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# Weak elastic anisotropy and the tube wave

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**ABSTRACT**

Tube-wave speed in the presence of a weakly anisotropic formation can be expressed in terms of an effective shear modulus for an equivalent isotropic formation. When combined with expressions for the speeds of the *SH*- and quasi-SV-waves along the borehole axis, a simple inversion procedure can be obtained to determine three of the five elasticities of a transversely isotropic (TI) formation tilted at some known angle with respect to the borehole axis. Subsequently, a fourth combination of elastic moduli can be estimated from the expression for the $qP$-wave speed along the borehole axis. The possibility of determining all five elasticities of a TI formation based on an assumed correlation between two anisotropy parameters is discussed.

## INTRODUCTION

It is well recognized that sedimentary rocks are not, in general, elastically isotropic, but suffer from some degree of anisotropy. Anisotropy may arise from intrinsic microstructural effects such as layering of thin zones, or from local biaxial or triaxial tectonic stresses within the formation. Thomsen (1986) provided a useful review of the measured anisotropy in many different rock types; based on the data, he concluded that most crustal rocks display weak anisotropy. The objectives of this paper are to examine the effect of weak anisotropy on the limiting low-frequency speed of the symmetric Stoneley mode, or the tube-wave speed, and to propose some possible uses for the resulting approximate formula.

In addition to Thomsen (1986), several other authors have discussed the implications of anisotropy for elastic wave propagation in applications relevant to exploration geophysics. Leveille and Seriff (1989) examined the possibility that the tube wave in a borehole might have a preferential polarization in the presence of azimuthal anisotropy, and concluded that the degree of polarization eccentricity was not significant. Nicoletis et al. (1990) used a combination of analytical and numerical methods to compute the tube-wave speed in anisotropic formations. Their analysis was based upon the fact (White, 1983) that the effect of the formation on tube-wave speed may be related to a purely static deformation in the formation. This observation is also useful in determining the influence of other parameters such as borehole eccentricity on tube-wave speed. Thus, Nicoletis et al. (1990) obtained an analytical expression for tube-wave speed in a borehole of elliptical cross section, while the more general formula for a borehole of arbitrary shape derived by Norris (1990) indicates that in an isotropic formation, the tube-wave speed is greatest for a circular bore. There has also been some recent work on the effect of anisotropy on the flexural and higher order modes in a borehole (Ellefsen et al., 1990, 1991a, b).

In this paper we examine the influence of anisotropy on tube-wave speed. We use a perturbation method that assumes that the anisotropy is weak. The method first estimates the speed of plane waves in an infinite homogeneous anisotropic medium. This application is simpler than the tube-wave speed analysis, but it illustrates the general method and also allows us to introduce some anisotropy parameters used later. The main result of the paper is described and illustrated for a transversely isotropic formation with its axis of symmetry tilted with respect to the borehole axis. Some implications of the tube-wave speed result are also discussed. Finally, we illustrate the application of this inversion procedure for determining the elastic constants of a TI formation in terms of the quasi-static tube wave and three head-wave speeds, assuming that the borehole inclination with respect to the TI symmetry axis is known.

## WAVE SPEEDS IN THE PRESENCE OF ANISOTROPY

Consider an elastic solid of mass density $\rho$ and arbitrary anisotropy, i.e., it may have as many as 21 independent

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elasticity parameters. The equations of motion at circular frequency $\omega$ are

$$\frac{\partial}{\partial x_i} C_{ijk\ell} e_{k\ell} + \rho\omega^2 u_i = 0. \tag{1}$$

Here $u_i$ are the components of displacement, $i = 1, 2, 3$, and the summation convention on repeated subscripts is assumed. The strain components are $e_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, and the elastic moduli $C_{ijk\ell}$ satisfy the general symmetries $C_{ijk\ell} = C_{jik\ell}$ and $C_{ijk\ell} = C_{k\ell ij}$, which are consequences of the symmetry of the stress tensor and the assumed existence of a strain energy function. The moduli can be succinctly represented by $C_{IJ}$, where the suffixes $I$ and $J$ run from 1 to 6, with ij $\leftrightarrow I$ according to 11, 22, 33, 23, 31, 12 $\leftrightarrow$ 1, 2, 3, 4, 5, 6.

Bulk waves

Ignoring the borehole problem for the moment, we consider the propagation of plane waves in the formation, which for simplicity is assumed to be spatially uniform. Substituting the plane-wave solution $u_i = a_i \exp(i\omega n_j x_j/v)$ into equation (1) where n is the unit direction of propagation, and then multiplying by $a_i$, where a is the unit polarization vector, gives an explicit expression for the phase speed v

$$\rho v^2 = a_i a_k C_{ijk\ell} n_j n_\ell . \tag{2}$$

The apparent simplicity of this expression is tempered by the difficulty of determining the polarization a, which requires solving a 3 x 3 matrix eigenvalue problem, also known as the Kelvin-Christoffel equation (Musgrave, 1970). However, if the anisotropy is *weak* then neither the eigenvalues nor the eigenvectors deviate much from their underlying isotropic counterparts. In particular, the polarization in equation (2) can be approximated by the equivalent isotropic polarization.

