Acoustic wave scattering from thin shell structures

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A general asymptotic theory is developed to describe the acoustic response of heavily fluid-loaded thin shells in the midfrequency regime between the ring and coincidence frequencies. The method employs the ideas of matched asymptotic expansions and represents the total response as the sum of an outer, or background response, plus an inner or resonant contribution. The theory is developed for thin shells with smoothly varying material and geometrical properties. First, a suitable background field is found which satisfies neither the rigid nor the soft boundary conditions that have been typically employed, but corresponds to an impedance boundary condition. The background field is effective throughout the midfrequency as well as the strong bending regimes. The corresponding inner or resonance field is also valid in the same range. The approach taken is to represent these fields as inverse power series in the asymptotically small parameter $1/kR$, where R is a typical radius of curvature of the shell and k is the fluid wave number. The leading-order terms in the series differ in the inner and outer expansions, in such a way that the displacement tangential to the surface is negligible in the outer (background) region, but dominates the scattering near resonances. The resonances can therefore be associated with compressional and shear waves in the shell. A uniform asymptotic solution is derived from the combined outer and inner fields. Numerical results are presented for the circular cylinder and the sphere and comparisons are made with exact results for these canonical geometries. The results indicate that the method is particularly effective in the midfrequency range. The strong bending regime is also well represented, especially for cylindrical scatterers.

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INTRODUCTION

It is both well known and intuitively clear that when a solid target, such as a metallic sphere, is subject to acoustic wave radiation in a fluid medium, such as water, the boundary condition on the scatterer surface may be accurately modeled by the so-called rigid condition for which the normal velocity is everywhere zero. However, it is also well established that the scattered field calculated by employing this boundary condition breaks down at or very near *every in-vacuo* resonant frequency of the scatterer. In physical terms, the coupling between the solid and the fluid is very strong in the vicinity of these resonances, and hence a submerged structure that is only weakly affected by a given excitation can in fact be subject to strong vibrations if the excitation frequency is at or close to one of its *in-vacuo* resonances. In mathematical terms, the approximation based upon the rigid boundary condition is singular at the resonances. For separable geometries, such as the sphere and the cylinder, exact solutions exist from which the structure of these resonances can be examined in detail. The analysis for these simple shapes serves as a guide for the study of more complicated shapes and also as a numerical test for approximate theories and computational schemes.

These observations concerning the separation into background plus resonant contributions has generated a substantial literature over the past decade or so on what is referred to as resonance scattering theory (RST),[1] a good account of which may be found in the review article by Gaunaurd.[2] The main utility of RST seems to be as a diagnostic and interpretive procedure, whereby one can deconstruct the total response by subtracting out the background field, which is relatively easy to compute, so that one may then clearly identify any underlying resonances. It does not lend itself to a *constructive* approach whereby one could use the separation into background plus resonances to generate an efficient and relatively simple means of computing the total response. A procedure for doing this was recently described by Norris.[3] The method is based upon the use of matched asymptotic expansions and can be applied, in principle, to targets with nonseparable geometries and complicated material properties. The matched asymptotic approach[3] splits the total field into the background, plus a sum of resonant contributions. The form of the background response comes out naturally from the asymptotic scaling, where the small parameter in the asymptotic expansion is the impedance ratio. In this approach the "outer" solution is the response sufficiently far away from a given *in-vacuo* resonance, while the "inner" solution is the rapidly varying response in the thin boundary layer region surrounding a resonance. The total response is a combination of the two solutions and displays, as a function of frequency, the general form of a smooth background or

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outer field punctuated by sharp fluctuations arising from the inner or resonance field.

A straightforward extension of the concept of a rigid background response is not equally rewarding when applied to thin shell structures.[4-7] As an example consider an infinitely long cylindrical thin shell in water subjected to a time harmonic acoustic plane wave. The far-field backscattered amplitude of the acoustic pressure field is plotted versus kR in Fig. 1, where $k = \omega/c$, ω is the circular excitation frequency, c is the fluid sound speed, and R is the radius of the cylinder. The sharp lines indicate resonances associated with extensional and flexural wave motion on the shell. It seems fairly clear that neither the rigid nor the soft boundary condition is adequate to model the shell response away from resonances, because, if either background were indeed representative of the actual field one would expect the response to drop almost down to zero between resonances, which is not the case from Fig. 1, although it is worth noting that at lower values of kR the soft or pressure release boundary actually performs better than its rigid counterpart. The same failure is found if the matched asymptotic algorithm[3] is used to generate the total response from a thin shell,[8] and the reason can be crudely explained by the presence of another small parameter in the problem, viz., the ratio h/R, where h is a typical shell thickness and R is the radius of curvature. In practice, and in the examples considered in this paper, this ratio is far smaller than the impedance ratio, and, hence, any asymptotic approximation based only upon the latter is doomed to failure. Recently, however, Gaunaurd and Werby[9] and Werby[10] have proposed an intermediate boundary condition for spherical shells which provides a "correct" background in the sense that when subtracted it yields sharp, isolated resonances. Furthermore, the boundary condition reduces to the soft and rigid conditions in the limits of low and high frequencies, as one might expect from Fig. 1. Another type of boundary condition which also appears to be "correct" has been recently proposed by Norris[8] and amounts to the approximation of the shell response away from resonances as being due only to flexural motion. The connections between these results will be addressed later in the paper.

The objective of this paper is to describe a rational approach for approximating the total response from heavily fluid-loaded thin shells. The methodology is similar to that of Norris,[3] which addressed only the case of scattering by solid targets, and for reasons mentioned above cannot be adapted to thin shells without major modifications. These modifications are described in detail in this paper. We begin in Sec. I with the general equations for an arbitrarily curved, smooth inhomogeneous thin shell. The outer or background field is derived and discussed in Sec. II, where the response is assumed to be a regular asymptotic series in inverse powers of kR. The form of this series is motivated by the observation that for frequencies between resonances the normal displacement is an order of magnitude greater than the in-surface displacements. Equating terms of like order leads to a boundary condition for the lowest-order pressure field which includes bending effects. Solving for the lowest order as well as the next higher-order pressure field we find that

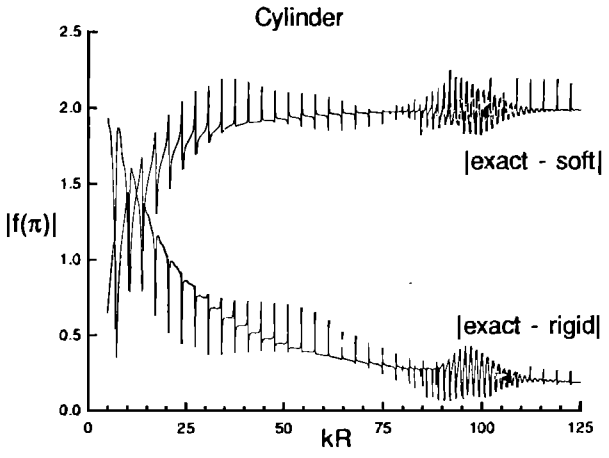

FIG. 1. The contrast between the soft and rigid backgrounds for broadside incidence on a steel cylinder in water with $h/R = 1/90$; h is the thickness and R is the radius of the cylinder. The curves show the difference between the backscattering amplitudes for the "exact" solution and the responses corresponding to rigid (Neumann) and pressure release (Dirichlet) boundary conditions on the surface. See Sec. VII for details.

this expansion breaks down in the vicinity of membrane resonances. We next turn to the inner field near a membrane resonance frequency and represent it once again as an inverse power series. This time, the form of the series is chosen to reflect the fact that at the compressional resonances the in-surface displacements are an order of magnitude higher than the normal displacements. Moreover, the shape of the leading-order in-surface displacements are now proportional to the membrane mode shape at that particular resonance. We complete the analysis by determining a uniform solution that is valid throughout the full frequency range for which kR is large, meaning in practice $kR > 5$, roughly. The details of the asymptotic analysis for the outer solution are given in Sec. II. The inner or resonant contributions are discussed in Sec. III, where they are combined with the outer solution to yield a uniform solution valid at all frequencies. Some of the general features and properties of the uniformly asymptotic solution are discussed in Sec. IV. The applicability of the procedure is demonstrated in Secs. V and VI for cylindrical and spherical thin shells, respectively, and comparisons are made with the corresponding exact analytical solutions for these canonical shapes. Numerical results are presented in Sec. VII which show that the asymptotic method is very accurate over the entire midfrequency range for the cylinder and sphere.

One of the motivations of the present work is to demonstrate that the response from thin shells can be profitably split into background and resonant parts, and that the determination of each of these separately is far less complicated than the solution of the total problem as a whole. In practice, the internal resonances can be very sharp, and any numerical procedure which does not take them into account explicitly may either give an incorrect amplitude or miss the resonance entirely. The theoretical development in this paper addresses among other topics the question of a suitable background field for thin shells.[8-10] It will become clear that the background response is far simpler to compute than the full re-

sponse, since it amounts simply to an effective impedance condition in the midfrequency range. Before commencing, we should mention some related works in the large literature on acoustic scattering by shells which have some bearing upon the present approach. Among these are the reviews of Gaunaurd and Werby[11] and Muzychenko and Rybak,[12] the latter being mainly a survey of the Russian literature. In a recent paper, Prikhod'ko[13] attempted to implement an asymptotic approach with the small parameter being the geometrical ratio of shell thickness to radius. The background field was identified as the one satisfying a soft boundary condition, which clearly cannot provide a suitable approximation except at very low frequencies (see Fig. 1). An interesting but more *ad-hoc* approach to describing the acoustic response of thin shells can be found in the papers by Veksler[14] and Veksler *et al.*;[15] however, these papers are restricted to cylindrical shells and deal only briefly with other geometries.

I. BOUNDARY CONDITIONS FOR THIN SHELLS

The general equations of motion for a smooth fluid-loaded shell are outlined in Appendix A, Eqs. (A9) and (A10). These are three equations, corresponding to the three displacement components w, v^α, $\alpha = 1,2$, which are coupled to the acoustic pressure p (see Appendix A for a definition of all the variables used here and a discussion of the notation). We will be concerned with scalings based upon a dimensionless frequency parameter. This is facilitated somewhat by working in terms of dimensionless variables $\hat{p}$, $\hat{w}$, and $\hat{v}^\alpha$, defined by

$$\hat{p} = \frac{p}{\rho c^2}, \quad \hat{w} = \frac{w}{R}, \quad \hat{v}^\alpha = \frac{v^\alpha}{R}, \tag{1}$$

where the fixed length R is a typical radius of curvature of the surface, c is the acoustic sound speed, and ρ is the inviscid fluid's density. These dimensionless parameters will be used for the remainder of the paper; however, to simplify notation we drop the carats. Thus, to convert back to dimensional variables, one simply multiplies p by ρc^2 and the shell displacements by R.

