

ONE-COMPONENT SURFACE WAVES IN MATERIALS WITH SYMMETRY

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ABSTRACT

IT HAS RECENTLY been established that the existence of one-component surface waves is compatible with material symmetry and that they can occur even in transversely isotropic materials. This paper deals primarily with the existence of one-component surface waves in materials with a plane of symmetry normal to the direction of propagation, with particular attention given to transversely isotropic materials in this class. Numerical results indicate that there are stable transversely isotropic materials for which one-component surface waves exist for any orientation of the symmetry axis relative to the free surface. A separate numerical search shows that the one-component surface wave is incompatible with cubic symmetry.

1. INTRODUCTION

A RAYLEIGH surface wave in an isotropic elastic solid can be thought of as a linear combination of evanescent longitudinal and transverse waves, each decaying into the solid and with the same speed of propagation along the surface. In this sense a Rayleigh wave is a two-component wave. In general, a surface wave propagating in a given direction on the free surface of an anisotropic material is composed of all three components, although in some circumstances and under certain assumptions about the symmetry of the material, the wave may consist of fewer than three components. Thus, surface waves polarized in a plane of material symmetry are two-component waves (CHADWICK, 1990). Recently, BARNETT *et al.* (1991) derived conditions under which a *one-component* surface wave can exist in an arbitrarily anisotropic material. The conditions amount to constraints upon the moduli, and are clearly not met by isotropic solids. However, it is a remarkable fact that moduli can be found which correspond to stable elastic materials and also satisfy the constraints of BARNETT *et al.* (1991). The first instance of such a material was provided by BARNETT and CHADWICK (1991) and consists of a triclinic family of materials with no inherent symmetry.

Although their existence is precluded in isotropic solids, it is natural to enquire whether one-component surface waves can exist in materials with symmetry. This question was answered in the affirmative by CHADWICK (1992) who demonstrated that there are stable transversely isotropic materials with this property. In addition

to providing a specific example of such a material, Chadwick also proved some negative results which have a bearing on the present study. Thus, one-component surface waves cannot exist if either the free surface or the plane spanned by the normal to the free surface and the direction of propagation is a plane of symmetry.

In this paper we explore in some detail the existence of one-component surface waves in materials with symmetry. Following some definitions in Section 2, the five existence conditions of BARNETT *et al.* (1991) are summarized in Section 3. Most of the paper is devoted to materials with a plane of symmetry normal to the propagation direction, for which two of the five conditions are automatically satisfied. Further simplification arises from assuming that the material is of orthorhombic or higher symmetry, but such that the free surface is not a plane of symmetry. The procedure adopted here for discussing this class of materials is defined in Section 5 and is applied to transversely isotropic materials in Section 6. An attempt to classify the range of transversely isotropic materials which support one-component surface waves is discussed in that section. Finally, in Section 7, some numerical results are discussed which indicate that one-component surface waves cannot exist in materials with cubic symmetry.

2. PRELIMINARIES

The fundamental measure of the anisotropy are the 21 elements of the fourth-order stiffness tensor, C , defined in some frame with coordinates x and an associated right-handed triad of orthonormal direction vectors $\{e_1, e_2, e_3\}$. The elements of the associated 6×6 stiffness matrix c are defined by $c_{IJ} = C_{ijkl}$, where capital and lower case subscripts run from one to six and one to three, respectively, with the correspondence $I \leftrightarrow ij$ defined by 1, 2, 3, 4, 5, 6 $\leftrightarrow$ 11, 22, 33, 23, 13, 12. The 6×6 compliance matrix s is the inverse of c . The usual symmetries are assumed, $C_{ijkl} = C_{jikl} = C_{klij}$, and the materials are assumed to be stable, implying, among other things, that the diagonal elements of s and c are positive.

We are concerned here with one-component surface waves in materials with symmetry, for which it is necessary to distinguish between the surface wave frame $\{e_1, e_2, e_3\}$ and the reference frame of the material symmetry. Let $\bar{s}$ be the matrix of compliances in the reference frame $\bar{x}$ with basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$. The coordinates transform from one system to the other according to $x = a\bar{x}a^T$, where a is the 3×3 unitary transformation matrix, while the compliances transform according to the equation

$$s = N\bar{s}N^T, \quad (2.1)$$

where N is a 6×6 matrix, defined by AULD (1973), and discussed later in the appropriate sections. We define for later use the reduced compliances (TING, 1988)

$$\hat{s}_{IJ} = s_{IJ} - \frac{s_{I3}s_{J3}}{s_{33}}, \quad (2.2)$$

and the Poisson's ratios,

$$v_{ij} = -\frac{s_{ij}}{s_{ii}}, \quad i \neq j; \text{ no summation on } i. \quad (2.3)$$

The Poisson's ratios in the reference frame are denoted by $\bar{v}_{ij}$.

