbol{D}_1(\partial) \\ +\{L_2(\partial)-L_3(\partial)\}M_2^{-1}(\partial)\boldsymbol{D}_2(\partial)\otimes\boldsymbol{D}_2(\partial).$$

The algebraic considerations in §2*b* affirm that $M_1(\partial)$ and $M_2(\partial)$ divide $L_3(\partial)-L_1(\partial)$ and $L_2(\partial)-L_3(\partial)$ respectively, so there exist C_1 and C_2 such that

$$L_3(\partial)-L_1(\partial) = C_1 M_1(\partial), \quad L_2(\partial)-L_3(\partial) = C_2 M_2(\partial). \tag{4.6}$$

Since $L_i(\partial)$ are second-order differential operators, C_1 (respectively C_2) is a constant or a second-order differential operator according as $\boldsymbol{D}_1(\partial)$ (respectively $\boldsymbol{D}_2(\partial)$) is a linear differential operator or a constant.

When $L_2(\partial) = L_3(\partial)$, equation (4.5) reduces to

$$\rho^{-1}\boldsymbol{Q}(\partial) = L_3(\partial)\boldsymbol{I}-\{L_3(\partial)-L_1(\partial)\}\boldsymbol{E}_1(\partial), \tag{4.7}$$

and the algebraic argument again produces $(4.6)_1$.

(*b*) *Representations of* $\boldsymbol{G}$

Suppose now that the fundamental tensor is generated by scalar functions ϕ_i through the representation

$$\boldsymbol{G} = \sum_{i=1}^{3} \boldsymbol{E}_i(\partial)\,\phi_i. \tag{4.8}$$

Introducing $(4.4)_1$ and (4.8) into equation $(4.1)_1$ and applying (4.3), we obtain

$$\sum_{i=1}^{3} \boldsymbol{E}_i(\partial)\{L_i(\partial)\,\phi_i-\partial^2\phi_i/\partial t^2\} = \boldsymbol{0}, \quad t>0, \tag{4.9}$$

while the conditions $(4.1)_2$, in conjunction with (4.8) and $(4.4)_2$, yield

$$\sum_{i=1}^{3} \boldsymbol{E}_i(\partial)\,\phi_i = \boldsymbol{0}, \quad \sum_{i=1}^{3} \boldsymbol{E}_i(\partial)\{\partial\phi_i/\partial t-\delta(\boldsymbol{x})\} = \boldsymbol{0}, \quad t=0. \tag{4.10}$$

It is evident from (4.9) and (4.10) that the initial-value problem (4.1) is solved if, for $i = 1,2,3, \phi_i$ satisfies the Cauchy problem

$$\left.\begin{aligned} L_i(\partial)\,\phi_i = \partial^2\phi_i/\partial t^2, \quad t>0, \\ \phi_i = 0, \quad \partial\phi_i/\partial t = \delta(\boldsymbol{x}), \quad t=0. \end{aligned}\right\} \tag{4.11}$$

Equation $(4.4)_2$ can be used to rewrite (4.8) as

$$\boldsymbol{G} = \phi_3\boldsymbol{I}+\boldsymbol{E}_1(\partial)\,(\phi_1-\phi_3)+\boldsymbol{E}_2(\partial)\,(\phi_2-\phi_3). \tag{4.12}$$

This variant of (4.8) avoids the calculation of $\boldsymbol{E}_3(\partial)\,\phi_3$ which becomes difficult in case C(i) where, as seen from $(3.15)_3$, $\boldsymbol{D}_3(\partial)$ is a second-order differential operator. It transpires, however, that (4.12) is more convenient than (4.8) whenever the eigenvectors of $\boldsymbol{Q}(\partial)$ are not all constant. We consequently use (4.12) in cases B and C(i), and (4.8) in case A.

When $L_2(\partial) = L_3(\partial)$, we adopt, in place of (4.8), the representation

$$\boldsymbol{G} = \phi_3\boldsymbol{I}+\boldsymbol{E}_1(\partial)\,(\phi_1-\phi_3). \tag{4.13}$$

The result of entering (4.7) and (4.13) into equation $(4.1)_1$ and using the relations

$$\{\boldsymbol{E}_1(\partial)\}^2 = \boldsymbol{E}_1(\partial), \quad \{\boldsymbol{I}-\boldsymbol{E}_1(\partial)\}^2 = \boldsymbol{I}-\boldsymbol{E}_1(\partial), \quad \boldsymbol{E}_1(\partial)\{\boldsymbol{I}-\boldsymbol{E}_1(\partial)\} = \{\boldsymbol{I}-\boldsymbol{E}_1(\partial)\}\boldsymbol{E}_1(\partial) = \boldsymbol{0},$$

supplied by (4.3), is

$$\boldsymbol{E}_1(\partial)\{L_1(\partial)\,\phi_1-\partial^2\phi_1/\partial t^2\}+\{\boldsymbol{I}-\boldsymbol{E}_1(\partial)\}\,\{L_3(\partial)\,\phi_3-\partial^2\phi_3/\partial t^2\}=\boldsymbol{0},\quad t>0.$$

The conditions $(4.1)_2$ combine with (4.13) to give

$$\begin{aligned}\boldsymbol{E}_1(\partial)\,\phi_1+\{\boldsymbol{I}-\boldsymbol{E}_1(\partial)\}\,\phi_3=\boldsymbol{0},\quad &\boldsymbol{E}_1(\partial)\,\{\partial\phi_1/\partial t-\delta(\boldsymbol{x})\}\\ &+\{\boldsymbol{I}-\boldsymbol{E}_1(\partial)\}\,\{\partial\phi_3/\partial t-\delta(\boldsymbol{x})\}=\boldsymbol{0},\quad t=0.\end{aligned}$$

The initial-value problem (4.1) is therefore solved by (4.13) if ϕ_1 and ϕ_3 satisfy (4.11). Equation (4.13) is the appropriate means of determining $\boldsymbol{G}$ in case C(ii).

5. The canonical problems

It has been shown in §4 that, in each of cases A, B, C(i) and C(ii), the fundamental tensor $\boldsymbol{G}$ is given by one of the formulae (4.8), (4.12) and (4.13), each of the scalar wave functions appearing in these representations satisfying the Cauchy problem (4.11). Remembering that $\boldsymbol{D}_i$, and hence $\boldsymbol{E}_i$, are constant in case A, we gather from (4.8), (4.12) and (4.13) that the construction of $\boldsymbol{G}$ requires only the solution of (4.11) and the evaluation of $\boldsymbol{E}_i(\partial)\,(\phi_i-\phi_3)$ for those values of i for which $\boldsymbol{D}_i(\partial)$ is a first-order differential operator. Using generic forms of $L_i(\partial)$ and $M_i(\partial)$, and referring to the crystallographic basis a, we proceed now to the solution of these canonical problems.

(*a*) *Solution of the scalar Cauchy problem*

We choose, as the archetype of (4.11), the problem

$$\left.\begin{aligned}L(\partial)\,\phi=\partial^2\phi/\partial t^2,\quad t>0,\\ \phi=0,\quad \partial\phi/\partial t=\delta(x_1)\,\delta(x_2)\,\delta(x_3),\quad t=0,\end{aligned}\right\}\tag{5.1}$$

where $x_i=\boldsymbol{a}_i\cdot\boldsymbol{x}$,

$$L(\partial)=\sum_{i=1}^{3}v_1^2\,\partial^2/\partial x_i^2,$$

and v_i are positive constants with the physical dimensions of speed. The values of v_j for each of the eigenvalues $L_i(\partial)$ in the four cases are provided by equations (3.7), (3.10), (3.14) and (3.17): the v_j for cases B, C(i) and C(ii) are displayed in table 1, wherein

$$c_\alpha=(c_{\alpha\alpha}/\rho)^{\frac{1}{2}}.\tag{5.2}$$

The anisotropy of the wave equation $(5.1)_1$ can be removed by making the change of variables $x_i=v_i\,\xi_i$. The Cauchy problem (5.1) then becomes

$$\left.\begin{aligned}\Delta_\xi\,\phi=\partial^2\phi/\partial t^2,\quad t>0,\\ \phi=0,\quad \partial\phi/\partial t=(v_1\,v_2\,v_3)^{-1}\,\delta(\xi_1)\,\delta(\xi_2)\,\delta(\xi_3),\quad t=0,\end{aligned}\right\}\tag{5.3}$$

Δ_ξ being the laplacian in the new variables. The solution of (5.3), as given, for example, by Courant & Hilbert (1962, pp. 737–740), is

$$\phi=(4\pi v_1\,v_2\,v_3\,\tau)^{-1}\,\delta(t-\tau),\tag{5.4}$$

with

$$\tau=\left(\sum_{i=1}^{3}\xi_i^2\right)^{\frac{1}{2}}=\left\{\sum_{i=1}^{3}(x_i/v_i)^2\right\}^{\frac{1}{2}}.\tag{5.5}$$

(*b*) *Evaluation of* $\boldsymbol{E}(\partial)\,(\phi-\tilde{\phi})$

Inspection of equations (3.11), (3.15), (3.18) and (4.12), (4.13) shows that the eigenvectors of $\rho^{-1}\boldsymbol{Q}(\partial)$ involved in the construction of $\boldsymbol{G}$ which are first-order

Table 1. *Constants relating to the eigenvalues and associated eigenvectors of* $\rho^{-1}\boldsymbol{Q}(\partial)$ *for cases* B, C(i) *and* C(ii)
(* denotes a repeated eigenvalue, † a constant eigenvector and ‡ an eigenvector not involved in the construction of $\boldsymbol{G}$)

	eigenvalues			eigenvectors			
case	v_1	v_2	v_3	m_1	m_2	m_3	C
B	c_4	c_4	c_3		†		
	c_1	c_2	c_4	$c_{11}+c_{12}$	$c_{12}+c_{22}$	0	$\{\rho(c_{11}+c_{22}+2c_{12})\}^{-1}$
	c_6	c_6	c_4		‡		
C(i)	c_1	c_1	c_3	$c_{11}+c_{13}$	$c_{11}+c_{13}$	$c_{13}+c_{33}$	$\{\rho(c_{11}+c_{33}+2c_{13})\}^{-1}$
	c_6	c_6	c_4	1	1	0	$(2\rho)^{-1}(c_{11}-c_{12}-2c_{44})$
	c_4	c_4	c_4		‡		
C(ii)	c_1	c_2	c_3	$c_{13}+c_{44}$	$c_{23}+c_{44}$	$c_{33}-c_{44}$	$\{\rho(c_{33}-c_{44})\}^{-1}$
*	c_4	c_4	c_4		‡		

differential operators are $\boldsymbol{D}_2(\partial)$ in case B, $\boldsymbol{D}_1(\partial)$ and $\boldsymbol{D}_2(\partial)$ in case C(i), and $\boldsymbol{D}_1(\partial)$ in case C(ii). By $(4.2)_1$, the associated $M_i(\partial)$ are each of the form

$$M(\partial) = \sum_{i=1}^{3} m_i^2\, \partial^2/\partial x_i^2, \tag{5.6}$$

where m_i are constants which are either dimensionless or have the physical dimensions of stress. The values of m_i for the first-order eigenvectors are set out in table 1. In two of the four entries, $m_3 = 0$. It turns out, however, that no special provision is needed and we proceed on the basis that $m_i \neq 0$.

It is seen from the representations (4.12) and (4.13) that, for each contribution to $\boldsymbol{G}$ of the form $\boldsymbol{E}(\partial)\,(\phi-\tilde{\phi})$, the eigenvalues appearing in the Cauchy problems for ϕ and $\tilde{\phi}$ and the eigenvector composing $\boldsymbol{E}(\partial)$ are related as in (4.6). We accordingly postulate, first, that $\tilde{\phi}$ satisfies (5.1), with

$$\tilde{L}(\partial) = \sum_{i=1}^{3} \tilde{v}_i^2\, \partial^2/\partial x_i^2$$

replacing $L(\partial)$, and, second, that

$$v_i^2 - \tilde{v}_i^2 = C m_i^2, \tag{5.7}$$

where C is a constant because of $L(\partial)$, $\tilde{L}(\partial)$ and $M(\partial)$ all being second-order operators. The final column of table 1 lists the values of C corresponding to the first-order eigenvectors: they have been obtained with the aid of $(3.9)_4$, $(3.12)_4$, (3.13) and $(3.16)_{1\text{-}4}$.