The acoustic pressure $p(\mathbf{x},t)$ satisfies the wave equation

$$\nabla^2 p - c^{-2} p_{tt} = 0, \tag{2}$$

in the infinite region exterior to the shell's surface S. In Eq. (2) and subsequent equations, the subscript t indicates the derivative with respect to time. The pressure may be decomposed into incident and scattered fields, but at this stage we will not distinguish between these separate parts of the total response. The remaining boundary condition requires that the fluid and shell normal accelerations are the same

$$\frac{R}{c^2} w_{tt} = \frac{-\partial p}{\partial n}. \tag{3}$$

Time harmonic motion of radial frequency ω is considered, and the term Re $\{...e^{-i\omega t}\}$ will be omitted from subsequent expressions. Then, Eqs. (A9), (A10), and (3) become, respectively,

$$\frac{R^2}{\rho_s h c_p^2} [\rho_s h c_p^2 H^{\alpha\beta\rho\lambda}(v_{\rho|\lambda} - b_{\rho\lambda} w)]_{|\beta} + \Omega^2 v^\alpha = 0; \quad \alpha = 1,2, \tag{4}$$

$$(kR)^2 w + \frac{c_p^2}{c^2} R^2 H^{\alpha\beta\rho\lambda}(v_{\rho/\lambda} - b_{\rho\lambda} w) b_{\alpha\beta} - \eta R^4 \left(\frac{\beta^2}{\eta} \frac{c_p^2}{c^2} H^{\alpha\beta\rho\lambda} w_{|\alpha\beta} \right)_{|\rho\lambda} = \eta p, \tag{5}$$

$$(kR)^2 w = R \frac{\partial p}{\partial n}, \tag{6}$$

where $k = \omega/c$ is the fluid wave number and $\Omega = \omega R / c_p$ is the dimensionless compressional wave number in the plate. Also, η and β are dimensionless numbers, the fluid loading parameter and the thickness parameter, respectively, defined as

$$\eta \equiv \frac{\rho R}{\rho_s h}, \quad \beta^2 \equiv \frac{h^2}{12 R^2}. \tag{7}$$

It is assumed that the fluid parameters ρ and c are constants but the geometrical and material parameters for the shell, such as the shell thickness h, may vary over its surface.

These are the basic boundary conditions for the acoustic scattering problem in the exterior region. As discussed in Appendix A, (4) and (5) are simply dynamic versions of the shell equations in the book by Green and Zerna,[16] and agree with a similar set of equations recently obtained by Pierce,[17] and by many others over the years. These equations are unique in the sense that they are the simplest set which describe the motion of arbitrarily curved shells and at the same time include both flexural and membrane effects.[18] The theory developed in this paper is in no way limited to these equations, but could be easily adapted to any linear set of shell equations; however, we choose to concentrate on these particular ones because they exhibit all the essential physics of the problem.

II. THE OUTER SOLUTION

A. The effective boundary condition

We now develop asymptotic approximations to the full set of boundary conditions in the limit of large kR. This choice of the asymptotic parameter is quite distinct from the ratio of acoustic impedances which turned out to be the natural parameter for solid targets,[3] and is independent of the geometrical parameter defined by the ratio h/R. However, it leads quite directly to scalings which define inner and outer solutions in the same way that the impedance ratio did for the solid target.[3] In practice, as demonstrated by the numerical examples, the fact that a high-frequency asymptotic method is employed to split the solution into outer and inner parts does not mean that we are restricted to large values of kR, but rather to values for which $kR = O(1)$ or greater.

We first summarize some results, that will be derived in a more rigorous fashion later, but are useful at this stage to understand the different physical effects and associated frequency ranges. It helps to distinguish two separate frequency regimes: first, the frequency range from $kR = O(1)$ to

slightly below coincidence is referred to as the *midfrequency regime*; while the range above and beyond this is called the *strong bending regime.*[14] In the midfrequency regime, the inertial effects of the shell dominate the elastic effects to first order over most of the frequency range, i.e., bending wave effects are small. This property is characteristic of the midfrequency range, as we will see. Bending or flexural wave effects become significant, if not dominant, in the higher frequency range, or the strong bending regime,[15] which starts roughly at the coincidence frequency ω_c defined by $k_c R = c/c_p\beta$.

Thus, in the midfrequency region, the condition (5) can be approximated simply by

$$(kR)^2 w = \eta p. \tag{8}$$

Equation (8) combined with (6) implies that the pressure satisfies a local impedance boundary condition

$$\frac{\partial p}{\partial n} - \frac{\eta}{R} p = 0. \tag{9}$$

This is equivalent to matching the pressure loading with the normal acceleration of the shell, ignoring the normal forces generated by the bending of the shell. On the other hand, when we need to account for the effects of the flexural wave motion, whether it is subsonic or supersonic, we must include the significant terms in (5) that were previously ignored, which are those involving the highest derivatives of w. This gives, instead of (8), the condition

$$(kR)^2 w - \eta R^4 \left(\frac{\beta^2}{\eta} \frac{c_p^2}{c^2} H^{\alpha\beta\rho\lambda} w_{\alpha\beta} \right)_{|\rho\lambda} = \eta p. \tag{10}$$

Using the velocity continuity, (6), the normal displacement w may be completely eliminated to yield a single boundary condition for the acoustic pressure

$$\frac{\partial p}{\partial n} - \frac{\eta}{R} p - \eta R^4 \left[\frac{\beta^2}{\eta\Omega^2} H^{\alpha\beta\rho\lambda} \left(\frac{\partial p}{\partial n} \right)_{|\alpha\beta} \right]_{|\rho\lambda} = 0. \tag{11}$$

Equations (9) and (11) are the boundary conditions which define the outer solutions or background fields in the midfrequency and strong bending regimes, respectively.

B. Asymptotic analysis

We now provide a justification for the conditions (9) and (11) based upon a formal asymptotic expansion in the small parameter $1/kR$, or equally well, Ω^{-1}. The preceding discussion suggests the following asymptotic perturbation scheme. We assume

$$\begin{aligned} p &= p^{(0)} + (kR)^{-2} p^{(1)} + (kR)^{-4} p^{(2)} + \cdots, \\ w &= (kR)^{-2} w^{(0)} + (kR)^{-4} w^{(1)} + \cdots, \\ v^\alpha &= (kR)^{-3} v^{\alpha(0)} + (kR)^{-5} v^{\alpha(1)} + \cdots. \end{aligned} \tag{12}$$

The motivation behind the scaling in (12) comes from a ray theory type of ansatz for each of the quantities on the right-hand side (rhs). Specifically, we assume that they depend upon the "fast" shell coordinates $\theta'_\alpha = \Omega\theta_\alpha$. We are not directly interested in applying the ray theory here, but use it purely as a basis for the scaling employed. In particular, each surface derivative of any of these physical quantities implies a magnification of order kR. Several explicit ray treatments of acoustic scattering from canonical thin shells are available,[19-22] and we refer to these papers for further details on ray applications.

Equation (6) implies that

$$w^{(m)} = R \frac{\partial p^{(m)}}{\partial n}, \quad m = 0,1,2,\ldots. \tag{13}$$

Substitution of (12) into (5), and recalling that a surface derivative is equivalent to a magnification of order kR in the asymptotic sequence, yields for $m = 0$

$$w^{(0)} = \eta p^{(0)} + \eta R^4 \left(\frac{\beta^2}{\eta\Omega^2} H^{\alpha\beta\rho\lambda} w^{(0)}_{|\alpha\beta} \right)_{|\rho\lambda}. \tag{14}$$

This relation, combined with (13), implies that the following condition must be satisfied by the leading-order pressure field, $p^{(0)}$,

$$\frac{\partial p^{(0)}}{\partial n} - \frac{\eta}{R} p^{(0)} - \eta R^4 \left[\frac{\beta^2}{\eta\Omega^2} H^{\alpha\beta\rho\lambda} \left(\frac{\partial p^{(0)}}{\partial n} \right)_{|\alpha\beta} \right]_{|\rho\lambda} = 0. \tag{15}$$

This is clearly equivalent to (11), while the corresponding condition for the midfrequency regime can be seen as a special case of the above condition if the following scaling holds

$$1 \ll kR \ll \beta^{-1}. \tag{16}$$

This scaling permits us to ignore the bending effects and leads to the point impedance condition (9).

The in-surface displacements, to leading order, follow from (4) and (12) as the solution to the forced system of equations

$$\begin{aligned} &(\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} v^{(0)}_{\rho|\lambda})_{|\beta} + \rho_s h \omega^2 v^{\alpha(0)} \\ &\quad = kR (\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} b_{\rho\lambda} w^{(0)})_{,\beta}. \end{aligned} \tag{17}$$

This system of two coupled equations describes the forced motion of a curved and closed membrane. Note that the coupling to the normal displacement equation has been eliminated and that all terms in (17) are of order $(kR)^2$. The subsequent term in the series for p follows by substituting the previously found values for the leading-order displacements in (5), yielding

$$\begin{aligned} \eta p^{(1)} = w^{(1)} &+ R^2 \frac{c_p^2}{c^2} H^{\alpha\beta\rho\lambda} \left(\frac{1}{kR} v^{(0)}_{\rho|\lambda} - b_{\rho\lambda} w^{(0)} \right) b_{\alpha\beta} \\ &- \eta R^4 \left(\frac{\beta^2}{\eta\Omega^2} H^{\alpha\beta\rho\lambda} w^{(1)}_{\alpha\beta} \right)_{|\rho\lambda}. \end{aligned} \tag{18}$$

The final bending term in this equation may or may not be of the same magnitude as the others, on the basis of the scaling (16) alone. The condition under which it matches the others is $\beta = O[(kR)^{-2}]$. If for the moment we assume for the sake of simplicity that bending effects are negligible, then $w^{(1)}$ can be eliminated from (18) by using (13) with $m = 1$ to give an equation for $p^{(1)}$,

$$\begin{aligned} R \frac{\partial p^{(1)}}{\partial n} &- \eta p^{(1)} \\ &= -R^2 \frac{c_p^2}{c^2} H^{\alpha\beta\rho\lambda} \left(\frac{1}{kR} v^{(0)}_{\rho\lambda} - b_{\rho\lambda} w^{(0)} \right) b_{\alpha\beta}. \end{aligned} \tag{19}$$

This may be solved using the Green's function for the exterior Helmholtz equation subject to the boundary condition (9). The Green's function will be discussed later.

In principle, this regular asymptotic perturbation procedure can be iterated to find the successively smaller terms, providing that the iterative process converges. However, it is clear that the iteration *must* break down at the membrane resonance frequencies. These are defined as the frequencies at which (17) has a solution for zero forcing on the rhs. These homogeneous equations do not involve the normal displacement ($w = 0$) and hence the modal solutions correspond roughly to in-surface compressional and shear waves, quite distinct from bending wave type of solutions. These are membrane modes, in the terminology of shell theory.[16] The generic structure of the asymptotic expansion therefore breaks down at or near these *membrane resonance* frequencies, implying that a different assumption on the form of the total response is required locally in the frequency domain. We call this, by analogy with boundary layer theory, the *inner solution*, and the solution just discussed is the *outer solution*. The full solution comprises both the inner and outer parts, and will be derived once the form of the inner solution has been determined. First, it is useful to define some quantities associated with the membrane modes.

