Acoustic reciprocity for fluid-structure problems

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A reciprocity relation for acoustic interaction with fluid-loaded structures is derived. The displacement in the direction $\mathbf{e}^{(2)}$ at any position $\mathbf{x}^{(2)}$ in the solid due to a monopole of strength $(\rho_f\omega^2)$ at any position $\mathbf{x}^{(1)}$ in the fluid is equal to the pressure at $\mathbf{x}^{(1)}$ caused by a unit force in the direction $\mathbf{e}^{(2)}$ applied at $\mathbf{x}^{(2)}$. An application to plane wave scattering is discussed.

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INTRODUCTION

Reciprocity relations in acoustics and elasticity are useful since they provide the solution to one problem in terms of another. Reciprocity in acoustics is well known and straightforward.[1] Thus, the acoustic pressure at position $\mathbf{x}^{(2)}$ due to a monopole source at position $\mathbf{x}^{(1)}$ is identical to the pressure at $\mathbf{x}^{(1)}$ resulting from a monopole of the same strength located at $\mathbf{x}^{(2)}$. The statement of reciprocity for dynamic elasticity pertains to vector quantities, force and displacement, and is also known as the dynamic Betti–Rayleigh theorem.[2] Specifically, if a force directed along $\mathbf{e}^{(1)}$, $|\mathbf{e}^{(1)}|=1$, is applied at position $\mathbf{x}^{(1)}$ in an elastic body, inducing a displacement at $\mathbf{x}^{(2)}$ in the direction $\mathbf{e}^{(2)}$, then an equal force applied at $\mathbf{x}^{(2)}$ in direction $\mathbf{e}^{(2)}$ produces an identical displacement at $\mathbf{x}^{(1)}$ in the $\mathbf{e}^{(1)}$ direction. In this Letter we derive a reciprocity relation for the case in which one point lies in the fluid and the other in the solid. The relation involves scalar quantities in the fluid (pressure) and vector quantities in the solid (displacement, force).

I. THE RECIPROCITY RELATION

Let V_f and V_s denote the inviscid fluid and solid regions, respectively, which are separated by the boundary or interface B. We consider time harmonic motion with the term $\mathrm{Re}\{\cdots e^{-i\omega t}\}$ omitted everywhere. The fundamental variables of interest are the pressure $p(\mathbf{x})$ in V_f and the displacement $\mathbf{u}(\mathbf{x})$ in V_s. The continuity conditions for the normal displacement and the traction on the boundary are

$$\left.\begin{array}{l} -p\mathbf{n}=\mathbf{T}\cdot\mathbf{n}, \\ (\rho_f\omega^2)^{-1}\mathbf{n}\cdot\nabla p=\mathbf{u}\cdot\mathbf{n}, \end{array}\right\} \quad \text{on } B. \tag{1}$$

Here, ρ_f is the fluid density and $\mathbf{n}$ is the unit normal to B directed into the fluid. The stress tensor, $\mathbf{T}=\mathbf{T}(\mathbf{u},\mathbf{x})$, is assumed to depend upon the strain according to the generalized Hooke's law for linear elasticity,

$$T_{ij}=C_{ijkl}u_{k,l}, \tag{2}$$

where C_{ijkl} are the elements of the elastic stiffness tensor, $u_{i,j}=\partial u_i/\partial x_j$, and the summation convention on suffices is understood. We assume the symmetries

$$C_{ijkl}=C_{jikl}, \quad C_{ijkl}=C_{klij}=C_{ijlk}, \tag{3}$$

the first of which guarantees that $\mathbf{T}$ is symmetric, and the remaining ones ensure the existence of a strain energy function.

Let $\{p^{(1)}, \mathbf{u}^{(1)}\}$ be the solutions for a unit monopole at $\mathbf{x}^{(1)}$ in V_f. These fields satisfy the equations

$$\Delta p^{(1)}+k^2p^{(1)}=\delta(\mathbf{x}-\mathbf{x}^{(1)}), \quad \text{in } V_f, \tag{4a}$$

$$\text{div } \mathbf{T}^{(1)}+\rho_s\omega^2\mathbf{u}^{(1)}=0, \quad \text{in } V_s, \tag{4b}$$

where $k=\omega/c$, c is the acoustic sound speed, and ρ_f is the solid density. Similarly, $\{p^{(2)}, \mathbf{u}^{(2)}\}$ are the solutions for a point force, $\mathbf{e}^{(2)}$, applied at $\mathbf{x}^{(2)}$ in V_s, which satisfy

$$\Delta p^{(2)}+k^2p^{(2)}=0, \quad \text{in } V_f, \tag{5a}$$

$$\text{div } \mathbf{T}^{(2)}+\rho_s\omega^2\mathbf{u}^{(2)}=\mathbf{e}^{(2)}\delta(\mathbf{x}-\mathbf{x}^{(2)}), \quad \text{in } V_s. \tag{5b}$$

