Third-order elastic constants for an inviscid fluid

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In dealing with nonlinear problems involving fluids and solids, whether of a static or dynamic nature, a common description of the fields in terms of Eulerian or Lagrangian variables is desirable. Usually the former is used for fluids and the latter for solids. The choice of primitive variables also differs when dealing with fluids or solids. The material constants describing the constitutive behavior of these media will thus depend on the description adopted. In this paper, explicit relations are provided between third-order elastic constants for an inviscid fluid and the more common coefficients, A and B, appearing in the Taylor expansion of the equation of state. The essential results are $c_{111}=-(5A+B)$, $c_{112}=-(A+B)$, and $c_{123}=A-B$.

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INTRODUCTION

Nonlinear problems in the acoustics of fluids are usually formulated in terms of an Eulerian description of the wave motion.[1] The elastodynamic of solids, on the other hand, is mostly formulated in terms of a Lagrangian description.[2] Also, the choice of fundamental (or primitive) variables is different depending on the type of media. Pressure and density are usually used in adiabatic processes in fluids, and stress (Cauchy or Piola–Kirchhoff) and strain (Green or Almansi) are normally used for solids. Therefore, the material constants that describe the constitutive behavior of such media will depend on the particular choices of the description and fundamental variables. In dealing with problems involving both fluids and solids, it is desirable to use the same description throughout. We next present a derivation of the relation between third-order elastic constants of an inviscid fluid and the more common parameters, A and B, appearing in the Taylor expansion of the equation of state.

I. THEORY

The adiabatic equation of state for a fluid, $p=p(\rho)$, can be expanded in a Taylor series about a given state, and is usually presented in the following form:[3]

$$p=p_0+A\left(\frac{\rho-\rho_0}{\rho_0}\right)+\frac{B}{2}\left(\frac{\rho-\rho_0}{\rho_0}\right)^2+O\left[\left(\frac{\rho-\rho_0}{\rho_0}\right)^3\right], \tag{1}$$

where p and ρ are pressure and density, respectively, with p_0 and ρ_0 being their reference values, $A=\rho_0\, dp/d\rho(\rho_0)$ and $B=\rho_0^2\, d^2p/d\rho^2(\rho_0)$. In terms of this pressure, the Cauchy stress tensor is given by

$$\tau=-p\mathbf{I}, \tag{2}$$

where $\mathbf{I}$ is the second-rank identity tensor. Equations (1) and (2) define the *constitutive relation* for an inviscid fluid.

In a stressed configuration, particles originally at $\mathbf{X}$ (X_K, $K=1,2,3$) are displaced to $\mathbf{x}$ (x_k, $k=1,2,3$), such that we can define the displacement vector as

$$\mathbf{u}=\mathbf{x}-\mathbf{X}. \tag{3}$$

The deformation gradient is defined as

$$\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}}=\mathbf{I}+\frac{\partial \mathbf{u}}{\partial \mathbf{X}}, \tag{4}$$

and the Lagrangian (Green's) strain tensor is given by

$$\mathbf{E}=\tfrac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I}). \tag{5}$$

The density in the stressed configuration can be expressed in terms of its value in the unstressed (reference) configuration through

$$\frac{\rho}{\rho_0}=\frac{1}{\det \mathbf{F}}=(1+2I_E+4II_E+8III_E)^{-1/2}, \tag{6}$$

where I_E, II_E, and III_E are the principal invariants of the Lagrangian strain tensor,[2] and are given by

$$I_E=\operatorname{tr}\mathbf{E},\quad II_E=\tfrac{1}{2}[(\operatorname{tr}\mathbf{E})^2-\operatorname{tr}\mathbf{E}^2],\quad III_E=\det \mathbf{E}. \tag{7}$$

Expanding Eq. (6) to second-order in strain, and substituting the result into Eq. (1) yields

$$p=-AI_E+\tfrac{1}{2}(3A+B)I_E^2-2AII_E+O(\mathbf{E}^3), \tag{8}$$

which upon substitution into Eq. (2) gives

$$\tau_{ij}=AI_E\delta_{ij}-\tfrac{1}{2}(3A+B)I_E^2\delta_{ij}+2AII_E\delta_{ij}. \tag{9}$$