C. The membrane modes

The breakdown of the outer solution is apparent if the solution to (17) is expressed as a modal series. Let ω_m, $m = 1,2,...$, be the modal frequencies and $V^{(m)\alpha}$ be the modal displacements, which are solutions to

$$(\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} V^{(m)}_{\rho|\lambda})_{|\beta} + \rho_s h \omega_m^2 V^{(m)\alpha} = 0. \tag{20}$$

We assume, for simplicity in later equations, that the modes are normalized according to

$$\langle \mathbf{V}^{(m)}, \mathbf{V}^{(m)} \rangle = 1, \tag{21}$$

where $\langle\ \rangle$ is the inner product on the surface S, defined by

$$\langle \mathbf{U}, \mathbf{V} \rangle \equiv \langle U^\alpha, V_\alpha \rangle \equiv \int_S U^\alpha V_\alpha \rho_s h \, dS. \tag{22}$$

It may then be demonstrated that the modes are orthonormal in the sense that

$$\langle \mathbf{V}^{(m)}, \mathbf{V}^{(n)} \rangle = \delta_{mn}. \tag{23}$$

Modes which are degenerate may be defined such that (23) holds, however, it is assumed for the sake of simplicity that the modes are not degenerate. The solution to (17) can then be formally expressed as

$$v^{\gamma(0)} = kR \sum_{m=1}^{\infty} \langle (\rho_s h)^{-1} (\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} b_{\rho\lambda} w^{(0)})_{|\beta}, V^{(m)}_\alpha \rangle \times [V^{(m)\gamma}/(\omega^2 - \omega_m^2)].$$

Integrating by parts, this becomes

$$v^{\gamma(0)} = -kR \sum_{m=1}^{\infty} \frac{\langle w^{(0)} f^{(m)} \rangle}{\omega^2 - \omega_m^2} V^{(m)\gamma}, \tag{24}$$

where in order to save space later the surface functions are now defined

$$f^{(m)} \equiv c_p^2 H^{\alpha\beta\rho\lambda} b_{\rho\lambda} V^{(m)}_{\alpha|\beta}. \tag{25}$$

The quantity $f^{(m)}$ is an inner product of interior or surface stress with the curvature tensor, and vanishes wherever either the curvature is zero (locally flat) or where the membrane stresses are zero. Note that although the mth mode is normalized according to (21), it is assumed to be a ray-type solution, which implies that the surface function $f^{(m)}$ is of order (kR). As mentioned before, ray theory is not used here to determine the modes explicitly, although that could certainly be done; our only purpose is to estimate the order of magnitude of terms in the asymptotic scheme. This is a rather unusual application of ray theory, but is not unknown in applications to the vibrations of shells. Gol'denveizer[23] and others[24] have used ray-type ansatzes to simplify the general shell equations, although the precise scaling considered here does not seem to fall into Gol'denveizer's classification.[23,24]

The fundamental consequence of the modal expansion (24) at this stage is that it clearly shows how the outer solution blows up at each and every membrane modal frequency. Specifically, this happens to the next term in the scattered pressure, since it follows from (19) and (24) that near the mth resonance frequency $p^{(1)}$ must satisfy the boundary condition

$$R \frac{\partial p^{(1)}}{\partial n} - \eta p^{(1)} = \frac{R^2}{c^2} \frac{\langle w^{(0)} f^{(m)} \rangle}{\omega^2 - \omega_m^2} f^{(m)}, \tag{26}$$

and hence it becomes singular as $\omega \to \omega_m$.

III. THE RESONANT CONTRIBUTIONS

A. The inner solution

The breakdown of the outer expansion is due to the fact that it allows the in-surface vibration to become unbounded. There is no feedback between the forcing of the leading-order approximation and the displacements $v^{\alpha(0)}$, which are explicitly assumed to be small in magnitude. In order to alleviate this shortcoming we propose an inner solution valid near the membrane resonance frequencies. Thus, near the resonance frequency ω_m we assume instead of (12) the new ansatz

$$p = P^{(0)} + (kR)^{-2} P^{(1)} + \cdots,$$
$$w = (kR)^{-2} W^{(0)} + \cdots, \tag{27}$$
$$v^\alpha = (kR)^{-1} A(\omega) V^{(m)\alpha} + (kR)^{-3} u^{\alpha(1)} + \cdots.$$

Comparing this with (12), we note that the in-surface displacements have leap frogged the normal displacement and are now one order greater, rather than an order smaller in magnitude. The present asymptotic analysis is based upon the assumption that the frequency is close to the resonance frequency in the sense that $(kR)^2 - (k_m R)^2 = O(1)$. The perturbation analysis could be performed in a more formal manner by rescaling the frequency variable according to this assumption, but the details tend to obscure the physical aspects of the solution. It is preferable to retain the original variables in so far as is possible so that the origin of each term is apparent.

In the present asymptotic approximation the in-surface solution is essentially just the mode under consideration

with a frequency-dependent modal amplitude $A(\omega)$ which must be determined. An equation for A can be found by considering the equation for $u^{\alpha(1)}$, which follows from (4), (20), and (27) as

$$(\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} u^{(1)}_{\rho|\lambda})_{|\beta} + \rho_s h \omega_m^2 u^{\alpha(1)} = kR(\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} b_{\rho\lambda} W^{(0)})_{|\beta} + (kR)^2 A \rho_s h(\omega_m^2 - \omega^2) V^{(m)\alpha}. \quad (28)$$

We have replaced ω by ω_m in the left member of (28), since the difference is assumed to be relatively small, but the difference is retained in the right member in order to arrive at a frequency-dependent expression for A. Equation (28) has a unique solution for $u^{\alpha(1)}$ if the inner product of the right member with the mode $\mathbf{V}^{(m)}$ vanishes, implying the relation

$$A(\omega_m^2 - \omega^2) + (kR)^{-1} \times \langle (\rho_s h)^{-1} (\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} b_{\rho\lambda} W^{(0)})_{|\beta}, V_\alpha^{(m)} \rangle = 0.$$

This may be transformed by integrating by parts on the surface, to give

$$A(\omega^2 - \omega_m^2) + (kR)^{-1} \langle W^{(0)} f^{(m)} \rangle = 0, \quad (29)$$

where $f^{(m)}$ is defined in (25). The factor of (kR) appears in these expressions because of the assumed scaling (27) and also because of the fact, noted previously, that $f^{(m)}$ is of order (kR). It remains to find $W^{(0)}$.

The equation for $W^{(0)}$ follows from (5) and (6). Now, in addition to the inertial term in the left member of (5), we must also include the possibility of a significant contribution from the term depending upon the in-surface components. Thus, instead of (5),

$$W^{(0)} - \eta R^4 \left(\frac{\beta^2}{\eta \Omega^2} H^{\alpha\beta\rho\lambda} W^{(0)}_{|\alpha\beta} \right)_{\rho\lambda} + (kR)^{-1} A(\omega) \frac{c_p^2}{c^2} R^2 H^{\alpha\beta\rho\lambda} b_{\alpha\beta} V^{(m)}_{\rho|\lambda} = \eta P^{(0)}. \quad (30)$$

Eliminating $W^{(0)}$ using the continuity condition (6) implies that the pressure satisfies the boundary condition

$$\frac{\partial P^{(0)}}{\partial n} - \frac{\eta}{R} P^{(0)} - \eta R^4 \left[\frac{\beta^2}{\eta \Omega^2} H^{\alpha\beta\rho\lambda} \left(\frac{\partial P^{(0)}}{\partial n} \right)_{\alpha\beta} \right]_{|\rho\lambda} = -\frac{A(\omega) R}{c^2 kR} f^{(m)}. \quad (31)$$

This should be compared with the equivalent boundary condition (15) for the outer solution. It is also instructive to compare (31) with (26), which does not properly account for the change in asymptotic behavior near resonance. Clearly, (31) gives a finite contribution as the frequency passes through resonance.

Referring to (15) and (31), the inner solution can be represented as the sum of the outer solution plus an additional part proportional to $A(\omega)$,

$$P^{(0)} = p^{(0)} - (kR)^{-1} (R/c^2) A \phi^{(m)}, \quad (32)$$

$$W^{(0)} = w^{(0)} - kR \frac{A}{\omega^2} \frac{\partial \phi^{(m)}}{\partial n}. \quad (33)$$

Here, $\phi^{(m)} \equiv G f^{(m)}$ where the Green's operator G maps an arbitrary function f from S to the exterior domain according to $G: f \to \phi \equiv Gf$, where ϕ satisfies the Helmholtz equation in the exterior region plus the radiation condition at infinity, and the following boundary condition on S

$$\frac{\partial \phi}{\partial n} - \frac{\eta}{R} \phi - \eta R^4 \left[\frac{\beta^2}{\eta \Omega^2} H^{\alpha\beta\rho\lambda} \left(\frac{\partial \phi}{\partial n} \right)_{\alpha\beta} \right]_{|\rho\lambda} = f. \quad (34)$$

Thus the operator is defined such that $\phi \equiv Gf$ is the radiated solution for the boundary condition (34) on S. Note that the boundary condition for the midfrequency regime is recovered if the last term on the left-hand side of (34) is dropped.

Substituting the normal displacement $W^{(0)}$ into the solvability condition (29) implies a linear equation for A, which may be solved to give

$$A = \frac{-\langle w^{(0)} f^{(m)} \rangle}{\omega^2 - \omega_m^2 + \alpha_m \omega_m^2 / \omega^2} (kR)^{-1}, \quad (35)$$

where

$$\alpha_m \equiv -\omega_m^{-2} \left\langle \frac{\partial \phi^{(m)}}{\partial n} f^{(m)} \right\rangle. \quad (36)$$

A further approximation may be made, which is entirely consistent with the high-frequency assumption, by removing the frequency dependence in the ratio $(\alpha_m \omega_m^2 / \omega^2)$ by evaluating its magnitude at $\omega = \omega_m$. This is justified by the weak dependence of α_m on frequency, about which we will say more later. Thus

$$A = \frac{-\langle w^{(0)} f^{(m)} \rangle}{\omega^2 - \omega_m^2 + \tilde{\alpha}_m} (kR)^{-1}, \quad (37)$$

where

$$\tilde{\alpha}_m = [\alpha_m]_{\omega = \omega_m}. \quad (38)$$

We reiterate that the appearance of the asymptotic parameter (kR) in (37) is a result of the definition of A as a dimensionless parameter of order unity. Finally, we define for later use dimensionless quantities analogous to α_m and $\tilde{\alpha}_m$,

$$a_m(\omega) \equiv (R^2/c_p^2) \alpha_m(\omega), \quad \tilde{a}_m \equiv (R^2/c_p^2) \tilde{\alpha}_m. \quad (39)$$

B. The uniform solution

Equation (37) provides us with the amplitude for a single resonance near the resonant frequency ω_m. Because the modes are assumed to be nondegenerate, the contribution from each may be summed as if it acts in ignorance of the others. Substituting from (37) into (32) and summing over the resonances gives

$$p^{\text{unif}} = p^{(0)} + \frac{1}{c^2} \sum_{m=1}^{\infty} \frac{\langle w_0 f^{(m)} \rangle}{\omega^2 - \omega_m^2 + \tilde{\alpha}_m} \phi^{(m)}, \quad (40)$$

where

$$w_0 \equiv (kR)^{-2} w^{(0)}, \quad (41)$$

is the leading-order approximation to the normal displacement according to the outer expansion. Recall that $p^{(0)}$ and w_0 are the total pressure and the normal displacement for the simplified scattering problem with boundary condition (15). In fact, using (13) w_0 can be eliminated from (40), giving

$$p^{\text{unif}} = p^{(0)} + \frac{1}{R\omega^2} \sum_{m=1} \frac{b_m}{\omega^2 - \omega_m^2 + \tilde{\alpha}_m} \phi^{(m)}, \qquad (42)$$

where

$$b_m = \left\langle \frac{\partial p^{(0)}}{\partial n} f^{(m)} \right\rangle. \qquad (43)$$

The relation in (42) is hereafter referred to as model I. If the quantity b_m is evaluated at $\omega = \omega_m$ and denoted by $\tilde{b}_m$ in the expression (42), then we refer to that as model II. Note also that the midfrequency solution is contained in the present solution by letting $\beta \to 0$. The limit of $\beta = 0$ is not, however, a regular limit in the strong bending regime. The singular nature of this limit arises from the possibility of supersonic or near-supersonic flexural modes on the shell which are insignificant in the mid-frequency regime. Comparing the total first-order inner expansion (42) with the combined first and second terms for the outer expansion, viz., (12), (18), and (24), we see that the solution (42) actually contains the correct form of the outer expansion away from the resonant frequencies. Thus p^{unif} of (40) or (42) is the desired *uniformly asymptotic solution*, correct to first order at all frequencies in the midfrequency range. This is the main result of the paper.

