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STATIC IMPLICATIONS OF THE EXISTENCE OF A PLANE OF SYMMETRY IN AN ANISOTROPIC ELASTIC SOLID

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SUMMARY

Several sets of conditions are presented each of which is related to the question of the existence of a plane of material symmetry in an anisotropic elastic solid. The conditions are defined in terms of possible static deformations that can exist in a sample in static equilibrium and subject to certain simple types of traction fields. In particular, a necessary and sufficient condition for the existence of a plane of symmetry is that simple shears can be maintained by pure shear forces in at least two planes.

1. Introduction

THE underlying anisotropy of an elastic solid can be characterized in terms of the *planes of material symmetry* (**1**,**2**). Thus, a material of the most general anisotropy, triclinic, has no symmetry planes. A material with a single plane of symmetry is known as a monoclinic material, while one with three planes corresponds to orthorhombic symmetry, etc. A full classification along these lines has been provided by Cowin and Mehrabadi (**2**) who derived a set of conditions necessary and sufficient for the existence of a plane of material symmetry. These conditions are in the form of identities involving the tensor of elastic moduli and the normal to the plane, and have been recast in a slightly different form by Cowin (**3**) and Norris (**4**). These forms of the conditions suggest an interpretation in terms of propagating plane waves (**3**,**4**); for instance it is required of the plane of symmetry that a longitudinal wave can propagate in the direction of its normal. It had previously been determined by Fedorov (**5**) that the wave conditions are necessary for the existence of a plane of symmetry; however, he does not appear to have noted that the same conditions are also sufficient.

The purpose of the present paper is to provide alternative interpretations of the conditions of Cowin (**3**) and Norris (**4**) in terms of static deformations as opposed to a dynamic interpretation involving waves. The wave conditions are defined in section 2 and are called L and T in view of their connection with longitudinal and transverse waves. The necessary and sufficient conditions for the existence of a plane of symmetry correspond to both L and T combined. However, each of L and T can be related to a state of static deformation, and these equivalences are discussed in sections 3 and 4, respectively. Probably the most interesting results are in section 5, where static interpretations are given for the conditions required for a plane of symmetry. Several equivalent possible deformations are discussed, each involving states of simple shear.

2. The longitudinal and transverse conditions

The components of the elastic stiffness tensor are C_{ijkl} relative to a rectangular basis and satisfy the standard symmetries

$$C_{ijkl} = C_{jikl}, \qquad C_{ijkl} = C_{ijlk}, \qquad C_{ijkl} = C_{klij}. \tag{2.1}$$

Suffixes i, j, k and l range from 1 to 3, and in (2.1) and subsequent equations the summation convention on repeated suffixes will be understood. Lower-case roman variables will denote unit vectors in three-dimensional space, with the exception of $\mathbf{u}$, $\mathbf{t}$ and $\mathbf{x}$ which denote displacement, traction and position vectors. The stiffness moduli C_{ijkl} enter into the generalized Hooke's law relating the symmetric, second-order stress and strain tensors, σ and ε respectively, according to

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{2.2}$$

where ε is the symmetric part of the spatial gradient of the displacement field $\mathbf{u}(\mathbf{x})$. The traction vector $\mathbf{t}(\mathbf{n})$ across a surface with normal $\mathbf{n}$ has components

$$t_i = \sigma_{ij}n_j. \tag{2.3}$$

It is assumed that the stiffnesses satisfy the strong-ellipticity condition

$$C_{ijkl}d_i d_k c_j c_l > 0 \quad \text{for all } \mathbf{d}, \mathbf{c} \neq \mathbf{0}. \tag{2.4}$$

The standard shorter notation c_{IJ} will be used occasionally instead of C_{ijkl}, where I and J range from 1 to 6, with the understanding that $(ij) \to I$ according to $\{11, 22, 33, 23, 31, 12\} \to \{1, 2, 3, 4, 5, 6\}$.