A more rigorous justification for this approximation may be developed by writing the exact equations (1) in a perturbative form, with the leading order operator being the isotropic equations and the perturbed part involving $C_{ijk\ell} - C^{(0)}_{...}$ where $C^{(0)}_{ijk\ell}$ are the isotropic moduli. One can then use the artillery of formal asymptotic expansions (Courant and Hilbert, 1962) to develop an asymptotic series for both v and a in terms of the "small parameter" $C_{ijk\ell} - C^{(0)}_{ijk\ell} \equiv \alpha$. This analysis indicates that the first correction to the isotropic speed is on the order of $\alpha$, while the change in a is of order $\alpha^2$. Hence, the first correction to the isotropic speed can be obtained from equation (2) using the isotropic polarization vector. This simple result is complicated by the degeneracy of the shear wave in isotropic solids. The degeneracy is broken by anisotropy, and it is often necessary to find the correct vector basis for the shear waves before using equation (2). Generally, this is not a major problem, and in fact, the appropriate directions are known for transverse isotropy, i.e., the $SH$ and $qSV$ directions are (not normalized) m A n and n A (m A n), respectively, where m is the symmetry axis and the prefix $q$ stands for quasi.

Consider a transversely isotropic (TI) material with axis of symmetry coincident with the $x_3$-direction. The five independent moduli are $C_{11}$, $C_{33}$, $C_{13}$, $C_{44}$ and $C_{66}$, such that

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix},$$

where $C_{66} = (C_{11} - C_{12})/2$. It is more convenient to work with the two moduli $C_{33}$ and $C_{44}$, and three dimensionless anisotropy parameters, $\varepsilon$, $\eta$ and $\gamma$, each of which vanishes when the medium is isotropic,

$$\varepsilon = \frac{C_{11} - C_{33}}{2C_{33}}, \quad \eta = \frac{C_{13} + 2C_{44} - C_{33}}{C_{33}}, \quad \gamma = \frac{C_{66} - C_{44}}{2C_{44}} \tag{3}$$

The parameters $\varepsilon$ and $\gamma$ were introduced by Thomsen (1986); however, $\eta$ is close to but not exactly the same as Thomsen's third anisotropy parameter $\delta$. The difference is discussed below.

The three wavespeeds in a TI medium can be expressed in closed form (Rudzki, 1911; Musgrave, 1970). For instance, if n = (sin $\theta$, 0, cos $\theta$), then the exact expression for the SH-phase speed is

$$\rho v_{SH}^2 = C_{44}(1 + 2\gamma \sin^2\theta). \tag{4}$$

The identity in equation (4) follows directly from equation (2) and the fact that the $SH$ polarization is $a = (0, 1, 0)$. The formulas for the $qSV$ and $qP$ speeds are slightly more complicated, but well known (Thomsen, 1986). Since the $qS\,V$ and $qP$ polarizations must be in the $x_1 - x_3$ plane, both may be expressed in the form $(a_1, 0, a_3)$. Substituting into equation (2) yields

$$\begin{aligned} \rho v^2 &= C_{33}(a_1 \sin\theta + a_3 \cos\theta)^2 \\ &\quad + C_{44}(a_1 \cos\theta - a_3 \sin\theta)^2 \\ &\quad + 2C_{33} a_1 \sin\theta(\varepsilon a_1 \sin\theta + \eta a_3 \cos\theta). \end{aligned} \tag{5}$$

If the anisotropy is *weak,* the qP polarization is almost (sin $\theta$, 0, cos $\theta$), while the $qSV$ is approximately (cos $\theta$, 0, -sin $\theta$). The discussion above implies that if these are used in equation (5), the result is a first-order approximation in $\varepsilon$ and $\eta$ to the phase speeds,

$$\rho v_{qP}^2 = C_{33}[1 + 2\varepsilon \sin^4\theta + 2\eta \sin^2\theta \cos^2\theta], \tag{6}$$

$$\rho v_{qSV}^2 = C_{44}\left[1 + 2\frac{C_{33}}{C_{44}}(\varepsilon - \eta)\sin^2\theta\cos^2\theta\right]. \tag{7}$$

Because $\varepsilon$, $\eta$, and $\gamma$ are small, one could use the approximation $(1 + x)^{1/2} \approx 1 + x/2$ for small x to get reasonable *approximations* to $v_{SH}$, $v_{qP}$ and $v_{qSV}$ in weakly anisotropic TI media. The resulting expression for $v_{SH}$ agrees with equation (16c) of Thomsen (1986), but those for $v_{qP}$ and $v_{qSV}$ do not agree with the corresponding formulas in Thomsen, equations (16a) and (16b). Perfect agreement is obtained if the substitution $\eta \rightarrow \delta$ is made, where

$$6 = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}. \tag{8}$$

Thomsen (1986) derived the approximate wavespeeds for the TI medium by explicit expansion of the known expressions for the speeds and was led by this route to the nondimensional parameter 6. It is clear from the algebraic identity

$$\delta = \eta + \frac{\eta^2}{2\left(\frac{C_{33}}{C_{44}} - 1\right)},$$

that $\eta$ is slightly smaller than $\delta$, but the two parameters are interchangeable in the limit of weak anisotropy; their difference is of second order. Hence, the differences between these results and Thornsen's are of second order in the anisotropy. In his paper, Thomsen (1986) demonstrated that $\delta$ (and hence $\eta$) is of critical significance to exploration geophysics, but that it is "an awkward combination of elastic parameters." Because of its simpler form [compare equations (3) and (S)], we suggest that $\eta$ rather than $\delta$ be used as a measure of anisotropy.

### Borehole modes

The same type of perturbative analysis may be performed for the modes of a borehole. The major difference between this situation and that of plane waves in an infinite homogeneous medium is that borehole modes are complicated functions of position and are also highly dispersive. No particular type of mode is assumed at this stage, and in fact the analysis below includes the possibility of a nonuniform formation, although it is assumed that all material parameters are invariant with respect to the axial direction.