3. THE ONE-COMPONENT SURFACE WAVE

The outward normal to the free surface of the anisotropic material is $\mathbf{e}_2$, and the propagation direction of the surface wave is $\mathbf{e}_1$. BARNETT *et al.* (1991) proved that the following conditions are both necessary and sufficient for the existence of a one-component surface wave

$$\hat{s}_{12} = \hat{s}_{15} = \hat{s}_{25} = 0, \quad \hat{s}_{11} = \hat{s}_{55}, \quad (3.1)$$

and

$$(\hat{s}_{16} - \hat{s}_{45})^2 + 4\hat{s}_{14}\hat{s}_{56} < 0. \quad (3.2)$$

This concise form of the conditions in terms of the reduced compliances is due to TING (1990), and is based directly upon the work of BARNETT and CHADWICK (1991).

The one-component surface wave may be represented by the velocity vector

$$\mathbf{u} = \mathbf{a} e^{i\kappa(\mathbf{e}_1 \cdot \mathbf{x} + p\mathbf{e}_2 \cdot \mathbf{x} - v_1 t)}, \quad (3.3)$$

where κ is arbitrary and the speed of propagation is given by

$$\rho v_1^2 = \hat{s}_{11}^{-1}, \quad (3.4)$$

where ρ is the mass density per unit volume. The complex-valued parameter p determines the rate of decay in the solid $\mathbf{e}_2 \cdot \mathbf{x} < 0$, and is

$$p = \frac{1}{2}\hat{s}_{11}^{-1}[\hat{s}_{16} + \hat{s}_{45} - i\{-(\hat{s}_{16} - \hat{s}_{45})^2 - 4\hat{s}_{14}\hat{s}_{56}\}^{1/2}]. \quad (3.5)$$

The polarization vector is (not normalized)

$$\mathbf{a} = \frac{\hat{s}_{56}}{\hat{s}_{11}}\mathbf{e}_1 + \left(p - \frac{\hat{s}_{16}}{\hat{s}_{11}}\right)\mathbf{e}_3. \quad (3.6)$$

These results are taken directly from the papers of BARNETT *et al.* (1991) and BARNETT and CHADWICK (1991), and use the concise notation of TING (1988).

The one-component surface wave obviously has zero traction on the free surface $\mathbf{e}_2 \cdot \mathbf{x} = 0$. However, since the dependence upon depth is simple and exponential, the traction is zero at all depths. Hence, the wave may exist in a slab of arbitrary thickness with free faces both normal to $\mathbf{e}_2$. The particle motion at all depths is in the plane parallel to the free surface and is clockwise when viewed from above if $\hat{s}_{56} > 0$, and counterclockwise if $\hat{s}_{56} < 0$. The particle motion describes an elliptical path in the plane parallel to the free surface. The axes of the ellipse are generally not aligned in any particular direction, except when $\hat{s}_{16} = \hat{s}_{45}$, in which case the principal axes are aligned with $\mathbf{e}_1$ and $\mathbf{e}_3$. The polarization vector is an example of a *bivector* and the one-component surface wave is an *inhomogeneous plane wave*. The general theory of

bivectors and inhomogeneous plane waves is described in detail by HAYES (1984), and the reader is referred to this paper for details. Briefly, a bivector is a complex-valued vector in three-dimensions, with an associated ellipse defined by the real and imaginary parts. Thus, if $\mathbf{b}_1$ and $\mathbf{b}_2$ are real vectors, the ellipse associated with the bivector $\mathbf{b}_1 + i\mathbf{b}_2$ is in the plane spanned by the real vectors and such that these vectors are conjugate radii of the ellipse. A useful alternative form for the polarization is as follows, again not normalized,

$$\mathbf{a} = \begin{cases} \mathbf{e}_1 - ia_0 e^{i\phi} \mathbf{e}_3, & \hat{s}_{56} > 0, \\ \mathbf{e}_1 + ia_0 e^{-i\phi} \mathbf{e}_3, & \hat{s}_{56} < 0, \end{cases} \quad (3.7)$$

where

$$a_0 = \left(\frac{-\hat{s}_{14}}{\hat{s}_{56}} \right)^{1/2} > 0, \quad \sin \phi = \frac{\hat{s}_{45} - \hat{s}_{16}}{2a_0 \hat{s}_{56}}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (3.8)$$

The exact orientation and aspect ratio of the polarization ellipse may be found using the methods outlined by HAYES (1984).