IV. DISCUSSION

The uniform solution in (42) has a simple form as the sum of a relatively smooth background solution, $p^{(0)}$, plus resonances at each of the frequencies of the membrane modes. The form of the resonance near a given modal frequency ω_m depends upon the parameter $\tilde{\alpha}_m$ of Eqs. (36) and (38). Using the fact that kR is assumed to be large, the perturbed resonant frequency may be approximated as

$$\omega = \omega_m - \tilde{\alpha}_m / 2\omega_m. \qquad (44)$$

It follows from the discussion in Appendix B that $\operatorname{Im} \tilde{\alpha}_m > 0$, and hence the imaginary part of the perturbed resonance frequency is strictly negative. This is in accord with the causality requirement that the total solution must be analytic in the upper half of the complex ω plane. The shift in both the real and imaginary parts of the frequency are small according to (44), indicating that the effect of fluid loading is not strong for these modes. This is not surprising, as the membrane modes are primarily caused by in-surface motion, which does not couple strongly to the exterior fluid. In fact, referring to the definition of α_m in (36), we see that it depends upon the surface function $f^{(m)}$ of (25), which in turns depends explicitly upon the curvature. When the curvature vanishes, as in a flat plate, the coupling disappears.

The coupling also vanishes for shear waves on regions with equal principal curvatures, or spherical regions. On such a region, the tensor **H** of Eq. (A7) simplifies considerably, and it may be shown, from (25), that

$$Rf^{(m)} \equiv -c_p^2 (1 + \nu) V^{(m)\alpha}{}_{|\alpha}, \qquad (45)$$

where R is the radius of curvature. The quantity $V^{(m)\alpha}{}_{|\alpha}$, which is the trace of the strain tensor, vanishes for a shear wave, and hence $f^{(m)} = 0$ if the mode is purely shear in nature.

At frequencies $O(1)$ away from the membrane resonant frequencies the contribution of each modal term in (40) is of order $1/\omega^2$ in comparison with $p^{(0)}$, and hence the uniform solution (40) reduces to the outer solution $p^{(0)}$. The mode m has on $O(1)$ contribution for frequencies such that

$$\omega - \omega_m = O(\omega_m^{-1} \operatorname{Re} \tilde{\alpha}_m), \qquad (46)$$

and the maximum occurs at $\omega = \omega_m - \operatorname{Re} \tilde{a}_m / 2\omega_m$. It is interesting to note that the total normal displacement right at ω_m is, using (32) and ignoring the other modal contributions which are small,

$$W^{(0)} = w^{(0)} - \frac{\langle w^{(0)} f^{(m)} \rangle}{\langle \partial \phi^{(m)} / \partial n f^{(m)} \rangle} \frac{\partial \phi^{(m)}}{\partial n}. \qquad (47)$$

If we think of the surface functions $f^{(m)}$ as basis functions for the normal displacement, then right at $\omega = \omega_m$ the *component* corresponding to mode m is identically zero.

The modes that arise in the present asymptotic expansion are not exactly the same as what is commonly understood as membrane modes. Equations (20) are a pair of simultaneous PDEs for the in-surface components of the shell displacement and are independent of the normal displacement. However, it is possible to supplement these equations with a third for the normal displacement. This is the conventional approach in discussions on membrane theories of shells. In terms of the general equations (4) and (5), it amounts to maintaining (4) exactly as shown, but to ignore the bending terms in (5), as well as the fluid pressure. Then, (5) provides an explicit expression for w in terms of the in-surface components,

$$w = \frac{L^{\rho\lambda} v_{\rho|\lambda}}{L^{\alpha\beta} b_{\alpha\beta} - \Omega^2 / R^2}, \qquad (48)$$

where

$$L^{\rho\lambda} = H^{\alpha\beta\rho\lambda} b_{\alpha\beta}. \qquad (49)$$

Eliminating w from (4) and assuming time harmonic motion, we deduce that the membrane modes must satisfy

$$\left[\rho_s h c_p^2 \left(H^{\alpha\beta\rho\lambda} + \frac{L^{\alpha\beta} L^{\rho\lambda}}{(\Omega^2 R^{-2} - L^{\alpha'\beta'} b_{\alpha'\beta'})} \right) V^{(m)}_{\rho|\lambda} \right]_{|\beta} + \rho_s h \omega_m^2 V^{(m)\alpha} = 0. \qquad (50)$$

Comparing these equations with (20), we see that the difference is associated with a term of order Ω^{-2}, which is small by assumption.

A regular perturbation analysis of (50) shows that the modal frequencies are shifted from those of (20) according to $\Omega^2 \to \Omega^2 + O(1)$. This is entirely consistent with the shift predicted by the uniformly asymptotic theory, viz., Eq. (46). Before proceeding to the examples, we mention that it is possible to improve upon the impedance condition (9) by the introduction of a modified impedance, which is described in Appendix C.

V. EXAMPLE: THE CYLINDRICAL SHELL

The theory outlined in the previous section will now be applied in detail to the specific example of a circularly cylindrical thin shell of infinite extent. The case of a thin spherical shell is discussed in the next section. For both of these canonical geometries, it is possible to compare the asymptotic solution with exact analytical solutions for the scattered field, thereby providing some general understanding of the range of validity of the asymptotic method. Numerical comparisons will be discussed in Sec. VII. The shell theories employed in each of the cases considered are consistent with the original equations, (4) and (5), upon which the theory is based. We reiterate that this does not imply that the asymptotic theory is restricted to this class of shell theories.

A. Equations of motion for the cylindrical shell

We consider a uniform thin cylindrical shell of radius R, thickness h and we introduce cylindrical polar coordinates r,θ,z. For the purposes of simplicity, we consider the case of broadside incidence thus eliminating the z dependence from the shell equations. The two pertinent rescaled shell equations are

$$\frac{d^2v}{d\theta^2}+\frac{dw}{d\theta}+\Omega^2 v=0, \tag{51}$$

$$-w-\frac{dv}{d\theta}-\beta^2\frac{d^4w}{d\theta^4}+\Omega^2 w=\frac{c^2}{c_p^2}\eta p, \tag{52}$$

where $\Omega=\omega R/c_p$. These equations are identical to the well-known Donnell equations and can be found in Junger and Feit,[25] Eqs. (7.80 a,b,c). The circumferential displacement is denoted by v and the radial displacement by w, and both are functions of the angle θ on the surface $r=R$. The remaining boundary condition (6) on the shell surface is

$$(kR)^2 w=R\frac{\partial p}{\partial r}. \tag{53}$$

The term in β^2 is significant in bending deformation and will also be incorporated into our asymptotic theory. The lowest-order approximation starts from (52). This can be written as

$$\beta^2\frac{c_p^2}{c^2}\frac{d^4w^{(0)}}{d\theta^4}-(kR)^2 w^{(0)}\approx\eta p^{(0)}. \tag{54}$$

By employing (53) we can write the boundary condition for the lowest-order approximation to the pressure field as

$$\frac{\partial p^{(0)}}{\partial r}-\frac{\eta}{R}p^{(0)}-\frac{\beta^2}{\Omega^2}\frac{\partial^5 p^{(0)}}{\partial\theta^4\,\partial r}=0,\quad r=R. \tag{55}$$

In summary, the outer or background field $p^{(0)}$ satisfies the Helmholtz equation in the exterior domain and (55) on the shell surface. Notice that the inclusion of the β^2 term in (55) indicates we are including the effects of strong bending. In the midfrequency range these effects turn out to be small and the surface condition reduces to the simpler local impedance condition (9).

B. Cylindrical membrane modes

The equation for the membrane modes is obtained from (51) and is

$$\frac{d^2V^{(m)}}{d\theta^2}+\Omega_m^2 V^{(m)}=0, \tag{56}$$

where $\Omega_m=m$, where $m=1,2,3...$ with corresponding normalized modes

$$V^{(m)}=\frac{\sin(m\theta)}{\sqrt{\pi R\rho_s h}}. \tag{57}$$

The appearance of the solid density is a consequence of the inner product definition (22). The corresponding $f^{(m)}$ follows from (25) as

$$f^{(m)}(\theta)=\frac{c_p^2}{R^2}\frac{m\cos(m\theta)}{\sqrt{\pi R\rho_s h}}. \tag{58}$$

C. The Green's operator for the cylindrical shell

As mentioned earlier in the strong bending regime the Green's operator must solve (34). When this equation is translated into cylindrical coordinates it reduces to

$$\frac{\partial\phi^{(m)}}{\partial r}-\frac{\eta}{R}\phi^{(m)}-\frac{\beta^2}{\Omega^2}\frac{\partial^5\phi^{(m)}}{\partial\theta^4\,\partial r}=f^{(m)},\quad r=R. \tag{59}$$

The solution to the exterior Helmholtz equation for $\phi^{(m)}$ which satisfies both the radiation condition and (59) is

$$\phi^{(m)}=\frac{c_p^2}{c^2}\frac{1}{\sqrt{\pi R\rho_s h}}\frac{m}{kH_m'(kR)}\frac{1}{\eta Z_m-E_m(\beta)}\times H_m(kr)\cos(m\theta). \tag{60}$$

Here, the prime denotes differentiation with respect to the argument and

$$Z_m=\frac{H_m(kR)}{kRH_m'(kR)}. \tag{61}$$

The bending effects are contained in the term $E_m(\beta)$,

$$E_m(\beta)=1-\frac{\beta^2 m^4}{\Omega^2}. \tag{62}$$

The quantity α_m then follows from (36), (58), and (60). For the present example, it is useful to work with the nondimensional version of (39), which simplifies to

$$a_m=\frac{1}{\eta Z_m-E_m(\beta)}. \tag{63}$$

The dependence of E_m upon β allows us to consider the midfrequency approximation separately by just replacing $E_m(\beta)$ by $E_m(0)=1$, which corresponds to the simpler impedance condition (9). Alternatively, we could *define* the midfrequency range for this problem to be the range of frequencies for which the contributing modes have $E_m(\beta)\simeq 1$. Flexural effects become significant only when the frequency and mode order are such that the second term in $E_m(\beta)$ is of order unity.