Let

$$\Phi = M^{-1}(\partial)\,(\phi-\tilde{\phi}).$$

Then, from (5.6) and (5.4), Φ satisfies the Poisson equation

$$M(\partial)\,\Phi = \sum_{i=1}^{3} m_i^2\, \Phi_{,ii} = (4\pi)^{-1}\{(v_1 v_2 v_3 \tau)^{-1}\delta(t-\tau) - (|\tilde{v}|_1 \tilde{v}_2 \tilde{v}_3 \tilde{\tau})^{-1}\,\delta(t-\tilde{\tau})\},$$

where

$$\tilde{\tau} = \left\{\sum_{i=1}^{3} (x_i/\tilde{v}_i)^2\right\}^{\frac{1}{2}} \tag{5.8}$$

and the comma notation is used from now on for partial derivatives with respect to x_i. As in $(5.1)_1$, the differential operator can be transformed into a laplacian by suitably scaling the variables. Here we put

$$x_i = m_i \eta_i, \tag{5.9}$$

whereupon

$$\Delta_\eta \Phi = (4\pi)^{-1}\{(v_1 v_2 v_3 \tau)^{-1}\delta(t-\tau) - (\tilde{v}_1 \tilde{v}_2 \tilde{v}_3 \tilde{\tau})^{-1}\delta(t-\tilde{\tau})\}, \tag{5.10}$$

and, from (5.5) and (5.8),

$$\tau = \left\{\sum_{i=1}^{3} (m_i \eta_i / v_i)^2\right\}^{\frac{1}{2}}, \quad \tilde{\tau} = \left\{\sum_{i=1}^{3} (m_i \eta_i / \tilde{v}_i)^2\right\}^{\frac{1}{2}}. \tag{5.11}$$

Equations (5.10) and (5.11) assert that Φ is the potential due to two surface 'charges' occupying at time $t(>0)$ the ellipsoids

$$\mathscr{E} : \sum_{i=1}^{3} (m_i \eta_i / v_i)^2 = t^2 \quad \text{and} \quad \tilde{\mathscr{E}} : \sum_{i=1}^{3} (m_i \eta_i / \tilde{v}_i)^2 = t^2$$

in η-space. Significantly, on account of (5.7), $\mathscr{E}$ and $\tilde{\mathscr{E}}$ are confocals and, by virtue of this relation, we refer to (5.7) as the *confocal property* of the acoustical tensor. The total charge on $\mathscr{E}$ is

$$e = -(16\pi^2 v_1 v_2 v_3)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^{-1}\delta(t-\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3,$$

or, with the change of variables

$$\eta_1 = (v_1/m_1)\tau \cos\varphi \sin\vartheta, \quad \eta_2 = (v_2/m_2)\tau \sin\varphi \sin\vartheta, \quad \eta_3 = (v_3/m_3)\tau\cos\vartheta,$$

$$e = -(4\pi m_1 m_2 m_3)^{-1} \int_{-\infty}^{\infty} \tau\delta(t-\tau)\,\mathrm{d}\tau = -(4\pi m_1 m_2 m_3)^{-1} t. \tag{5.12}$$

The total charge on $\tilde{\mathscr{E}}$ is plainly $-e$.

The continuous solution of equation (5.10) which behaves appropriately at infinity and on the surface charges $\mathscr{E}$ and $\tilde{\mathscr{E}}$ is formed by superposition from a standard result in potential theory (see Kellogg 1929, pp. 184–190). Let

$$f(s) = \sum_{i=1}^{3} \{(v_i t/m_i)^2 + s\}^{-1}\eta_i^2 - 1, \quad g(s) = \prod_{i=1}^{3} \{(v_i t/m_i)^2 + s\},$$

and let S be the algebraically largest real root of the cubic equation

$$f(s)\,g(s) = 0. \tag{5.13}$$

Then, assuming for definiteness that $C>0$, so that $\mathscr{E}$ lies outside $\tilde{\mathscr{E}}$,

$$\Phi = \begin{cases} 0, & 0 < t \leqslant \tau, \\ -\tfrac{1}{2}e \displaystyle\int_S^0 \{g(s)\}^{-\frac{1}{2}}\,\mathrm{d}s, & \tau \leqslant t \leqslant \tilde{\tau}, \\ -\tfrac{1}{2}e \displaystyle\int_{-Ct^2}^0 \{g(s)\}^{-\frac{1}{2}}\,\mathrm{d}s, & \tilde{\tau} \leqslant t, \end{cases} \tag{5.14}$$

the interiors of the intervals of t describing, in turn, the outside of $\mathscr{E}$, the region between $\mathscr{E}$ and $\tilde{\mathscr{E}}$, and the inside of $\tilde{\mathscr{E}}$.

The solution (5.14) can be simplified by means of the substitution $k = t^{-2}s$. Defining

$$F(k) = t^2 f(t^2 k) = \sum_{i=1}^{3} \{V_i(k)\}^{-1} x_i^2 - t^2, \tag{5.15}$$

$$G(k) = (m_1 m_2 m_3)^2 t^{-6} g(t^2 k) = \prod_{i=1}^{3} \{V_i(k)\}, \tag{5.16}$$

with

$$V_i(k) = v_i^2 + m_i^2 k, \tag{5.17}$$

and invoking (5.12), we find that

$$\Phi = (8\pi)^{-1} \int_K^0 \{G(k)\}^{-\frac{1}{2}} \mathrm{d}k, \quad \tau < t < \tilde{\tau}, \tag{5.18}$$

where $K = t^{-2} S$, the largest zero of $F(k)\,G(k)$, lies between $-C$ and 0. If, for some $i, m_i \to 0$, one semi-axis of $\mathscr{E}$ and $\tilde{\mathscr{E}}$ tends to infinity and one root of (5.13) tends to $-\infty$. No singularity occurs, therefore, in (5.14) or (5.18) and, as anticipated earlier, the limit is regular.

For the typical eigenvector $\boldsymbol{D}(\partial)$ introduced through (5.6), each non-zero component of $\boldsymbol{D}(\partial) \otimes \boldsymbol{D}(\partial)$ is a constant multiple of $\partial^2/\partial x_i \partial x_j$ for some i, j. To complete the determination of

$$\boldsymbol{E}(\partial)\,(\phi - \tilde{\phi}) = M^{-1}(\partial)\,\boldsymbol{D}(\partial) \otimes \boldsymbol{D}(\partial)(\phi - \tilde{\phi}) = \boldsymbol{D}(\partial) \otimes \boldsymbol{D}(\partial)\,\boldsymbol{\Phi},$$

we thus have to calculate $\boldsymbol{\Phi}_{,ij}$.

According to (5.14), $\boldsymbol{\Phi}$ depends on x_i only when $\tau < t < \tilde{\tau}$. Hence, from (5.18),

$$8\pi \boldsymbol{\Phi}_{,i} = -\{G(K)\}^{-\frac{1}{2}} K_{,i} \{H(t-\tau) - H(t-\tilde{\tau})\}, \tag{5.19}$$

where H is the unit step function. A second differentiation yields

$$\begin{aligned} 8\pi \boldsymbol{\Phi}_{,ij} = {} & -\{G(K)\}^{-\frac{1}{2}} [K_{,ij} - \tfrac{1}{2}\{G'(K)/G(K)\} K_{,i} K_{,j}] \{H(t-\tau) - H(t-\tilde{\tau})\} \\ & + \{G(K)\}^{-\frac{1}{2}} K_{,i}\, x_j \{(v_j^2 \tau)^{-1} \delta(t-\tau) - (\tilde{v}_j^2 \tilde{\tau})^{-1} \delta(t-\tilde{\tau})\}, \end{aligned} \tag{5.20}$$

use being made of (5.11) and (5.9). When $C < 0$ in (5.7), τ and $\tilde{\tau}$ are effectively interchanged in the preceding argument and, instead of $-C < K < 0$, we have $0 < K < -C$. To cover both possibilities the right-hand sides of (5.19) and (5.20) must be multiplied by sgn C.

Since $G(k)$ does not normally vanish at the zeros of $F(k), F(K) = 0$ and we deduce from (5.15) that

$$K_{,i} = -2\{V_i(K)\,F'(K)\}^{-1} x_i, \tag{5.21}$$

$$\begin{aligned} K_{,ij} = {} & -4[V_i(K)\,V_j(K)\,\{F'(K)\}^2]^{-1} [\{F''(K)/F'(K)\} \\ & + \{m_i^2/V_i(K)\} + \{m_j^2/V_j(K)\}]\, x_i x_j - 2\{V_i(K)\,F'(K)\}^{-1} \delta_{ij}. \end{aligned} \tag{5.22}$$

The derivatives of F and G appearing in (5.20) to (5.22) are given by (5.15) to (5.17) as

$$\left.\begin{aligned} F'(k) = -\sum_{i=1}^{3} \{V_i(k)\}^{-2} m_i^2 x_i^2, \quad F''(k) = 2\sum_{i=1}^{3} \{V_i(k)\}^{-3} m_i^4 x_i^2, \\ G'(k) = G(k) \sum_{i=1}^{3} \{V_i(k)\}^{-1} m_i^2. \end{aligned}\right\} \tag{5.23}$$

When $t = \tau$, K is zero and when $t = \tilde{\tau}, K = -C$. The coefficients of $\delta(t-\tau)$ and $\delta(t-\tilde{\tau})$ in (5.20) can therefore be evaluated at $K = 0$ and $K = -C$ respectively (Jones 1982, p. 167) and they are found, with the aid of (5.16), (5.17), (5.21) and (5.7), to be

$$-2\{v_1 v_2 v_3 v_i^2 v_j^2 F'(0)\,\tau\}^{-1} x_i x_j \quad \text{and} \quad 2\{\tilde{v}_1 \tilde{v}_2 \tilde{v}_3 \tilde{v}_i^2 \tilde{v}_j^2 F'(-C)\,\tilde{\tau}\}^{-1} x_i x_j. \tag{5.24}$$

Combining equations (5.20) to (5.22) and making the substitutions (5.24), we conclude that

$$\begin{aligned}4\pi\Phi_{,ij} = {} & [\![2\{V_i(K)\,V_j(K)\,F'(K)\}^{-1}[\{F''(K)/F'(K)\}+\tfrac{1}{2}\{G'(K)/G(K)\}+\{m_i^2/V_i(K)\}\\ & +\{m_j^2/V_j(K)\}]\,x_i x_j+\{V_i(K)\}^{-1}\delta_{ij}]\!]\,\{F'(K)\}^{-1}\{G(K)\}^{-\frac{1}{2}}\{H(t-\tau)-H(t-\tilde{\tau})\}\operatorname{sgn} C\\ & -[\{v_1 v_2 v_3 v_i^2 v_j^2 F'(0)\,\tau\}^{-1}\delta(t-\tau)-\{\tilde{v}_1\tilde{v}_2\tilde{v}_3\tilde{v}_i^2\tilde{v}_j^2F'(-C)\,\tilde{\tau}\}^{-1}\delta(t-\tilde{\tau})]\,x_i x_j \operatorname{sgn} C.\end{aligned} \tag{5.25}$$

(c) *Simplifications*

We take note here of two cases in which equation (5.25) is considerably simplified.

(i) When

$$v_1 = v_2 = v_3, \quad m_1 = m_2 = m_3, \tag{5.26}$$

equation (5.17) yields

$$V_1(k) = V_2(k) = V_3(k) = v_1^2+m_1^2 k, \tag{5.27}$$

and it follows from (5.15), (5.16), (5.23) and (5.7) that

$$\left.\begin{aligned}F(k) = \{V_1(k)\}^{-1}r^2-t^2, \quad F'(k) = -\{V_1(k)\}^{-2}m_1^2 r^2, \quad G(k) = \{V_1(k)\}^3,\\ \{F''(K)/F'(K)\}+\tfrac{1}{2}\{G'(K)/G(K)\} = -\tfrac{1}{2}\{V_1(K)\}^{-1}m_1^2,\\ F'(0) = -m_1^2 v_1^{-4} r^2, \quad F'(-C) = -m_1^2\tilde{v}_1^{-4} r^2,\end{aligned}\right\} \tag{5.28}$$

where

$$r = (x_p x_p)^{\frac{1}{2}} \tag{5.29}$$

is distance from the origin. The unique zero of $F(k)$ is K, whence

$$V_1(K) = r^2 t^{-2}, \tag{5.30}$$

and, from (5.5) and (5.8),

$$\tau = v_1^{-1} r, \quad \tilde{\tau} = \tilde{v}_1^{-1} r. \tag{5.31}$$

With the use of (5.26) to (5.31), equation (5.25) reduces to

$$\begin{aligned}\Phi_{,ij} = (4\pi m_1^2 r^2)^{-1}[-r^{-1}t\{H(t-v_1^{-1}r)-H(t-\tilde{v}_1^{-1}r)\}(\delta_{ij}-3\hat{r}_i\hat{r}_j)\\ +r\{v_1^{-2}\delta(t-v_1^{-1}r)-\tilde{v}_1^{-2}\delta(t-\tilde{v}_1^{-1}r)\}\hat{r}_i\hat{r}_j]\operatorname{sgn} C,\end{aligned} \tag{5.32}$$

with $\hat{r}_i = r^{-1}x_i$.

