

High-frequency nonlinear acoustic beams and wave packets

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Gaussian beams and wave packets are formally equivalent to high-frequency solutions of the linear wave equation with complex-valued phase functions. The theory of weakly nonlinear high-frequency waves is extended in this paper to allow for complex phase solutions. The procedure is similar to nonlinear geometrical optics and the nonlinearity causes the phase to vary, leading to the development of weak shocks. However, unlike standard geometrical optics that describes the evolution of wave fronts, complex phase solutions correspond to Gaussian decay away from a central ray, and the associated curvatures are complex valued. Geometrical singularities due to caustics and foci do not occur, and the only singularities in the theory are purely nonlinear. The theory is developed specifically for nonlinear acoustic waves in a homogeneous inviscid fluid. The time taken for nonlinear singularities to develop is compared for a real spherical wave front and a Gaussian beam with Gaussian radius equal to the real radius of curvature of the wave front. The time to blowup for the Gaussian beam is intermediate between the shortest possible, which is for the converging wave front, and the longest possible, for the diverging front.

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INTRODUCTION

Nonlinear acoustics of focused beams has been analyzed in depth over the past few decades, with most of the emphasis on the study of solutions to model equations such as the KZK equation. There is a particularly large literature on the subject within this journal, see for instance Refs. 1–3, and elsewhere, e.g., Refs. 4 and 5. An alternative approach is to start from the exact, underlying equations but to assume that the solution depends upon a small parameter and use appropriate asymptotic methods of analysis. This procedure is well developed for the evolution of high-frequency, weakly nonlinear waves. The basic mathematical methods were outlined by Choquet-Bruhat⁶ with subsequent development by Hunter and Keller,⁷ and independently by Parker.⁸ These methods are closely related to the theory of acceleration waves,⁹ and also to the linear theory of geometrical optics.^{10–12} The effects of refraction enter the weakly nonlinear theories through the rays of geometrical optics while nonlinear effects enter via the transport equation for the amplitude. In both the model equation approach, e.g., the KZK equation, and the weakly nonlinear methods, solutions can develop from initially continuous data into discontinuous solutions, or weak shocks. This is an inevitable consequence of nonlinear terms in the equations of motion and is displayed by both the unidirectional theory, such as the Burgers equations, and also by the more general theories which take refraction effects into account. Weakly nonlinear theories can also develop geometrical singularities associated with the convergence of rays at caustics. These are quite distinct from nonlinear effects, but signify a breakdown of the geometrical optics ansatz, and may be treated by boundary layer techniques. Hunter and Keller¹³ have shown that the evolution of weakly nonlinear solutions is governed by linear theo-

ry near caustics, and hence there is no physical singularity associated with caustics, in contrast to the purely nonlinear generation of shocks.

The focus of this paper is on the development and generalization of the weakly nonlinear theory to account for not only the possibility of curved wave fronts but also the decay of the amplitude away from a central ray. The types of general solutions include, but are not restricted to, Gaussian beams and Gaussian wave packets, which have been discussed at great length as solutions to the linear equations of acoustics and elasticity.^{14–18} The motivation for the present work comes from previous work on the propagation of Gaussian beams and wave packets within the context of linear equations of motion.^{15–17} In these references the beam and wave packets solutions were derived as asymptotic solutions in the sense of geometrical optics; not as solutions to a parabolic equation but as solutions to the standard eiconal and transport equations of hyperbolic wave equations. The point of departure with “standard” geometrical optics is that the phase function is allowed to be complex valued, with the result that the curvature matrix normally associated with curved wave fronts may be complex, yielding Gaussian decay about the central ray. The evolution and formal description of the solutions with complex phase is identical in many respects to the theory for curved wave fronts. There are some important differences, of course; one of the most significant being that the complex phase solutions do not exhibit the usual breakdown of geometrical optics solutions at caustics. This feature, which will be discussed later, means we do not need to provide boundary layer corrections near caustics, and it also means that the only singularities that arise are those associated with purely nonlinear effects.

As mentioned in the beginning, the present approach is

quite distinct from that of many previous articles in this Journal dealing with the KZK equation, e.g., Refs. 1–3 and references therein. For instance, Du and Breazeale¹ discuss a series solution to the KZK equation in the form $\bar{p} = \bar{p}_1 + \bar{p}_2 + \dots$, where $\bar{p}_1$ is a uniformly smooth Gaussian solution to the linearized equation, and $\bar{p}_2$ a regular perturbation which describes nonlinear effects. Numerical algorithms for solving the KZK equation^{2,3} are based upon a similar perturbation scheme extended to higher orders. The weakly nonlinear theory, however, deals with a single Gaussian term $\bar{p} = \bar{p}_1$, where $\bar{p}_1$ is not uniformly smooth but can become discontinuous as the wave progresses. The asymptotics, or perturbation analysis, is performed on the phase of $\bar{p}_1$. Cates and Crighton⁴ provide an informative discussion on the connection between the KZK equation and geometrical techniques, as exemplified by Hunter and Keller.⁷ The other major differences are that no model equations are used here, everything follows from the primitive equations and a high-frequency ansatz, and finally, it will become clear that the present approach is above all else a simple way to look at a fairly complex phenomenon.

The motivation for the weakly nonlinear theory is discussed in the next section, where the scaling between the amplitude and frequency is outlined in detail. In Sec. II the weakly nonlinear theory is developed for solutions with real phase functions, similar to the theory of Hunter and Keller.⁷ Then in Sec. III the results of the Sec. II are generalized to allow for the possibility of complex phase solutions. The precise conditions required for initially smooth solutions to evolve into shocks are outlined and illustrated by several examples, including Gaussian beams. A relatively simple formula is derived that gives the time to shock for an initial wave that may be either a plane wave, a curved wave front, or a Gaussian beam, and any intermediate wave type.