The energy flux at any point is in the plane of polarization, although it is not necessarily aligned with the propagation direction $\mathbf{e}_1$. The energy propagation velocity is the velocity vector associated with the energy flux, and may be written as

$$\mathbf{c} = v_1 \mathbf{e}_1 + c_3 \mathbf{e}_3, \quad (3.9)$$

where the equality $\mathbf{e}_1 \cdot \mathbf{c} = v_1$ follows from a general result of HAYES (1984). The component c_3 can be related to the ratio of the mean flux in the $\mathbf{e}_3$ -direction over a cycle, to the mean energy density. The energy flux component is $-(\sigma_{13}^+ u_1^+ + \sigma_{33}^+ u_3^+)$, where the $+$ denotes the real part. Taking the average over a cycle we find that

$$c_3 = \frac{-\text{Re}(\sigma_{13} u_1^* + \sigma_{33} u_3^*)}{\rho(u_1 u_1^* + u_3 u_3^*)}, \quad (3.10)$$

where the $*$ denotes the complex conjugate. This may be simplified using the following identities,

$$\sigma_{11} = -\rho v_1 u_1, \quad \sigma_{13} = -\rho v_1 u_3, \quad s_{13}\sigma_{11} + s_{35}\sigma_{13} + s_{33}\sigma_{33} = 0, \quad (3.11)$$

the first two of which follow from the equations of motion in the $\mathbf{e}_1$ - and $\mathbf{e}_3$ -directions, respectively, while the third is a consequence of the fact that the strain component e_{33} is identically zero for one-component surface waves. Eliminating the stresses from (3.10) using (3.11), and then substituting from (3.3) and (3.6) we find that

$$c_3 = \left[\frac{1}{2} \left(1 - \frac{s_{13}}{s_{33}} \right) \left(\frac{\hat{s}_{45} - \hat{s}_{16}}{\hat{s}_{56} - \hat{s}_{14}} \right) + \frac{s_{35}}{s_{33}} \frac{\hat{s}_{14}}{(\hat{s}_{56} - \hat{s}_{14})} \right] v_1. \quad (3.12)$$

Finally, we note that some idea of the nature of the materials which might exhibit one-component surface waves can be gained from rewriting (3.1)₁ as

$$v_{12} + v_{13} v_{32} = 0. \quad (3.13)$$

Thus, at least one of the Poisson's ratios must be negative or zero.

4. PROPAGATION NORMAL TO A PLANE OF SYMMETRY

If the material is presumed to possess monoclinic symmetry with $\mathbf{e}_1$ normal to the plane of symmetry, then the compliance matrix has the form

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & 0 \\ s_{12} & s_{22} & s_{23} & s_{24} & 0 & 0 \\ s_{13} & s_{23} & s_{33} & s_{34} & 0 & 0 \\ s_{14} & s_{24} & s_{34} & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & s_{56} \\ 0 & 0 & 0 & 0 & s_{56} & s_{66} \end{pmatrix}. \quad (4.1)$$

Hence, $\hat{s}_{15} = \hat{s}_{25} = 0$ and the remaining conditions in (3.1) become

$$\hat{s}_{12} = 0, \quad \hat{s}_{11} = s_{55}. \quad (4.2)$$

Also, $\hat{s}_{16} = \hat{s}_{45} = 0$, and (3.2) reduces to the inequality

$$\hat{s}_{14}s_{56} < 0. \quad (4.3)$$

The polarization becomes

$$\mathbf{a} = s_{55}^{-1}s_{56}\mathbf{e}_1 + p\mathbf{e}_3, \quad (4.4)$$

where now p is purely imaginary,

$$p = -is_{55}^{-1}(-\hat{s}_{14}s_{56})^{1/2}. \quad (4.5)$$

The wave speed reduces to $\rho v_1^2 = s_{55}^{-1}$, which can be related to the speed of an SH wave on the elliptical SH sheet of the slowness surface. Specifically, the SH wave is polarized in the $\mathbf{e}_1$ -direction, with slowness vector in the direction $(s_{56}/s_{55})\mathbf{e}_2 + \mathbf{e}_3$, and corresponds to the unique wave which has energy propagation velocity in the direction $\mathbf{e}_3$ of magnitude v_1 . This connection was first noted by CHADWICK (1992), to whom we refer for further discussion. The form of the polarization in (4.4) indicates that the particle motion is elliptical in the planes parallel to the surface, with major and minor axes aligned with the coordinate axes, and the sense of motion is clockwise viewed from above if $s_{56} > 0$. We note, from (3.12), that the energy propagation velocity is codirectional with the phase velocity, and is of the same magnitude, v_1 .

It should be noted that there exists a preferred frame for monoclinic materials which is obtained by rotation about the normal to the plane of symmetry and within which both s_{56} and c_{56} vanish (MUIR, 1990). The existence of the one-component surface wave requires that s_{56} be non-zero, and hence it cannot exist if these axes coincide with the directions $\mathbf{e}_2$ and $\mathbf{e}_3$. Similarly, it cannot exist if the SH slowness sheet is circular rather than strictly elliptical.