(ii) When

$$v_1 = v_2, \quad m_1 = m_2, \quad m_3 = 0, \tag{5.33}$$

we obtain, by the same steps as in case (i),

$$V_1(k) = V_2(k) = v_1^2+m_1^2 k, \quad V_3(k) = v_3^2,$$

and

$$F'(k) = -\{V_1(k)\}^{-2}m_1^2 R^2, \quad G(k) = \{V_1(k)\}^2 v_3^2,$$

$$\{F''(K)/F'(K)\}+\tfrac{1}{2}\{G'(K)/G(K)\} = -\{V_1(K)\}^{-1}m_1^2,$$

$$F'(0) = -m_1^2 v_1^{-4} R^2, \quad F'(-C) = -m_1^2\tilde{v}_1^{-4}R^2,$$

where

$$R = (x_1^2+x_2^2)^{\frac{1}{2}} \tag{5.34}$$

is distance from the x_3 axis. When these expressions are entered into (5.25), $V_1(K)$ cancels out and there is no need to evaluate K. We find that

$$\begin{aligned}\Phi_{,\alpha\beta} = (4\pi m_1^2 v_3)^{-1}[-R^{-2}\{H(t-\tau)-H(t-\tilde{\tau})\}(\delta_{\alpha\beta}-2\hat{R}_\alpha\hat{R}_\beta) \\ +\{(v_1^2\tau)^{-1}\delta(t-\tau)-(\tilde{v}_1^2\tilde{\tau})^{-1}\delta(t-\tilde{\tau})\}\hat{R}_\alpha\hat{R}_\beta]\operatorname{sgn} C, \quad \alpha,\beta=1,2, \quad (5.35)\end{aligned}$$

where $\hat{R}_\alpha = R^{-1}x_\alpha$ and, from (5.5) and (5.8),

$$\tau = (v_1^{-2}R^2+v_3^{-2}x_3^2)^{\frac{1}{2}}, \quad \tilde{\tau} = (\tilde{v}_1^{-2}R^2+\tilde{v}_3^{-2}x_3^2)^{\frac{1}{2}}.$$

Since $m_3 = 0$, (5.35) are the only derivatives of Φ required in the construction of $\boldsymbol{G}$.

6. The structure of the fundamental solutions

It is now apparent from the representations (4.8), (4.12) and (4.13) that, when $\mathscr{S}$ is ellipsoidal, $\boldsymbol{G}$ is a bilinear combination of generalized functions of the forms (5.4) and (5.25) with the elementary tensors $\boldsymbol{a}_i \otimes \boldsymbol{a}_j$. Equation (5.4) depicts a single wavefront, the arrival of which at $t = \tau$ produces an instantaneous singularity in the displacement. We refer to the associated contribution to $\boldsymbol{G}$ as the *τ-wavefront*. The motion described by equation (5.25) consists of two wavefronts, arriving at $t = \tau$ and $t = \tilde{\tau}$ and each carrying displacement singularities, and a continuous disturbance, sandwiched between the wavefronts and characterized by the terms involving step functions. The order in which the wavefronts arrive is decided by the sign of C in the confocal property (5.7): $\tau \lesseqgtr \tilde{\tau}$ according as $C \gtreqless 0$. The contribution to $\boldsymbol{G}$ provided by (5.25) is referred to as the *$(\tau,\tilde{\tau})$-pair*. The fundamental solution as a whole contains either two or three wavefronts and either no pair, one pair, or two pairs with a wavefront in common. The presence of pairs is a direct consequence of the confocal property and is, in turn, responsible for $\boldsymbol{G}$ being causal in the sense of being identically zero up to the first and after the last of the wavefront arrivals.

With these facts in mind we can infer the structure of the fundamental solution for each of cases A, B, C(i) and C(ii) from the relevant representation of $\boldsymbol{G}$ and the information collected in §3 and table 1.

(a) *Case* A

All the eigenvectors of $\boldsymbol{Q}(\partial)$ are constant in this case and, from (3.8) and (4.2),

$$\boldsymbol{D}_i = \boldsymbol{a}_i, \quad \boldsymbol{E}_i = \boldsymbol{a}_i \otimes \boldsymbol{a}_i.$$

Equations (4.8), (5.4), (3.7) and (5.2) then lead to

$$\boldsymbol{G} = (4\pi c_4 c_5 c_6)^{-1} \sum_{i=1}^{3} c_{i+3}(c_i t_i)^{-1}\delta(t-t_i)\,\boldsymbol{a}_i \otimes \boldsymbol{a}_i,$$

with

$$t_1 = (c_1^{-2}x_1^2+c_6^{-2}x_2^2+c_5^{-2}x_3^2)^{\frac{1}{2}}, \quad t_2 = (c_6^{-2}x_1^2+c_2^{-2}x_2^2+c_4^{-2}x_3^2)^{\frac{1}{2}},$$
$$t_3 = (c_5^{-2}x_1^2+c_4^{-2}x_2^2+c_3^{-2}x_3^2)^{\frac{1}{2}}.$$

The fundamental solution consists of t_1-, t_2- and t_3-wavefronts, the structure of minimal complexity. All three wavefronts are ellipsoidal, the order of the arrivals depending on the relative magnitudes of $c_{\alpha\alpha}$ and, when the wavefronts intersect, on position.

(b) *Case* B

The appropriate form of $\boldsymbol{G}$ is, in this case, (4.12). From equations $(3.11)_{1,2}$ and table 1, the eigenvectors involved are

$$\boldsymbol{D}_1 = \boldsymbol{a}_3, \quad \boldsymbol{D}_2(\partial) = m_1(\partial/\partial x_1)\,\boldsymbol{a}_1 + m_2(\partial/\partial x_2)\,\boldsymbol{a}_2, \qquad (6.1)$$

with $m_1 = c_{11}+c_{12}, m_2 = c_{12}+c_{22}$, and, from (4.2),

$$\boldsymbol{E}_1(\phi_1-\phi_3) = (\phi_1-\phi_3)\,\boldsymbol{a}_3\otimes\boldsymbol{a}_3, \tag{6.2}$$

$$\boldsymbol{E}_2(\partial)\,(\phi_2-\phi_3) = \boldsymbol{D}_2(\partial)\otimes\boldsymbol{D}_2(\partial)\,\boldsymbol{\Phi}^{(23)}, \tag{6.3}$$

where

$$\boldsymbol{\Phi}^{(23)} = M_2^{-1}(\partial)\,(\phi_2-\phi_3).$$

Substitution from (6.2), (6.3) and $(6.1)_2$ into equation (4.12) results in

$$\boldsymbol{G} = \phi_1\,\boldsymbol{a}_3\otimes\boldsymbol{a}_3+\phi_3\boldsymbol{J}+\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} m_\alpha m_\beta\,\boldsymbol{\Phi}^{(23)}_{,\alpha\beta}\,\boldsymbol{a}_\alpha\otimes\boldsymbol{a}_\beta, \tag{6.4}$$

with

$$\boldsymbol{J} = \boldsymbol{I}-\boldsymbol{a}_3\otimes\boldsymbol{a}_3 = \boldsymbol{a}_1\otimes\boldsymbol{a}_1+\boldsymbol{a}_2\otimes\boldsymbol{a}_2. \tag{6.5}$$

The fundamental solution is made up of t_1- and t_3-wavefronts, specified by ϕ_1 and ϕ_3, and a (t_2, t_3)-pair generated by $\boldsymbol{\Phi}^{(23)}$. The times of arrival, derived from equation (5.5) and table 1, are

$$t_1 = (c_4^{-2}R^2+c_3^{-2}x_3^2)^{\frac{1}{2}},\quad t_2 = (c_1^{-1}x_1^2+c_2^{-2}x_2^2+c_4^{-2}x_3^2)^{\frac{1}{2}},\quad t_3 = (c_6^{-2}R^2+c_4^{-2}x_3^2)^{\frac{1}{2}}, \tag{6.6}$$

with R defined by (5.34), and, from (5.4),

$$\phi_1 = (4\pi c_3\,c_4^2\,t_1)^{-1}\,\delta(t-t_1),\quad \phi_3 = (4\pi c_4\,c_6^2\,t_3)^{-1}\,\delta(t-t_3). \tag{6.7}$$

In view of $(3.4)_2$, the value of C given in table 1 is positive. Hence $t_2 < t_3$: the leading wavefront of the (t_2, t_3)-pair is ellipsoidal and the trailing wavefront spheroidal. The t_1-wavefront is also spheroidal.

(c) *Case* C(i)

Equation (4.12) again applies. Calling on $(3.15)_{1,2}$, $(4.2)_2$, $(3.4)_2$ and the relevant rows of table 1, we find that

$$\begin{aligned}\boldsymbol{G} = {}&\phi_3\boldsymbol{I}+m_1^2\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\boldsymbol{\Phi}^{(13)}_{,\alpha\beta}\,\boldsymbol{a}_\alpha\otimes\boldsymbol{a}_\beta+m_1m_3\sum_{\alpha=1}^{2}\boldsymbol{\Phi}^{(13)}_{,\alpha3}(\boldsymbol{a}_\alpha\otimes\boldsymbol{a}_3+\boldsymbol{a}_3\otimes\boldsymbol{a}_\alpha)+m_3^2\,\boldsymbol{\Phi}^{(13)}_{,33}\,\boldsymbol{a}_3\otimes\boldsymbol{a}_3\\ &+\{\boldsymbol{\Phi}^{(23)}_{,22}\,\boldsymbol{a}_1\otimes\boldsymbol{a}_1+\boldsymbol{\Phi}^{(23)}_{,11}\,\boldsymbol{a}_2\otimes\boldsymbol{a}_2-\boldsymbol{\Phi}^{(23)}_{,12}(\boldsymbol{a}_1\otimes\boldsymbol{a}_2+\boldsymbol{a}_2\otimes\boldsymbol{a}_1)\}\,\operatorname{sgn}\,(c_{11}-c_{12}-2c_{44}),\end{aligned}$$

with $m_1 = c_{11}+c_{13}, m_3 = c_{13}+c_{33}$ and

$$\boldsymbol{\Phi}^{(13)} = M_1^{-1}(\partial)\,(\phi_1-\phi_3),\quad \boldsymbol{\Phi}^{(23)} = M_2^{-1}(\partial)\,(\phi_2-\phi_3). \tag{6.8}$$

From (5.4) and (5.5),

$$\phi_3 = (4\pi c_4^3\,t_3)^{-1}\,\delta(t-t_3), \tag{6.9}$$

and the arrival times are

$$t_1 = (c_1^{-2}R^2+c_3^{-2}x_3^2)^{\frac{1}{2}},\quad t_2 = (c_6^{-2}R^2+c_4^{-2}x_3^2)^{\frac{1}{2}},\quad t_3 = c_4^{-1}\,r,$$

r being defined by (5.29).

The fundamental solution has the structure of maximal complexity, comprising a spherical t_3-wavefront, represented by ϕ_3, and (t_1, t_3)- and (t_2, t_3)-pairs, described by $\boldsymbol{\Phi}^{(13)}$ and $\boldsymbol{\Phi}^{(23)}$ respectively, in which the wavefronts arriving at $t = t_1$ and $t = t_2$ are spheroidal. The upper value of C in table 1 is positive, so $t_1 < t_3$, and since, for a transversely isotropic elastic material, the condition $(3.4)_2$ gives $c_{11}+c_{12} > 0$, we deduce from equations (5.2), $(3.12)_4$ and (3.13) that $t_1 < t_2$. Reference to the lower value of C in table 1 thus shows that $t_1 < t_2 < t_3$ or $t_1 < t_3 < t_2$ according as $c_{11}-c_{12}-2c_{44} \gtrless 0$. In the former case, the (t_2, t_3)-pair is superimposed on the (t_1, t_3)-pair and the disturbance is terminated by the t_3-wavefront. In the latter case, the (t_2, t_3)-pair follows the (t_1, t_3)-pair and the entire motion lies between the spheroidal wavefronts.

An alternative method of constructing $\boldsymbol{G}$ in case C(i), based on integral transforms, has been developed by Payton (1975).

(d) Case C(ii)

The representation (4.13), in conjunction with equation (3.18) and table 1, provides the fundamental solution

$$\boldsymbol{G} = \phi_3 \boldsymbol{I} + \sum_{i=1}^{3} \sum_{j=1}^{3} m_i m_j \Phi^{(13)}_{,ij} \boldsymbol{a}_i \otimes \boldsymbol{a}_j, \tag{6.10}$$

where $m_1 = c_{13}+c_{44}, m_2 = c_{23}+c_{44}, m_3 = c_{33}-c_{44}$, and ϕ_3 and $\Phi^{(13)}$ are given by equations (6.9) and $(6.8)_1$. The times of arrival, acquired from (5.5), are

$$t_1 = (c_1^{-2} x_1^2 + c_2^{-2} x_2^2 + c_3^{-2} x_3^2)^{\frac{1}{2}}, \quad t_3 = c_4^{-1} r.$$

The disturbance consists of an ellipsoidal t_1-wavefront and a (t_1, t_3)-pair in which the wavefront arriving at $t = t_3$ is spherical. Equations (3.16) and (5.2) imply that c_1, c_2 and c_3 all exceed c_4. The value of C in table 1 is therefore positive and $t_1 < t_3$: the ellipsoidal wavefront arrives first and the spherical wavefront last.