The equations of motion (1) are valid at every point in the formation. Similar equations can be written for the motion in the fluid in the bore, supplemented by the continuity conditions at the fluid/solid interface for pressure and normal particle velocity. A *mode* is defined as a solution of these equations that propagates in the direction of the borehole axis and decays exponentially away from the axis. Let us assume that u of equation (1) is a particular mode at frequency $\omega$. Multiply equation (1) by $u_i$ and then integrate the resulting identity over an arbitrary plane perpendicular to the borehole axis. The integral involving $C_{ijk\ell}$ may be expressed in a more symmetric form by integrating by parts and using the decay far from the bore and the continuity conditions at the fluid/solid interface. We eventually deduce that

$$\omega^2 = \frac{\int C_{ijk\ell} e_{ij} e_{k\ell}\, dS}{\int \rho u_i u_i\, dS}, \tag{9}$$

where the integration is over the entire plane. This identity is the analog of equation (2) and is our starting point for a calculation of the mode speed in the presence of anisotropy.

In the same way that equation (2) is correct to first order in the presence of weak anisotropy if the isotropic polarization vector a is used, so it can also be demonstrated using formal asymptotic methods that equation (9) is also correct to leading order when the isotropic *mode* is used in the integrals. This modal perturbation method is standard and has been recently used by Ellefsen et al. (1991a, b) to analyze Stoneley and flexural waves in anisotropic formations. The modal integrals in equation (9) are generally very complicated. However, at very low frequencies some modes have particularly simple spatial dependence and the integrals can be evaluated explicitly. For example, the tube or Stoneley mode, the flexural mode, and the torsional mode, all behave in a relatively simple manner in the quasi-static limit. The limiting speeds of the tube and torsional modes in an anisotropic formation are considered in the next section, but discussion of the low-frequency behavior of the flexural mode in the presence of anisotropy is deferred to a separate paper.

## QUASI-STATIC TUBE- AND TORSIONAL-WAVE SPEEDS

### The tube-wave speed

Consider a circular borehole, $r < a$ in cylindrical coordinates $(r, \phi, x_3)$, which is occupied by an inviscid fluid of density $\rho_f$ and bulk modulus $K_f = \rho_f v_f^2$, where $v_f$ is the fluid-wave speed. The formation, $r > a$, is an arbitrary anisotropic solid, and for simplicity is assumed to be spatially uniform. The tube wave is the quasistatic or limiting low-frequency form of the azimuthally symmetric Stoneley wave mode in an isotropic formation (White, 1983), with speed

$$v_T = v_f \left(1 + \frac{K_f}{\mu}\right)^{-1/2}, \tag{10}$$

where $\mu$ is the formation shear modulus. The displacement field in the formation is proportional to the plane strain displacement that results from an applied uniform pressure, say $p$, on $r = a$. The static displacement for $r > a$ is (White, 1983)

$$u_\alpha = \frac{pa^2}{2\mu} \frac{x_\alpha}{r^2}, \quad \alpha = 1, 2; \; u3 = 0. \tag{11}$$

Although this displacement has a simple form, substituting it directly into equation (9) leads to some difficulty, since the integral $\int_{r=a}^{r=R} \rho u_i u_i \, dS$ becomes unbounded as $R \rightarrow \infty$. This problem is solved in Appendix A where it is shown that the proper form for the integral in the denominator of equation (9) is

$$\int \rho u_i u_i \, dS = \frac{p^2 \pi a^2}{\omega^2 \rho_f v_T^2} 1 \tag{12}$$

The apparent blowup of the integral is due to the fact that it is of order $1/\omega^2$ in the quasistatic limit. We can now calculate the shift in frequency for an arbitrary perturbation in the formation elastic moduli, i.e., for arbitrary $\Delta C_{ijk\ell}$. Combining equations (9), (11), and (12), and eliminating the arbitrary parameter $p$, which has no influence on the answer, we can arrive at a formula for $\Delta\omega^2$ for fixed axial wavenumber $k$. Alternatively, the shift in the tube-wave speed is

$$\frac{\Delta v_T}{v_T} = \frac{\rho_f v_T^2 a^2}{8\pi\mu^2} \int_{r>a} \Delta C_{ijk\ell} E_{ij}(\phi) E_{k\ell}(\phi) \frac{dS}{r^4}, \tag{13}$$

where the strain components $E_{ij}$ are

$$E_{\alpha\beta}(\phi) = \delta_{\alpha\beta} - 2\frac{x_\alpha x_\beta}{r^2}, \; \mathrm{a}, \beta = 1, 2; \; E_{i3} = 0, \; i = 1, 2, 3. \tag{14}$$

The integral in equation (13) is performed in Appendix A. It follows from the result in equation (A-6) that the change in the tube-wave speed for arbitrarily weak anisotropy is

$$\frac{\Delta v_T}{v_T} = \frac{\rho_f v_T^2}{16\mu^2} \Delta(C_{11} + C_{22} - 2C_{12} + 4C_{66}). \qquad (15)$$

This is the main result of the paper.

The explicit nature of equation (15) permits selecting the background shear modulus of the isotropic formation in such a way that $\Delta v_T \equiv 0$. The proper choice for the effective modulus follows by comparison of equation (15) with the isotropic equivalent, equation (A-2), which implies $\mu = \mu^*$, where

$$\mu^* = \frac{1}{8}(C_{11} + C_{22} - 2C_{12} + 4C_{66}). \qquad (16)$$

Explicit calculation shows that $\mu^* = C^*_{66}$, where $C^*_{ijk\ell}$ are the average moduli obtained by rotating the frame of $x_1$- and $x_2$-axes about the $x_3$-axis. $C^*_{ijk\ell}$ are the effective moduli of the rotationally averaged transversely isotropic medium. Of course, the resulting equation (16) is not restricted to any particular material symmetry and is equally valid for a triclinic or a TI formation.

In summary, the tube-wave speed in a weakly anisotropic formation is $v_T$ given by equation (10) where the effective shear modulus for the formation is $\mu^*$.