I. BASIC EQUATIONS AND SCALING

The equations of motion for a homogeneous, isentropic, compressible, and inviscid fluid are

$$\rho_t + \nabla \cdot \rho \mathbf{v} = 0, \quad \rho \mathbf{v}_t + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = 0, \quad \rho = \rho(p), \quad (1)$$

where ρ , p , and $\mathbf{v}$ are the density, pressure, and fluid velocity, respectively. The independent variables are position $\mathbf{x}$ and time t , and the subscript t , or any other variable, indicates a partial derivative. The equation of state (1)₃ will be left unspecified, but we assume that the density and pressure are within the range for which the relation is uniquely invertible, i.e., $p = p(\rho)$. The base state in which the fluid is at rest under constant pressure p_0 is $\mathbf{v} = 0$, $\rho = \rho_0 = \rho(p_0)$. In order to motivate the subsequent analysis, suppose that the equilibrium state is disturbed in such a way that at $t = 0$,

$$\rho(\mathbf{x}, 0) = \rho_0 \{ 1 + \text{Re } A_0 \exp [i \omega (\mathbf{n} \cdot \mathbf{x} / c + \frac{1}{2} \mathbf{x} \cdot \mathbf{D} \cdot \mathbf{x})] \}, \\ \mathbf{v}(\mathbf{x}, 0) = c \mathbf{n} \text{Re } A_0 \exp [i \omega (\mathbf{n} \cdot \mathbf{x} / c + \frac{1}{2} \mathbf{x} \cdot \mathbf{D} \cdot \mathbf{x})], \quad (2)$$

where A_0 is a complex number of magnitude $|A_0| = \epsilon \ll 1$, $\mathbf{n}$ is a unit vector, and c is the sound speed of infinitesimal waves

$$c^{-2} = \left. \frac{dp}{d\rho} \right|_{p=p_0}. \quad (3)$$

The initial disturbance (2) represents a wave that would propagate in the $\mathbf{n}$ direction according to standard linear acoustics. The wave is modulated by the quadratic phase term $\mathbf{x} \cdot \mathbf{D} \cdot \mathbf{x}$, which if $\mathbf{D}$ were real and symmetric with $\mathbf{D} \cdot \mathbf{n} = 0$, would represent an initially curved wave front. We take $\mathbf{D}$ symmetric with no loss in generality, and furthermore, we permit $\mathbf{D}$ to be complex valued such that $\text{Im } \mathbf{D}$ remains positive definite. Then (2) represents a localized disturbance that decays as a Gaussian away from the center $\mathbf{x} = 0$. Within the realm of linear acoustics this set of initial conditions would generate a propagating Gaussian wave packet.¹⁶ Alternatively, if we loosen the restrictions upon $\mathbf{D}$ so that $\text{Im } \mathbf{x} \cdot \mathbf{D} \cdot \mathbf{x} > 0$ only for those $\mathbf{x}$ perpendicular to $\mathbf{n}$, then (2) is localized only in the direction perpendicular to $\mathbf{n}$ but is of infinite extent in the $\mathbf{n}$ direction. This may be viewed as a particular limiting case of a GWP, and is known as a Gaussian beam.^{14,15,17,18}

We are concerned with the evolution of initial wave profiles like (2) for times and distances of propagation such that nonlinear effects are significant. The units of time and length are T and L , defined as

$$T = 1/\epsilon \omega, \quad L = cT = (\epsilon k)^{-1}, \quad (4)$$

where $\epsilon \ll 1$ is defined by the initial density perturbation, i.e., for the example (2), $\epsilon = |A_0|$, and $k = \omega/c$ is the acoustic wave number. The weakly nonlinear wave theory is valid for times on the order of T and propagation distances on the order of L , which in turn is $(2\pi\epsilon)^{-1}$ times the central wavelength of the initial disturbance.

We close this section by defining dimensionless variables $\mathbf{x}' = \mathbf{x}/L$ and $t' = t/T$, so that (1) becomes

$$c\rho_{t'} + \nabla' \cdot (\rho \mathbf{v}) = 0, \\ c\rho \mathbf{v}_{t'} + \rho \mathbf{v} \cdot \nabla' \mathbf{v} + \nabla' p = 0. \quad (5)$$

The following ansatz is assumed:

$$\rho = \rho_0 [1 + \epsilon R^{(0)}(\mathbf{x}', t', \theta) + \epsilon^2 R^{(1)}(\mathbf{x}', t', \theta) + \dots], \\ \mathbf{v} = c [\epsilon \mathbf{V}^{(0)}(\mathbf{x}', t', \theta) + \epsilon^2 \mathbf{V}^{(1)}(\mathbf{x}', t', \theta) + \dots], \quad (6)$$

where θ is a “fast” phase function that is related to the phase function ϕ of order unity by

$$\theta = \epsilon^{-1} \phi(\mathbf{x}', t'). \quad (7)$$

The remainder of this paper concentrates on developing solutions in the form of (6) and (7), subject to general initial conditions discussed later. For the moment we note that by necessity $R^{(0)}$, $\mathbf{V}^{(0)}$, etc., must be real-valued functions; however, the phase θ may be complex depending upon the initial conditions. We will first consider the phase to be real valued, as this case corresponds closely to many previous treatments of weakly nonlinear waves, and defer until Sec. III the possibility of θ being complex valued.

II. REAL PHASE SOLUTIONS

The results of this section are essentially the same as those of Hunter and Keller,⁷ but are necessary in order to develop the general theory for complex-valued phase functions in the next section.

A. The eiconal equation

Substitution of (6) and (7) into the equations of motion (5) and setting $\epsilon = 0$ yields the first of a hierarchy of equations

$$\begin{aligned} R_{\theta}^{(0)} \phi_t + \mathbf{V}_{\theta}^{(0)} \cdot \nabla \phi &= 0, \\ R_{\theta}^{(0)} \nabla \theta + \mathbf{V}_{\theta}^{(0)} \phi_t &= 0. \end{aligned} \quad (8)$$

In (8) and subsequent equations the primes on t' and $\mathbf{x}'$ are omitted. We deduce from (8) the eiconal equation

$$\phi_t^2 - (\nabla \phi)^2 = 0, \quad (9)$$

which has two roots, corresponding to waves propagating in opposite directions. We focus our attention on one root,

$$\phi_t + (\nabla \phi \cdot \nabla \phi)^{1/2} = 0. \quad (10)$$

The PDE (10) for ϕ may be solved by considering the Hamilton-Jacobi equations, which possess parametric solutions along one-parameter curves in space-time, or rays. Let the initial value of ϕ be

$$\phi(\mathbf{x}, 0) = \Phi(\mathbf{x}), \quad (11)$$

and select a starting point $\mathbf{x}_0$. With no less generality, let

$$\mathbf{n} = \nabla \Phi(\mathbf{x}_0), \quad |\mathbf{n}| = 1, \quad (12)$$

then for $t > 0$ the central ray $(\bar{\mathbf{x}}(\tau), \tau)$ is

$$\bar{\mathbf{x}}(\tau) = \mathbf{x}_0 + \mathbf{n}\tau, \quad (13)$$

where τ is the ray-length parameter. Along the ray we have the usual solutions for a homogeneous medium,

$$\begin{aligned} \phi(\bar{\mathbf{x}}(\tau), \tau) &= \phi_0 = \Phi(\mathbf{x}_0), \\ \phi_t(\bar{\mathbf{x}}(\tau), \tau) &= -1, \\ \nabla \phi(\bar{\mathbf{x}}(\tau), \tau) &= \mathbf{n}. \end{aligned} \quad (14)$$

Note that τ is identical to the elapsed time t along the ray.