Assuming the fluid is "hyperelastic," we can postulate the existence of a strain energy density function U, defined per unit mass in the reference or Lagrangian description. The strain energy is assumed to be a function of the deformation gradient tensor. Consequently, it depends solely on the strain, and as such admits the following expansion:[2]

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$$\rho_0 U(\mathbf{E}) = \tfrac{1}{2} C_{KLMN} E_{KL} E_{MN} + \tfrac{1}{6} C_{KLMNPQ} E_{KL} E_{MN} E_{PQ} + O(\mathbf{E}^4), \tag{10}$$

where C_{KLMN} and C_{KLMNPQ} are, respectively, the second- and third-order adiabatic elastic coefficients evaluated at zero strain.[2,4] These possess the symmetries $C_{KLMN}=C_{LKMN}=C_{MNKL}$ and $C_{KLMNPQ}=C_{LKMNPQ}=C_{MNKLPQ}=C_{KLPQMN}$. The adiabatic Piola–Kirchhoff stress tensor of the second kind T_{KL} is defined by

$$T_{KL} = \rho_0 \frac{\partial U}{\partial E_{KL}}. \tag{11}$$

The Cauchy and Piola–Kirchhoff stress tensors are related through

$$\tau_{ij} = (\rho/\rho_0) F_{iK} F_{jL} T_{KL}. \tag{12}$$

Making use of Eqs. (10)–(12), we can rewrite the latter as

$$\begin{aligned}\tau_{ij} &= (1 - I_E + \cdots)(\delta_{iK} + u_{i,K})(\delta_{jL} + u_{j,L})(C_{KLMN} E_{MN} + \tfrac{1}{2} C_{KLMNPQ} E_{MN} E_{PQ}) \\ &= C_{ijKL} E_{KL} + \tfrac{1}{2} C_{ijKLMN} E_{KL} E_{MN} - C_{ijKL} E_{KL} I_E \\ &\quad + (u_{i,K} C_{KjMN} + u_{j,L} C_{iLMN} + u_{i,K} u_{j,L} C_{KLMN}) E_{MN}.\end{aligned} \tag{13}$$

Comparing the terms linear in $\mathbf{E}$ in Eqs. (9) and (13), we get

$$A I_E \delta_{ij} = C_{ijKL} E_{KL}. \tag{14}$$

Assuming C_{KLMN} to be isotropic, i.e.,

$$C_{KLMN} = \lambda \delta_{KL} \delta_{MN} + \mu(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM}), \tag{15}$$

leads to

$$\lambda = A \quad \text{and} \quad \mu = 0. \tag{16}$$

Equating the nonlinear terms in Eqs. (9) and (13), and using Eq. (16) gives the following identity:

$$\tfrac{1}{2} C_{ijKLMN} E_{KL} E_{MN} - A I_E^2 \delta_{ij} + (u_{i,j} + u_{j,i} + u_{i,K} u_{j,K}) A I_E = -[\tfrac{1}{2}(3A + B) I_E^2 - 2A I I_E] \delta_{ij}. \tag{17}$$

The quantity $(u_{i,j} + u_{j,i} + u_{i,K} u_{j,K})/2 \equiv \hat{E}_{ij}$ can be rewritten as

$$\hat{\mathbf{E}} = \mathbf{E} + \tfrac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{F}\mathbf{F}^T) = \mathbf{E} + (\boldsymbol{\Omega}\mathbf{E} - \mathbf{E}\boldsymbol{\Omega}) + O(\mathbf{E}^2), \tag{18}$$

where $\boldsymbol{\Omega} = (\mathbf{F} - \mathbf{F}^T)/2$ is the infinitesimal rotation tensor. However, in deriving both sides of Eq. (17) we have implicitly neglected terms of order $\mathbf{E}^2\boldsymbol{\Omega}$ and smaller. We will say more about this below, but note for the present that to the same degree of approximation, Eq. (17) becomes

$$\tfrac{1}{2} C_{ijKLMN} E_{KL} E_{MN} = -2A I_E E_{ij} + [2A I I_E - \tfrac{1}{2}(A + B) I_E^2] \delta_{ij}. \tag{19}$$