The fundamental relations of interest are the following set of identities first laid out by Cowin (**3**), based upon the original work by Cowin and Mehrabadi (**2**) and also discussed by Norris (**4**). The moduli are said to satisfy the *longitudinal* or *L-condition* for the direction $\mathbf{a}$ if $\mathbf{a}$ is an eigenvector of the second-order acoustical tensor with components $C_{ijkl}a_j a_l$, that is, if

$$C_{ijkl}a_j a_k a_l = \lambda a_i. \tag{L}$$

The L-condition means that a wave polarized in its direction of propagation can exist in the solid if it propagates in the direction $\pm\mathbf{a}$. Kolodner (**6**) has shown that there exist at least three distinct directions satisfying L in every anisotropic solid. The other constraint upon the moduli is the *transverse* or

T-condition, which is satisfied for the direction **a** if

$$C_{ijkl}b_jb_la_k = \mu(\mathbf{b})a_i \quad \text{for all } \mathbf{b} \text{ such that } \mathbf{b}.\mathbf{a} \neq 0. \tag{T}$$

The transverse condition implies that a wave polarized in the direction **a** may propagate in all directions orthogonal to **a** (**3, 4, 5**).

Conditions L and T are together equivalent to the statement that **a** is *perpendicular to a plane of material symmetry* (**3, 4**), which in turn is identical to the result that the material possesses monoclinic symmetry in the direction **a**. If, for instance, **a** is the unit vector $\mathbf{e}_3$, then the material is monoclinic with respect to this direction if and only if

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0. \tag{M}$$

For the same choice of **a** the conditions L and T can be readily shown to be separately equivalent to

$$c_{34} = c_{35} = 0, \tag{L}$$

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{46} = c_{56} = 0. \tag{T}$$

Cowin's identity (**3**) that L and T together form necessary and sufficient conditions for the existence of a plane of symmetry is obvious from the previous three equations; hence

$$L + T = M.$$

3. Static interpretation of the longitudinal condition L

Consider the static displacement field

$$\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{a}.\mathbf{x})\mathbf{a}, \tag{3.1}$$

where γ is a constant. The deformation defined by (3.1) will be called a *pure extension* because the only strain component which is non-zero is the strain in the **a** direction. The associated stress tensor is

$$\boldsymbol{\sigma} = \gamma\boldsymbol{\phi}, \tag{3.2}$$

where $\boldsymbol{\phi}$ is a symmetric tensor with components

$$\phi_{ij} = C_{ijkl}a_ka_l. \tag{3.3}$$

The L-condition implies that **a** is an eigenvector of $\boldsymbol{\phi}$ and hence is a principal direction of stress.

The constant stress field (3.2) associated with the deformation (3.1) trivially satisfies the homogeneous equations of static equilibrium at every position. The associated tractions on the surface of a given body are non-zero and for certain bodies the form of the tractions required to maintain the deformation is relatively simple when L holds. Thus, let $\mathbf{a}'$ and $\mathbf{a}''$ be the other two directions of principal stress, which are also the other eigenvectors of $\boldsymbol{\phi}$. Consider a rectangular block such that its faces are normal to the directions of principal stress, **a**, **a**$'$ and **a**$''$. Then the necessary tractions on each face are also in the directions of principal stress, that is, the tractions are all normal and no shear tractions are required.

Let us define the *static L-condition* (SL): 'A state of pure extension can be maintained by the application of only normal tractions to a rectangular block with one of its faces normal to the direction of extension'. We have just seen that SL is a consequence of L. It may be easily checked that the reverse is also true, that is, if SL holds for some direction for a given material then L is also true for the same direction. In short

$$L = SL. \qquad (SL)$$

4. Static interpretation of the transverse condition T

A material is defined to be in a state of *simple shear* if it is deformed according to

$$\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{b}.\mathbf{x})\mathbf{a}, \qquad \mathbf{b}.\mathbf{a} = 0, \qquad (4.1)$$