Away from the central ray we expand ϕ in a Taylor series,

$$\begin{aligned} \phi(\bar{\mathbf{x}}(\tau) + \Delta \mathbf{x}, \tau + \Delta \tau) \\ = \phi_0 + \nabla \phi \cdot \Delta \mathbf{x} + \phi_t \Delta \tau + \frac{1}{2} \phi_{tt} (\Delta \tau)^2 + \nabla \phi_t \cdot \Delta \mathbf{x} \Delta \tau \\ + \frac{1}{2} \Delta \mathbf{x} \cdot \nabla \nabla \phi \cdot \Delta \mathbf{x} + \dots \end{aligned} \quad (15)$$

Define the second-order symmetric tensor $\mathbf{M}$ by

$$\mathbf{M}(\tau) = \nabla \nabla \phi(\bar{\mathbf{x}}(\tau), \tau), \quad (16)$$

then using (14) and the following relations that result from differentiating (10):

$$\nabla \phi_t(\bar{\mathbf{x}}(\tau), \tau) = -\mathbf{M} \cdot \mathbf{n}, \quad \phi_{tt}(\bar{\mathbf{x}}(\tau), \tau) = \mathbf{n} \cdot \mathbf{M} \cdot \mathbf{n}, \quad (17)$$

Eq. (15) simplifies to

$$\begin{aligned} \phi = \phi_0 + (\mathbf{n} \cdot \Delta \mathbf{x} - \Delta \tau) \\ + \frac{1}{2} (\Delta \mathbf{x} - \mathbf{n} \Delta \tau) \cdot \mathbf{M}(\tau) \cdot (\Delta \mathbf{x} - \mathbf{n} \Delta \tau) + \dots \end{aligned} \quad (18)$$

If we retain only the terms shown in (18), i.e., up to and including quadratic, the resulting approximation to the phase may be called the paraxial approximation.

The evolution equation for $\mathbf{M}(\tau)$ can be derived from the eiconal equation (10) as¹⁶

$$\frac{d(\mathbf{M}^{-1})}{d\tau} = \mathbf{I} - \mathbf{n} \times \mathbf{n} = \mathbf{P}_{\perp}, \quad (19)$$

with solution

$$\mathbf{M}(\tau) = \mathbf{M}(0) [\mathbf{I} + \tau \mathbf{P}_{\perp} \mathbf{M}(0)]^{-1}. \quad (20)$$

where $\mathbf{I}$ is the identity, and $\mathbf{P}_{\perp}$ is a tensor that projects vectors onto the plane perpendicular to the ray direction $\mathbf{n}$.

B. The transport equation

The next in the sequence of asymptotic equations is obtained by substituting (6) and (7) into (5) and then setting to zero $d/d\epsilon$ of the resulting equations at $\epsilon = 0$. This gives

$$\begin{aligned} R_{\theta}^{(1)} \phi_t + \mathbf{V}_{\theta}^{(1)} \cdot \nabla \phi + R_t^{(0)} + \nabla \cdot \mathbf{V}^{(0)} \\ + (R^{(0)} \mathbf{V}^{(0)})_{\theta} \cdot \nabla \phi = 0, \\ R_{\theta}^{(1)} \nabla \phi + \mathbf{V}_{\theta}^{(1)} \phi_t + \mathbf{V}_t^{(0)} + \nabla R^{(0)} + R^{(0)} \mathbf{V}_{\theta} \phi_t \\ + (\mathbf{V}^{(0)} \cdot \nabla \phi) \mathbf{V}_{\theta}^{(0)} + \frac{B}{A} R^{(0)} R_{\theta}^{(0)} \nabla \phi = 0, \end{aligned} \quad (21)$$

where A and B are the usual parameters encountered in nonlinear acoustics¹⁹

$$A = \rho_0 c^2, \quad B = \rho_0^2 \left. \frac{d^2 p}{d\rho^2} \right|_{p=\rho_0}. \quad (22)$$

The unknown quantities $R^{(1)}$ and $\mathbf{V}^{(1)}$ can be eliminated from (21) by using the eiconal equation, to yield

$$\begin{aligned} \nabla \phi \cdot [\mathbf{V}_t^{(0)} + \nabla R^{(0)} + R^{(0)} \mathbf{V}_{\theta}^{(0)} \phi_t + (\nabla \phi \cdot \mathbf{V}^{(0)}) \mathbf{V}_{\theta}^{(0)} \\ + (B/2A) (R^{(0)})^2 \nabla \phi] - \phi_t [R_t^{(0)} + \nabla \cdot \mathbf{V}^{(0)} \\ + (R^{(0)} \mathbf{V}^{(0)})_{\theta} \cdot \nabla \phi] = 0. \end{aligned} \quad (23)$$

The total derivative along the central ray $(\bar{\mathbf{x}}(\tau), \tau)$ is

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \mathbf{n} \cdot \nabla. \quad (24)$$

The velocity follows from (8) as

$$\mathbf{V}^{(0)} = R^{(0)} \mathbf{n}, \quad (25)$$

and its divergence is, from (12) and (16),

$$\nabla \cdot \mathbf{V}^{(0)} = \mathbf{n} \cdot \nabla R^{(0)} + R^{(0)} \text{tr}(\mathbf{P}_{\perp} \mathbf{M}). \quad (26)$$

Therefore, when evaluated along the ray the transport equation (23) becomes the PDE

$$2 \frac{\partial R^{(0)}}{\partial \tau} + R^{(0)} \text{tr}(\mathbf{P}_{\perp} \mathbf{M}) + \beta (R^{(0)})_{\theta} = 0, \quad (27)$$

where β is the nonlinearity parameter¹⁹

$$\beta = 1 + B/2A. \quad (28)$$

For example, $\beta = (1 + \gamma)/2$ for an adiabatic perfect gas, $p = p_0(\rho/\rho_0)^{\gamma}$.

C. The general solution to the nonlinear transport equation

The transport equation (27) may be rewritten as an ordinary differential equation in τ

$$2 \frac{dR^{(0)}}{d\tau} + R^{(0)} \text{tr}(\mathbf{P}_{\perp} \mathbf{M}(\tau)) = 0, \quad (29)$$

along the characteristic

$$\frac{d\theta}{d\tau} = \beta R^{(0)}. \quad (30)$$

Equation (29) is the usual transport equation of linear geometrical acoustics, modified in the present circumstances by

the relation (30), which implies that the phase θ varies as the ray position moves. The general solution to (29) follows from (20),

$$R^{(0)} = g(\tau)b(\xi), \quad (31)$$

where $g(\tau)$ is the geometrical spreading factor for a homogeneous medium,

$$g(\tau) = \{\det[\mathbf{I} + \tau\mathbf{P}_1\mathbf{M}(0)]\}^{1/2}, \quad (32)$$

$b(\cdot)$ is a function defined by the initial amplitude of the density perturbation, discussed below, and ξ is a phase variable related to θ . To be more specific, we choose the initial value such that $\xi = \theta$ at $\tau = 0$, then Eqs. (30) and (31) imply the following nonlinear equation for ξ as a function of θ for $\tau > 0$:

$$\xi + \beta b(\xi)G(\tau) = \theta, \quad (33)$$

where $G(\tau)$ is the integral of $g(\tau)$, with initial condition $G(0) = 0$.