D. Comparison of the exact and asymptotic solutions

We consider an incident plane wave of unit amplitude propagating in the direction $\theta = 0$,

$$p^{\text{inc}} = e^{ikr\cos(\theta)} = \sum_{m=0}^{\infty} C_m J_m(kr)\cos(m\theta), \tag{64}$$

where $C_0 = 1$ and $C_m = 2i^m$, $m \geqslant 1$. The total exact response may be split into the sum of incident plus scattered where the scattered field is given by

$$p^{\text{sc}} = \sum_{m=0}^{\infty} D_m H_m(kr)\cos(m\theta), \tag{65}$$

where

$$D_m = -C_m \frac{J'_m(kR)}{H'_m(kR)} \left(\frac{\eta Q_m - E^{\text{ex}}_m}{\eta Z_m - E^{\text{ex}}_m} \right), \tag{66}$$

and

$$Q_m = \frac{J_m(kR)}{kRJ'_m(kR)}, \quad E^{\text{ex}}_m = E_m(\beta) - (\Omega^2 - \Omega^2_m)^{-1}. \tag{67}$$

The scattered component of the outer solution is very similar in form and is found by replacing $E^{\text{ex}}_m(\beta)$ by $E_m(\beta)$. Hence, we can write

$$p^{(0)} = p^{\text{inc}} + p^{\text{sc}(0)}, \tag{68}$$

where $p^{\text{sc}(0)}$ is the leading-order term in the scattered portion of the outer field.

The uniformly asymptotic field now follows from the inner and outer solution as

$$p^{\text{unif}} = p^{(0)} + \Omega^{-2} \sum_{m=1}^{\infty} \frac{d_m}{\Omega^2 - \Omega^2_m + \tilde{a}_m} \phi^{(m)}(r,\theta), \tag{69}$$

where $\tilde{a}_m$ follows from (39) and (63), the radiated field $\phi^{(m)}$ is given in (60), and

$$d_m = \frac{c^4}{c_p^4} \frac{2i}{m} \frac{\eta \rho_s h}{\sqrt{\pi R \rho_s h}} \frac{kRC_m}{H'_m(kR)(\eta Z_m - E_m)}. \tag{70}$$

Combining these terms we find

$$p^{\text{unif}} = p^{(0)} + \frac{2i\eta}{\pi kR} \sum_{m=1}^{\infty} \frac{C_m Z_m a_m^2}{H'_m(kR)} \times \frac{1}{\Omega^2 - \Omega^2_m + \tilde{a}_m} \frac{H_m(kr)}{H_m(kR)} \cos(m\theta). \tag{71}$$

The above expression defines model I. If we evaluate the d_m at $\omega = \omega_m$, then we obtain model II. In this paper, all our calculations for the cylinder are based on model I.

Finally, we note that the exact solution for the cylinder can be manipulated into a form which is remarkably like the uniform solution,

$$p = p^{(0)} + \frac{2i\eta}{\pi kR} \sum_{m=1}^{\infty} \frac{C_m Z_m a_m^2}{H'_m(kR)} \times \frac{1}{\Omega^2 - \Omega^2_m + a_m} \frac{H_m(kr)}{H_m(kR)} \cos(m\theta). \tag{72}$$

The only difference between (71) and (72) is that in the uniform approximation the denominator contains $\tilde{a}_m$ whereas in the exact solution the associated frequency-dependent parameter a_m appears. We therefore see that the asymptotic solution has a very similar analytical form to the exact solution. Before discussing numerical comparisons, we turn first to the case of the sphere.

VI. EXAMPLE: THE SPHERICAL SHELL

A. Equations of motion

The shell radius is R, with surface $r = R$ in spherical polar coordinates (r,θ,ϕ). We assume that the shell is subject to plane-wave incidence in the direction $\theta = 0$ so that there is no dependence upon the azimuthal angle ϕ. Only one in-surface component is zero, the one associated with θ and we denote it by v. Let $\mu \equiv \cos\theta$ and ν be the Poisson's ratio, then the two shell equations follow from (4) and (5) as

$$Lv - (1+\nu)\sqrt{1-\mu^2}\frac{dw}{d\mu} + \Omega^2 v = 0, \tag{73}$$

$$(1+\nu)\frac{d}{d\mu}\sqrt{1-\mu^2}v - 2(1+\nu)w - \beta^2\nabla^2_\mu(\nabla^2_\mu + 1 - \nu)w + \Omega^2 w = (c^2/c_p^2)\eta p, \tag{74}$$

where

$$\nabla^2_\mu = \frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu},$$

$$L = \sqrt{1-\mu^2}\frac{d^2}{d\mu^2}\sqrt{1-\mu^2} + (1-\nu). \tag{75}$$

These equations are similar to those of Junger and Feit,[25] Eqs. (7.102), (7.103) *et seq.*, which contain additional terms proportional to β^2. The general shell theory of (4) and (5) does not include these effects, but they could be considered without much difficulty. Our purpose here is to develop the asymptotic theory within the context of shell equations that exhibit all of the essential features, including both membrane and bending effects. Both the present equations and those in Junger and Feit satisfy these criteria, therefore the omission of some terms here does not alter the significance of the results, but rather makes the algebra more transparent and simpler to follow. It is not our contention that the shell equations presented here are best in any sense other than in their simplicity and generality.

B. Spherical membrane modes

The equation for the membrane modes is

$$Lv + \Omega^2 v = 0. \tag{76}$$

Therefore, the modal frequencies are $\Omega_m = \omega_m R/c_p$, where

$$\Omega^2_m = m(m+1) - 1 + \nu; \quad m = 1,2,3,..., \tag{77}$$

and the normalized modes [see Eq. (21)] are

$$V^{(m)} = B^{(m)}\sqrt{1-\mu^2}\frac{dP_m}{d\mu}, \tag{78}$$

where $P_m(\mu)$ is the mth Legendre polynomial, and the normalization constant is

$$B^{(m)} = \left(\frac{1}{2\pi R^2 \rho_s h} \frac{(2m+1)}{2m(m+1)} \right)^{1/2}. \tag{79}$$

The associated surface function $f^{(m)}$ of (25) follows, using (78), as

$$f^{(m)} = (1+\nu) \frac{c_p^2}{R^2} m(m+1) B^{(m)} P_m(\mu). \tag{80}$$

C. The Green's operator for the spherical shell

Consider an arbitrary azimuthally symmetric function g on the surface of the sphere. Then, g may be expanded in a series of Legendre polynomials, and the Green's operator for each term may be found separately. The boundary condition (34) reduces to the condition

$$\frac{\partial \phi}{\partial r} - \frac{\eta}{R}\phi - \frac{\beta^2}{\Omega^2} \nabla_\mu^2 (\nabla_\mu^2 + 1 - \nu) \frac{\partial \phi}{\partial r} = g, \quad r = R. \tag{81}$$

The Green's operator for arbitrary $g = \Sigma_{n=0}^{\infty} g_n P_n(\mu)$ is therefore

$$Gg = \sum_{n=0}^{\infty} \frac{-h_n(kr)}{kh_n'(kR)} [\eta Z_n - E_n(\beta)]^{-1} g_n P_n(\mu), \tag{82}$$

where $h_n(z) \equiv h_n^{(1)}(z)$ is the spherical Hankel function of order n, the prime denotes a derivative, and now

$$Z_n = \frac{h_n(kR)}{kRh_n'(kR)}. \tag{83}$$

The function $E_m(\beta)$ for the sphere is

$$E_m(\beta) = 1 - \beta^2 (\Omega_m^2/\Omega^2)(\Omega_m^2 + 1 - \nu). \tag{84}$$

The Green's operator for the simpler impedance condition (9) follows by putting $\beta = 0$, for which we have simply $E_m(0) = 1$.

The radiated field for mode n follows from (34) and (80)–(82) as

$$\phi^{(n)}(r,\theta) = (1+\nu) n(n+1) \frac{c_p^2}{kR^2} \frac{B^{(n)}}{(\eta Z_n - E_n)} \times \frac{h_n(kr)}{h_n'(kR)} P_n(\mu). \tag{85}$$

The quantity α_m of (36) then follows from (80) and (85), while the associated dimensionless quantity a_m of (39) is

$$a_m(\omega) = (1+\nu)^2 \frac{1 + (1-\nu)\Omega_m^{-2}}{\eta Z_m - E_m(\beta)}. \tag{86}$$

D. Comparison of the exact and asymptotic solutions

The incident plane wave of unit amplitude propagating in the direction $\theta = 0$ can be represented as

$$p^{\text{inc}} = e^{ikr\cos\theta} = \sum_{n=0}^{\infty} C_n j_n(kr) P_n(\mu), \tag{87}$$

where $C_n = (2n+1)i^n$. The total, exact response may be split into the sum of the incident plus the scattered, where the exact scattered pressure is

$$p^{\text{sc}} = \sum_{n=0}^{\infty} D_n h_n(kr) P_n(\mu), \tag{88}$$

where

$$D_n = -C_n \frac{j_n'(kR)}{h_n'(kR)} \left(\frac{\eta Q_n - E_n^{\text{ex}}}{\eta Z_n - E_n^{\text{ex}}} \right), \tag{89}$$

and

$$Q_n = \frac{j_n(kR)}{kRj_n'(kR)},$$

$$E_n^{\text{ex}} = E_n(\beta) - (1+\nu)^2 \frac{(\Omega_n^2 + 1 - \nu)}{\Omega^2(\Omega^2 - \Omega_n^2)} - 2\frac{(1+\nu)}{\Omega^2}. \tag{90}$$

If the outer response is written in the same way; i.e.,

$$p^{(0)} = p^{\text{inc}} + p^{\text{sc}(0)}, \tag{91}$$

then $p^{\text{sc}(0)}$ is the leading-order term in the regular perturbation approximation to the scattered field. It is very similar in form to the exact solution, and in fact can be obtained from (88)–(90) by simply replacing E_m^{ex} with $E_m(\beta)$.

We are now in a position to compute the quantities associated with the inner solution. The uniformly asymptotic field of (40) now follows from Eqs. (79), (82), (84), and (86), which combined with the outer solution just discussed, yields

$$p^{\text{unif}} = p^{(0)} + \Omega^{-2} \sum_{n=0}^{\infty} \frac{d_n}{\Omega^2 - \Omega_n^2 + \tilde{a}_n} \phi^{(n)}(r,\theta), \tag{92}$$

where $\tilde{a}_n$ follows from (38) and (86), $\phi^{(n)}$ is the radiated field of (85), and

$$d_n = \frac{i\eta}{c_p^2} \frac{(1+\nu)C_n}{k^2 R h_n'(kR) B^{(n)}} (\eta Z_n - E_n)^{-1}. \tag{93}$$

Combining the various elements in (92), the uniformly asymptotic solution can be written as

$$p^{\text{unif}} = p^{(0)} + \frac{i\eta}{(kR)^2} \sum_{n=1}^{\infty} \frac{\Omega^2}{\Omega_n^2} \frac{C_n Z_n a_n}{h_n'(kR)} \times \frac{[\eta Z_n - E_n(\beta)]^{-1}}{\Omega^2 - \Omega_n^2 + \tilde{a}_n} \frac{h_n(kr)}{h_n(kR)} P_n(\mu). \tag{94}$$

This result corresponds to model I, defined in Sec. III. Model II results from evaluating d_m at $\omega = \omega_m$. In order to appreciate the precision in either case, we note that the exact solution can be manipulated into a somewhat similar form by using the identities in Appendix D, to give

$$p = p^{(0)} + \frac{i\eta}{(kR)^2} \sum_{n=1}^{\infty} \frac{C_n Z_n a_n}{h_n'(kR)}$$

$$\times \frac{[\eta Z_n - E_n(\beta)]^{-1}}{\Omega^2 - \Omega_n^2 + a_n + X_n} \frac{h_n(kr)}{h_n(kR)} P_n(\mu), \tag{95}$$

where all the parameters are the same as for the uniform asymptotic solution, with the additional term

$$X_n = \frac{(1+\nu)^2}{E_n(\beta) - E_n^{ex}} \left(1 + \frac{(1-\nu)}{\Omega_n^2}\right) - (\Omega^2 - \Omega_n^2). \tag{96}$$

Note that the sum in (95) should really be from $n = 0$, but this term is identically zero because of the fact that $1/X_0 = 0$. In comparing the uniform and the exact solutions note the difference in the denominator between the $\tilde{a}_m$ and a_m terms. The quantity X_n is real and its effect is therefore to shift the resonance, but its influence upon the height and width of the resonance is weak. This is apparent if X_n is calculated explicitly for the case in which bending effects are included in the outer solution, for which it reduces to

$$X_n = \frac{(\Omega^2 - \Omega_n^2)^2}{\Omega_n^2} \left(1 - \frac{2\Omega^2}{(1-\nu)(\Omega_n^2 - 1 - \nu)}\right)^{-1}. \tag{97}$$

Near resonance, the difference $(\Omega^2 - \Omega_n^2)$ is of order unity and therefore the quantity X_n is asymptotically small, of order $1/\Omega^2$. The complex-valued parameter a_n is of order unity, and therefore the difference between the exact and uniformly asymptotic solutions is indeed asymptotically small for the spherical shell.