(e) Two special cases

It is seen from table 1 and equation (5.2) that the simplifying relations (5.26) and (5.33) apply respectively to case C(ii) when

$$c_{11} = c_{22} = c_{33}, \quad c_{13} = c_{23} = c_{33} - 2c_{44}, \tag{6.11}$$

and to case B when

$$c_{11} = c_{22}. \tag{6.12}$$

(i) Equations (6.11), in conjunction with (3.16), imply isotropy, with characteristic speeds c_1 for longitudinal and c_4 for transverse waves. Replacing $\Phi^{(13)}_{,ij}$ in (6.10) by the right-hand side of (5.32), substituting for ϕ_3 from (6.9) and setting

$$m_1 = m_2 = m_3, \quad v_1 = c_1, \quad \tilde{v}_1 = c_4, \quad \tau = t_1 = c_1^{-1} r, \quad \tilde{\tau} = t_3 = c_4^{-1} r, \quad C > 0,$$

we recover the fundamental tensor

$$\begin{aligned}\boldsymbol{G} = (4\pi r^2)^{-1} [& -r^{-1} t\{H(t - c_1^{-1} r) - H(t - c_4^{-1} r)\} (\boldsymbol{I} - 3\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}) \\ & + r\{c_1^{-2} \delta(t - c_1^{-1} r) \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + c_4^{-2} \delta(t - c_4^{-1} r) (\boldsymbol{I} - \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}})\}],\end{aligned}$$

due to Stokes (1849; see also Gurtin 1984, §68), $\hat{\boldsymbol{r}} = \hat{r}_p \boldsymbol{a}_p$ being the unit vector directed radially outwards from the origin.

(ii) When the relation (6.12) is adjoined to (3.9), the symmetry becomes transversely isotropic and subject to the additional condition

$$c_{13} + c_{44} = 0. \tag{6.13}$$

This is one of the special cases mentioned in the opening paragraph of §1, case C(i) being the other. The fundamental tensor is secured by replacing $\Phi^{(23)}_{,\alpha\beta}$ in (6.4) by the right-hand side of (5.35), substituting for ϕ_1 and ϕ_3 from (6.7), and putting

$$m_1 = m_2, \quad v_1 = c_1, \quad v_3 = c_4, \quad \tilde{v}_1 = c_6, \quad \tau = t_2, \quad \tilde{\tau} = t_3, \quad C > 0.$$

The result is

$$\begin{aligned}\boldsymbol{G} = (4\pi c_4)^{-1} [& -R^{-2}\{H(t - t_2) - H(t - t_3)\} (\boldsymbol{J} - 2\hat{\boldsymbol{R}} \otimes \hat{\boldsymbol{R}}) + (c_1^2 t_2)^{-1} \delta(t - t_2) \hat{\boldsymbol{R}} \otimes \hat{\boldsymbol{R}} \\ & + (c_6^2 t_3)^{-1} \delta(t - t_3) (\boldsymbol{J} - \hat{\boldsymbol{R}} \otimes \hat{\boldsymbol{R}}) + (c_3 c_4 t_1)^{-1} \delta(t - t_1) \boldsymbol{a}_3 \otimes \boldsymbol{a}_3],\end{aligned} \tag{6.14}$$

with t_1 and t_3 given by $(6.6)_{1,3}$ and $(6.6)_2$ reducing to

$$t_2 = (c_1^{-2} R^2 + c_4^{-2} x_3^2)^{\frac{1}{2}}.$$

The tensor $\boldsymbol{J}$ is defined by (6.5) and $\hat{\boldsymbol{R}} = \hat{R}_1 \boldsymbol{a}_1 + \hat{R}_2 \boldsymbol{a}_2$ is the unit vector directed radially outwards from the axis of material symmetry. As noted in §6*b*, $t_2 < t_3$. Otherwise, the order of arrival of the three wavefronts contained in (6.14) depends on the signs of $c_{11} - c_{44}$, $c_{33} - c_{44}$ and $c_{11} - c_{12} - 2c_{44}$, and possibly on arctan $(R^{-1} x_3)$ as well. A complete listing of possibilities may be drawn from table 6 of Chadwick (1989). (In the third column of this table, the third $c_d(\boldsymbol{n})$ from the top should be $c_e(\boldsymbol{n})$. In the fourth column, C_{25} should be D_{25}.)

When $c_{33} \to \infty$, with the other elastic moduli held fixed, the transversely isotropic material becomes inextensible along the axis of symmetry. As pointed out by Chadwick (1989, §5*f*), the mechanical properties of the special material satisfying (6.13) and the unrestricted transversely isotropic elastic material coincide in this limit, so the fundamental solution for an inextensible medium can be obtained by proceeding to the limit $c_{33} \to \infty$ in (6.14). Only the final term is affected and, with reference to $(6.6)_1$,

$$(c_3 c_4 t_1)^{-1} \delta(t - t_1) \to \begin{cases} 0, & R \neq 0, \\ (c_4 |x_3|)^{-1} \delta(t), & R = 0, \quad x_3 \neq 0. \end{cases}$$

The t_1-wavefront therefore disappears except on the x_3 axis, where a signal is received simultaneously at each point at $t = 0$. This behaviour is consistent with the speed of longitudinal plane waves in the direction of symmetry becoming unbounded as $c_{33} \to \infty$ (Chadwick 1989, §5*f*) and with examples of the channelling of disturbances along inextensible fibres that have been encountered in other contexts (Pipkin 1984; Captain & Chadwick 1986, §6).

This work was supported, in part, by the award of fellowships to P.C. and A.N.N. by the Leverhulme Trust and the Science and Engineering Research Council respectively.

Appendix. Conditions under which the slowness surface of an anisotropic elastic material is the union of concentric ellipsoids

(*a*) *Preliminaries*

The definition (2.3) of the components of the acoustical tensor relative to b can be written as

$$Q_{ij}(\boldsymbol{n}) = \operatorname{tr}\{(\boldsymbol{n} \otimes \boldsymbol{b}_i)\, \boldsymbol{C}[\boldsymbol{n} \otimes \boldsymbol{b}_j]\} = \operatorname{tr}\{\boldsymbol{B}_{pi}\, \boldsymbol{C}[\boldsymbol{B}_{qj}]\}\, n_p n_q,$$

use being made of (3.3). It follows from (3.1) and (3.2) that

$$\left.\begin{aligned} Q_{11}(\boldsymbol{n}) &= c_{11} n_1^2 + c_{66} n_2^2 + c_{55} n_3^2 + 2c_{56} n_2 n_3 + 2c_{15} n_3 n_1 + 2c_{16} n_1 n_2, \\ Q_{22}(\boldsymbol{n}) &= c_{66} n_1^2 + c_{22} n_2^2 + c_{44} n_3^2 + 2c_{24} n_2 n_3 + 2c_{46} n_3 n_1 + 2c_{26} n_1 n_2, \\ Q_{33}(\boldsymbol{n}) &= c_{55} n_1^2 + c_{44} n_2^2 + c_{33} n_3^2 + 2c_{34} n_2 n_3 + 2c_{35} n_3 n_1 + 2c_{45} n_1 n_2, \end{aligned}\right\} \qquad \text{(A 1)}$$

$$\left.\begin{aligned} Q_{23}(\boldsymbol{n}) &= c_{56} n_1^2 + c_{24} n_2^2 + c_{34} n_3^2 + (c_{23} + c_{44}) n_2 n_3 + (c_{36} + c_{45}) n_3 n_1 + (c_{25} + c_{46}) n_1 n_2, \\ Q_{31}(\boldsymbol{n}) &= c_{15} n_1^2 + c_{46} n_2^2 + c_{35} n_3^2 + (c_{36} + c_{45}) n_2 n_3 + (c_{13} + c_{55}) n_3 n_1 + (c_{14} + c_{56}) n_1 n_2, \\ Q_{12}(\boldsymbol{n}) &= c_{16} n_1^2 + c_{26} n_2^2 + c_{45} n_3^2 + (c_{25} + c_{46}) n_2 n_3 + (c_{14} + c_{56}) n_3 n_1 + (c_{12} + c_{66}) n_1 n_2. \end{aligned}\right\} \text{(A 2)}$$

If $\boldsymbol{Q}(\boldsymbol{n})$ has the spherical form (2.15), we deduce from equations $(\text{A } 1)_{1,2}$ and $(\text{A } 2)_3$ that $c_{11} = c_{22} = -c_{12} = c_{66}$, in violation of the inequality $(3.4)_2$. This justifies the statement in the final sentence of §2*c*.

Let $b' = \{\boldsymbol{b}'_1, \boldsymbol{b}'_2, \boldsymbol{b}'_3\}$ be the orthonormal basis related to b by

$$\boldsymbol{b}'_1 = \cos\theta \boldsymbol{b}_1 + \sin\theta \boldsymbol{b}_2, \quad \boldsymbol{b}'_2 = -\sin\theta \boldsymbol{b}_1 + \cos\theta \boldsymbol{b}_2, \quad \boldsymbol{b}'_3 = \boldsymbol{b}_3, \tag{A 3}$$

and let $\boldsymbol{B}'_{ij}$ and $\boldsymbol{B}'_\alpha$ be derived from $\boldsymbol{b}'_i$ by relations analogous to (3.3) and (3.2). Then, as may easily be verified,

$$\left.\begin{array}{c} \boldsymbol{B}'_1 + \boldsymbol{B}'_2 = \boldsymbol{B}_1 + \boldsymbol{B}_2, \quad \boldsymbol{B}'_3 = \boldsymbol{B}_3, \\ (\boldsymbol{B}'_1 - \boldsymbol{B}'_2) \otimes (\boldsymbol{B}'_1 - \boldsymbol{B}'_2) + 4\boldsymbol{B}'_6 \otimes \boldsymbol{B}'_6 = (\boldsymbol{B}_1 - \boldsymbol{B}_2) \otimes (\boldsymbol{B}_1 - \boldsymbol{B}_2) + 4\boldsymbol{B}_6 \otimes \boldsymbol{B}_6, \\ \boldsymbol{B}'_4 \otimes \boldsymbol{B}'_4 + \boldsymbol{B}'_5 \otimes \boldsymbol{B}'_5 = \boldsymbol{B}_4 \otimes \boldsymbol{B}_4 + \boldsymbol{B}_5 \otimes \boldsymbol{B}_5. \end{array}\right\} \tag{A 4}$$

We require for later use the following result.