5. ROTATED ORTHORHOMBIC MATERIALS

We next consider materials which are formally of the type considered in the previous section, but possess a symmetry higher than monoclinic. The formal equivalence arises

from rotation of the principal axes relative to those of the surface wave frame. The most general case is that of an orthorhombic material with principal axes $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ which are related to the surface wave frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by rotation about the coincident $\bar{\mathbf{e}} = \mathbf{e}_1$ direction through an angle θ , $0 < \theta < \pi/2$. The results of CHADWICK (1992) exclude consideration of the limiting angles $\theta = 0$, for which the frames coincide, and $\theta = \pi/2$, for which $\bar{\mathbf{e}}_3 = \mathbf{e}_2$ and $\bar{\mathbf{e}}_2 = -\mathbf{e}_3$. The compliance matrix $\mathbf{s}$ in the surface wave frame can be found by applying (2.1) with

$$\bar{\mathbf{s}} = \begin{Bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} & 0 & 0 & 0 \\ \bar{s}_{12} & \bar{s}_{22} & \bar{s}_{23} & 0 & 0 & 0 \\ \bar{s}_{13} & \bar{s}_{23} & \bar{s}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{s}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{s}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{s}_{66} \end{Bmatrix} \quad (5.1)$$

and

$$\mathbf{N} = \begin{Bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2 \theta & \sin^2 \theta & \cos \theta \sin \theta & 0 & 0 \\ 0 & \sin^2 \theta & \cos^2 \theta & -\cos \theta \sin \theta & 0 & 0 \\ 0 & -\sin 2\theta & \sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & 0 & 0 & \sin \theta & \cos \theta \end{Bmatrix}. \quad (5.2)$$

By computing the elements s_{11} , s_{12} , s_{13} , s_{23} , s_{33} and s_{55} , the two conditions (4.2) become

$$\begin{aligned} 4 \cos 2\theta [(\sin^2 \theta \bar{s}_{23} + \cos^2 \theta \bar{s}_{33})\bar{s}_{12} - (\sin^2 \theta \bar{s}_{22} + \cos^2 \theta \bar{s}_{33})\bar{s}_{13}] \\ + \sin^2 2\theta (\bar{s}_{12} + \bar{s}_{13})\bar{s}_{44} = 0, \\ (\bar{s}_{11} - \cos^2 \theta \bar{s}_{55} - \sin^2 \theta \bar{s}_{66})(\sin^4 \theta \bar{s}_{22} + 2 \sin^2 \theta \cos^2 \theta \bar{s}_{23} + \cos^4 \theta \bar{s}_{33} \\ + \sin^2 \theta \cos^2 \theta \bar{s}_{44}) - (\sin^2 \theta \bar{s}_{12} + \cos^2 \theta \bar{s}_{13})^2 = 0. \end{aligned} \quad (5.3)$$

The inequality (4.3) follows after computing the additional compliances s_{14} , s_{34} and s_{56} , as

$$\begin{aligned} [(\sin^2 \theta \bar{s}_{23} + \cos^2 \theta \bar{s}_{33} + \frac{1}{2} \sin^2 \theta \bar{s}_{44})\bar{s}_{12} \\ - (\sin^2 \theta \bar{s}_{22} + \cos^2 \theta \bar{s}_{33} + \frac{1}{2} \cos^2 \theta \bar{s}_{44})\bar{s}_{13}](\bar{s}_{55} - \bar{s}_{66}) > 0. \end{aligned} \quad (5.4)$$

In summary, (5.3) and (5.4) are the necessary and sufficient conditions for the existence of a one-component surface wave.

For instance, if $\theta = \pi/4$, using the fact that both $(\bar{s}_{22} + \bar{s}_{33} + 2\bar{s}_{23})$ and $\bar{s}_{44}$ are positive, the conditions (5.3) reduce to

$$\bar{s}_{12} + \bar{s}_{13} = 0, \quad \bar{s}_{11} = \frac{1}{2}(\bar{s}_{55} + \bar{s}_{66}) \quad (5.5)$$

and the inequality (5.4) becomes

$$(\bar{s}_{55} - \bar{s}_{66})\bar{s}_{12} > 0. \quad (5.6)$$

When (5.5) and (5.6) are satisfied, the speed and decay are

$$\rho v_1^2 = \bar{s}_{11}^{-1}, \quad p = -i\bar{s}_{11}^{-1}[(\bar{s}_{55} - \bar{s}_{66})\bar{s}_{12}]^{1/2}. \quad (5.7)$$

The unnormalized polarization direction is simply

$$\mathbf{a} = p\mathbf{e}_1 + 2\bar{v}_{12}\mathbf{e}_3 \quad (5.8)$$

and the motion is clockwise if $\bar{v}_{12} < 0$.