Again, if we assume that C_{KLMNPQ} is isotropic, we can express it as

$$\begin{aligned}C_{KLMNPQ} = {} & \alpha \delta_{KL} \delta_{MN} \delta_{PQ} + \beta[\delta_{KL}(\delta_{MP}\delta_{NQ} + \delta_{MQ}\delta_{NP}) \\ & + \delta_{MN}(\delta_{KP}\delta_{LQ} + \delta_{KQ}\delta_{LP}) + \delta_{PQ}(\delta_{KM}\delta_{LN} \\ & + \delta_{LM}\delta_{KN})] + \gamma[\delta_{LM}(\delta_{KP}\delta_{NQ} \\ & + \delta_{KQ}\delta_{NP}) + \delta_{KM}(\delta_{LP}\delta_{NQ} + \delta_{LQ}\delta_{NP}) \\ & + \delta_{KN}(\delta_{LP}\delta_{MQ} + \delta_{LQ}\delta_{MP}) \\ & + \delta_{LN}(\delta_{KP}\delta_{MQ} + \delta_{KQ}\delta_{MP})],\end{aligned} \tag{20}$$

where

$$\begin{aligned}\alpha &= C_{112233} = c_{123}, \\ \beta &= \tfrac{1}{2}(C_{111122} - C_{112233}) = \tfrac{1}{2}(c_{112} - c_{123}), \\ \gamma &= \tfrac{1}{8}(C_{111111} - 3C_{111122} + 2C_{112233}) \\ &= \tfrac{1}{8}(c_{111} - 3c_{112} + 2c_{123}),\end{aligned} \tag{21}$$

and c_{KLM} is the tensor of elastic constants in the abbreviated Voigt notation. Substituting Eq. (20) into (19) gives the following:

$$\begin{aligned}&\tfrac{1}{2} c_{123} I_E^2 \delta_{KL} + \tfrac{1}{2}(c_{112} - c_{123})(\mathrm{tr}\, \mathbf{E}^2 \delta_{KL} + 2 I_E E_{KL}) \\ &\quad + \tfrac{1}{2}(c_{111} - 3c_{112} + 2c_{123}) E_{KP} E_{PL} \\ &\quad = -\tfrac{1}{2}(A + B) I_E^2 \delta_{KL} + 2A I I_E \delta_{KL} - 2A I_E E_{KL}.\end{aligned} \tag{22}$$

Using the second of Eqs. (7) to eliminate $\mathrm{tr}\, \mathbf{E}^2$, and equating the coefficients of similar terms on the left- and right-hand sides of Eq. (22), we get

$$\begin{aligned}c_{112} &= -(A + B), \\ -(c_{112} - c_{123}) &= 2A, \\ c_{111} - 3c_{112} + 2c_{123} &= 0.\end{aligned} \tag{23}$$

Notice that the second of Eqs. (23) is obtained twice in this process, thus ascertaining the consistency of the derivation. We thus obtain

$$c_{111} = -(5A + B), \quad c_{112} = -(A + B), \quad c_{123} = A - B. \tag{24}$$

We note that had we retained terms of order $\mathbf{E}^2\boldsymbol{\Omega}$ in the expansion of Eq. (13), it can be shown, using Eqs. (15)–(18), and the isotropic form of C_{KLMNPQ} in Eq. (20), that they contribute $(2A + c_{112} - c_{123}) I_E (\boldsymbol{\Omega}\mathbf{E} - \mathbf{E}\boldsymbol{\Omega})$ to the Cauchy stress. However, it is clear from the second of Eqs. (23) that this contribution vanishes. Hence, we have shown that the hyperelastic and equation of state derivations are consistent, neglecting terms of order $\mathbf{E}^3$.

The first of Eqs. (24) was derived in Ref. 5 [see the first of Eqs. (43)] by comparing the one-dimensional nonlinear equations of motion derived from the Eulerian and Lagrangian descriptions. Table I relates these constants to other sets of constants which appear in the literature.[6–8]

For water at a temperature of 30 °C and at atmospheric pressure,[3,9] $B/A = 5.2$ and $A = 2.277$ GPa, giving the following third-order elastic constants:

TABLE I. Relation between third-order elastic constants for isotropic solids.