Lemma. *Suppose that, in b, the matrix $[c_{\alpha\beta}]$ of elastic moduli takes the form*

$$\begin{bmatrix} r+y_1^2 & -r+y_1y_2 & -s+y_1y_3 & y_1y_4 & y_1y_5 & y_1y_6 \\ \cdot & r+y_2^2 & -s+y_2y_3 & y_2y_4 & y_2y_5 & y_2y_6 \\ \cdot & \cdot & t+y_3^2 & y_3y_4 & y_3y_5 & y_3y_6 \\ \cdot & \cdot & \cdot & s+y_4^2 & y_4y_5 & y_4y_6 \\ \cdot & \cdot & \cdot & \cdot & s+y_5^2 & y_5y_6 \\ \cdot & \cdot & \cdot & \cdot & \cdot & r+y_6^2 \end{bmatrix}. \tag{A 5}$$

Then, if r, s and t are invariant under the change of basis $b \rightarrow b'$ and y_α are transformed into

$$\left.\begin{array}{c} y'_1 = y_1\cos^2\theta + y_2\sin^2\theta + 2y_6\sin\theta\cos\theta, \\ y'_2 = y_1\sin^2\theta + y_2\cos^2\theta - 2y_6\sin\theta\cos\theta, \quad y'_3 = y_3, \\ y'_4 = y_4\cos\theta - y_5\sin\theta, \quad y'_5 = y_4\sin\theta + y_5\cos\theta, \\ y'_6 = -(y_1 - y_2)\sin\theta\cos\theta + y_6(\cos^2\theta - \sin^2\theta), \end{array}\right\} \tag{A 6}$$

the matrix of elastic moduli in b' is given by (A 5) *with y'_α replacing y_α. Furthermore,*

$$y_1'^2 + y_2'^2 + 2y_6'^2 = y_1^2 + y_2^2 + 2y_6^2, \quad y_3'^2 = y_3^2, \quad y_4'^2 + y_5'^2 = y_4^2 + y_5^2, \tag{A 7}$$

and the tensors

$$\left.\begin{array}{c} y_1\boldsymbol{B}_{11} + y_2\boldsymbol{B}_{22} + y_6(\boldsymbol{B}_{12} + \boldsymbol{B}_{21}), \quad y_1\boldsymbol{B}_{21} - y_2\boldsymbol{B}_{12} - y_6(\boldsymbol{B}_{11} - \boldsymbol{B}_{22}), \\ y_3\boldsymbol{B}_{33}, \quad y_4\boldsymbol{B}_{23} + y_5\boldsymbol{B}_{13}, \quad y_4\boldsymbol{B}_{32} + y_5\boldsymbol{B}_{31}, \quad y_4\boldsymbol{B}_{13} - y_5\boldsymbol{B}_{23}, \end{array}\right\} \tag{A 8}$$

are unchanged by the replacement of $y_\alpha, \boldsymbol{b}_i$ by *$y'_\alpha, \boldsymbol{b}'_i$.*

Proof. Confirmation of (A 7) and the form invariance of (A 8) is a matter of straightforward calculation based on (A 3) and (A 6). The linear elasticity tensor, as assembled from (A 5) and (3.1), is

$$\begin{aligned} \boldsymbol{C} = r\{(\boldsymbol{B}_1 - \boldsymbol{B}_2) \otimes (\boldsymbol{B}_1 - \boldsymbol{B}_2) + 4\boldsymbol{B}_6 \otimes \boldsymbol{B}_6\} + s\{4(\boldsymbol{B}_4 \otimes \boldsymbol{B}_4 + \boldsymbol{B}_5 \otimes \boldsymbol{B}_5) \\ - (\boldsymbol{B}_1 + \boldsymbol{B}_2) \otimes \boldsymbol{B}_3 - \boldsymbol{B}_3 \otimes (\boldsymbol{B}_1 + \boldsymbol{B}_2)\} + t\boldsymbol{B}_3 \otimes \boldsymbol{B}_3 + \boldsymbol{Y} \otimes \boldsymbol{Y}, \end{aligned} \tag{A 9}$$

where

$$\boldsymbol{Y} = \sum_{\alpha=1}^{3} y_\alpha \boldsymbol{B}_\alpha + 2\sum_{\alpha=4}^{6} y_\alpha \boldsymbol{B}_\alpha. \tag{A 10}$$

It is plain from (A 4) that the tensors multiplied by r, s and t in (A 9) are unaltered by the addition of primes to $\boldsymbol{B}_\alpha$ and from (A 10), (A 8) and (3.2) that $\boldsymbol{Y}$ is unchanged when y_α and $\boldsymbol{B}_\alpha$ are both modified in this way. ∎

(*b*) *Case* A

The eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ are constant and mutually orthogonal, so we can take

$$\boldsymbol{q}_i = \boldsymbol{b}_i. \tag{A 11}$$

Then, from (2.4),

$$Q_{ii}(\boldsymbol{n}) = \lambda_i(\boldsymbol{n}), \quad Q_{ij}(\boldsymbol{n}) = 0, \quad i \neq j, \quad \forall \quad \boldsymbol{n} \in \mathscr{U} \tag{A 12}$$

in b. Equations $(\mathrm{A}\ 12)_2$ and (A 2) entail the conditions (3.5) for orthorhombic symmetry, together with the relations (3.6). We can thus identity the base vectors $\boldsymbol{b}_i$ and $\boldsymbol{a}_i$. Equations (3.8) follow from (A 11) and (3.7) from $(\mathrm{A}\ 12)_1$, (A 1) and (3.5).

(*c*) *Case* B

Without loss of generality we can set

$$\boldsymbol{q}_1 = \boldsymbol{b}_3, \quad \boldsymbol{q}_2(\boldsymbol{n}) = \alpha(\boldsymbol{n})\,\boldsymbol{b}_1 + \beta(\boldsymbol{n})\,\boldsymbol{b}_2, \quad \boldsymbol{q}_3(\boldsymbol{n}) = -\beta(\boldsymbol{n})\,\boldsymbol{b}_1 + \alpha(\boldsymbol{n})\,\boldsymbol{b}_2, \tag{A 13}$$

with

$$\alpha(\boldsymbol{n}) = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3, \quad \beta(\boldsymbol{n}) = \beta_1 n_1 + \beta_2 n_2 + \beta_3 n_3. \tag{A 14}$$

From (2.8),

$$m_1 = 1, \quad m_2(\boldsymbol{n}) = m_3(\boldsymbol{n}) = \{\alpha(\boldsymbol{n})\}^2 + \{\beta(\boldsymbol{n})\}^2. \tag{A 15}$$

On substituting (A 13) to (A 15) into the spectral representation (2.9) of $\boldsymbol{Q}(\boldsymbol{n})$ and reading off the coefficients of $\boldsymbol{B}_\alpha$, we obtain expressions for the components of $\boldsymbol{Q}(\boldsymbol{n})$ relative to b. As shown in §2b, $m_2(\boldsymbol{n})$ divides $\lambda_2(\boldsymbol{n}) - \lambda_3(\boldsymbol{n})$ and, from (A 15) and (A 14), $m_2(\boldsymbol{n})$ is, like $\lambda_2(\boldsymbol{n}) - \lambda_3(\boldsymbol{n})$, of degree 2 in n_i. Hence,

$$\lambda_2(\boldsymbol{n}) - \lambda_3(\boldsymbol{n}) = c[\{\alpha(\boldsymbol{n})\}^2 + \{\beta(\boldsymbol{n})\}^2], \tag{A 16}$$

where c is a constant. The seven equations consisting of (A 16) and the expressions for $Q_{ij}(\boldsymbol{n})$ provide the formulae

$$\lambda_1(\boldsymbol{n}) = Q_{33}(\boldsymbol{n}), \quad \lambda_2(\boldsymbol{n}) = Q_{11}(\boldsymbol{n}) + c\{\beta(\boldsymbol{n})\}^2, \quad \lambda_3(\boldsymbol{n}) = Q_{22}(\boldsymbol{n}) - c\{\beta(\boldsymbol{n})\}^2, \tag{A 17}$$

for the eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ together with the relations

$$Q_{11}(\boldsymbol{n}) - Q_{22}(\boldsymbol{n}) = c[\{\alpha(\boldsymbol{n})\}^2 - \{\beta(\boldsymbol{n})\}^2], \quad Q_{12}(\boldsymbol{n}) = c\alpha(\boldsymbol{n})\,\beta(\boldsymbol{n}), \tag{A 18}$$

and

$$Q_{23}(\boldsymbol{n}) = 0, \quad Q_{31}(\boldsymbol{n}) = 0, \tag{A 19}$$

valid for all $\boldsymbol{n} \in \mathscr{U}$. We infer from (A 19) and $(\mathrm{A}\ 2)_{1,2}$ that

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0, \tag{A 20}$$

$$c_{36} + c_{45} = 0, \quad c_{23} + c_{44} = 0, \quad c_{13} + c_{55} = 0, \tag{A 21}$$

and from (A 18), (A 14), $(\mathrm{A}\ 1)_{1,2}$, $(\mathrm{A}\ 2)_3$ and (A 20) that

$$c(\alpha_1^2 - \beta_1^2) = c_{11} - c_{66}, \quad c(\alpha_2^2 - \beta_2^2) = c_{66} - c_{22}, \quad c(\alpha_1\beta_2 + \alpha_2\beta_1) = c_{12} + c_{66}, \tag{A 22}$$

$$c(\alpha_1\alpha_2 - \beta_1\beta_2) = c_{16} - c_{26}, \quad c\alpha_1\beta_1 = c_{16}, \quad c\alpha_2\beta_2 = c_{26}, \tag{A 23}$$

$$\alpha_2\alpha_3 - \beta_2\beta_3 = 0, \quad \alpha_3\alpha_1 - \beta_3\beta_1 = 0, \quad \alpha_2\beta_3 + \alpha_3\beta_2 = 0, \quad \alpha_3\beta_1 + \alpha_1\beta_3 = 0, \tag{A 24}$$

$$c(\alpha_3^2 - \beta_3^2) = c_{55} - c_{44}, \quad c\alpha_3\beta_3 = c_{45}. \tag{A 25}$$

Equations (A 24) yield

$$\alpha_1(\alpha_3^2 + \beta_3^2) = 0, \quad \alpha_2(\alpha_3^2 + \beta_3^2) = 0, \quad \beta_1(\alpha_3^2 + \beta_3^2) = 0, \quad \beta_2(\alpha_3^2 + \beta_3^2) = 0,$$

implying that either

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \tag{A 26}$$

or
$$\alpha_3 = \beta_3 = 0. \tag{A 27}$$

But equations (A 26), together with (A 22), give $c_{11}+c_{22}+2c_{12} = 0$, incompatibly with $(3.4)_2$. Equations (A 27) therefore apply and we see from (A 25) and (A 21) that

$$c_{36} = c_{45} = 0, \quad c_{44} = c_{55} = -c_{13} = -c_{23}. \tag{A 28}$$

Next, from (A 22) and (A 23),

$$c\{(\alpha_1+\beta_2)^2-(\alpha_2-\beta_1)^2\} = c_{11}+c_{22}+2c_{12}, \quad c(\alpha_1+\beta_2)(\alpha_2-\beta_1) = 0,$$

presenting the mutually exclusive alternatives

$$\beta_1 = \alpha_2, \qquad c > 0, \tag{A 29}$$

and
$$\beta_2 = -\alpha_1, \quad c < 0. \tag{A 30}$$

Supposing that (A 29) hold, we put

$$y_1 = c^{\frac{1}{2}}\alpha_1, \quad y_2 = c^{\frac{1}{2}}\beta_2, \quad y_6 = c^{\frac{1}{2}}\alpha_2. \tag{A 31}$$

Then, with reference to equations (A 22), $(\text{A } 23)_{2,3}$, (A 20) and (A 28), we find that the matrix of elastic moduli in b has the structure (A 5) with $y_3 = y_4 = y_5 = 0$ and

$$r = c_{66}-y_6^2, \quad s = c_{44}, \quad t = c_{33}. \tag{A 32}$$

From (A 13), (A 14), (A 27), $(\text{A } 29)_1$, (A 31) and (3.3),

$$\left.\begin{aligned} \boldsymbol{q}_2(\boldsymbol{n}) &= c^{-\frac{1}{2}}\{y_1\boldsymbol{B}_{11}+y_2\boldsymbol{B}_{22}+y_6(\boldsymbol{B}_{12}+\boldsymbol{B}_{21})\}\boldsymbol{n}, \\ \boldsymbol{q}_3(\boldsymbol{n}) &= c^{-\frac{1}{2}}\{y_1\boldsymbol{B}_{21}-y_2\boldsymbol{B}_{12}-y_6(\boldsymbol{B}_{11}-\boldsymbol{B}_{22})\}\boldsymbol{n}. \end{aligned}\right\} \tag{A 33}$$

Provided that r, s and t are invariant under the change of basis $b \rightarrow b'$, we can therefore make use of the Lemma. The modulus c_{33} is unchanged by the transformation and, because of $(\text{A } 28)_{2,3}$, so is c_{44}: c_{66} becomes

$$(c_{11}+c_{22}-2c_{12})\sin^2\theta\cos^2\theta-2(c_{16}-c_{26})\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta) \\ +c_{66}(\cos^2\theta-\sin^2\theta)^2$$

(see, for example, Hearmon 1961, pp. 12, 13), and it can easily be checked from (A 5) and $(\text{A } 6)_6$ that this equals $r+y_6'^2$. The Lemma now enables us to transform to b', retaining the forms of the elastic moduli and the eigenvectors (A 33). By $(\text{A } 6)_{3\text{-}5}$, $y_3' = y_4' = y_5' = 0$ and, by $(\text{A } 6)_6$, the particular choice

$$\theta = \tfrac{1}{2}\arctan\{2(y_1-y_2)^{-1}y_6\} \tag{A 34}$$

of rotation angle makes y_6' zero as well. From (A 31), $(\text{A } 29)_1$, (A 22) and (A 23) we then have, in b',

$$y_1'^2 = c_{11}-c_{66}, \quad y_2'^2 = c_{22}-c_{66}, \quad y_1'y_2' = c_{12}+c_{66}, \quad c_{16} = c_{26} = 0, \tag{A 35}$$

and $(\text{A } 14)_2$ reduces to

$$\beta(\boldsymbol{n}) = \beta_2 n_2 = c^{-\frac{1}{2}}(c_{22}-c_{66})^{\frac{1}{2}}n_2. \tag{A 36}$$

Equations $(\text{A } 35)_{4,5}$ complete the requirements (3.5) for orthorhombic symmetry, thus allowing b' to be identified with the crystallographic basis a. Equations $(\text{A } 35)_{1\text{-}3}$ imply the relations

$$\left.\begin{aligned} y_1'(y_1'+y_2') = c_{11}+c_{12}, \quad y_2'(y_1'+y_2') = c_{12}+c_{22}, \\ (c_{11}-c_{66})(c_{22}-c_{66})-(c_{12}+c_{66})^2 = 0. \end{aligned}\right\} \tag{A 37}$$

We now collect results. Equations (A 28)$_{3\text{-}5}$ and (A 37)$_3$ supply the conditions (3.9). Substitution from (A 1), (A 28)$_3$ and (A 36) into (A 17) produces the eigenvalues (3.10). Lastly, combining (A 37)$_{1,2}$ with (A 33), transformed to b', omitting the factor $c^{-\frac{1}{2}}(y_1'+y_2')^{-1}$ and replacing $\boldsymbol{b}_i'$ by $\boldsymbol{a}_i$, we obtain the eigenvectors (3.11).