In closing this section, we note that Eqs. (72) for the cylinder and (95) for the sphere are apparently the first instances in which the resonant scattering from a thin shell has been explicitly decomposed into a classical resonant form, i.e., the background $p^{(0)}$ plus the resonant terms. Previous attempts along these lines invariably used an incorrect form for the background, which inevitably leads to an incorrect form for the resonant terms.

VII. NUMERICAL RESULTS AND DISCUSSION

The numerical results presented here are for cylindrical and spherical steel shells in water such that the radius to thickness ratio is always $R/h = 90$. The material parameters are (in mks units) $\rho = 1000$, $c = 1482$, $\rho_s = 2700$, $c_p = 5435$, and $\nu = 0.289$. The plots shown here are all for the far-field backscattering amplitude ($\theta = \pi$), where the far-field amplitude is defined by

$$f(\theta) = \lim_{r \to \infty} \{(2r)^{(D-1)/2} e^{-ikr} (p - p^{inc})\}, \tag{98}$$

where D is the spatial dimension, i.e., $D = 2$ for the cylinder, and $D = 3$ for the sphere. The scattering amplitude for the cylinder and sphere, for both the uniformly asymptotic and the exact solutions, can thus be calculated by substituting from (69), (72), (94), or (95) into (98). The limit is performed using the large argument asymptotic formulas for the cylindrical and spherical Hankel functions (see Appendix D). The ring frequency for both the cylinder and sphere is at $\Omega = 1$, or $kR = c_p/c = 3.67$, while the coincidence frequency is at $kR = c/c_p\beta = 85.01$. The midfrequency range for the present purposes will be defined roughly as the range from about twice the ring frequency to slightly below the coincidence frequency. The numerical results were all computed for the range $5 < kR < 125$, which includes the entire midfrequency range plus the strong bending regime,[14,15] which starts where the midfrequency range ends and continues to about $kR = 120$.

We first discuss the numerical results for the cylindrical thin shell. All the results presented here correspond to model I, i.e., all the quantities in the resonant terms in (71) are fully frequency dependent, except $\tilde{a}_m$ which is evaluated at $\omega = \omega_m$ according to (38) and (39). The exact and asymptotic results are shown together in Fig. 2, from which we note generally good agreement except at low frequencies and at an apparently spurious resonance close to $kR = 75$. The lack of accuracy at low frequencies, i.e., less than about twice the ring frequency, is not surprising since the asymptotic theory is based upon a high-frequency ansatz. However, the spurious resonance is unexpected, but on closer examination it can be associated with a narrow, subcoincidence flexural resonance. This conclusion is verified by inspection of Fig. 3 which shows both the outer and total asymptotic responses, each of which displays a narrow resonance at the location of the spurious resonance. Hence, the appearance of the spurious resonance is due to the fact that the outer expansion exhibits resonance structure itself, and these resonances are due to flexural wave motion. There are many flexural modes over the midfrequency range but they have little or no effect upon the scattered response because they remain nonradiating until close to the coincidence frequency, above which they are radiating and we then have the so-called strong bending regime[14] (note that the actual transition from nonradiating to radiating is complicated by the presence of surface curvature, and has been discussed recently by Pierce[17]).

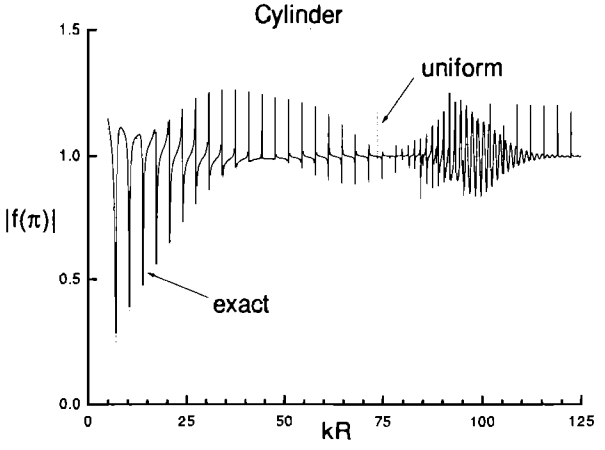

FIG. 2. Comparison of the exact solution and the matched asymptotic solution for the cylinder. Note the dotted line before $kR = 75$. When magnified the exact solution also has a small, narrow resonance here which is thought to be a flexural resonance.

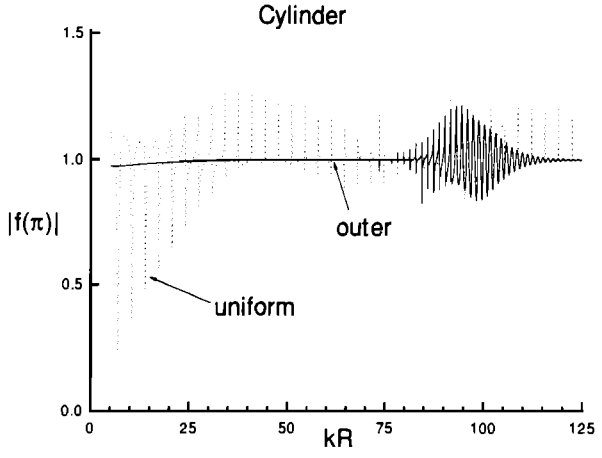

FIG. 3. The outer or background is here compared with the full asymptotic solution. Note the presence of flexural resonances in the outer field.

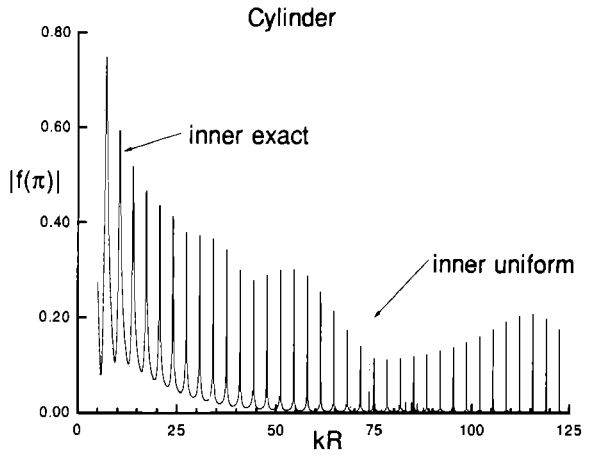

FIG. 5. A comparison of the exact inner field and the asymptotic version. Note the slight deviation at lower frequencies and the spurious resonance near $kR = 75$.

Thus the outer or background response is smooth over the entire midfrequency range, but it includes the effect of flexural resonances in the strong bending regime. In a sense, this *violates* our assumption used in deriving the matched asymptotic expansion that the outer solution is smooth near the membrane resonances. The appearance of the spurious resonance in the asymptotic response in Fig. 2 therefore results from *rapid* variation in the outer solution, and the variation is particularly rapid in the precursor region of the strong bending regime, where the flexural resonances are particularly narrow. Further into the strong bending regime the outer response is again sufficiently smooth, and no spurious resonances are produced. In summary, the erroneous spike in Fig. 2 is an early flexural resonance, magnified by the inner expansion to give an unphysical response. We will return to this point later.

The exact inner solution, defined by subtracting the outer response from the exact total scattered field, is shown in Fig. 4, from which it is clear that the outer solution provides an adequate background over the entire midfrequency range, in the sense that the rigid and soft backgrounds of Fig. 1 did not. Almost all of the resonances in the inner solution of Fig. 4 are attributable to compressional waves, whereas the major portion of the flexural wave content is in the outer response. The only exceptions to this categorization are the few small and narrow flexural modes which are just discernible in the inner response between $kR = 70$ and 90. The exact inner field and the asymptotically derived inner solutions are compared in Fig. 5, which again shows excellent agreement as low as $kR = 5$ and over the entire midfrequency range. The real measure of the accuracy of the asymptotic theory is indicated in Fig. 6. Finally, Fig. 7 shows the outer solution generated by ignoring bending effects in $p^{(0)}$, i.e., by using $E_n(0)$ rather than $E_n(\beta)$ in the outer solution. Now the outer solution is smooth at all frequencies, especially throughout the strong bending regime, while the exact inner found by subtracting the outer from the exact displays both membrane and flexural resonances. This form of background is similar to that recently discussed by Gaunaurd

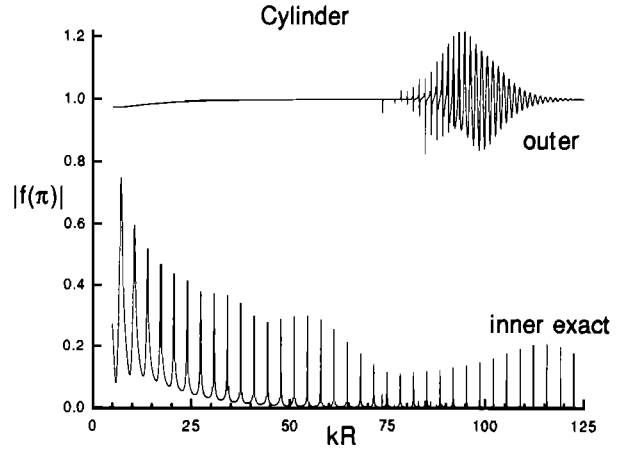

FIG. 4. The exact inner solution, defined as the exact minus the outer field, is compared here with the outer contribution. Note the regular membrane resonances in the inner field, even in the strong bending regime where flexural effects dominate the response.

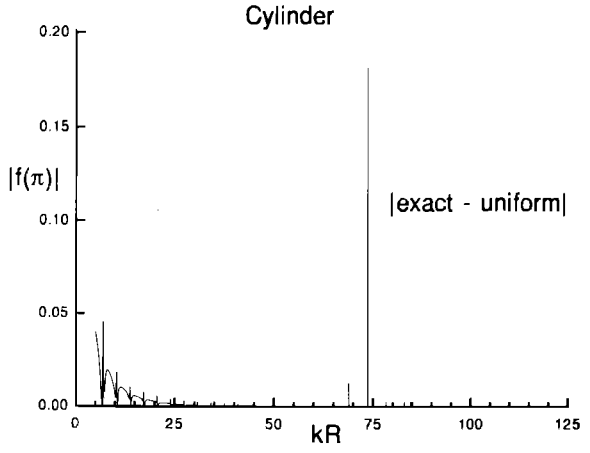

FIG. 6. The total error of the asymptotic solution is shown here on a magnified scale.