## A tilted TI formation

The moduli appearing in the effective modulus $\mu^*$ are defined relative to the borehole axis, coincident with the $x_3$-axis, and any pair of orthogonal axes. The compass orientation of the $x_1$- and $x_2$-axes is arbitrary, since the combination of moduli involved in $\mu^*$ is invariant with respect to the orientation of these axes; the invariance follows from the fact that $\mu^*$ is defined by a rotational average about the borehole axis.

Consider the particular case of a TI medium with its axis of symmetry tilted through an angle $\theta$ with respect to the borehole axis. The moduli $C_{11}$, $C_{22}$, $C_{12}$, and $C_{66}$ in the borehole coordinate system can be related to the five TI moduli by simple transformation rules (Auld, 1973). The resulting form of the effective modulus $\mu^*$ is

$$\mu^*(\theta) = C_{44} + 2\gamma C_{44}\cos^2\theta + \frac{1}{4}(\varepsilon - \eta)C_{33}\sin^4\theta, \qquad (17)$$

where $\varepsilon$, $\gamma$, and $\eta$ are the anisotropy parameters of equation (3), and $C_{33}$ and $C_{44}$ are the TI moduli defined with respect to the TI symmetry axis, not the borehole axis. The variation of $\mu^*$ with angle depends upon two dimensionless parameters, $\gamma$ and $\xi$ where

$$\xi = \frac{N}{2C_{44}}, \qquad (18)$$

and the "modulus" N is $2(\varepsilon - \eta)C_{33}$, or

$$N = C_{11} + C_{33} - 2C_{13} - 4C_{44}. \qquad (19)$$

The range of these parameters based upon the compilation of data in Thomsen (1986) is illustrated in Figure 1. The larger range in values for $\xi$ is caused, to a great extent, by the fact that the ratio $C_{33}/C_{44}$ is approximately $(v_P/v_S)^2$, which is often quite large. The parameter $\gamma$ is obviously related to the difference in shear properties, but the parameter $\xi$, or equivalently, the modulus N, is a more complicated combination of moduli.

Leveille and Seriff (1989) derived the tube-wave speed for a particular type of transversely isotropic medium with its axis perpendicular to the borehole. The general perturbation theory is compared with this exact case in Appendix B. Based upon the analytical results of Appendix B we conclude that the approximate theory is accurate for mildly anisotropic formations.

## The torsional mode

The torsional mode in a borehole is another mode that has a zero cutoff frequency (White, 1983). It is characterized by pure torsional motion in the formation and no motion of the borehole fluid. We now prove that its behavior in a weakly anisotropic formation depends on the same modulus that defines the tube wave speed. We first note that the quasistatic displacement field is of the form

$$\mathrm{U} = \frac{A}{r}\mathbf{e}_\phi,$$

where $A$ is a constant and $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction. The torsional wave speed in an isotropic formation is the shear speed, $v_{TOR} = (\mu/\rho)^{1/2}$. As before, we can use equation (9) to obtain an expression for the speed in an anisotropic formation. The integral in the denominator again becomes unbounded at low frequency, but this can be handled in the same way that equation (12) was deduced, with the result, by analogy to equation (13), that the shift in the torsional-wave speed in a weakly anisotropic formation becomes

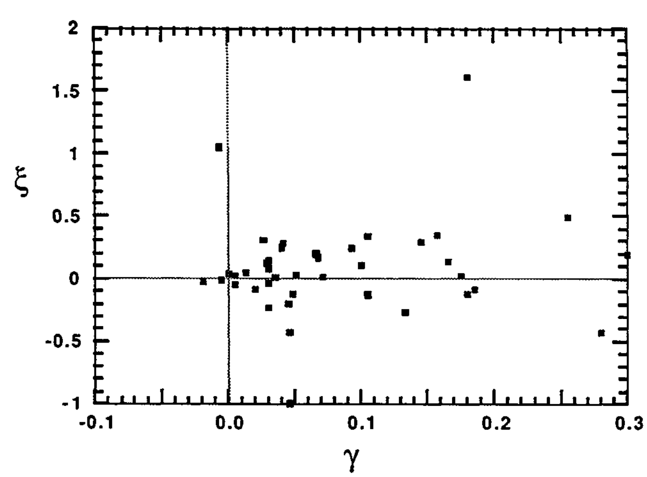


FIG. 1. The correlation, or lack thereof, between the two anisotropy parameters $\xi$ and $\gamma$. The points correspond to the 44 cases of anisotropy in sedimentary rocks listed by Thomsen (1986).

$$\frac{\Delta v_{TOR}}{v_{TOR}} = \frac{a^2}{8\pi\mu} \int_{r>a} \Delta C_{ijk\ell} F_{ij}(\phi) F_{k\ell}(\phi) \frac{dS}{r^4}, \quad (20)$$

where $F_{ij}$ are strain components for the torsional displacement field analogous to the strains $E_{ij}$ of equation (14). In fact, it may be easily seen that $F_{ij}(\phi) = E_{ij}(\phi - \pi/2)$; therefore the integral in equation (20) can be directly evaluated using the methods of Appendix A to give

$$\frac{\Delta v_{TOR}}{v_{TOR}} = \frac{1}{16\mu}(C_{11} + C_{22} - 2C_{12} + 4C_{66}). \quad (21)$$

By comparison with the shift in speed for an isotropic formation we conclude that the correct choice of effective shear modulus for the anisotropic formation is again $\mu^*$ of equation ( 16).

## AN INVERSION PROCEDURE

We now examine the specific case of a borehole in a tilted TI formation, for which the tube-wave speed follows from equations (10) and (16) with $\mu = \mu^*$. Assuming that the properties of the bore fluid are known, then $\mu^*$ can, in principle, be determined from measurement of the tube-wave speed.