In addition, both sets of solutions must satisfy the continuity conditions (1) on B. Now consider the sequence of identities

$$\begin{aligned}
\mathbf{e}^{(2)}\cdot\mathbf{u}^{(1)}(\mathbf{x}^{(2)}) &= \int_{V_s} \mathbf{u}^{(1)}(\mathbf{x})\cdot\mathbf{e}^{(2)}\delta(\mathbf{x}-\mathbf{x}^{(2)})dV \\
&= \int_{V_s} [\mathbf{u}^{(1)}(\mathbf{x})\cdot(\text{div } \mathbf{T}^{(2)}+\rho_s\omega^2\mathbf{u}^{(2)}) \\
&\quad -\mathbf{u}^{(2)}(\mathbf{x})\cdot(\text{div } \mathbf{T}^{(1)}+\rho_s\omega^2\mathbf{u}^{(1)})]dV \\
&= \int_B (\mathbf{u}^{(1)}\cdot\mathbf{T}^{(2)}-\mathbf{u}^{(2)}\cdot\mathbf{T}^{(1)})\cdot\mathbf{n}\, dS \\
&= \frac{1}{\rho_f\omega^2}\int_B [p^{(2)}\nabla p^{(1)}-p^{(1)}\nabla p^{(2)}] \\
&\quad \cdot(-\mathbf{n})dS \\
&= \frac{1}{\rho_f\omega^2}\int_{V_f} [p^{(2)}(\Delta p^{(1)}+k^2p^{(1)}) \\
&\quad -p^{(1)}(\Delta p^{(2)}+k^2p^{(2)})]dV.
\end{aligned} \tag{6}$$

The development should be clear to the reader; it uses Eq. (5), the divergence theorem, the continuity conditions (1), and some of the symmetry properties of the elastic stiffness, (3). Finally, using Eq. (4) gives the desired relation,

$$\mathbf{e}^{(2)}\cdot\mathbf{u}^{(1)}(\mathbf{x}^{(2)})=(1/\rho_f\omega^2)p^{(2)}(\mathbf{x}^{(1)}), \tag{7}$$

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which is equivalent to the statement of reciprocity given in the Abstract.

II. APPLICATION

As an illustration of how the reciprocity relation can be applied we present an example of relevance to acoustic scattering from elastic targets. We consider a finite elastic structure—the target—in an infinite fluid. The reciprocity relation allows us to express the far-field Green's function for the target in terms of the target response for an incident plane wave. The former is the pressure in the fluid at large distances caused by a point force on the target, and it satisfies the Sommerfeld radiation condition. To be specific, consider a point force of unit magnitude applied at a point $\mathbf{x}^{(2)}$ on the surface B, and acting in the normal direction $\mathbf{n}$. The resulting far field may be written as

$$p^{(2)}(\mathbf{x})=f(\hat{\mathbf{x}})\,\frac{e^{ikr}}{r^{(d-1)/2}},\quad r\to\infty, \tag{8}$$

where $r=|\mathbf{x}|$, $\hat{\mathbf{x}}=\mathbf{x}/r$ and $d=2$ or 3 is the dimension. At the same time, a monopole of strength $(\rho_f\omega^2)$ at a point $\mathbf{x}^{(1)}$ in the far field produces an *incident* plane wave near the target of the following form:

$$p_{\text{inc}}^{(1)}(\mathbf{x})=-\rho_f\omega^2 e^{-ik\hat{\mathbf{x}}^{(1)}\cdot\mathbf{x}}\times\frac{e^{ikr^{(1)}}}{(r^{(1)})^{(d-1)/2}}\begin{cases}e^{i\pi/4}/\sqrt{8\pi k}, & d=2,\\ 1/4\pi, & d=3.\end{cases} \tag{9}$$

Therefore, if we define $w(\mathbf{x},\mathbf{e})$ as the normal displacement on the target surface due to an incident plane wave of the form $p_{\text{inc}}=e^{ik\mathbf{e}\cdot\mathbf{x}}$, then the reciprocity relation combined with Eqs. (8) and (9) implies the identity,

$$w(\mathbf{x},\mathbf{e})=\frac{-f(-\mathbf{e})}{\rho_f\omega^2}\begin{cases}\sqrt{8\pi k}\,e^{-i\pi/4}, & d=2,\\ 4\pi, & d=3.\end{cases} \tag{10}$$

Hence, the normal displacement for plane wave incidence can be related to the far-field directivity for a normal point force applied at the same point on the target surface. A similar relation follows for moments applied on the structure. Let $\mathbf{m}$ be a unit vector in the tangent plane at a point on B, and define a point moment with axis $\mathbf{m}\wedge\mathbf{n}$ as the resultant couple from a pair of forces, $\pm\mathbf{n}/\epsilon$ at $\mathbf{x}\pm(\epsilon/2)\mathbf{m}$, in the limit $\epsilon\to 0$. If the far-field directivity f is again defined as in (8), then the associated reciprocity relation is given by (10) with the right member replaced by $\mathbf{m}\cdot\nabla w(\mathbf{x},\mathbf{e})$. Hence, the reciprocal quantity for the point moment is the rotation.

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[1] M. C. Junger and D. Feit, *Sound, Structures, and Their Interaction* (MIT, Cambridge, MA, 1986).

[2] J. D. Achenbach, *Wave Propagation in Elastic Solids* (North-Holland, Amsterdam, 1973).