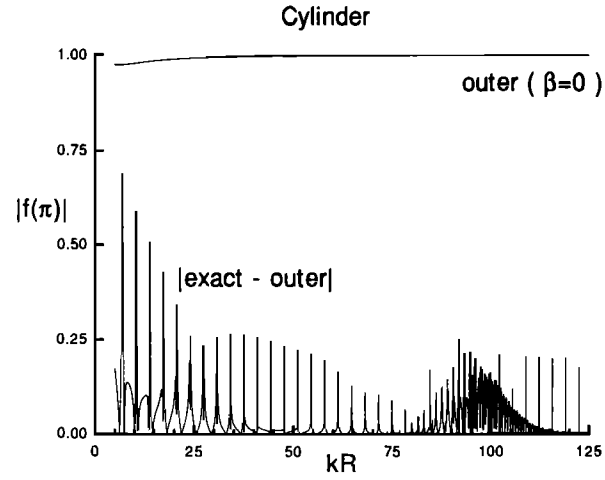

FIG. 7. The outer or background field without flexural contributions. This is computed from Eq. (65) by using $E_m(0)$ rather than E_m^{ex}. The lower curve is the difference between the exact and this background response. Note the presence of both compressional and flexural resonances in this "inner" field.

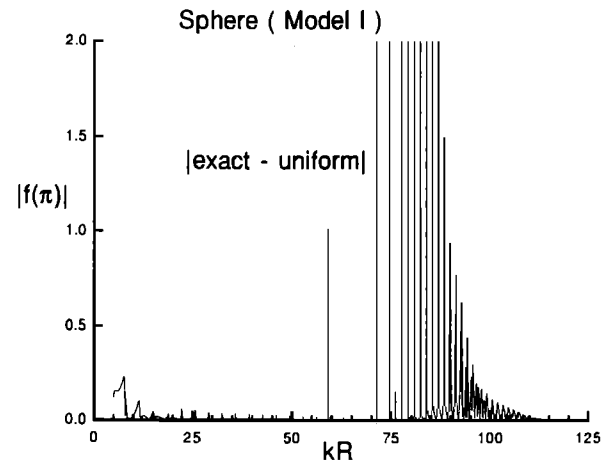

FIG. 9. The exact and uniform solutions for the sphere are compared here. Note that the error is very small over the midfrequency range $10 < kR < 50$.

and Werby[9,10] for the specific case of spherical shells, to which we now turn.

The exact and asymptotic backscattered amplitude for the spherical shell are compared in Figs. 8 and 9. As with the cylindrical shell, the agreement is very good over the midfrequency range, say $5 < kR < 70$, with understandable deviations at the lower end. The major feature which distinguishes the asymptotic solution is the appearance of many spurious resonances in the early part of the strong bending region. The explanation is the same as for the cylinder, i.e., they arise from the rapid variation in the outer solution which becomes magnified and enhanced in the inner response, because the latter is obtained by iteration of the operator or Green's function of the outer solution. However, the uniform solution, (42), was based upon the explicit assumption of a locally smooth outer response. This is why we made the distinction between models I and II. In a similar treatment of scattering from solid targets,[8] the background or outer response was obtained from the rigid boundary condition, and was always smooth as a function of frequency. Hence, there was no need to arbitrarily decide how to evaluate the matched solution for solid targets. In theory, if the solution is a proper uniform solution for all frequencies, it should not matter whether model I, II, or whatever is used. However, as can be seen by comparing Fig. 10 with Fig. 8, the difference between models I and II is significant in the strong bending region, where model II fares noticeably better. This is not to suggest that model II is to be preferred over model I; our point is that there is some sensitivity in the solution which indicates a nonuniform behavior. In fact, model II is less accurate at the low-frequency end, which is to be expected since it does not have the same flexibility as model I. Further details of the structure of the asymptotic solution are shown

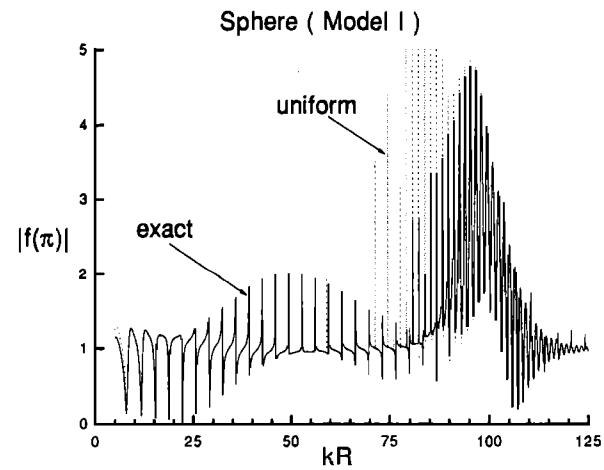

FIG. 8. The exact and uniform solutions compared for the spherical steel shell with $h/R = 1/90$. Model I leads to a good match in the midfrequency regime, and most of the error is confined to spurious flexural resonances in the early stages of the strong bending region.

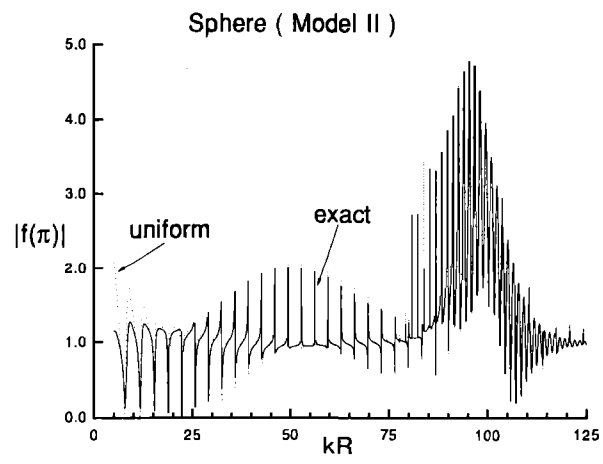

FIG. 10. A comparison of the exact and asymptotic solutions, where the latter is computed using model II. This is to be contrasted with Fig. 8. The error in the midfrequency regime is now significant whereas it is decreased considerably in the strong bending region.

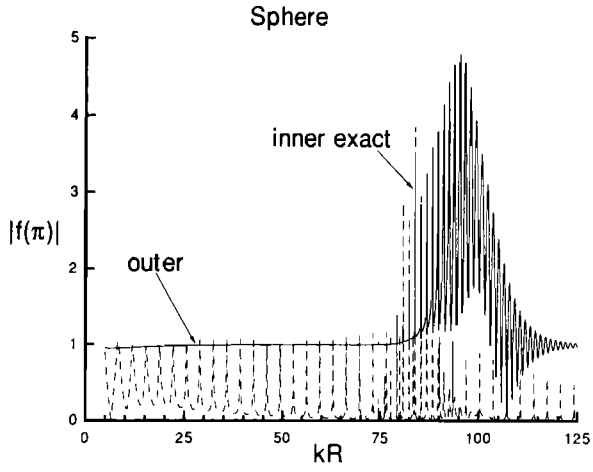

FIG. 11. The two separate constituents of the exact solution for the spherical shell: the outer solution and the exact inner solution, defined as the exact minus the outer.

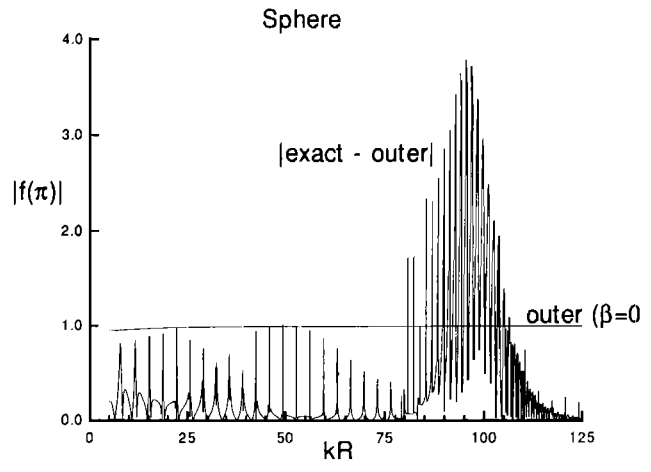

FIG. 13. The outer or background field for the sphere, without flexural contributions, and the remainder after subtraction from the exact solution. These curves are similar to the curves for the cylinder in Fig. 7. Again, both compressional and flexural resonances are present in the "inner" field.

in Figs. 11 and 12, which are similar to Figs. 4 and 5 for the cylindrical shell.

Finally, the background obtained by setting β to zero in the outer solution for the spherical shell, i.e., ignoring bending, is plotted in Fig. 13, which also shows the remainder after subtraction from the exact solution. The latter is similar to the residual computed and discussed by Gaunaurd and Werby for the specific case of the spherical shell.[9,10] As discussed above, this background does not contain any flexural effects, but depends only upon the inertial reaction of the shell. The distinction between this background ($\beta = 0$) and the one which contains bending effects ($\beta > 0$) is irrelevant in the midfrequency range, as demonstrated by the figures, although it is to be preferred on the grounds of simplicity, since it reduces to an effective impedance condition (9).

The results for both the cylinder and the sphere show that the asymptotic solution is not quite the uniform expansion we would like. It is perfectly satisfactory throughout the midfrequency region but suffers from sensitivity, or more precisely, nonuniformity, in the strong bending region. The basic difficulty arises from the attempt to construct a matched asymptotic solution from an outer solution which itself possesses resonances. There are, of course, different remedies to this which could be explored, but any attempt at a "better" uniform asymptotic solution would probably not be as simple as the present scheme, particularly in the strong bending regime. These questions will be addressed in future publications.

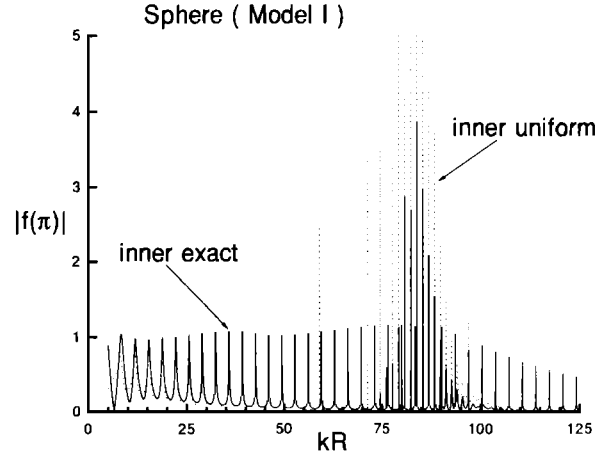

FIG. 12. A comparison of the inner solution for the asymptotic and exact theories.

VIII. CONCLUSIONS

In this study, we have presented a general theory of acoustic wave scattering from thin shells of varying material and geometrical properties. The theory is based upon an asymptotic expansion of the total response, and yields as part of the answer a new background field which contains the inertial and flexural effects of the shell. The remainder of the response comes from the lightly fluid-loaded membrane resonances. Each of the two separate parts is far simpler to obtain than the total response, and in combination they provide a new representation for the scattered field. The outer, or background, field simplifies over the midfrequency range in the sense that the flexural contributions become negligible and the effective boundary condition is a simple impedance condition on the shell surface. Analytical results and numerical computations show that the combined asymptotic solution is very accurate over the mid-frequency range for the canonical examples of the cylindrical and spherical thin shells.