When the alternative (A 30) is selected, the definitions

$$y_1 = -(-c)^{\frac{1}{2}}\beta_1, \quad y_2 = (-c)^{\frac{1}{2}}\alpha_2, \quad y_6 = (-c)^{\frac{1}{2}}\alpha_1,$$

lead to the same matrix of elasticities as before and, by steps exactly parallel to those described in the last two paragraphs, the same conclusions are reached, with $\lambda_2(\boldsymbol{n})$ and $\lambda_3(\boldsymbol{n})$ interchanged and $\boldsymbol{q}_2(\boldsymbol{n}), \boldsymbol{q}_3(\boldsymbol{n})$ replaced by $-\boldsymbol{q}_3(\boldsymbol{n}), \boldsymbol{q}_2(\boldsymbol{n})$.

(*d*) *Case* C(i)

The two eigenvectors of degree 1 in n_i are of the form

$$\boldsymbol{q}_1(\boldsymbol{n}) = n_1\boldsymbol{k}_1+n_2\boldsymbol{k}_2+n_3\boldsymbol{k}_3, \quad \boldsymbol{q}_2(\boldsymbol{n}) = n_1\boldsymbol{l}_1+n_2\boldsymbol{l}_2+n_3\boldsymbol{l}_3,$$

and, on account of these vectors being orthogonal for all $\boldsymbol{n}\in\mathscr{U}$,

$$\boldsymbol{k}_i\cdot\boldsymbol{l}_i = 0, \quad \boldsymbol{k}_i\cdot\boldsymbol{l}_j = -\boldsymbol{k}_j\cdot\boldsymbol{l}_i, \quad i\neq j.$$

The identity

$$[\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3][\boldsymbol{l}_1,\boldsymbol{l}_2,\boldsymbol{l}_3] = \begin{vmatrix} \boldsymbol{k}_1\cdot\boldsymbol{l}_1 & \boldsymbol{k}_1\cdot\boldsymbol{l}_2 & \boldsymbol{k}_1\cdot\boldsymbol{l}_3 \\ \boldsymbol{k}_2\cdot\boldsymbol{l}_1 & \boldsymbol{k}_2\cdot\boldsymbol{l}_2 & \boldsymbol{k}_2\cdot\boldsymbol{l}_3 \\ \boldsymbol{k}_3\cdot\boldsymbol{l}_1 & \boldsymbol{k}_3\cdot\boldsymbol{l}_2 & \boldsymbol{k}_3\cdot\boldsymbol{l}_3 \end{vmatrix}$$

therefore shows that either $[\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3]$ or $[\boldsymbol{l}_1,\boldsymbol{l}_2,\boldsymbol{l}_3]$ is zero. When the second option is chosen there is a unit vector orthogonal to each of $\boldsymbol{l}_1,\boldsymbol{l}_2,\boldsymbol{l}_3$, which we take to be $\boldsymbol{b}_3$. We can then set

$$\boldsymbol{q}_1(\boldsymbol{n}) = \alpha(\boldsymbol{n})\,\boldsymbol{b}_1+\beta(\boldsymbol{n})\,\boldsymbol{b}_2+\gamma(\boldsymbol{n})\,\boldsymbol{b}_3, \quad \boldsymbol{q}_2(\boldsymbol{n}) = -\beta(\boldsymbol{n})\,\boldsymbol{b}_1+\alpha(\boldsymbol{n})\,\boldsymbol{b}_2, \tag{A 38}$$

with $\alpha(\boldsymbol{n})$ and $\boldsymbol{b}(\boldsymbol{n})$ defined by (A 14) and

$$\gamma(\boldsymbol{n}) = \gamma_1 n_1+\gamma_2 n_2+\gamma_3 n_3. \tag{A 39}$$

The third eigenvector is given by (2.6) and (A 38) as

$$\boldsymbol{q}_3(\boldsymbol{n}) = -\gamma(\boldsymbol{n})\{\alpha(\boldsymbol{n})\,\boldsymbol{b}_1+\beta(\boldsymbol{n})\,\boldsymbol{b}_2\}+[\{\alpha(\boldsymbol{n})\}^2+\{\beta(\boldsymbol{n})\}^2]\,\boldsymbol{b}_3. \tag{A 40}$$

From (2.8),

$$\left.\begin{aligned} m_1(\boldsymbol{n}) = \{\alpha(\boldsymbol{n})\}^2+\{\beta(\boldsymbol{n})\}^2+\{\gamma(\boldsymbol{n})\}^2, \quad m_2(\boldsymbol{n}) = \{\alpha(\boldsymbol{n})\}^2+\{\beta(\boldsymbol{n})\}^2, \\ m_3(\boldsymbol{n}) = m_1(\boldsymbol{n})\,m_2(\boldsymbol{n}). \end{aligned}\right\} \tag{A 41}$$

It should be noted that none of $\alpha(\boldsymbol{n}), \beta(\boldsymbol{n}), \gamma(\boldsymbol{n})$ can vanish for all $\boldsymbol{n}\in\mathscr{U}$: otherwise, either $\boldsymbol{q}_2(\boldsymbol{n})$ or $\boldsymbol{q}_3(\boldsymbol{n})$ would be reducible to a constant vector and possibility C would not apply.

The components of $\boldsymbol{Q}(\boldsymbol{n})$ relative to b are found by entering (A 38), (A 40) and (A 41) into the spectral representation (2.9) and distinguishing the coefficients of $\boldsymbol{B}_\alpha$. As proved in §2*b*, $m_1(\boldsymbol{n})$ and $m_2(\boldsymbol{n})$ divide $\lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n})$ and $\lambda_2(\boldsymbol{n})-\lambda_3(\boldsymbol{n})$ respectively and, by (A 41)$_{1,2}$, (A 14) and (39), $m_1(\boldsymbol{n})$ and $m_2(\boldsymbol{n})$ are, like $\lambda_i(\boldsymbol{n})$, of degree 2 in n_i. There consequently exist constants c_1 and c_2 such that

$$\left.\begin{aligned} \lambda_3(\boldsymbol{n})-\lambda_1(\boldsymbol{n}) &= c_1[\{\alpha(\boldsymbol{n})\}^2+\{\beta(\boldsymbol{n})\}^2+\{\gamma(\boldsymbol{n})\}^2], \\ \lambda_2(\boldsymbol{n})-\lambda_3(\boldsymbol{n}) &= c_2[\{\alpha(\boldsymbol{n})\}^2+\{\beta(\boldsymbol{n})\}^2]. \end{aligned}\right\} \tag{A 42}$$

The eight equations consisting of (A 42) and the expressions for $Q_{ij}(\boldsymbol{n})$ can be arranged, in parallel with (A 17) to (A 19), as

$$\left.\begin{aligned}&\lambda_1(\boldsymbol{n}) = Q_{11}(\boldsymbol{n}) - (c_1+c_2)\{\beta(\boldsymbol{n})\}^2 - c_1\{\gamma(\boldsymbol{n})\}^2,\\ &\lambda_2(\boldsymbol{n}) = Q_{22}(\boldsymbol{n}) + (c_1+c_2)\{\beta(\boldsymbol{n})\}^2, \quad \lambda_3(\boldsymbol{n}) = Q_{33}(\boldsymbol{n}) + c_1\{\gamma(\boldsymbol{n})\}^2,\end{aligned}\right\} \tag{A 43}$$

together with

$$\left.\begin{aligned}&Q_{11}(\boldsymbol{n}) - Q_{22}(\boldsymbol{n}) = -(c_1+c_2)[\{\alpha(\boldsymbol{n})\}^2 - \{\beta(\boldsymbol{n})\}^2],\\ &Q_{33}(\boldsymbol{n}) - Q_{11}(\boldsymbol{n}) = c_1[\{\alpha(\boldsymbol{n})\}^2 - \{\gamma(\boldsymbol{n})\}^2] - c_2\{\beta(\boldsymbol{n})\}^2, \quad Q_{23}(\boldsymbol{n}) = -c_1\,\beta(\boldsymbol{n})\,\gamma(\boldsymbol{n}),\\ &Q_{31}(\boldsymbol{n}) = -c_1\,\alpha(\boldsymbol{n})\,\gamma(\boldsymbol{n}), \quad Q_{12}(\boldsymbol{n}) = -(c_1+c_2)\,\alpha(\boldsymbol{n})\,\beta(\boldsymbol{n}).\end{aligned}\right\} \tag{A 44}$$

If $c_1 = 0$, equations $(\mathrm{A}\ 44)_{3,4}$ deliver the false conclusion that $\boldsymbol{Q}(\boldsymbol{n})$ has the constant eigenvector $\boldsymbol{b}_3$. If $c_1+c_2 = 0$, equations $(\mathrm{A}\ 44)_{1,5}$, in conjunction with $(\mathrm{A}\ 1)_{1,2}$ and $(\mathrm{A}\ 2)_3$, contradict $(3.4)_2$ by giving $c_{11} = c_{22} = -c_{12} = c_{66}$. Hence,

$$c_1 \neq 0, \quad c_1+c_2 \neq 0. \tag{A 45}$$

Introducing (A 1), (A 2), (A 14) and (A 39) into equations (A 44) and equating the coefficients of the squares and products of n_i provides 30 equations that can be rearranged into sets of 20 and 10. The first set, specifying the elastic moduli other than c_{66} in terms of c_{66}, c_1, c_2 and $\alpha_1, \ldots, \gamma_3$, is

$$\left.\begin{aligned}&c_{11} = -(c_1+c_2)(\alpha_1^2-\beta_1^2)+c_{66}, \quad c_{22} = (c_1+c_2)(\alpha_2^2-\beta_2^2)+c_{66},\\ &c_{33} = c_1(\alpha_3^2+\beta_1^2-\gamma_1^2-\gamma_3^2) - c_2(\alpha_1^2+\beta_3^2)+c_{66},\\ &c_{12} = -(c_1+c_2)(\alpha_1\beta_2+\alpha_2\beta_1)-c_{66},\\ &c_{13} = -c_1(\alpha_1\gamma_3+\alpha_3\gamma_1+\beta_1^2-\gamma_1^2)+c_2\alpha_1^2-c_{66},\\ &c_{23} = -c_1(\beta_2\gamma_3+\beta_3\gamma_2+\alpha_2^2-\gamma_2^2)+c_2\beta_2^2-c_{66},\\ &c_{14} = -c_1(\alpha_1\gamma_2+\alpha_2\gamma_1-\beta_1\gamma_1), \quad c_{15} = -c_1\alpha_1\gamma_1, \quad c_{16} = -(c_1+c_2)\alpha_1\beta_1,\\ &c_{24} = -c_1\beta_2\gamma_2, \quad c_{25} = -c_1(\beta_1\gamma_2+\beta_2\gamma_1-\alpha_2\gamma_2), \quad c_{26} = -(c_1+c_2)\alpha_2\beta_2,\\ &c_{34} = -c_1\beta_3\gamma_3, \quad c_{35} = -c_1\alpha_3\gamma_3, \quad c_{36} = -c_1(\beta_1\gamma_3+\beta_3\gamma_1)+(c_1+c_2)\alpha_3\beta_3,\\ &c_{44} = c_1(\alpha_2^2-\gamma_2^2)-c_2\beta_2^2+c_{66}, \quad c_{55} = c_1(\beta_1^2-\gamma_1^2)-c_2\alpha_1^2+c_{66},\\ &c_{45} = -(c_1+c_2)\alpha_3\beta_3, \quad c_{46} = -c_1\alpha_2\gamma_2, \quad c_{56} = -c_1\beta_1\gamma_1.\end{aligned}\right\} \tag{A 46}$$