where again γ is a constant. The plane spanned by **a** and **b** is called the *plane of shear* and **a** and **b** are defined as the associated *shear directions*. The corresponding strain tensor is deviatoric ($\varepsilon_{kk} = 0$) and the only non-zero elements are the symmetric components $\mathbf{a}.(\varepsilon\mathbf{b}) = \mathbf{b}.(\varepsilon\mathbf{a})$. The stress is also constant, with components

$$\sigma_{ij} = \gamma C_{ijkl} a_k b_l, \qquad (4.2)$$

and the equations of equilibrium are automatically satisfied.

Tractions must be applied to the surface of a given body to maintain a simple shear, and in particular, let us consider the traction on an element of surface with normal **n** orthogonal to **a**. Using the decomposition

$$\mathbf{n} = n'\mathbf{b} + n''\mathbf{c}, \qquad (4.3)$$

where $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ form an orthonormal triad, the traction vector becomes

$$t_i(\mathbf{n}) = \gamma n' C_{ijkl} b_j b_l a_k + \gamma n'' C_{ijkl} c_j b_l a_k. \qquad (4.4)$$

Assuming that T holds, the first term on the right-hand side of (4.4) is $\gamma n' \mu(\mathbf{b}) a_i$, that is, it represents a vector parallel to **a**. The second term is also seen to be in the direction **a** by successively contracting $C_{ijkl} c_j b_l a_k$ with b_i and c_i, using T and the orthogonality of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ to show that the results are zero. Hence, the tractions required on the lateral surfaces of a cylinder of material with axis **a** are pure shears in the axial direction.

On the basis of the previous discussion, define the *static transverse condition* (ST) as follows: 'A cylindrical sample of material with generator in the direction **a** is in simple shear, $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{b}.\mathbf{x})\mathbf{a}$, for at least two distinct **b** orthogonal to **a**. In each case the tractions on the lateral surfaces required to maintain the sample in static equilibrium are pure shears in the axial direction'. We have seen that ST is implied by T. In order to prove that ST implies T let $\mathbf{d}$ and $\mathbf{e}$ be two distinct directions orthogonal to $\mathbf{a}$ for which the simple shears $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{d}.\mathbf{x})\mathbf{a}$ and $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{e}.\mathbf{x})\mathbf{a}$ can be maintained by tractions in the axial direction on the lateral surfaces. Let $\mathbf{n}$ and $\mathbf{m}$ be two distinct normals to the lateral surface (the cross-section of any cylinder has at least two distinct normals). The traction on the face with normal $\mathbf{n}$ has components $\sigma_{ij}n_j = \gamma C_{ijkl}n_j a_k d_l$ for the simple shear $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{d}.\mathbf{x})\mathbf{a}$. The ST condition implies that this traction is orthogonal to both $\mathbf{n}$ and $\mathbf{m}$. By successive consideration of the different deformations and tractions we conclude that the following six quantities vanish:

$$C_{ijkl}n_i n_j a_k d_l, \qquad C_{ijkl}n_i m_j a_k d_l, \qquad C_{ijkl}m_i m_j a_k d_l,$$
$$C_{ijkl}n_i n_j a_k e_l, \qquad C_{ijkl}n_i m_j a_k e_l, \qquad C_{ijkl}m_i m_j a_k e_l.$$

For simplicity, but with no loss in generality, let $\mathbf{a} = \mathbf{e}_3$, so that $\mathbf{n}$ and $\mathbf{m}$ span the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$. The first three and second three of the above identities imply, respectively, that

$$C_{\alpha\beta 3l}d_l = 0, \qquad C_{\alpha\beta 3l}e_l = 0,$$

for all α and β which range over the values 1 and 2. Furthermore, because $\mathbf{d}$ and $\mathbf{e}$ by assumption also span the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$, we conclude that the moduli $C_{\alpha\beta 3\delta}$ vanish for all α, β and δ equal to 1 or 2. This is identical to the T-condition, and thus we have shown that