For angles other than $\pi/4$ the inequality (5.4) can be simplified by multiplying by $\cos 2\theta$ and using (5.3)₁ and the fact that $\bar{s}_{44} > 0$, to give

$$(\sin^2 \theta \bar{s}_{12} + \cos^2 \theta \bar{s}_{13})(\bar{s}_{55} - \bar{s}_{66}) \cos 2\theta < 0. \quad (5.9)$$

This is even simpler when expressed in terms of the rotated compliances,

$$s_{13}s_{56} \cos 2\theta < 0. \quad (5.10)$$

The conditions for the existence of a one-component surface wave cannot be met by specializing the orthorhombic material to a cubic material. This result was found by CHADWICK (1991) and follows from the fact that $\bar{s}_{55} = \bar{s}_{66}$ for a cubic material and hence $s_{56} = 0$ for any rotation. Neither can waves exist if the material is transversely isotropic with the symmetry axis coincident with the propagation direction. However, it is possible to have one-component surface waves in transversely isotropic materials when the symmetry axis is in some other direction. In the next section we consider the particular case of transverse isotropy with the axis perpendicular to the propagation direction.

6. ROTATED TRANSVERSELY ISOTROPIC MATERIALS

The procedure of the previous section is now applied to a transversely isotropic material, with principal axis $\bar{\mathbf{e}}_3$ in the orthonormal frame $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$. The compliance matrix is

$$\bar{\mathbf{s}} = \begin{Bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} & 0 & 0 & 0 \\ \bar{s}_{12} & \bar{s}_{11} & \bar{s}_{13} & 0 & 0 & 0 \\ \bar{s}_{13} & \bar{s}_{13} & \bar{s}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{s}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{s}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(\bar{s}_{11} - \bar{s}_{12}) \end{Bmatrix} \quad (6.1)$$

and the conditions required for positive definiteness are

$$\bar{s}_{11} > 0, \quad \bar{s}_{11} - \bar{s}_{12} > 0, \quad (\bar{s}_{11} + \bar{s}_{12})\bar{s}_{33} - \bar{s}_{13}^2 > 0, \quad \bar{s}_{44} > 0. \quad (6.2)$$

6.1. The case of $\theta = \pi/4$

We first consider the case $\theta = \pi/4$, for which the conditions (5.5) become

$$\bar{s}_{12} + \bar{s}_{13} = 0, \quad \bar{s}_{44} = 2\bar{s}_{12}. \quad (6.3)$$

The remaining condition (5.6) becomes, using (6.1), (6.1)₂, and the definition (2.3) of the Poisson's ratios,

$$\bar{\nu}_{12} < -\frac{1}{2}. \quad (6.4)$$

Note that the stability condition (6.2)₂ requires

$$-1 < \bar{\nu}_{12}. \quad (6.5)$$

The speed and polarization are the same as those of (5.7)₁ and (5.8), whereas the decay, given by (5.7)₂, simplifies to

$$p = i2\bar{\nu}_{12} \left(1 + \frac{1}{2\bar{\nu}_{12}} \right)^{1/2}. \quad (6.6)$$

A one parameter family of materials which is stable and meets the conditions (6.3) and (6.4) is defined and discussed by CHADWICK (1992). Translated into the present compliance notation, the material can be parameterized in terms of $\bar{\nu}_{12}$ as

$$\bar{s}_{12} = -\bar{s}_{13} = \bar{s}_{33} = \frac{1}{2}\bar{s}_{44} = -\bar{\nu}_{12}\bar{s}_{11}, \quad (6.7)$$

where the Poisson's ratio can take any value consistent with (6.4) and (6.5), i.e. $-1 < \bar{\nu}_{12} < -\frac{1}{2}$. The parameter used by CHADWICK (1992) is $\tau = -\frac{1}{2}(1 + \bar{\nu}_{12}^{-1})$.

6.2. Angles other than $\pi/4$

For a given value of $\theta \neq \pi/4$, the conditions (5.3) and (5.9) for the existence of a one-component surface wave can be expressed in terms of the transversely isotropic moduli as

$$4 \cos 2\theta [(\bar{s}_{13} \sin^2 \theta + \bar{s}_{33} \cos^2 \theta)\bar{s}_{12} - (\bar{s}_{11} \sin^2 \theta + \bar{s}_{13} \cos^2 \theta)\bar{s}_{13}] + (\bar{s}_{12} + \bar{s}_{13})\bar{s}_{44} \sin^2 2\theta = 0, \quad (6.8)$$

$$[\bar{s}_{11} \cos 2\theta + 2\bar{s}_{12} \sin^2 \theta - \bar{s}_{44} \cos^2 \theta][\bar{s}_{11} \sin^4 \theta + \bar{s}_{33} \cos^4 \theta + \frac{1}{4}(\bar{s}_{44} + 2\bar{s}_{13}) \sin^2 2\theta] - (\bar{s}_{12} \sin^2 \theta + \bar{s}_{13} \cos^2 \theta)^2 = 0 \quad (6.9)$$

and

$$(\bar{s}_{12} \sin^2 \theta + \bar{s}_{13} \cos^2 \theta)(\bar{s}_{11} - \bar{s}_{12} - \frac{1}{2}\bar{s}_{44}) \cos 2\theta > 0. \quad (6.10)$$