The second set, comprising relations between c_1, c_2 and $\alpha_1, \ldots, \gamma_3$, is

$$\left.\begin{aligned}&c_1(\alpha_1\gamma_1-\alpha_2\gamma_2)-(c_1+c_2)(\alpha_1\alpha_3-\beta_1\beta_3) = 0,\\ &c_1(\beta_1\gamma_2+\beta_2\gamma_1)-(c_1+c_2)(\alpha_2\beta_3+\alpha_3\beta_2) = 0,\end{aligned}\right\} \tag{A 47}$$

$$\left.\begin{aligned}&c_1(\beta_1\gamma_1-\beta_2\gamma_2)-(c_1+c_2)(\alpha_2\alpha_3-\beta_2\beta_3) = 0,\\ &c_1(\alpha_1\gamma_2+\alpha_2\gamma_1)-(c_1+c_2)(\alpha_1\beta_3+\alpha_3\beta_1) = 0,\end{aligned}\right\} \tag{A 48}$$

$$(c_1+c_2)(\alpha_1+\beta_2)(\alpha_2-\beta_1) = 0, \quad c_1\{\alpha_3\gamma_2-\beta_3\gamma_1+(\alpha_2-\beta_1)\gamma_3\} = 0, \tag{A 49}$$

$$\left.\begin{aligned}c_1(\alpha_1\alpha_3+\beta_1\beta_3-\alpha_1\gamma_1+\alpha_3\gamma_3-\gamma_1\gamma_3)-(c_1+c_2)\beta_1\beta_3 &= 0,\\ c_1(\alpha_2\alpha_3+\beta_2\beta_3-\beta_1\gamma_1+\beta_3\gamma_3-\gamma_2\gamma_3)-(c_1+c_2)\beta_2\beta_3 &= 0,\\ c_1(\alpha_1^2-\alpha_2^2+\beta_1^2-\beta_2^2-\gamma_1^2+\gamma_2^2)-(c_1+c_2)(\alpha_1^2-\alpha_3^2-\beta_2^2+\beta_3^2) &= 0,\\ c_1(\alpha_1\alpha_2+\beta_1\beta_2-\gamma_1\gamma_2)-(c_1+c_2)(\alpha_1\beta_1-\alpha_3\beta_3+\beta_1\beta_2) &= 0.\end{aligned}\right\} \tag{A 50}$$

Adding together equations $(\mathrm{A}\ 46)_{1,2,4}$ and equations (A 47) and subtracting (A 48) results in

$$\left.\begin{aligned} c_{11}+c_{22}+2c_{12} &= (c_1+c_2)\{(\alpha_2-\beta_1)^2-(\alpha_1+\beta_2)^2\},\\ (\alpha_1+\beta_2)\{c_1\gamma_1-(c_1+c_2)\alpha_3\}-(\alpha_2-\beta_1)\{c_1\gamma_2+(c_1+c_2)\beta_3\} &= 0,\\ (\alpha_1+\beta_2)\{c_1\gamma_2-(c_1+c_2)\beta_3\}+(\alpha_2-\beta_1)\{c_1\gamma_1+(c_1+c_2)\alpha_3\} &= 0.\end{aligned}\right\} \qquad (\mathrm{A}\ 51)$$

In view of the inequalities $(3.4)_2$ and (A 45), the relations (A 49) and (A 51) offer the mutually exclusive alternatives

$$\text{(i)} \qquad \beta_1=\alpha_2, \quad \gamma_1=(1+\eta)\alpha_3, \quad \gamma_2=(1+\eta)\beta_3, \quad c_1+c_2<0, \qquad (\mathrm{A}\ 52)$$

and

$$\text{(ii)} \quad \beta_2=-\alpha_1, \quad \gamma_1=-(1+\eta)\alpha_3, \quad \gamma_2=-(1+\eta)\beta_3, \quad \gamma_3=0, \quad c_1+c_2>0, \quad (\mathrm{A}\ 53)$$

where

$$\eta=c_2/c_1. \qquad (\mathrm{A}\ 54)$$

Equations (A 52) satisfy (A 47) to (A 49) in full and, with the aid of (A 54), reduce (A 50) to

$$\left.\begin{aligned} c_2\{\alpha_3(\alpha_1+\gamma_3)+\alpha_2\beta_3\}=0, &\quad c_2\{\alpha_2\alpha_3+\beta_3(\beta_2+\gamma_3)\}=0,\\ c_2\{\alpha_1^2-\beta_2^2+(1+\eta)(\alpha_3^2-\beta_3^2)\}=0, &\quad c_2\{\alpha_2(\alpha_1+\beta_2)+(1+\eta)\alpha_3\beta_3\}=0.\end{aligned}\right\} \qquad (\mathrm{A}\ 55)$$

Equations (53) reduce (A 47) and (A 48) to

$$2\alpha_1\alpha_3-\beta_3(\alpha_2+\beta_1)=0, \quad \alpha_3(\alpha_2+\beta_1)+2\alpha_1\beta_3=0,$$

linear combinations of which are

$$\alpha_1(\alpha_3^2+\beta_3^2)=0, \quad (\alpha_2+\beta_1)(\alpha_3^2+\beta_3^2)=0.$$

If $\alpha_3=\beta_3=0$, equations $(\mathrm{A}\ 53)_{2\text{-}4}$ and (A 39) give $\gamma(\boldsymbol{n})=0$, which is inadmissible. Hence

$$\alpha_1=\beta_2=0, \quad \beta_1=-\alpha_2, \qquad (\mathrm{A}\ 56)$$

and these relations, with $(\mathrm{A}\ 53)_{2\text{-}4}$ and $(\mathrm{A}\ 45)_2$, simplify (A 50) and $(\mathrm{A}\ 51)_1$ to

$$c_2\alpha_2\beta_3=0, \quad c_2\alpha_2\alpha_3=0, \quad c_2(\alpha_3^2-\beta_3^2)=0, \quad c_2\alpha_3\beta_3=0,$$
$$c_{11}+c_{22}+2c_{12}=4(c_1+c_2)\alpha_2^2.$$

By virtue of $(3.4)_2$ and $(\mathrm{A}\ 45)_2$, $\alpha_2\neq 0$, and, as already proved, α_3 and β_3 are not both zero. Hence $c_2=0$, which, from (A 55), is also a possible outcome of alternative (i). When $c_2=0$, equations $(\mathrm{A}\ 43)_{2,3}$, combined with $(\mathrm{A}\ 1)_{2,3}$, (A 46) and either (A 52) or (A 53) and (A 56), lead to

$$\lambda_2=\lambda_3=c_1\alpha_2^2+c_{66}, \qquad (\mathrm{A}\ 57)$$

indicating that $\mathscr{S}$ has two identical spherical sheets. Because of the coincidence of eigenvalues we are concerned here with case C(ii) and further discussion is postponed to section (*e*).

Returning to alternative (i) and assuming that $c_2\neq 0$, we define

$$\left.\begin{aligned} y_1=(-c_1-c_2)^{\frac12}\alpha_1, \quad y_2=(-c_1-c_2)^{\frac12}\beta_2, \quad y_3=(1+\eta)^{-1}(-c_1-c_2)^{\frac12}\gamma_3,\\ y_4=(-c_1-c_2)^{\frac12}\beta_3, \quad y_5=(-c_1-c_2)^{\frac12}\alpha_3, \quad y_6=(-c_1-c_2)^{\frac12}\alpha_2.\end{aligned}\right\} \qquad (\mathrm{A}\ 58)$$

Equations (A 55) then yield the relations

$$\left.\begin{aligned} y_1y_5+y_4y_6+(1+\eta)y_3y_5=0, &\quad y_2y_4+y_5y_6+(1+\eta)y_3y_4=0,\\ y_1^2-y_2^2-(1+\eta)(y_4^2-y_5^2)=0, &\quad (y_1+y_2)y_6+(1+\eta)y_4y_5=0.\end{aligned}\right\} \qquad (\mathrm{A}\ 59)$$

It is found from (A 46), (A 52) and (A 59)$_3$ that the matrix of elastic moduli assumes the form (A 5) with r given by (A 32)$_1$ and

$$s = r+\tfrac{1}{2}\eta\{(1+\eta)^{-1}(y_1^2+y_2^2+2y_6^2)+y_4^2+y_5^2\}, \quad t = s+\eta\{y_3^2+(1+\eta)^{-1}(y_4^2+y_5^2)\}. \quad \text{(A 60)}$$

With the use of (A 14), (A 39), (A 52) and (3.3), the eigenvectors (A 38) can be expressed as

$$\left.\begin{aligned} \boldsymbol{q}_1(\boldsymbol{n}) &= (-c_1-c_2)^{-\frac{1}{2}}\{y_1\boldsymbol{B}_{11}+y_2\boldsymbol{B}_{22}+y_6(\boldsymbol{B}_{12}+\boldsymbol{B}_{21})+y_4\boldsymbol{B}_{23}+y_5\boldsymbol{B}_{13} \\ &\quad +(1+\eta)(y_4\boldsymbol{B}_{32}+y_5\boldsymbol{B}_{31}+y_3\boldsymbol{B}_{33})\}\boldsymbol{n}, \\ \boldsymbol{q}_2(\boldsymbol{n}) &= (-c_1-c_2)^{-\frac{1}{2}}\{y_1\boldsymbol{B}_{21}-y_2\boldsymbol{B}_{12}-y_6(\boldsymbol{B}_{11}-\boldsymbol{B}_{22})-(y_4\boldsymbol{B}_{13}-y_5\boldsymbol{B}_{23})\}\boldsymbol{n}. \end{aligned}\right\} \quad \text{(A 61)}$$

It was shown in section (c) that r is unaffected by the change of basis $b\to b'$ and we see from (A 60) and (A 7) that this property extends successively to s and t. The form invariance of $\boldsymbol{q}_1(\boldsymbol{n})$ and $\boldsymbol{q}_2(\boldsymbol{n})$ is likewise a consequence of (A 8) and it is easily confirmed from (A 6) that equations (A 59) remain valid when y_α is replaced by y'_α.

The application of the Lemma is thus justified and, as before, y'_6 is made zero by choosing the particular value (A 34) of θ. By (A 59)$_4$, one or both of y'_4 and y'_5 is also zero. Suppose that $y'_4 \neq 0$, $y'_5 = 0$. Then, from (A 59), (A 32)$_1$ and (A 60), transformed to b',

$$\left.\begin{aligned} y'_3 &= -(1+\eta)^{-1}y'_2, \quad y_4'^2 = (1+\eta)^{-1}(y_1'^2-y_2'^2), \\ r &= c_{66}, \quad s = r+\eta(1+\eta)^{-1}y_1'^2, \quad t = r+\eta(2+\eta)(1+\eta)^{-2}y_1'^2. \end{aligned}\right\} \quad \text{(A 62)}$$

The principal minor formed by the first four rows and columns of (A 5) is

$$M = s[-4rs^2+rt(y_1+y_2)^2-s^2(y_1-y_2)^2+4rs\{(y_1+y_2)y_3-y_4^2\}], \quad \text{(A 63)}$$

and on mapping into b', substituting from (A 62) and noting that $s = c_{55}$, we obtain

$$M = -c_{55}(y_1'^2+c_{66})\{2c_{66}+\eta(1+\eta)^{-1}y'_1(y'_1-y'_2)\}^2.$$

Due to (3.4)$_1$, M is non-positive, contrary to the positive definiteness condition (2.2). The assumption that $y'_4 = 0$, $y'_5 \neq 0$ similarly leads to a contradiction, so

$$y'_4 = y'_5 = y'_6 = 0. \quad \text{(A 64)}$$

In the particular basis b', $\beta_2 \neq -\alpha_1$ and hence $y'_1+y'_2 \neq 0$ (see (A 52), (A 53) and (A 58)$_{1,2}$). Equations (A 59) thus contract to the single relation

$$y'_1 = y'_2. \quad \text{(A 65)}$$

The result of applying (A 64) and (A 65), in b', to equations (A 32)$_1$, (A 60) and (A 61) is

$$r = c_{66}, \quad s = r+\eta(1+\eta)^{-1}y_1'^2, \quad t = s+\eta y_3'^2, \quad \text{(A 66)}$$

and

$$\left.\begin{aligned} \boldsymbol{q}_1(\boldsymbol{n}) &= (-c_1-c_2)^{-\frac{1}{2}}\{y'_1(n_1\boldsymbol{b}'_1+n_2\boldsymbol{b}'_2)+(1+\eta)y'_3n_3\boldsymbol{b}'_3\}, \\ \boldsymbol{q}_2(\boldsymbol{n}) &= (-c_1-c_2)^{-\frac{1}{2}}y'_1(-n_2\boldsymbol{b}'_1+n_1\boldsymbol{b}'_2). \end{aligned}\right\} \quad \text{(A 67)}$$

Equations (A 64) imply, via (A 5), that the elastic moduli in b' meet the requirements (3.5) for orthorhombic symmetry. We can accordingly equate b' to a, in which basis the non-zero moduli, other than c_{66}, are given by (A 5) and (A 64) to (A 66) as

$$\left.\begin{aligned} c_{11} &= c_{22} = y_1'^2+c_{66}, \quad c_{33} = \eta(1+\eta)^{-1}y_1'^2+(1+\eta)y_3'^2+c_{66}, \\ c_{12} &= y_1'^2-c_{66}, \quad c_{13} = c_{23} = -\eta(1+\eta)^{-1}y_1'^2+y'_1y'_3-c_{66}, \\ c_{44} &= c_{55} = \eta(1+\eta)^{-1}y_1'^2+c_{66}. \end{aligned}\right\} \quad \text{(A 68)}$$

The connexions (3.12) and (3.13) evidently apply and

$$y_1' = N(c_{11}+c_{13}), \quad (1+\eta)\, y_3' = N(c_{13}+c_{33}), \tag{A 69}$$

with

$$N = \{(1+\eta)^{-1}\, y_1' + y_3'\}^{-1}.$$

Entering (A 69) and the substitutions $\boldsymbol{b}_i' = \boldsymbol{a}_i$ into (A 67), removing the factors $(-c_1-c_2)^{-\frac{1}{2}}N$ and $(-c_1-c_2)^{-\frac{1}{2}}y_1'$ from $\boldsymbol{q}_1(\boldsymbol{n})$ and $\boldsymbol{q}_2(\boldsymbol{n})$ respectively, and calculating $\boldsymbol{q}_3(\boldsymbol{n})$ from (2.6) delivers the eigenvectors (3.15).