An acoustic measurement in a borehole generates head waves as well as borehole modes. The head waves are caused by coupling to bulk elastic waves in the formation that propagate in the direction of the borehole axis. Specifically, there are three head waves for the borehole in the tilted TI formation, corresponding to the *SH*, the *qP* and the *qSV* waves previously discussed, with speeds given by equations (4), (6), and (7) when the anisotropy is weak. We note that only three elastic parameters are involved in $\mu^*$ of equation (17), and in $\rho v_{SH}^2$ and $\rho v_{SV}^2$, i.e., $C_{44}$, $\gamma$, and $\xi$. The three equations for $\mu^*$, $\rho v_{qSV}^2$ and $\rho v_{SH}^2$ may be written in a matrix form that emphasizes dependence upon the three elastic parameters:

$$\begin{bmatrix} 1 & \cos^2\theta & \frac{1}{4}\sin^2\theta \\ 1 & 0 & 2\cos^2\theta \\ 1 & \sin^2\theta & 0 \end{bmatrix} \begin{bmatrix} C_{44} \\ 2\gamma C_{44} \\ \xi C_{44}\sin^2\theta \end{bmatrix} = \begin{bmatrix} \mu^* \\ \rho v_{qSV}^2 \\ \rho v_{SH}^2 \end{bmatrix}. \quad (22)$$

This may be formally inverted and rearranged into the form

$$\begin{bmatrix} C_{44} \\ C_{66} \\ N \end{bmatrix} = \frac{1}{D(\theta)} \times \begin{bmatrix} \cos^4\theta & -\sin^2\theta\cos^2\theta & \frac{1}{8}\sin^4\theta \\ (\frac{1}{8} - \cos^2\theta)\sin^2\theta & \cos^4\theta & -\frac{1}{8}\sin^2\theta\cos^2\theta \\ -\cot^2\theta & 1 & \cot^2\theta - 1 \end{bmatrix} \times \begin{bmatrix} \rho v_{SH}^2 \\ \mu^* \\ \rho v_{qSV}^2 \end{bmatrix}, \quad (23)$$

where

$$D(\theta) = \cos^4\theta - \sin^2\theta\cos^2\theta + \frac{1}{8}\sin^4\theta, \quad (24)$$

and N was defined in equation (19).

The explicit and simple form of equation (22) suggests a means to infer three of the five TI moduli if both the tilt angle and the formation density are known. These must be obtained independently, and the issues involved in their determination will not be addressed here. We note some properties of the inversion formula (23). First, if $\theta = 0$, then both the *SH* and *SV* waves have identical speeds, and only two elastic parameters can be obtained from the tube-wave speed and the shear head-wave speed: $C_{44}$ and $C_{66}$. The third modulus N can be found if $\theta > 0$, except at the two angles for which $D(\theta) = 0$: $\theta \approx 47$ degrees and $\theta \approx 69$ degrees. At these angles the "measured" quantities satisfy

$$\mu^* = \rho v_{qSV}^2 + \frac{\rho}{2}(v_{SH}^2 - v_{qSV}^2)[1 \pm 1/\sqrt{2}] \quad (25)$$

for any combination of $C_{44}$, $C_{66}$, and N, according to the weak anisotropy approximations for $\mu^*$ and $\rho v_{qSV}^2$. Therefore, at these angles only one independent elastic modulus may be determined, using the identity (4) for example, as opposed to three moduli in general.

Transverse isotropy can be caused by the existence of finely layered isotropic constituents (Backus, 1962). This leads to certain constraints upon the effective TI moduli, beyond the fundamental requirement that they be positive definite. Berryman (1979) and Helbig (1979) have demonstrated that $\varepsilon > \delta$ for TI media formed from laminated isotropic materials. Since $\eta < \delta$, it follows that $\varepsilon - \eta > 0$, or equivalently $N > 0$, for a layered TI medium. This can be seen more directly by using Backus's (1962) formula for the effective constants $C_{11}$, $C_{33}$, $C_{13}$, and $C_{44}$, which give

$$\frac{N}{4} = \left\langle \frac{1}{\mu} \right\rangle^{-1} \left[ \left\langle \frac{1}{\mu} \right\rangle \langle \mu \rangle - 1 \right] - \left\langle \frac{A}{\mu} \right\rangle^{-1} \left[ \left\langle \frac{A}{\mu} \right\rangle \langle A\mu \rangle - \langle A \rangle^2 \right], \quad (26)$$

where $\langle\,\rangle$ denotes the spatial average, $A = \mu/(\lambda + \mu)$, and $\lambda$ and $\mu$ are the isotropic Lame moduli. The inequality $N > 0$, then follows from the observation that the two terms in the right member of equation (26) are non-negative, while the second is the smaller because $A \le 1$. This digression implies that it is feasible, in principle, to exclude the possibility of microstructural isotropic layering simply by determining the sign of the anisotropy parameter $N = C_{11} + C_{33} - 2C_{13} - 4C_{44}$.