A major motivation behind the development of the present approach is the idea that the total scattering from a thin shell in the midfrequency range can be profitably split into physically and mathematically simpler constituents. In this theory, the two which fall out quite naturally are the outer response, which amounts to solving an impedance condition on the shell, and a contribution due to membrane resonances. The structural details are contained in the latter,

through the mode shapes of the compressional resonances. The tasks involved in determining both constituents are indeed nontrivial, but together they are guaranteed to provide the correct form of the response near the frequencies of structural resonance. The same cannot be said of explicit methods which ignore the physics of the problem and may not fully capture the resonant response. The major difficulty with the proposed scheme is that it is essentially a modal method, and therefore requires determining all possible membrane modes in the frequency range of interest. This in itself may be a very formidable task for large structures, since the number of modes which may contribute can increase very rapidly. Some techniques will probably be required to filter out those modes which will not contribute significantly, see Refs. 26, 3 for instance, for preliminary attempts along these lines. Statistical approaches to classifying both the density of modes[27] and the modal shapes[28] will also be very useful, if not essential, for considering large and complex structures.

ACKNOWLEDGMENTS

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APPENDIX A: SHELL THEORY

We follow the development of Green and Zerna[16] who provide two derivations of the asymptotic theory of smooth shells, both methods yielding the same set of equations. The first derivation uses a direct method, analogous to the method used for plate theory, to obtain approximate equations for the bending of shells. In the second procedure Green and Zerna use asymptotic expansions to obtain the same equations from the three-dimensional equations of elasticity in a more consistent manner. The same set of equations were obtained by, among others, Koiter[18] who also used asymptotic methods to find a consistent theory for small deflections.

The equations defined below are for an arbitrarily curved, smooth shell, and are the simplest set of shell equations which includes both membrane effects and bending effects, each of which is important for the scattering problem. The basic assumptions are: (i) the shell is thin; i.e.,

$$h/R_{\min} \ll 1,$$

where h is the thickness and $R_{\min}$ the smallest principal radius of the undeformed middle surface; (ii) the strains are small and hence Hooke's law applies everywhere, and (iii) the state of stress is approximately plane, i.e., the traction in the direction normal to the undeformed middle surface is small in comparison with the remaining components of stress, which lie in the tangent plane. We first review the static theory from Green and Zerna.[16]

The curvilinear coordinates on the shell are θ_1 and θ_2, with corresponding direction vectors $\mathbf{a}_\alpha \equiv \mathbf{x}_\alpha$, $\alpha = 1,2$, not necessarily of unit magnitude, and unit surface normal $\mathbf{a}_3$ directed out of the shell. Greek sub- or superscripts assume the values 1 or 2, and the suffix, α denotes differentiation with respect to θ_α. The symmetric metric surface tensor has contravariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, and covariant components such that $a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma$. The covariant derivative is distinguished by $|$ and is defined such that $v^\alpha{}_{|\beta} \equiv \mathbf{v}_{,\beta} \cdot \mathbf{a}_\lambda a^{\alpha\lambda}$, or $v^\alpha{}_{|\beta} = v^\alpha{}_{,\beta} + \Gamma^\alpha_{\lambda\beta} v^\lambda$, where $\Gamma^\alpha_{\lambda\beta} = a^{\alpha\gamma} \mathbf{a}_\gamma \cdot \mathbf{a}_{\lambda,\beta}$ are the Christoffel symbols of the second kind.

The in-plane stress resultant has components $n^{\alpha\beta}$, and the stress couples are $m^{\alpha\beta}$. The shearing forces q^α are related to the couples by

$$q^\beta = m^{\alpha\beta}{}_{|\alpha}. \tag{A1}$$

The equilibrium equations are

$$n^{\alpha\beta}{}_{|\alpha} + p^\beta = 0, \tag{A2}$$

$$n^{\alpha\beta} b_{\alpha\beta} + q^\alpha{}_{|\alpha} + p_3 = 0, \tag{A3}$$

where p^α are the applied shear forces and p_3 the applied normal force, and $b_{\beta\alpha}$ define the surface curvature

$$b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta}. \tag{A4}$$

The displacement vector of a point originally on the middle surface is $v_\alpha \mathbf{a}^\alpha + w\mathbf{a}_3$. The constitutive relations for an elastic shell are

$$n^{\alpha\beta} = \rho_s h c_p^2 H^{\alpha\beta\rho\lambda} (v_{\rho|\lambda} - b_{\rho\lambda} w), \tag{A5}$$

$$m^{\alpha\beta} = -\frac{h^3}{12} \rho_s c_p^2 H^{\alpha\beta\rho\lambda} w_{|\rho\lambda}, \tag{A6}$$

where $\mathbf{H}$ depends upon the symmetry of the material comprising the shell, and simplifies for isotropic materials to

$$H^{\alpha\beta\rho\lambda} = \tfrac{1}{2}[(1-\nu)(a^{\alpha\lambda}a^{\beta\rho} + a^{\alpha\rho}a^{\beta\lambda}) + 2\nu a^{\alpha\beta}a^{\rho\lambda}]. \tag{A7}$$

Here, ν is the Poisson's ratio, c_p is the plate compressional wave speed, $c_p^2 = E/\rho_s(1-\nu^2)$, where E is the Young's modulus. The dynamic equations for a fluid-loaded shell are obtained by substituting

$$p^\alpha = -\rho_s h v^\alpha_{tt}, \quad p_3 = -\rho_s h w_{tt} - p, \tag{A8}$$

where p is the acoustic pressure in the external fluid and ρ_s is the mass density per unit volume of the solid. The resulting shell equations are

$$[\rho_s h c_p^2 H^{\alpha\beta\rho\lambda}(v_{\rho|\lambda} - b_{\rho\lambda} w)]_{|\beta} - \rho_s h v^\alpha_{tt} = 0; \quad \alpha = 1,2, \tag{A9}$$

$$\rho_s h c_p^2 H^{\alpha\beta\rho\lambda}(v_{\rho|\lambda} - b_{\rho\lambda} w) b_{\alpha\beta} - \tfrac{1}{12}(\rho_s h^3 c_p^2 H^{\alpha\beta\rho\lambda} w_{|\alpha\beta})_{|\rho\lambda} - \rho_s h w_{tt} = p. \tag{A10}$$

APPENDIX B: INTEGRAL IDENTITIES

Let ϕ be a radiating solution to the Helmholtz equation in the region exterior to S with far-field behavior defined by the radiation amplitude B such that

$$\phi = B\left(\frac{\mathbf{x}}{r}\right)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r \to \infty. \tag{B1}$$

Integrating the identity $\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^* = 0$ over the exterior region, and using Green's theorem to cast it as surface

integrals on S and the surface at infinity, implies the identity

$$\int_S \left(\phi^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial \phi^*}{\partial n}\right) dS = 2ik \int_{4\pi} |B|^2 \, d\Omega. \tag{B2}$$

Alternatively,

$$\mathrm{Im} \int_S \phi^* \frac{\partial \phi}{\partial n} \, dS = k \int_{4\pi} |B|^2 \, d\Omega. \tag{B3}$$

Now, assume that ϕ is the solution to the radiation problem (34) *for real-valued f*. Then, using (34) implies that

$$-\left\langle \frac{\partial \phi}{\partial n}, f \right\rangle = \rho \left\langle \frac{\partial \phi}{\partial n}, (\rho_s h)^{-1} \phi^* \right\rangle - \left\langle \frac{\partial \phi}{\partial n}, \frac{\partial \phi^*}{\partial n} \right\rangle + \left\langle \frac{\partial \phi}{\partial n}, \eta \left[\frac{\beta^2}{\eta \Omega^2} H^{\alpha\beta\rho\lambda} \left(\frac{\partial \phi^*}{\partial n} \right)_{|\alpha\beta} \right]_{|\rho\lambda} \right\rangle.$$

Integrating by parts, or using the self-adjoint property of the boundary condition (34), we may convert the final term in the right member so that it becomes

$$-\left\langle \frac{\partial \phi}{\partial n}, f \right\rangle = \rho \left\langle \frac{\partial \phi}{\partial n}, (\rho_s h)^{-1} \phi^* \right\rangle - \left\langle \frac{\partial \phi}{\partial n}, \frac{\partial \phi^*}{\partial n} \right\rangle + \left\langle \left(\frac{\partial \phi}{\partial n} \right)_{|\rho\lambda}, \frac{\beta^2}{\Omega^2} H^{\alpha\beta\rho\lambda} \left(\frac{\partial \phi^*}{\partial n} \right)_{|\alpha\beta} \right\rangle. \tag{B4}$$

It therefore follows from (B3) and (B4) that

$$-\mathrm{Im}\left\langle \frac{\partial \phi}{\partial n}, f \right\rangle = \rho k \int_{4\pi} |B|^2 \, d\Omega > 0. \tag{B5}$$

APPENDIX C: A MODIFIED IMPEDANCE

The impedance condition (9) depends only upon the ratio of inertial effects, since $\eta/R = \rho/\rho_s h$. The impedance condition was motivated by identifying the terms in the left member of Eq. (5) which are largest under the assumption of high-frequency, subject to (16), but all the while ignoring bending effects. One result of this particular asymptotic scaling was that the terms dependent upon the in-surface displacement were ignored, as was the term $-\rho_s h c_p^2 H^{\alpha\beta\rho\lambda} b_{\alpha\beta} b_{\rho\lambda} w$. By including this term, which is linear in w, but still ignores membrane effects, we obtain a slightly different impedance, i.e., $\eta \to \tilde{\eta}$, where

$$\tilde{\eta} = \frac{\eta}{1 - (R^2/\Omega^2) H^{\alpha\beta\rho\lambda} b_{\alpha\beta} b_{\rho\lambda}} \geqslant \eta. \tag{C1}$$

The inequality (C1) follows from the positive definite nature of the elasticity tensor, and the equality prevails only at a locally flat region.

The difference between η and $\tilde{\eta}$ is small, of order $1/\Omega^2$, on account of the high-frequency assumption. Therefore, the modified impedance $\tilde{\eta}$ will not differ much from η. It is possible to simplify the term in the denominator of (C1) by choosing the local coordinates to coincide with the principal directions of curvature. Let the signed principal radii of curvature be R_1 and R_2, then it follows from the definition of $H^{\alpha\beta\rho\lambda}$ in (A7) that

$$H^{\alpha\beta\rho\lambda} b_{\alpha\beta} b_{\rho\lambda} = \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{2\nu}{R_1 R_2}. \tag{C2}$$

The values for the sphere ($R_1 = R_2$) and the cylinder ($1/R_2 = 0$) follow immediately.

APPENDIX D: SOME USEFUL IDENTITIES

The surface integrals for the spherical shell simplify using the following identities[29]

$$\int_{-1}^{1} (1 - \mu^2) \left(\frac{dP_m}{d\mu} \right)^2 d\mu = \frac{2m(m+1)}{2m+1}, \tag{D1}$$

$$\int_{-1}^{1} P_m^2 \, d\mu = \frac{2}{2m+1}. \tag{D2}$$

The Wronskian relation for the spherical Bessel functions is

$$j_n(z) h_n'(z) - j_n'(z) h_n(z) = i/z^2. \tag{D3}$$

The asymptotic behavior of spherical Hankel functions of large argument is also used,

$$h_n(z) = (-i)^{n+1} (e^{iz}/z) + O(1/z^2). \tag{D4}$$

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