$$T = ST. \tag{ST}$$

It should be noted that no mention was made in the definition of the ST condition of the tractions on the end faces of the cylindrical sample. These tractions are not necessarily zero, and may in fact be in any direction. They do, however, simplify if the material satisfies both L and T for the direction of interest, which case will be discussed next. Before concluding this section, we note that the ST condition can be equally well defined in terms of the simple shear $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{a}.\mathbf{x})\mathbf{b}$, for $\mathbf{b}.\mathbf{a} = 0$, because this displacement field induces the same constant stress tensor as (4.1).

5. Static interpretation of the monoclinic condition M

We have seen that $M = L + T$, and therefore it is possible to interpret the monoclinic condition M as a static condition similar to $ST = T$, but with some additional physical constraints. These arise from the requirement that certain tractions must be maintained on the end faces of the cylindrical sample of section 4. Since the surface normals on these faces are $\pm\mathbf{a}$, it follows from the definition of the displacement and stress in (4.1) and (4.2) that the tractions on the end faces are given by

$$t_i(\mathbf{a}) = \gamma\phi_{ik}b_k, \tag{5.1}$$

where ϕ is defined in (3.3). Assuming that L holds for direction $\mathbf{a}$ implies that the normal component, $\gamma\phi_{ik}a_i b_k$, vanishes, and hence the traction on each end of the cylindrical sample is a pure shear. Thus, L and T together imply the *first static monoclinic condition* ($SM1$): 'A cylinder of finite length and axis $\mathbf{a}$ can be maintained in the state of simple shear $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{b}.\mathbf{x})\mathbf{a}$ for at least two directions $\mathbf{b}$ orthogonal to $\mathbf{a}$ by the application of shear tractions on all surfaces and such that the tractions on the lateral surfaces are in the axial direction'. It may be shown by the same procedure used in the previous section that $SM1$ implies M, so that we have the equivalence

$$M = SM1. \tag{SM1}$$

Further understanding and slight variations on the condition $SM1$ can be obtained by the introduction of the symmetric, positive definite tensor $\psi(\mathbf{a})$ with components

$$\psi_{ik} = C_{ijkl}a_j a_l. \tag{5.2}$$

The L-condition implies that $\mathbf{a}$ is an eigenvector of ψ; suppose, further, that the remaining eigenvectors are $\mathbf{b}'$ and $\mathbf{b}''$. Consider the simple shear deformation of (4.1) with $\mathbf{b} = \mathbf{b}'$ specifically. It then follows as a consequence of L and T and the definition of the eigenvectors of ψ that (i) the traction across a face with normal $\mathbf{a}$ is in the direction of $\mathbf{b}'$, (ii) the traction on a face with normal $\mathbf{b}'$ is parallel to $\mathbf{a}$, and (iii) the traction vanishes on a face with normal $\mathbf{b}''$. This prompts the definition of a *second static monoclinic condition* ($SM2$): 'There exists an orthonormal triad $\{\mathbf{a}, \mathbf{b}', \mathbf{b}''\}$ for which a rectangular block of material with faces normal to each of these directions can be maintained in a state of simple shear $\mathbf{u}(\mathbf{x}) = \gamma(\mathbf{b}.\mathbf{x})\mathbf{a}$ by the application of shear tractions to the faces. Here $\mathbf{b}$ is either of $\mathbf{b}'$ or $\mathbf{b}''$; the tractions on those faces with normal $\pm\mathbf{a}$ are parallel to $\mathbf{b}$, those on the faces with normal $\pm\mathbf{b}$ are directed along $\mathbf{a}$, and the tractions vanish on the faces with normals $\pm\mathbf{a} \wedge \mathbf{b}$'.