The function $g(\tau)$ represents the growth or decay in amplitude of a curved wave front and is in agreement with the amplitude obtained by considering the conservation of energy flux in a ray tube bundle. By assumption, the phase ϕ is real valued and therefore the matrix $\mathbf{M}(0)$ is also. The

rank 2 matrix $\mathbf{P}_1\mathbf{M}(0)\mathbf{P}_1$ represents the initial wave-front curvature. Hence, unless the initial wave front is flat, corresponding to a plane wave and $\mathbf{P}_1\mathbf{M}(0)\mathbf{P}_1 = 0$, there will be at least one value of τ , $-\infty < \tau < \infty$, for which $g(\tau)$ of Eq. (32) becomes singular. These are the times at which the wave front focuses down to a point or a line, and the geometrical singularity in the ray amplitude signifies the breakdown of the original ray ansatz, which must be replaced by a local boundary layer analysis. This can be done, but at the cost of significant effort and a loss of simplicity.¹³ We will see in the next section that this problem does not arise for Gaussian beam solutions, which is one of the major factors in their favor.¹⁴⁻¹⁷

The integral of $g(\tau)$ may be easily evaluated to find $G(\tau)$ as follows. The initial curvature matrix, although it need not have $\mathbf{n}$ as a principal direction, defines two real numbers α_1 and α_2 such that $\mathbf{P}_1\mathbf{M}(0)\mathbf{P}_1 = \alpha_1\mathbf{m}^{(1)} \times \mathbf{m}^{(1)} + \alpha_2\mathbf{m}^{(2)} \times \mathbf{m}^{(2)}$, where $(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{n})$ form an orthonormal triad of vectors, and so from (32)

$$g(\tau) = [(1 + \alpha_1\tau)(1 + \alpha_2\tau)]^{-1/2}. \quad (34)$$

This integrates to yield

$$\begin{aligned} \tau, \quad \alpha_1 = \alpha_2 = 0, \\ (2/\alpha_1)[1 + \alpha_1\tau]^{1/2} - 1, \quad \alpha_1 \neq 0, \alpha_2 = 0, \\ G(\tau) = \begin{cases} (1/\alpha_1)\log(1 + \alpha_1\tau), & \alpha_1 = \alpha_2 \neq 0 \\ \frac{1}{(\alpha_1\alpha_2)^{1/2}} \left| \operatorname{ch}^{-1} \left[\frac{\alpha_1 + \alpha_2 + 2\alpha_1\alpha_2\tau}{\alpha_1 - \alpha_2} \right] - \operatorname{ch}^{-1} \left[\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} \right] \right|, & \alpha_1 \neq \alpha_2, \alpha_1\alpha_2 > 0, \\ \frac{1}{|\alpha_1\alpha_2|^{1/2}} \left| \cos^{-1} \left[\frac{\alpha_1 + \alpha_2 + 2\alpha_1\alpha_2\tau}{\alpha_1 - \alpha_2} \right] - \cos^{-1} \left[\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} \right] \right|, & \alpha_1 \neq \alpha_2, \alpha_1\alpha_2 < 0 \end{cases} \end{aligned} \quad (35)$$

Each case describes a different type of wave front: $\alpha_1 = \alpha_2 = 0$ corresponds to a plane wave front; $\alpha_1 \neq \alpha_2 = 0$ is a cylindrical wave front; $\alpha_1 = \alpha_2 \neq 0$ is a spherical wave front, and when $\alpha_1 \neq \alpha_2$ with $\alpha_1\alpha_2 \neq 0$, the principle wave-front curvatures are distinct and nonzero. The forms given in (35) for $G(\tau)$ are valid only until the earliest times $\tau_F > 0$ at which $g(\tau)$ is singular. The time to focusing is finite only if either of α_1 or α_2 is negative in which case $\tau_F = -1/\min(\alpha_1, \alpha_2)$, otherwise focusing does not occur and (34) and (35) are valid for $0 \leq \tau < \infty$. We note that $G(\tau)$ is monotonically increasing for $0 < \tau < \tau_F$. The solution for times after focusing may be considered within the more general theory of Sec. III, which permits the phase to be complex valued.

Finally we return to the amplitude function in the general solution (31). By assumption, the initial conditions depend upon both the phase $\Phi(\mathbf{x})$ and the amplitude function, such that the initial density perturbation is of the general form

$$\rho(\mathbf{x}, 0) - \rho_0 = \epsilon \rho_0 b \left(\frac{\Phi(\mathbf{x})}{\epsilon} \right), \quad (36a)$$

with corresponding velocity

$$\mathbf{v}(\mathbf{x}, 0) = \epsilon c b \left(\frac{\Phi(\mathbf{x})}{\epsilon} \right) \nabla \Phi(\mathbf{x}). \quad (36b)$$

The functions Φ and b are independent in general: the former determines the ray direction and the initial values for the elements of the wave-front curvature matrix $\mathbf{M}(0)$. The amplitude function b is normally assumed to be sinusoidal, corresponding to time harmonic waves, but it may take on any form in the present theory. Some examples of different functional forms for b are discussed in subsection E where it is shown that the amplitude function strongly influences the development of shocks.

D. Summary of the theory for real phase

The procedure for applying the present theory is thus: for a given starting point at $\mathbf{x} = \mathbf{x}_0$, find the central ray $\bar{\mathbf{x}}(\tau)$ of (12) and (13) and the initial curvature tensor $\mathbf{M}(0) = \nabla \nabla \Phi(\mathbf{x}_0)$, from which the geometrical functions $g(\tau)$ and $G(\tau)$ are determined by (32) and (35). Then for each $0 < \tau < \tau_F$, (33) is solved for ξ , with initial phase $\theta = \theta_0 = \epsilon^{-1} \Phi(\mathbf{x}_0)$. The quantities $R^{(0)}$ and $\mathbf{V}^{(0)}$ associated with density and velocity are given by (31) and (25), where $g(\tau)$ defines the refraction of the wave front and $b(\xi)$

is the amplitude function, defined by the initial data through Eqs. (36). It is interesting to note that the particle acceleration is of order unity,

$$\rho \mathbf{v}_t = a(\tau) \rho_0 c \mathbf{n} + O(\epsilon), \quad (37)$$

where

$$a = -R_{\theta}^{(0)} = \frac{-g(\tau)b'(\xi)}{1 + \beta G(\tau)b'(\xi)}, \quad (38)$$

which follows from (31) and (33).