Finally, a fourth combination of elastic moduli can be estimated from equation (6) in terms of the anisotropy parameter N and the compressional head-wave speed. The resulting expression takes the form

$$C_{11}\sin^2\theta + C_{33}\cos^2\theta = \rho v_{qP}^2 + N\sin^2\theta\cos^2\theta. \quad (27)$$

## EXAMPLE

An example helps demonstrate the accuracy of the simple expressions for the $qP$- and $qSV$-wave velocities presented in the preceding sections, and illustrate their application in inverting for four of the five elastic moduli in terms of the three head wave, quasi-static tube-wave (or equivalently, torsional wave) velocities, provided the tilt of the borehole with respect to the TI symmetry axis is also known. We examine the case of a liquid-filled borehole surrounded by a formation characterized by Cotton Valley shale (a fast TI medium). When referred to the Cartesian axes with $x_3$ as the TI symmetry axis, the elastic moduli have the values

(Thomsen, 1986): $C_{11} = 74.73 \times 10^9 \ N/m^2$, $C_{12} = 14.75 \times 10^9$, $C_{13} = 25.29 \times 10^9$, $C_{33} = 58.84 \times 10^9$, and $C_{44} = 22.05 \times 10^9$. The mass density $\rho = 2640$ kg/m$^3$. The anisotropy parameters defined by equation (3) are $\varepsilon = 0.135$, $\eta = 0.179$, and $\gamma = 0.180$.

Figure 2 shows a comparison of the qP-wave velocity obtained from equation (6) (dotted line) and its exact value (solid line) from the solution of the Kelvin-Christoffel equations as a function of propagation direction measured from the TI symmetry axis. Agreement between the two values is excellent, with the maximum difference on the order of 0.1 percent. Figure 3 illustrates a comparison of SH- and *qSV-* wave velocities determined from the approximate expressions (4) and (7), and the exact results from the solution of the Kelvin-Christoffel equations. While the results for the SH-wave velocity are the same in the two cases, the maximum difference between the values for the *qSV*-wave velocity from the approximate expression and that from the exact result is less than 1 percent.

Figure 3 also displays the torsional-wave velocity obtained from the equivalent shear modulus $\mu^*$ for the anisotropic formation given by equation (17). Note that this equivalent shear modulus is the same as that appearing in the expression for the quasi-static tube-wave velocity (10). The resulting quasi-static tube-wave velocity as a function of the propagation direction is shown in Figure 4.

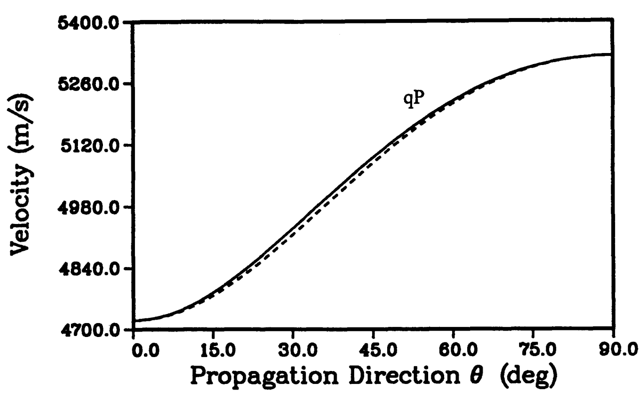


FIG. 2. The *qP-wave* velocity as a function of propagation direction from the TI symmetry axis. Solid and dotted lines denote the exact and approximate results, respectively.

On the assumption that the inclination of the borehole with respect to the TI symmetry axis is known, the two shear moduli $C_{44}$, $C_{66}$, and the third elasticity parameter N can be estimated from equation (22) in terms of the available quasi-static tube, *SH-*, *qS* V-wave velocities along the borehole axis. The results for the three elastic moduli thus obtained from equation (22) are plotted in Figure 5 as a function of the borehole inclination. The solid lines represent the results when the borehole inclination $\theta$ is known exactly. When we introduce an error of $\pm 1$ degree in the actual value of 8, the corresponding results for the inverted three moduli are shown by dotted lines. These results demonstrate that of the three elastic parameters, the third moduli N is most sensitive to an error in the borehole inclination $\theta$ with respect to the TI symmetry axis. In addition, the error in the estimate of N is significantly larger in the vicinity of $\theta = 47$ degrees and 69 degrees when the denominator $D(\theta)$ tends to zero (in fact, the error becomes unbounded at these angles and the numerical results shown in Figures 5 and 6 have been clipped accordingly). On the other hand, the estimates for $C_{44}$ and $C_{66}$ remain quite good even when N is significantly in error. High sensitivity of N to an error in $\theta$ is not surprising in view of its value being rather small and hovering around zero, its value for isotropic formations. Figure 6 shows the results for the inverted moduli as the error in the borehole inclination is increased to $\pm 2$ degrees. A comparison of Figures 5 and 6 indicates that the errors in the estimated values of the moduli are proportional to the inaccuracy in the tilt angle $\theta$ for small values of the latter.

The fourth elastic parameter can be obtained directly from equation (27) in terms of qP-wave velocity, borehole inclination $\theta$ and the elastic moduli N. Thus, any error in the estimate of N will mitigate the accuracy of the fourth elastic parameter except when $\theta = 0$ degrees or 90 degrees, insofar as this quantity is independent of the value of N. Finally, if we accept a correlation between the two anisotropy parameters $\xi$ and $\gamma$ as indicated in Figure 1, we have in principle, the fifth relationship to estimate all five elastic constants of a TI formation.

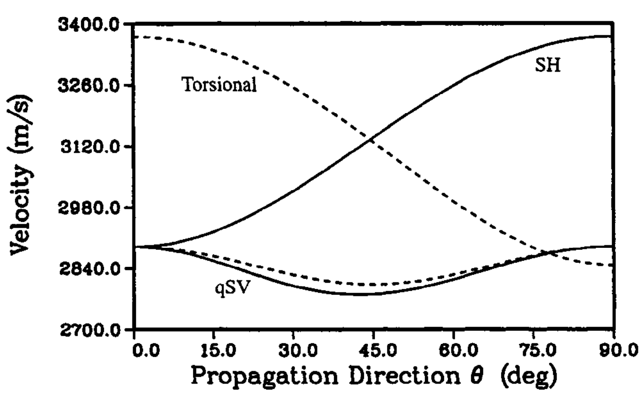


FIG. 3. Torsional-, *SH-* and qSV-wave velocities as a function of propagation direction from the TI symmetry axis. Solid and dotted lines represent the exact and approximate results, respectively.