As an example, consider the compliance matrix

$$\bar{s} = c^{-1} \begin{Bmatrix} 2\tau+1 & \tau+1 & 0 & 0 & 0 & 0 \\ \tau+1 & 2\tau+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\tau \end{Bmatrix}, \quad (6.11)$$

where c is an arbitrary positive stiffness. This family of materials satisfies the conditions (6.8), (6.9) and (6.10) for the specific case of $\theta = \pi/3$, and is stable for $0 < \tau < 1$. The wave parameters are

$$\rho v_1^2 = \frac{2c}{1+3\tau}, \quad p = \frac{-2i}{1+3\tau} (1-\tau)^{1/2}, \quad \mathbf{a} = (1-\tau)^{1/2} \frac{\sqrt{3}}{2} \mathbf{e}_1 - i\mathbf{e}_3. \quad (6.12)$$

6.3. Some numerical explorations

The example provided by CHADWICK (1992) for $\theta = \pi/4$ and the previous example for $\theta = \pi/3$ suggest that there is a large class of transversely isotropic materials which can support a one-component surface wave. In order to appreciate the extent of the range of materials which exhibit this phenomenon, a numerical search was performed to classify those materials for which the axis of symmetry is perpendicular to the direction of propagation, i.e. the class of rotated TI materials considered above. Such materials must satisfy the three conditions (6.8), (6.9) and (6.10), and the stability requirements (6.2).

The task of searching over this multi-dimensional space can be simplified in the following manner. First rewrite (6.8) as

$$\frac{\bar{s}_{44}}{\bar{s}_{11}} = \frac{4 \cos 2\theta}{(\bar{v}_{12} + \bar{v}_{13}) \sin^2 2\theta} [(\bar{v}_{12} + \bar{v}_{13}) \sin^2 \theta + (\bar{v}_{12} \sin^2 \theta - \bar{v}_{13} \cos^2 \theta) \bar{v}_{13} - \bar{v}_{12} q], \quad (6.13)$$

where

$$q = \sin^2 \theta + \frac{\bar{s}_{33}}{\bar{s}_{11}} \cos^2 \theta. \quad (6.14)$$

Then, using this relation to eliminate $\bar{s}_{44}$ from (6.9), we obtain

$$\cos 2\theta \bar{v}_{12} q^2 - (2 \sin^4 \theta \bar{v}_{12}^2 - \cos^2 \theta \cos 2\theta \bar{v}_{13}^2 + \cos^2 \theta \bar{v}_{12} \bar{v}_{13}) q - (\bar{v}_{12}^2 \sin^2 \theta + \bar{v}_{13}^2 \cos^2 \theta)(\bar{v}_{12} \sin^2 \theta + \bar{v}_{13} \cos^2 \theta) = 0. \quad (6.15)$$

The inequality (6.10) becomes

$$\left(1 + \bar{v}_{12} - \frac{\bar{s}_{44}}{2\bar{s}_{11}}\right)(\bar{v}_{12} \sin^2 \theta + \bar{v}_{13} \cos^2 \theta) \cos 2\theta < 0. \quad (6.16)$$

For any given values of θ , $\bar{v}_{12}$ and $\bar{v}_{13}$, (6.15) can be solved to give two values for q ,

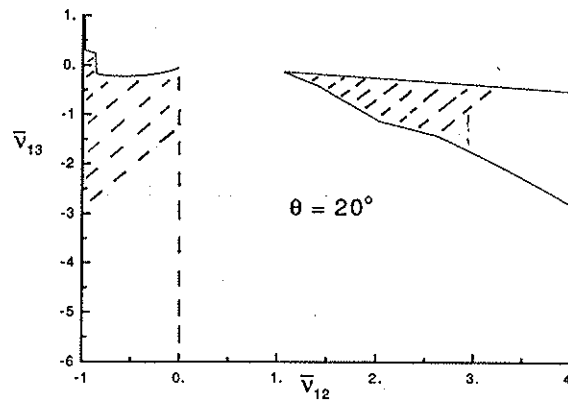


FIG. 1. The shaded regions show the ranges of possible values for the Poisson's ratios $\bar{\nu}_{12}$ and $\bar{\nu}_{13}$ for which a transversely isotropic material may support one-component surface waves with the axis of symmetry perpendicular to the propagation direction and at an angle $\theta = 20^\circ$ with respect to the free surface. The numerical search was performed over the region $-1 < \bar{\nu}_{12} < 4$, $-6 < \bar{\nu}_{13} < 4$.

each of which defines possible values of the parameter $\bar{s}_{33}/\bar{s}_{11}$, and corresponding possible values of $\bar{s}_{44}/\bar{s}_{11}$ follow directly from (6.13). We may assume $\bar{s}_{11} > 0$ and $\bar{\nu}_{12} > -1$, so that (6.2)₁ and (6.2)₂ are automatically satisfied. It then remains to check whether or not the parameters satisfy the three remaining inequalities: (6.2)₃, (6.2)₄ and (6.16).