E. Development of discontinuities and examples

Although the present theory is premised on the assumption that the solutions remain smooth, it may under certain circumstances predict the onset of a discontinuous solution. This is a purely nonlinear effect in contrast to the possibility of geometrical singularities associated with focusing in the linear theory, discussed above. The latter is an artifact of the asymptotic ansatz and may be corrected for, but the emergence of discontinuities indicates a significant change in the physical nature of the solution. The subsequent evolution of discontinuities can be analyzed by the theory of weak shocks.^{7,10,12} A discontinuity or weak shock will arise when (33) admits multivalued solutions. Therefore, the requirement for a shock is that there is a simultaneous solution to (33) and the condition obtained by equating its derivative with respect to ξ to zero, viz.

$$\beta G(\tau)b'(\xi) + 1 = 0. \quad (39)$$

For a given $\theta = \theta_0$, let $\tau = \tau_0$ be the earliest time for which (33) and (39) possess a simultaneous solution. Because the left member of (39) is the denominator in (38), it is clear that the acceleration becomes unbounded as $\tau \rightarrow \tau_0$. More precisely, eliminating $\beta'(\xi)$ between (38) and (39), and assuming $g(\tau)$ is sufficiently smooth as τ approaches τ_0 , implies

$$a \sim 1/\beta(\tau_0 - \tau), \quad \tau \rightarrow \tau_0. \quad (40)$$

The first shock appears for that value ξ_0 which both satisfies (33) and (39) and maximizes $|b'(\xi_0)|$, such that $\text{sgn } b'(\xi_0) = -\text{sgn } \beta G(\tau_0)$. This follows from the monotonic property of $G(\tau)$, and since β is usually positive,¹⁹ the two conditions $b'(\xi_0) < 0$ and $b''(\xi_0) = 0$ must be simultaneously satisfied. If multiple values of ξ exist for which $b'(\xi_0) < 0$, the one selected is that which makes $b'(\xi)$ an absolute minimum. The associated time then follows from (39) and the initial phase from (33). For example, if $b(\xi) = \cos \xi$ and $\beta > 0$, the critical value of ξ is $\xi_0 = \pi/2$ corresponding to the point on the initial velocity profile of greatest slope, since $\theta = \xi_0$ for this initial profile. Also, the time to shock is τ_0 , where $G(\tau_0) = \beta^{-1}$. Alternatively, if $b(\xi) = \exp(-\xi^2/2)$, then $\xi_0 = 1$, $\theta = 2$, and the time to shock is given by the solution to $G(\tau_0) = \beta^{-1}e^{1/2}$, which is $e^{1/2}$ times larger than the previous case, with corresponding larger τ_0 . Finally, suppose $b(\xi) = -\xi h(-\xi)$, where $h(x)$ is zero for $x < 0$, $h(x) = 1$ for $x > 1$, and $h(x)$ is smoothly varying for x between 0 and 1. It follows from (38) that the initial acceleration at $\tau = 0+$ is in the form of a step func-

tion of width ϵL in dimensional coordinates. This is a smoothed version of an acceleration wave⁹ which is by definition a traveling discontinuity. The solution to (33) for this particular amplitude function is simply

$$\xi = \begin{cases} \theta, & \theta > 0, \\ \frac{\theta}{1 - \beta G(\tau)}, & \theta < -1, \end{cases} \quad (41)$$

and is smooth for $-1 \leq \theta \leq 0$. The time to shock is again τ_0 where $G(\tau_0) = \beta^{-1}$, but in this case all values of $\theta < -1$ develop a shock simultaneously at $\tau = \tau_0$.

The results for these examples of different initial amplitude functions are valid for any initial wave-front curvature $M(0)$. Suppose, for instance, that the initial curvature is spherical, i.e., $M(0) = \alpha P_1$, α real. Then the time at which the sinusoidal wave $b(\xi) = \cos \xi$ develops a discontinuity is τ_0 such that $G(\tau_0) = \beta^{-1}$ (assuming again that $\beta > 0$), or from (35), $\tau_0 = \alpha^{-1}[e^{(\alpha/\beta)} - 1]$. This value of τ_0 is positive and finite independent of whether the wave diverges ($\alpha > 0$) or converges ($\alpha < 0$), and for a given magnitude of α , the time τ_0 is shortest for the converging wave. Comparison of Eqs. (2), (14), and (36) shows that $|\alpha| = \epsilon L^2/a^2$, where a is the initial wave-front radius of curvature. Taking the present example of a sinusoidal wave with spherical curvature as a canonical case, it may be stated that the role played by nonlinear effects versus geometrical effects is governed by the magnitude of $|\alpha|/\beta = (\epsilon \beta k^2 a^2)^{-1}$, where $k = \omega/c$ is the wave number. Thus, if $|\alpha|/\beta \gg 1$, then τ_0 is either very large, if $\alpha > 0$, or it is approximately $\tau_0 \approx -1/\alpha$ if $\alpha < 0$. In the former case the discontinuity does not develop until the wave has diverged significantly, and in the latter case it develops only just before geometrical focusing occurs. Conversely, if $|\alpha|/\beta \ll 1$, then $\tau_0 \approx \beta^{-1}$ independent of α , which means that the discontinuity develops before any geometrical spreading or focusing occurs.

The parameters used here can also be related to the nonlinearity parameter N of Bakhvalov *et al.*,⁴ defined as a $N = L_d/L_p$, where L_d is the diffraction length and L_p the plane wave discontinuity length. The former is $L_d = ka^2$, where a is a typical radius of curvature for the initial data. Bakhvalov *et al.*⁴ were specifically concerned with initial data of the form $\exp(-r^2/2a^2)$, which generates a Gaussian beam and is really outside the purview of real phase solutions, but will be considered in the next Section. If the initial dimensionless complex curvature matrix is $M(0) = \alpha(\mathbf{I} - \mathbf{n} \times \mathbf{n})$, where α is real, then the initial data corresponds to a spherically curved wave front with radius of curvature a , where $a^2 = \epsilon L^2/|\alpha|$ and L is the basic length defined in (4). The distance to shock for a plane wave is $L_p = (\beta k \epsilon)^{-1} = L/\beta$, where β is the nonlinearity parameter of order unity, defined in (28). The number N is therefore $N = \beta/|\alpha| = \epsilon \beta k^2 a^2$. Since β is typically of order unity, and by assumption, α is also, we conclude that the present analysis is relevant to initial data for which $N = O(1)$. Another way of looking at it is that the number of wavelengths in the distance to shock is of order $kL_p = (\epsilon \beta)^{-1}$. Alternatively, let δ be the inverse of the number of wavelengths in L_p , then the present scaling corresponds to $\delta \sim \epsilon$.