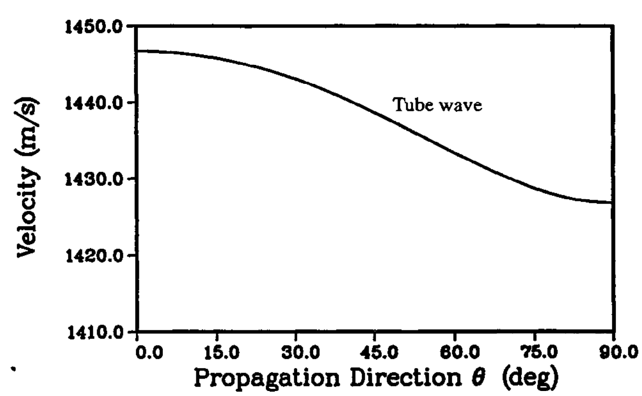


FIG. 4. Quasi-static tube-wave velocity as a function of the propagation direction from the TI symmetry axis. The borehole liquid is assumed to have a compressional speed of 1500 m/s and mass density of 1000 kg/m .

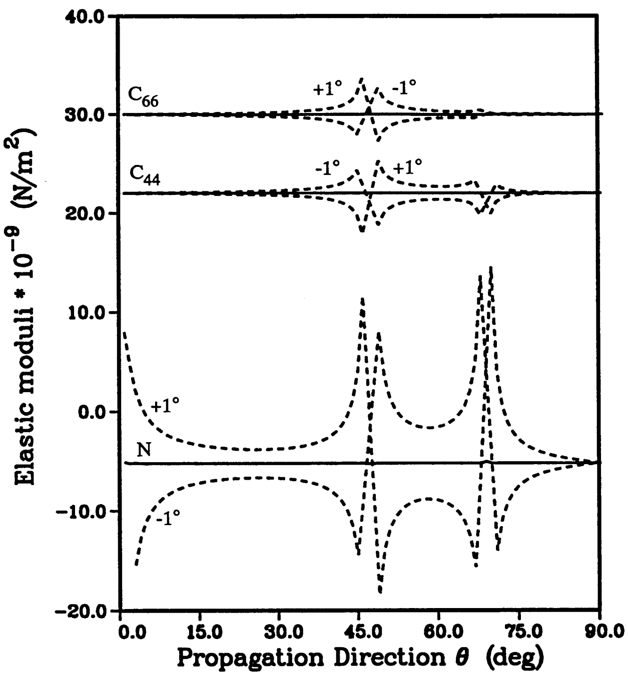


FIG. 5. Inverted elastic moduli. Solid lines denote the results for actual borehole inclination $\theta$ with respect to the TI symmetry axis. Dotted lines show the corresponding results when the error in $\theta$ is $\pm$ 1 degree.

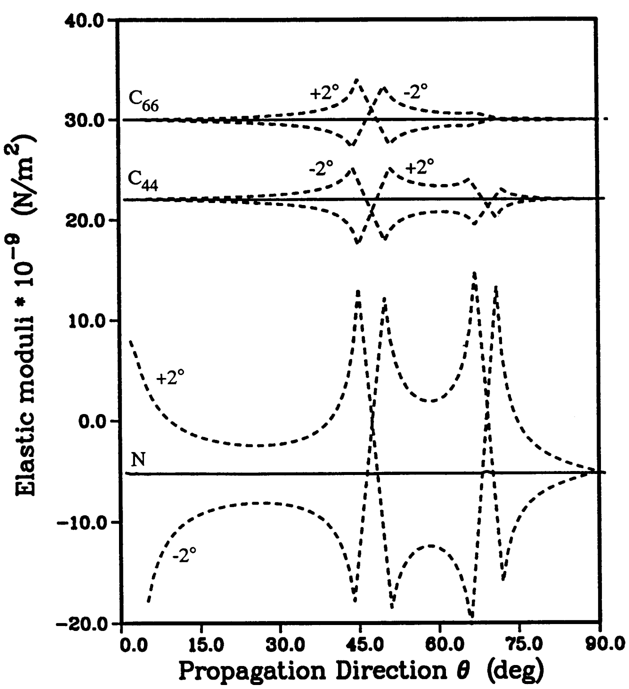


FIG. 6. The same as in Figure 5, but for an assumed error in $\theta$ of $\pm 2$ degrees.

## CONCLUSIONS

The geometry of the borehole severely restricts the ability to determine the intrinsic anisotropy of the surrounding formation. Furthermore, acoustical measurement techniques in the borehole obtain indirect information such as the speed of head waves in the axial direction, but they do not provide the freedom of access necessary to fully define the formation anisotropy. Despite these environmental disadvantages it is possible, in principle, to completely determine three of the five elastic parameters of a tilted TI formation. The inversion procedure relies upon knowledge of the two shear head-wave speeds and the effective formation shear modulus $\mu^*$, which in turn can be related to the speed of either the tube wave or the torsional wave. The formation density and the tilt angle of the TI medium must be determined independently. The numerical examples shown here suggest that the moduli $C_{44}$ and $C_{66}$ of the TI medium can be determined fairly accurately despite some error in the value of the tilt angle. However, the third anisotropy parameter $N$ is more sensitive to such errors. In addition, it is possible to estimate a fourth combination of elastic moduli from the expression for the quasi-compressional wave speed along the borehole axis. These inversion algorithms are based on the assumption that the TI formation is weakly anisotropic in the sense that it does not deviate appreciably from isotropy.