Based upon this algorithm, a computer program was written to perform a two-dimensional search over pairs $(\bar{\nu}_{12}, \bar{\nu}_{13})$ with $\bar{\nu}_{12} > -1$, for given values of θ in the range $0-\pi/2$. The results indicate that there are continuous ranges of $(\bar{\nu}_{12}, \bar{\nu}_{13})$ and θ for which one-component surface waves exist. Some of the numerical results are summarized in Figs 1-5, each of which shows the range in the two-dimensional space of $(\bar{\nu}_{12}, \bar{\nu}_{13})$ for different values of θ . At $\theta = 20^\circ$ there are two distinct regions, for $-1 < \bar{\nu}_{12} < 0$ and $\bar{\nu}_{12} > 1$. The latter region is present for all angles less than $\pi/4$, as

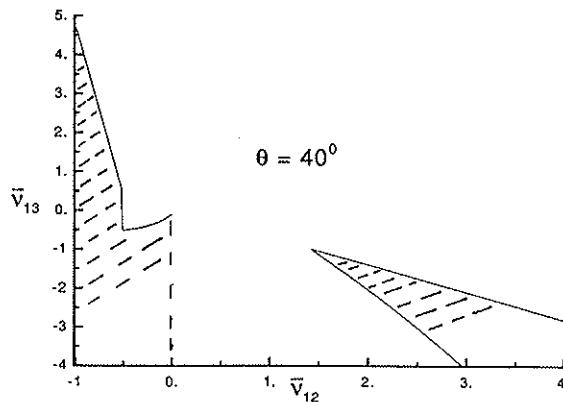


FIG. 2. The same as Fig. 1 but with $\theta = 40^\circ$, and for the search region $-1 < \bar{\nu}_{12} < 4$, $-4 < \bar{\nu}_{13} < 6$.

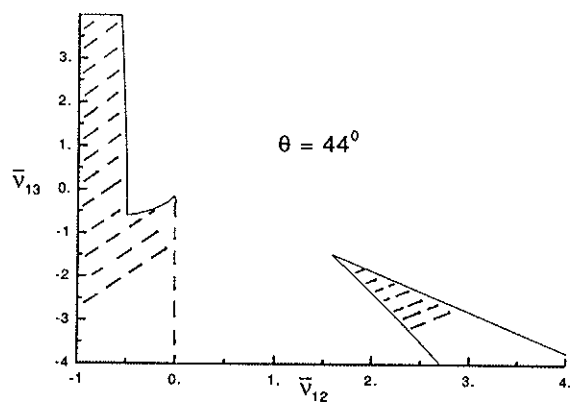


FIG. 3. The same as Fig. 1 but with $\theta = 44^\circ$, and $-4 < \bar{v}_{13} < 4$.

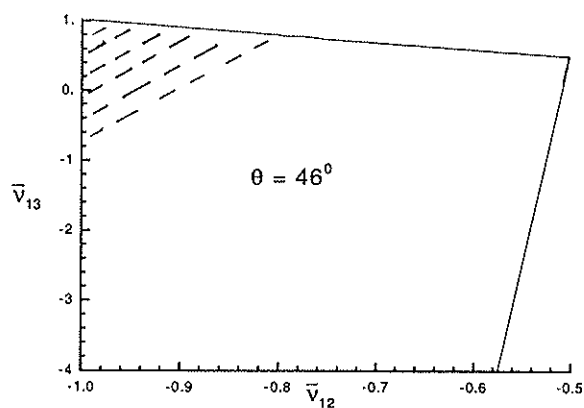


FIG. 4. The same as Fig. 3 but with $\theta = 46^\circ$.

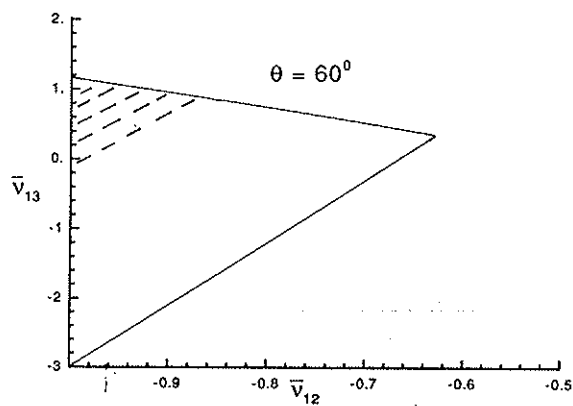


FIG. 5. The same as Fig. 1 but with $\theta = 60^\circ$.

illustrated in Fig. 2 for instance, but it disappears for $\pi/4 < \theta < \pi/2$. There is an abrupt transition at $\theta = \pi/4$, illustrated in Figs 3 and 4. The region for $-1 < \bar{\nu}_{12} < 0$ still exists beyond this angle, but its extent gradually diminishes as θ increases, until it vanishes at $\pi/2$. Figure 5 shows the range for $\theta = \pi/3$, and it can be seen that the materials defined by (6.11) fall within this region. Notice that the small region of positive $\bar{\nu}_{13}$ in Fig. 1 gradually increases with θ , as can be seen in Figs 2 and 3. The numerical results also show that there is no transversely isotropic material in this class with $\bar{\nu}_{12}$ between zero and unity, and that at least one of the two Poisson's ratio values $\bar{\nu}_{12}$ and $\bar{\nu}_{13}$ must always be negative.