The earlier observations about the time to shock may be summarized as follows. If

$$N \ll 1, \text{ geometrical effects dominate shock evolution,} \quad (42)$$

and if

$$N \gg 1, \text{ nonlinear effects dominate,} \quad (43)$$

and if N is of order unity, then the two effects are intimately coupled.

III. COMPLEX PHASE SOLUTIONS

The formalism for real phase solutions is now generalized to allow the possibility that the phase is complex valued. This permits us to include Gaussian beam and wave packet solutions among the general class of weakly nonlinear high-frequency waves, and to discuss the development of shocks for these wave types.

A. Evolution equation for the complex phase

The types of solution discussed in Sec. II do not include solutions corresponding to initial conditions in the form of a localized Gaussian beam or Gaussian wave packet, i.e., initial data of the form (2) with $\text{Im } \mathbf{D}$ positive semi-definite or definite, respectively. We therefore generalize the ansatz (6) to

$$\rho = \rho_0 [1 + \epsilon \text{Re } R^{(0)}(\mathbf{x}, t, \theta) + \epsilon^2 \text{Re } R^{(1)}(\mathbf{x}, t, \theta) + \dots], \quad (44)$$

$$\mathbf{v} = c[\epsilon \text{Re } \mathbf{V}^{(0)}(\mathbf{x}, t, \theta) + \epsilon^2 \text{Re } \mathbf{V}^{(1)}(\mathbf{x}, t, \theta) + \dots],$$

where θ is complex and $R^{(0)}, R^{(1)}, \dots, \mathbf{V}^{(0)}, \mathbf{V}^{(1)}, \dots$, are complex-valued functions. The complex-valued fast phase θ is related to ϕ by (7), where again ϵ is real and small.

Substituting (44) into the equations of motion (5) again produces an asymptotic hierarchy of equations in ϵ . We consider ϕ and its complex conjugate $\bar{\phi}$ as independent variables, implying that the first set of equations reduces to (9), which gives the eiconal equation, and we again concentrate upon the root (10). The ray equations (12) and (13) follow once an initial phase $\Phi(\mathbf{x})$ is given, although Φ must be such that the ray direction, $\nabla\Phi(\mathbf{x})$, at $\mathbf{x} = \mathbf{x}_0$ is real. This is always true for Gaussian beams and Gaussian wave packets if $\mathbf{x}_0$ is the central point. One could develop the present theory to include rays in complex space, as has been done over the past 20 years by numerous authors for the linear theory, but it will be assumed for simplicity that the ray is real in this paper.

The phase ϕ can be locally expanded about the central ray by defining the complex valued tensor, $\mathbf{M}(\tau)$, through (16). The evolution of $\mathbf{M}(\tau)$ is again given by (20) and some general properties of $\mathbf{M}(\tau)$ are proved in the Appendix. In particular, it is shown that the imaginary part of $\mathbf{M}$ remains positive definite for all $\tau > 0$ if it is initially positive definite. A Gaussian wave packet corresponds to the imaginary part of $\mathbf{M}$ being positive definite. On the other hand, in a Gaussian beam only that part of $\text{Im } \mathbf{M}$ associated with lateral dependence is positive definite, i.e., $\text{Im } \mathbf{M}$ may only be positive semidefinite of rank 2. In this case the results of the Appendix also imply that a beam with initial conditions

of this type maintains these conditions. Thus, Gaussian wave packets and beams remains localized about the moving central point $\bar{\mathbf{x}}(\tau)$.

The other important conclusion to be drawn from the Appendix is that neither Gaussian wave packets nor Gaussian beams can develop the type of geometrical singularities common to real phase solutions, i.e., singularities associated with caustics and foci. This is a well-known result for Gaussian beams and may be explained in terms of real and complex rays. Real phase solutions propagate as a bundle of adjacent rays in real space, caustics and foci occurring when congruent rays coalesce at a point. Gaussian beams and wave packets, on the other hand, have only a single real ray associated with them, which is the central ray. The adjacent rays define curves in complex space and are accordingly called complex rays. They remain in complex space, and whether or not they form caustics in complex space, they cannot produce geometrical singularities in real space, the space in which the physical quantities are defined.

Transport equations similar to, but not quite the same as, (21) follow from the next order in ϵ by invoking the linear independence of ϕ and $\bar{\phi}$

$$\begin{aligned} R_\theta^{(1)} \phi_t + \mathbf{V}_\theta^{(1)} \cdot \nabla \phi + R_t^{(0)} + \nabla \cdot \mathbf{V}^{(0)} + (\mathbf{V}_\theta^{(0)} \cdot \nabla \phi) \\ \times \text{Re } R^{(0)} + \nabla \phi \cdot \mathbf{R}_\theta^{(0)} \text{Re } \mathbf{V}^{(0)} = 0, \\ R_\theta^{(1)} \nabla \phi + \mathbf{V}_\theta^{(1)} \phi_t + \mathbf{V}_t^{(0)} + \nabla R^{(0)} + (\text{Re } R^{(0)}) \mathbf{V}_\theta^{(0)} \phi_t \\ + \mathbf{V}_\theta^{(0)} \nabla \phi \cdot \text{Re } \mathbf{V}^{(0)} + B/A (\text{Re } R^{(0)}) R_\theta^{(0)} \nabla \phi = 0. \end{aligned} \quad (45)$$

These may be combined to form a single equation similar to (23)

$$\begin{aligned} \nabla \phi \cdot [\mathbf{V}_t^{(0)} + \nabla R^{(0)} + (\text{Re } R^{(0)}) \mathbf{V}_\theta^{(0)} \phi_t \\ + (\text{Re } \mathbf{V}^{(0)}) \cdot \nabla \phi \mathbf{V}_\theta^{(0)} + (B/A) (\text{Re } R^{(0)}) R_\theta^{(0)} \nabla \phi] \\ - \phi_t [R_t^{(0)} + \nabla \cdot \mathbf{V}^{(0)} + (\text{Re } R^{(0)}) \nabla \phi \cdot \mathbf{V}_\theta^{(0)} \\ + (\text{Re } \mathbf{V}^{(0)}) \cdot \nabla \phi R_\theta^{(0)}] = 0, \end{aligned} \quad (46)$$

and this simplifies along the ray to become, analogous to (37),

$$2 \frac{\partial R^{(0)}}{\partial \tau} + R^{(0)} \text{tr}(\mathbf{P}_\perp \mathbf{M}) + 2\beta (\text{Re } R^{(0)}) R_\theta^{(0)} = 0. \quad (47)$$

This PDE may in turn be written in the same form as the ODE (29) along the characteristic

$$\frac{d\theta}{d\tau} = \beta \text{Re } R^{(0)}. \quad (48)$$

The solution to the transport equation is

$$R^{(0)} = g(\tau) b(\xi), \quad (49)$$

where $b(\theta)$ is the complex valued initial amplitude function. Integration of (48) implies

$$\xi + \beta \text{Re}[b(\xi)G(\tau)] = \theta, \quad (50)$$

where $G(\tau)$ is the integral of $g(\tau)$ such that $G(0) = 0$. Note the similarity of the phase equations (50) and (33). In particular, (50) implies that the change in phase is *real*, independent of the initial pulse profile.