	Murnaghan[6]	Eringen and Suhubi[2]	Toupin and Bernstein[7]	Landau and Lifshitz[8]	Eq. (21)
c_{111}	$2l+4m$	$6l_E$	$\nu_1+6\nu_2+8\nu_3$	$2\mathscr{A}+6\mathscr{B}+2\mathscr{C}$	$\alpha+6\beta+8\gamma$
c_{112}	$2l$	$6l_E+2m_E$	$\nu_1+2\nu_2$	$2\mathscr{B}+2\mathscr{C}$	$\alpha+2\beta$
c_{123}	$2l-2m+n$	$6l_E+3m_E+n_E$	ν_1	$2\mathscr{C}$	α

$$c_{111}=-23.23 \text{ GPa}, \quad c_{112}=-14.12 \text{ GPa}, \quad c_{123}=-9.56 \text{ GPa}. \tag{25}$$

The adiabatic wave speed c is given by[1]

$$c^2(\rho)=\frac{dp}{d\rho}, \tag{26}$$

which for a fluid with equation of state given by Eq. (1) is

$$c^2(\rho)=A/\rho_0+(B/\rho_0^2)(\rho-\rho_0). \tag{27}$$

The natural wave speed c_0 is thus given by

$$c_0^2=c^2(\rho_0)=A/\rho_0. \tag{28}$$

A measure of the nonlinearity of the fluid is given by the dimensionless parameter $\rho_0\, dc^2/dp$, which by using Eqs. (1) and (26) gives

$$\rho_0\frac{dc^2}{dp}=\frac{B}{A}. \tag{29}$$

For isotropic solids there are two parameters which measure the degree of nonlinearity and are given by[7,10]

$$\rho_0\frac{dv_c^2}{dp}=-\frac{7\lambda+10\mu+c_{111}+2c_{112}}{3\lambda+2\mu}, \qquad \rho_0\frac{dv_s^2}{dp}=-\frac{3\lambda+6\mu+c_{111}/2-c_{123}/2}{3\lambda+2\mu}, \tag{30}$$

where v_c and v_s are the speeds of compressional and shear waves, respectively. If we substitute in these expressions the elastic constants as given in Eqs. (16) and (24), we get

$$\rho_0\frac{dv_c^2}{dp}=\frac{B}{A}, \quad \rho_0\frac{dv_s^2}{dp}=0, \tag{31}$$

which are in agreement with Eq. (29) and the fact that inviscid fluids do not support shear waves.

The approach taken here compares the Cauchy stress tensor according to the equation of state and that from the hyperelastic strain energy density. Alternatively, one can start from the equation of state and find the strain energy, from which the third- and higher-order elastic coefficients could be determined. Thus Eqs. (2), (5), (11), and (12) imply

$$\rho_0\frac{\partial U}{\partial E_{KL}}=-\frac{\rho_0}{\rho}p(\mathbf{I}+2\mathbf{E})^{-1}. \tag{32}$$

Integration then yields

$$\rho_0 U=-\int_0^{\mathbf{E}} p\frac{\rho_0}{\rho}\operatorname{tr}(\mathbf{I}+2\mathbf{E})^{-1}\,d\mathbf{E} =-\frac{1}{2}\int_0^{\mathbf{E}} p\frac{\rho_0}{\rho}d\,[\log\det(\mathbf{I}+2\mathbf{E})]. \tag{33}$$

Then using Eqs. (5) and (6) this reduces to the familiar form,

$$U=-\int_{\rho_0}^{\rho} p\,d\rho^{-1}. \tag{34}$$

Substituting Eq. (1) into (33), and ignoring the higher-order terms, gives

$$\rho_0 U=(A-B)\left(\frac{\rho_0}{\rho}-1-\log\frac{\rho_0}{\rho}\right)-\frac{B}{2}\frac{\rho}{\rho_0}\left(\frac{\rho_0}{\rho}-1\right)^2+\cdots. \tag{35}$$

Then, ρ_0/ρ can be expressed in terms of the strain using Eq. (6), or the alternative form,

$$\rho_0/\rho=[1+2I_1+2(I_1^2-I_2)+\tfrac{4}{3}(2I_3-3I_1I_2+I_1^3)]^{1/2}, \tag{36}$$

where $I_n=\operatorname{tr}\mathbf{E}^n$, for $n=1$, 2, or 3. Combining Eqs. (35) and (36) and expanding yields

$$\rho_0 U=(A/2)I_1^2+[(A-B)/6]I_1^3-AI_1I_2+O(\mathbf{E}^4). \tag{37}$$

This is the same form of the strain energy expansion used by Landau and Lifshitz,[8] for instance, and allows us to read off the values of the third-order moduli. Fourth- and higher-order moduli could be determined in the same manner.

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