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## APPENDIX A
## SOME TUBE WAVE INTEGRALS

The problem of the infinite limit for the integral in the denominator of equation (9) using equation (11) for the displacement can be circumvented by using a variational argument. In the purely isotropic case, we have

$$\omega^2 = v_T^2 k^2, \tag{A-1}$$

where $k$ is the axial wavenumber. Let $k$ be fixed, and consider a small variation $\Delta\mu$ in the formation shear modulus. The shift in frequency follows from equations (10) and (A-l) as

$$\Delta\omega^2 = k^2 \rho_f \frac{v_T^4}{\mu^2} \Delta\mu. \tag{A-2}$$

The frequency shift can also be estimated from equation (9). The integral in the numerator then becomes proportional to $\Delta\mu$, and can be worked out quite easily using equation (11), as

$$\int \Delta C_{ijk\ell} e_{ij} e_{k\ell} \, dS = \pi a^2 \frac{p^2}{\mu^2} \Delta\mu. \tag{A-3}$$

Combining equations (9) and (A-l) through (A-3) we obtain equation (12).

Referring to the integral in equation (13), we note that the strain tensor $E_{ij}$ of equation (14) is independent of $r$, and so the $r$ integration can be performed readily, yielding

$$\frac{1}{\pi} \int_{r>a} \Delta C_{ijk\ell} E_{ij}(\phi) E_{k\ell}(\phi) \frac{dS}{r^4}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Delta C_{\alpha\beta\gamma\delta} E_{\alpha\beta}(\phi) E_{\gamma\delta}(\phi) \, d\phi. \tag{A-4}$$

Here we have also used the fact that the strain is planar, and the repeated Greek indices only assume the values 1 and 2. Do not confuse the Greek letters here for the anisotropy parameters elsewhere. Substituting the explicit form for the strain tensor, the right member of equation (A-4) can be expanded as

$$\Delta C_{\alpha\alpha\beta\beta} - \frac{4}{2\pi} \int_0^{2\pi} \Delta C_{\alpha\alpha\gamma\delta} \hat{x}_\gamma \hat{x}_\delta \, d\phi$$

$$+ \frac{4}{2\pi} \int_0^{2\pi} \Delta C_{\alpha\beta\gamma\delta} \hat{x}_\alpha \hat{x}_\beta \hat{x}_\gamma \hat{x}_\delta \, d\phi, \tag{A-5}$$

where $\hat{x}_1 = \cos\phi$ and $\hat{x}_2 = \sin\phi$. Substituting for the components of $\hat{x}_\alpha$ gives integrals of powers of $\sin\phi$ and $\cos\phi$ which can be integrated easily. The final result is

$$\frac{1}{2} \Delta(C_{11} + C_{22} - 2C_{12} + 4C_{66}). \tag{A-6}$$

## APPENDIX B
## COMPARISON WITH AN EXACT RESULT

The formula in equation (17) reduces to the correct modulus for a TI formation with its symmetry axis coincident with the borehole axis, i.e., $\mu^*(0) = C_{44}(1 + 2\gamma) \equiv C_{66}$ (White, 1983). This case exhibits axial symmetry in the displacement field, and it is not surprising that the perturbation analysis provides the correct result since it is based on a symmetric trial field. A more stringent test of our result occurs when there is true azimuthal anisotropy in the formation. This situation was considered by Leveille and Seriff (1989) who drew upon results of Savin (1961).

In general, the tube-wave speed can be expressed in the form of equation (10) with the modulus $\mu$ of the isotropic formation replaced by an effective formation modulus $\mu^{eff}$, which may be related to the average radial displacement at the bore wall due to an applied pressure (White, 1983). Thus, for the circular bore

$$\frac{1}{\mu^{eff}} = \frac{1}{\pi a p} \int_0^{2\pi} u_r(\phi) \, d\phi. \tag{B-1}$$

Savin (1961) presents the radial displacement $u_r(\phi)$ for a material that exhibits azimuthal anisotropy. The material corresponds to a TI medium with its axis of symmetry perpendicular to the borehole ($\theta = \pi/2$) and such that the moduli satisfy the constraint $\xi = 0$, or from equations (3) and (18), $C_{11} + C_{33} - 2C_{13} - 4C_{44} = 0$. In this case the approximate effective modulus according to equation (17) with $\theta = \pi/2$ and $\varepsilon = \eta$, is $\mu^* = C_{44}$, whereas the exact formation modulus for the tube-wave speed follows from Savin (1961) (and also from Leveille and Seriff (1989) after some sign errors are corrected) as

$$\mu^{eff} = C_{44} \left[ \frac{1 + 2\varepsilon(1 - \sigma)}{1 + \varepsilon} \right]$$

$$\times \left\{ \sigma + (1 - \sigma) \left[ 1 + 2\varepsilon \left( \frac{1 - 2\sigma}{1 - \sigma} \right) \right]^{1/2} \right\}^{-1}, \tag{B-2}$$

where $\sigma$ is a Poisson's ratio, $\sigma = C_{13}/C_{11} + C_{13}$. Expanding equation (B-2) gives

$$\mu^{eff} = C_{44} \left[ 1 - \frac{\varepsilon^2}{2} \left( \frac{1 - 2\sigma}{1 - \sigma} \right) + \cdots \right]. \tag{B-3}$$

Assuming that $-1 \le \sigma < 1/2$, the relative error in the weakly anisotropic theory is at most $3\varepsilon^2/4$. The range of $\varepsilon$ is from 0 to about 0.2 for most of the rocks listed by Thomsen (1986), and hence we conclude that the approximate theory is generally accurate for mildly anisotropic formations.