It should be clear that $SM2$ follows from $SM1$, and the reverse can be shown without difficulty, implying that

$$SM2 = SM1. \tag{SM2}$$

It is instructive to see exactly how $SM1$, and hence M, follows from the premises of $SM2$. Let the directions $\mathbf{a}$, $\mathbf{b}'$ and $\mathbf{b}''$ coincide with $\mathbf{e}_3$, $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, and consider first the case of $\mathbf{b} = \mathbf{e}_1$. The stress tensor follows from (4.2) as $\sigma_{ij} = \gamma C_{ij13}$, and the traction conditions of $SM2$ imply the relations

$$C_{i113} = C_{3113}\delta_{i3}, \qquad C_{i213} = 0, \qquad C_{i313} = C_{1313}\delta_{i1}, \tag{5.3}$$

for $i = 1, 2, 3$. These in turn yield the identities

$$c_{15} = c_{25} = c_{35} = c_{45} = c_{56} = 0. \tag{5.4}$$

The same procedure for $\mathbf{b} = \mathbf{e}_2$ yields

$$c_{14} = c_{24} = c_{34} = c_{45} = c_{46} = 0, \tag{5.5}$$

and therefore, (5.4) and (5.5) together imply that the material possesses monoclinic symmetry. Thus it satisfies condition M, and we deduce the identity $SM1 = SM2$.

We note that this choice of basis also implies that additionally the modulus c_{45} vanishes. This is consistent with the observation for materials with monoclinic symmetry (**4, 5**) that there are at least three coordinate systems in which the number of non-zero components of the stiffness tensor is less than or equal to 12. Recently, F. Muir pointed out (in a private communication) that the constraints (5.4) and (5.5) imply that the components of the *compliance* tensor, S_{ijkl} or s_{IJ} for short, satisfy similar identities, viz.

$$s_{14} = s_{15} = s_{24} = s_{25} = s_{34} = s_{35} = s_{45} = s_{46} = s_{56} = 0. \tag{5.6}$$

Thus, for this particular coordinate system, *both* the stiffness and compliance tensors have at most 12 non-zero elements.

Finally, we note that the shear tractions of $SM2$ are all in the plane of shear and they must be of equal magnitude because the sample is in static equilibrium. This can also be seen as follows. Let $\mathbf{b}$ be either of the eigenvectors $\mathbf{b}'$ or $\mathbf{b}''$, with eigenvalue η, that is,

$$\psi\mathbf{b} = \eta\mathbf{b}. \tag{5.7}$$

The traction on an end face follows from (4.2) and (5.7) as

$$\mathbf{t}(\mathbf{a}) = \gamma\eta\mathbf{b}. \tag{5.8}$$

The traction on the face with normal $\mathbf{b}$ is in the direction $\mathbf{a}$, and because of the identity $\mathbf{a}.\mathbf{t}(\mathbf{b}) = \mathbf{b}.\mathbf{t}(\mathbf{a})$, it follows from (5.8) that

$$\mathbf{t}(\mathbf{b}) = \gamma\eta\mathbf{a}. \tag{5.9}$$

Hence, we have

$$\mathbf{t}(\mathbf{a}+\mathbf{b}) = \gamma\eta(\mathbf{a}+\mathbf{b}), \qquad \mathbf{t}(\mathbf{a}-\mathbf{b}) = -\gamma\eta(\mathbf{a}-\mathbf{b}), \qquad \mathbf{t}(\mathbf{a}\wedge\mathbf{b}) = 0, \tag{5.10}$$

where the final identity follows from $SM2$. It is evident from (5.10) that a state of plane stress exists in the plane of shear and that the stress is a pure shear. Denoting this state of stress and strain a state of *pure shear*, we arrive at the following general result.

THEOREM. *A necessary and sufficient condition that a material possesses a plane of material symmetry is that there are at least two orthogonal planes of pure shear with one shear direction in common (see section* 4)*, and for every orthogonal pair the plane of symmetry is spanned by the normals to the planes of pure shear. Equivalently, the normal to the plane of symmetry is the common shear direction.*

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