7. MATERIALS WITH CUBIC SYMMETRY

It was noted previously in Section 5 that one-component surface waves cannot exist if we specialize the rotational procedure to materials with cubic symmetry (CHADWICK, 1992). Therefore, if such waves are to be found in cubic materials none of the cube axes can be coincident with the surface wave axes. In fact, it would seem very unlikely that any stable cubic material can satisfy the requirements (3.1) and (3.2). In order to prove the conjecture of nonexistence one must consider all possible three-dimensional rotations of the cube axes relative to the surface frame. This presents formidable obstacles to any attempt at an analytical answer to the question. Here we will adopt the simpler, more expedient, route of performing a numerical search over the full space of possible cubic materials.

The numerical search was performed in the following manner. Any cubic material may be defined by three independent compliances, $\bar{s}_{11}$, $\bar{s}_{12}$ and $\bar{s}_{44}$,

$$\bar{s} = \begin{Bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{12} & 0 & 0 & 0 \\ \bar{s}_{12} & \bar{s}_{11} & \bar{s}_{12} & 0 & 0 & 0 \\ \bar{s}_{12} & \bar{s}_{12} & \bar{s}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{s}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{s}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{s}_{44} \end{Bmatrix}. \quad (7.1)$$

The compliance, and hence the stiffness, is positive definite if and only if

$$\bar{s}_{11} > 0, \quad \bar{s}_{44} > 0, \quad -1 < \bar{\nu}_{12} < \frac{1}{2}. \quad (7.2)$$

Two separate searches were performed. In the first $\bar{s}_{11}$ was set to unity, while $\bar{s}_{44}$ and $\bar{\nu}_{12}$ were allowed to vary such that $0 < \bar{s}_{44} \leq 1$ and $-1 < \bar{\nu}_{12} < \frac{1}{2}$. In the second search $\bar{s}_{44}$ was fixed at unity and $\bar{s}_{11}$ and $\bar{\nu}_{12}$ were varied, with the same range for $\bar{\nu}_{12}$ and $0 < \bar{s}_{11} \leq 1$. In each case the compliance matrix relative to the surface wave frame was computed using (2.1) and the general form of the transformation matrix (AULD, 1973)

$N^T =$

$$\left\{ \begin{array}{cccccc} a_{11}^2 & a_{21}^2 & a_{31}^2 & 2a_{21}a_{31} & 2a_{31}a_{11} & 2a_{11}a_{21} \\ a_{12}^2 & a_{22}^2 & a_{32}^2 & 2a_{22}a_{32} & 2a_{32}a_{12} & 2a_{12}a_{22} \\ a_{13}^2 & a_{23}^2 & a_{33}^2 & 2a_{23}a_{33} & 2a_{33}a_{13} & 2a_{13}a_{23} \\ a_{12}a_{13} & a_{22}a_{23} & a_{32}a_{33} & a_{23}a_{32} + a_{22}a_{33} & a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} + a_{13}a_{22} \\ a_{31}a_{11} & a_{23}a_{21} & a_{33}a_{31} & a_{21}a_{33} + a_{23}a_{31} & a_{13}a_{31} + a_{11}a_{33} & a_{13}a_{21} + a_{11}a_{23} \\ a_{11}a_{12} & a_{21}a_{22} & a_{31}a_{32} & a_{22}a_{31} + a_{21}a_{32} & a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} + a_{12}a_{21} \end{array} \right\} \quad (7.3)$$

Arbitrary rotations were considered using three Eulerian angles, (θ, Φ, ψ) , as defined in MORSE and FESHBACH (1953). Because of the cubic symmetry, it was only necessary to search over the cube $0 < \theta, \Phi, \psi < \pi/2$.

Each search was therefore over a five-dimensional box, and at every point in this space the value of the parameter [see (3.1)]

$$\lambda = |\delta_{12}| + |\delta_{15}| + |\delta_{25}| + |\delta_{11} - \delta_{55}| \quad (7.4)$$

was evaluated and its minimum over all rotations was recorded for each value of the material parameters considered. The sign of the left member of (3.2) was also recorded at each minimum. The searches were computed using 30 points in each of the five dimensions. At no point was it found that the left member of (3.2) was negative at a minimum of λ . Programs were also run which found the minimum of λ over all possible materials for each rotation, and in some cases it was found that the left member of (3.2) became negative at a minimum of λ , but the latter always exceeded 0.12 while the former never went below -10^{-5} in value at these points. Based upon these findings, we conclude that *one-component surface waves cannot exist in stable materials with cubic symmetry*.

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