In the light of (A 52), (A 64) and (A 65), we see from the definitions (A 58) that, in a, $\alpha_1 = \beta_2$ and γ_3 are the only non-zero members of $\alpha_1, \dots, \gamma_3$. It follows from $(\text{A } 14)_2$ and (A 39) that

$$\beta(\boldsymbol{n}) = \alpha_1\, n_2 = (-c_1-c_2)^{-\frac{1}{2}} y_1'\, n_2, \quad \gamma(\boldsymbol{n}) = \gamma_3\, n_3 = (1+\eta)\,(-c_1-c_2)^{-\frac{1}{2}} y_3'\, n_3,$$

whence, with reference to (A 68) and (A 54),

$$(c_1+c_2)\,\{\beta(\boldsymbol{n})\}^2 = (c_{66}-c_{11})\, n_2^2, \quad c_1\{\gamma(\boldsymbol{n})\}^2 = (c_{44}-c_{33})\, n_3^2.$$

Inserting these expressions, together with (A 1), into the formulae (A 43) and utilizing (3.5) and $(3.12)_{1,3}$, we arrive at the eigenvalues (3.14).

(*e*) *Case* C(ii)

We know from equations (A 57) that when $c_2 = 0, \mathscr{S}$ has coincident spherical sheets. It is convenient in this case to make b the basis in which the eigenvalues of $\boldsymbol{Q}(\boldsymbol{n})$ take the form

$$\lambda_1(\boldsymbol{n}) = q_1\, n_1^2 + q_2\, n_2^2 + q_3\, n_3^2, \quad \lambda_2 = \lambda_3 = r. \tag{A 70}$$

Equations $(\text{A } 38)_1$ and $(\text{A } 42)_1$ again hold and on introducing them into the spectral representation (2.13) we obtain

$$\boldsymbol{Q}(\boldsymbol{n}) = \lambda_3(\boldsymbol{n})\,\boldsymbol{I} - c_1\{\alpha(\boldsymbol{n})\,\boldsymbol{b}_1+\beta(\boldsymbol{n})\,\boldsymbol{b}_2+\gamma(\boldsymbol{n})\,\boldsymbol{b}_3\} \otimes \{\alpha(\boldsymbol{n})\,\boldsymbol{b}_1+\beta(\boldsymbol{n})\,\boldsymbol{b}_2+\gamma(\boldsymbol{n})\,\boldsymbol{b}_3\}. \tag{A 71}$$

Since $\boldsymbol{Q}(\boldsymbol{n})$ has at most two coincident eigenvalues,

$$c_1 \neq 0. \tag{A 72}$$

The seven equations consisting of $(\text{A } 42)_1$ and the expressions for $Q_{ij}(\boldsymbol{n})$ supplied by (A 71) can be written in terms of n_i by means of (A 1), (A 2), (A 14), (A 39) and (A 70). Equating the coefficients of squares and products then gives rise to 42 equations, made up of the formulae

$$q_i = r - c_1(\alpha_i^2+\beta_i^2+\gamma_i^2) \tag{A 73}$$

for q_i, 21 equations for the elastic moduli in terms of c_1, r and $\alpha_1, \dots, \gamma_3$, and 18 relations between $\alpha_1, \dots, \gamma_3$. The moduli are given by

$$\left.\begin{aligned}
&c_{11} = r - c_1\,\alpha_1^2, \quad c_{22} = r - c_1\,\beta_2^2, \quad c_{33} = r - c_1\,\gamma_3^2,\\
&c_{12} = -r - c_1\{\alpha_2(\beta_1-\alpha_2)+\alpha_1\,\beta_2\}, \quad c_{13} = -r - c_1\{\alpha_3(\gamma_1-\alpha_3)+\alpha_1\,\gamma_3\},\\
&c_{23} = -r - c_1\{\beta_3(\gamma_2-\beta_3)+\beta_2\,\gamma_3\},\\
&c_{14} = -c_1\{\alpha_3(\beta_1-\alpha_2)+\alpha_1\,\beta_3\}, \quad c_{15} = -c_1\,\alpha_1\,\alpha_3, \quad c_{16} = -c_1\,\alpha_1\,\alpha_2,\\
&c_{24} = -c_1\,\beta_2\,\beta_3, \quad c_{25} = c_1\{\beta_3(\beta_1-\alpha_2)-\alpha_3\,\beta_2\}, \quad c_{26} = -c_1\,\alpha_2\,\beta_2,\\
&c_{34} = -c_1\,\beta_3\,\gamma_3, \quad c_{35} = -c_1\,\alpha_3\,\gamma_3, \quad c_{36} = -c_1\{\alpha_3(\gamma_2-\beta_3)+\alpha_2\,\gamma_3\},\\
&c_{44} = r - c_1\,\beta_3^2, \quad c_{55} = r - c_1\,\alpha_3^2, \quad c_{66} = r - c_1\,\alpha_2^2,\\
&c_{45} = -c_1\,\alpha_3\,\beta_3, \quad c_{46} = -c_1\,\beta_1\,\beta_3, \quad c_{56} = -c_1\,\alpha_2\,\alpha_3,
\end{aligned}\right\} \tag{A 74}$$

and, using (A 72), we can set out the additional relations as

$$\beta_1^2 = \alpha_2^2, \quad \gamma_1^2 = \alpha_3^2, \quad \gamma_2^2 = \beta_3^2, \quad \alpha_2\alpha_3 = \beta_1\gamma_1, \quad \beta_1\beta_3 = \alpha_2\gamma_2, \quad \alpha_3\beta_3 = \gamma_1\gamma_2, \tag{A 75}$$

$$\left.\begin{array}{l} \alpha_1(\beta_1-\alpha_2) = 0, \quad \beta_2(\beta_1-\alpha_2) = 0, \quad \alpha_1(\gamma_1-\alpha_3) = 0, \quad \gamma_3(\gamma_1-\alpha_3) = 0, \\ \beta_2(\gamma_2-\beta_3) = 0, \quad \gamma_3(\gamma_2-\beta_3) = 0, \quad \beta_3\gamma_1-\alpha_3\gamma_2+\gamma_3(\beta_1-\alpha_2) = 0, \\ \beta_1\gamma_2-\alpha_2\beta_3+\beta_2(\gamma_1-\alpha_3) = 0, \quad \alpha_2\gamma_1-\alpha_3\beta_1+\alpha_1(\gamma_2-\beta_3) = 0, \end{array}\right\} \tag{A 76}$$

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 = 0, \quad \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3 = 0, \quad \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 = 0. \tag{A 77}$$

We observe from $(\mathrm{A}\ 74)_{1,2,4}$ that

$$c_{11}+c_{22}+2c_{12} = -c_1\{(\alpha_1+\beta_2)^2+2\alpha_2(\beta_1-\alpha_2)\}. \tag{A 78}$$

Equations (A 75) imply that either

$$\beta_1 = \alpha_2, \quad \gamma_1 = \alpha_3, \quad \gamma_2 = \beta_3, \tag{A 79}$$

or

$$\beta_1 = -\alpha_2, \quad \gamma_1 = -\alpha_3, \quad \gamma_2 = -\beta_3. \tag{A 80}$$

The first solution, (A 79), satisfies (A 76) in full, reduces (A 77) to

$$\alpha_3(\alpha_1+\gamma_3)+\alpha_2\beta_3 = 0, \quad \beta_3(\beta_2+\gamma_3)+\alpha_2\alpha_3 = 0, \quad \alpha_2(\alpha_1+\beta_2)+\alpha_3\beta_3 = 0, \tag{A 81}$$

and requires, through (A 78) and $(3.4)_2$, that $c_1 < 0$. Adopting the definitions (A 58), with $c_2 = 0$, we find from equations (A 79) that the moduli (A 74) can be compiled into the matrix (A 5) with $r = s = t$. Equations (A 81) reproduce $(\mathrm{A}\ 59)_{1,2,4}$, with $\eta = 0$, and $(\mathrm{A}\ 61)_1$ holds with the same simplification. The invariance of r under an arbitrary change of basis is obvious from $(\mathrm{A}\ 70)_{2,3}$. We can therefore make a third application of the Lemma. The details are much the same as in section (*d*). The rotation $b \to b'$ with the choice of angle (A 34) makes y_6' zero and, by $(\mathrm{A}\ 59)_4$, one or both of y_4' and y_5' is zero. If $y_4' \neq 0, y_5' = 0$, equation $(\mathrm{A}\ 59)_2$ gives $y_3' = -y_2'$ and, from $(\mathrm{A}\ 74)_{18}$, $r\,(= s = t) = c_{66}$. We thus infer from $(\mathrm{A}\ 63)_1$, transformed to b', that, contrary to the basic condition (2.2),

$$M = -4c_{66}^3(y_2'^2+y_4'^2+c_{66}) < 0.$$

The assumption that $y_4' = 0, y_5' \neq 0$ is similarly untenable, forcing the conclusion (A 64) and the corollary that orthorhombic symmetry prevails in b'. The non-zero elastic moduli in this basis are specified by equations (A 74), (A 79) and (A 58) as

$$c_{ii} = r+y_i'^2, \quad c_{ij} = -r+y_i'y_j', \quad c_{44} = c_{55} = c_{66} = r, \tag{A 82}$$

and, from (A 73),

$$q_i = r+y_i'^2 = c_{ii}. \tag{A 83}$$

The conditions (3.16) follow directly from (A 82) and the eigenvalues (3.17) from (A 70), $(\mathrm{A}\ 82)_3$ and (A 83). The eigenvector (3.18) is derived from $(\mathrm{A}\ 61)_1$, transformed to a, by setting $c_2 = 0$, inserting the factor $(-c_1)^{\frac{1}{2}}y_3'$ and using the relations

$$y_1'y_3' = c_{13}+c_{44}, \quad y_2'y_3' = c_{23}+c_{44}, \quad y_3'^2 = c_{33}-c_{44},$$

furnished by (A 82), together with (A 64).

Equations (A 80) reduce (A 76) and (A 77) to

$$\begin{aligned} \alpha_1\alpha_2 = \alpha_1\alpha_3 = \alpha_2\alpha_3 = \alpha_1\beta_3 = \alpha_2\beta_2 = \alpha_2\beta_3 \\ = \alpha_3\beta_2 = \beta_2\beta_3 = \alpha_3\beta_3 = \alpha_2\gamma_3 = \alpha_3\gamma_3 = \beta_3\gamma_3 = 0. \end{aligned} \tag{A 84}$$

If $\alpha_2 \neq 0$, we deduce from (A 84), (A 80) and (A 39) that $\gamma(\boldsymbol{n}) = 0$: if $\alpha_3 \neq 0$

(respectively $\beta_3 \neq 0$), we see from (A 14) that $\beta(\boldsymbol{n}) = 0$ (respectively $\alpha(\boldsymbol{n}) = 0$). None of $\alpha(\boldsymbol{n}), \beta(\boldsymbol{n}), \gamma(\boldsymbol{n})$ can vanish identically, however, as noted in the opening paragraph of section (*d*), so $\alpha_2 = \alpha_3 = \beta_3 = 0$ is the only solution of (A 84). This means, *vide* (A 58), that the alternative (A 80) induces the same situation in b as applies in b' when (A 79) holds. Equations (3.16) to (3.18) follow as before.

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Received 27 July 1992; accepted 23 September 1992