The condition for the development of weak shocks is

that $d\theta/d\xi = 0$, i.e., in addition to (50), the following must be satisfied

$$1 + \beta \operatorname{Re}[b'(\xi)G(\tau)] = 0. \quad (51)$$

B. Example: Gaussian wave packets

The initial amplitude function for a Gaussian beam or wave packet is $b(\xi) = e^{i\xi}$, and thus both of the real valued equations (50) and (51) can be combined into a single complex-valued condition

$$\beta G(\tau)e^{i\xi} = \theta - \xi + i, \quad (52)$$

with the constraint that $\xi - \theta$ is real. Now, $G(0) = 0$, and as τ increases the magnitude of $G(\tau)$ will increase. At the same time, the positive definite requirements on $M(\tau)$ imply $|e^{i\xi}| \leq 1$, and therefore, a shock cannot occur until τ_0 such that

$$|\beta G(\tau_0)| = 1. \quad (53)$$

At the time that the shock develops (52) and (53) imply the dual relations

$$\xi = \theta, \quad \beta G(\tau_0)e^{i\theta} = i. \quad (54)$$

Further, since (53) holds, the identities (54) are satisfied only if the initial phase θ is real and

$$\theta = (\pi/2)\operatorname{sgn} \beta - \arg G(\tau_0). \quad (55)$$

In summary, the part of a GWP or a Gaussian beam that first develops a shock is initially at the position $\mathbf{x}$, which has θ given by (55) where τ_0 satisfies the shock condition (53).

To be more specific, assume the wave is propagating in the x_3 direction with

$$\mathbf{M}(0) = \operatorname{diag}[\alpha, \alpha, \gamma]. \quad (56)$$

This represents an axially symmetric initial disturbance, and it is assumed that $\operatorname{Im} \alpha \geq 0$, $\operatorname{Im} \gamma \geq 0$. A Gaussian beam corresponds to $\gamma = 0$, and the beam or wave packet is tapered in the x_1 and x_2 directions only if $\operatorname{Im} \alpha > 0$. A non-Gaussian wave-front results if α is real and the wave front is initially convergent (divergent) if $\alpha < (\alpha > 0)$. A plane wave front is given by $\alpha = 0$. The function $G(\tau)$ is

$$G(\tau) = (1/\alpha)\log(1 + \alpha\tau), \quad (57)$$

and the earliest blowup time is τ_0 of (53), which is the earliest root of

$$|\log(1 + \alpha\tau_0)| = |\alpha/\beta|. \quad (58)$$

Equation (58) is a very general result for spherically symmetric weakly nonlinear high-frequency waves of complex phase, and includes, of course, real phase solutions with curved wave fronts, and also includes the case of a plane wave as the special limit of $\alpha \rightarrow 0$. The equation may be transformed to a more canonical form by eliminating the explicit dependence of the nonlinearity parameter β through the introduction of characteristic parameters,

$$\hat{\tau}_0 = |\beta|\tau_0, \quad \hat{\alpha} = \alpha/|\beta|. \quad (59)$$

Both parameters are still dimensionless and $\hat{\alpha}$ may be complex valued such that its imaginary part is positive semidefinite. Equation (58) becomes

$$|\log(1 + \hat{\alpha}\hat{\tau}_0)| = |\hat{\alpha}|. \quad (60)$$

The plane wave solution, for $\alpha = 0$, is $\hat{\tau}_0 = 1$ or $\tau_0 = 1/|\beta|$, and for α real (60) admits the explicit solution seen in Sec. III

$$\hat{\tau}_0 = (1/\hat{\alpha})(e^{\hat{\alpha}} - 1). \quad (61)$$

Thus, as might be expected, the dimensionless blowup time satisfies

$$0 < \hat{\tau}_0 < 1, \quad \text{for a converging wave front} \\ (\hat{\alpha} \text{ real, negative}), \quad (62)$$

$$\hat{\tau}_0 > 1, \quad \text{for a diverging wave front} \\ (\hat{\alpha} \text{ real, positive}).$$

Figure 1 compares the minimum blowup time for spherically converging and diverging waves and for a Gaussian beam. For a given $|\alpha|$ the converging wave always blows up first, the diverging wave last, and the Gaussian beam is intermediate between a plane wave ($\alpha = 0$) and a divergent spherical wave. Note that for small α , the Gaussian beam solution is like the plane wave, i.e., $\hat{\tau}_0 \approx 1$. Generally, α may be complex valued such that $\operatorname{Im} \alpha \geq 0$. If we write $\alpha = |\alpha|e^{i\psi}$, $0 \leq \psi \leq \pi$, then it is a simple matter to show

$$\hat{\tau}_0 = 1 + (|\hat{\alpha}|/2)\cos \psi + O(|\hat{\alpha}|^2), \quad |\hat{\alpha}| \ll 1, \quad (63)$$

and thus spherically converging ($\psi = \pi$) and spherically diverging ($\psi = 0$) waves bound the initial slopes of curves like those of Fig. 1 for arbitrary complex α .

The relationships of $\hat{\alpha}$ and $\hat{\tau}_0$ to the actual physical quantities are

$$\hat{\tau}_0 = \epsilon|\beta|kct_0, \quad (64)$$

where t_0 is the actual time to blowup and $k = \omega/c$. Also,

$$\hat{\alpha} = \pm (\epsilon|\beta|k^2a^2)^{-1}, \quad (65)$$

for a spherically convergent (−) or divergent (+) wave, where a is the initial radius of curvature. Finally, for a Gaussian beam or wave packet, if the initial lateral decay is of the form $\exp(-\frac{1}{2}r^2/a^2)$, then

$$\hat{\alpha} = i(\epsilon|\beta|k^2a^2)^{-1}. \quad (66)$$

IV. CONCLUSIONS

The weakly nonlinear theory of high-frequency waves was originally developed with the explicit assumption that the phase function is real valued.^{6,7} This is by no means a severe restriction, since it includes the possibility of consid-

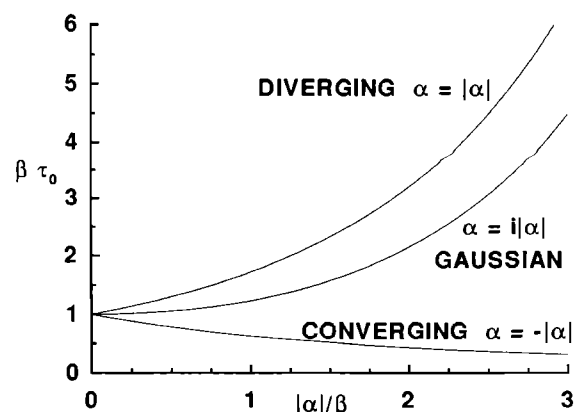


FIG. 1. The time to blowup for three types of spherical wave fronts.

ering both plane waves and waves with arbitrarily curved wave fronts. In this paper we have seen that the same theory can be quite easily adapted to include the possibility of solutions with complex-valued phase functions. This enables us to extend many of the results of weakly nonlinear geometrical optics to include Gaussian beams and wave packets, in the same way that consideration of complex phase in the linear theory leads to beams and wave packets.

The formal application of the theory is almost identical to the treatment of classical high-frequency waves, but some of the final results are not at all obvious, to the author at least. For instance, the generalization of the phase modulation equation (33) to its complex counterpart, Eq. (50), is remarkably simple. This is one of the major results of the paper, since it provides the basis for discussing the conditions for development of weak shocks, which are contained in Eqs. (53) and (55). These are relatively simple conditions and may be applied to initial data which are either a plane wave, a curved wave front, a purely Gaussian beam, or a combination of a Gaussian beam with initial wave-front curvature. For instance, the shock condition for an axially symmetric disturbance is given in Eq. (58). The simplicity of the final results, such as (58), should be stressed as one of the foremost advantages of the weakly nonlinear theory. It is capable of providing insight into difficult nonlinear phenomena through the use of geometrical techniques.

Some remarks were made in the Introduction about the distinctions between the present approach and that of model equations, such as the KZK equation. A few further comments are in order. In their paper Cates and Crighton⁵ obtained similarity solutions of the KZK equation and generalizations of it. These solutions exhibit, among other things, the phenomenon of ray curvature due to transverse amplitude variation, and Cates and Crighton note, correctly, that this type of ray path curvature cannot be obtained within the framework of the geometrical theory of Hunter and Keller,⁷ which explicitly assumes the amplitude is slowly varying along the wave front. Cates and Crighton summarize their view of the limits of geometrical methods with the comment that "there seems to be no obvious way of extending the geometrical optics scheme to cover cases when amplitude variation along wave fronts contributes to signal modulation..." The present results, however, appear to offer a simple means of including transverse amplitude variation that is sufficient to provide modulation. Gaussian beams and wave packets vary on length scales of the order of $\epsilon^{1/2}$ in the transverse direction, in agreement with the lateral dependence that Cates and Crighton noted would be necessary for wave-front amplitude variations to influence modulation over long distances. It must be pointed out that the present theory does not provide a means for the rays to bend, since the solutions are paraxial in nature. Any amplitude variation in the transverse direction is Gaussian in form and must therefore be symmetric.

APPENDIX: PROPERTIES OF M FOR GAUSSIAN BEAM SOLUTIONS

The two main results are derived here, both related to the evolution of Gaussian wave packets. The decay of the

initial Gaussian away from its center depends upon $\text{Im } \mathbf{M}(0)$, which is assumed to be positive definite. Actually, in order to include Gaussian beams we need not be this restrictive about the initial data, but just require that the rank 2 matrix $\mathbf{P}_1 \mathbf{M}(0) \mathbf{P}_1$ have positive definite imaginary part. This is equivalent to the requirement that

$$\mathbf{x} \cdot \text{Im } \mathbf{M}(0) \cdot \mathbf{x} > 0, \quad \text{with } \mathbf{x} \cdot \text{Im } \mathbf{M}(0) \cdot \mathbf{x} > 0, \quad \text{for all } \mathbf{x} \neq 0$$

$$\text{such that } \mathbf{x} \cdot \mathbf{n} = 0. \quad (\text{A1})$$

We will now show that the following are direct consequences of (A1): For any value of the time parameter τ ,

$$\mathbf{x} \cdot \text{Im } \mathbf{M}(\tau) \cdot \mathbf{x} > 0, \quad \text{with } \mathbf{x} \cdot \text{Im } \mathbf{M}(\tau) \cdot \mathbf{x} > 0, \quad \text{for all } \mathbf{x} \neq 0$$

$$\text{such that } \mathbf{x} \cdot \mathbf{n} = 0, \quad (\text{A2})$$

$$g(\tau) \text{ is finite.} \quad (\text{A3})$$

We will derive (A3) first. Let

$$\mathbf{A}(\tau) = \mathbf{I} + \tau \mathbf{P}_1 \mathbf{M}(0), \quad (\text{A4})$$

then the condition (A3) is equivalent to

$$\det \mathbf{A}(\tau) \neq 0. \quad (\text{A5})$$

This in turn is equivalent to the requirement that the determinant of the 2×2 matrix with complex-valued elements,

$$\begin{bmatrix} 1 + \tau M_{11}(0) & \tau M_{12}(0) \\ \tau M_{12}(0) & 1 + \tau M_{22}(0) \end{bmatrix}, \quad (\text{A6})$$

be non-zero. Let the real 2×2 matrix $\mathbf{Q}$ be the imaginary part of this matrix. Then, by the initial condition (A1) $\mathbf{Q}$ must be positive definite, with eigenvalues q_1^2 and q_2^2 , say. Let us now rotate the coordinate axes to the principal axes of $\mathbf{Q}$, so that the matrix (A6) is of the form

$$\begin{bmatrix} a_{11} + iq_1^2 & a_{12} \\ a_{12} & a_{22} + iq_2^2 \end{bmatrix}, \quad (\text{A7})$$

where the elements a_{11} , a_{12} , and a_{22} are real and arbitrary. Setting the determinant of (A7) to zero gives

$$a_{11}a_{22} - a_{12}^2 - q_1^2q_2^2 + i(a_{22}q_1^2 + a_{11}q_2^2) = 0. \quad (\text{A8})$$

Both the real and imaginary parts of the left member must be zero, implying the identity

$$a_{12}^2 = -(q_1^2q_2^2 + a_{11}^2q_2^2/q_1^2) < 0. \quad (\text{A9})$$

But this implies that a_{12} cannot be real, and we have therefore shown by contradiction that (A5), and hence (A3), are indeed true.

We now turn to the proof of (A2), first noting the identity

$$\bar{\mathbf{M}}^{-1}(\text{Im } \mathbf{M})\mathbf{M}^{-1} = (1/2i)(\bar{\mathbf{M}}^{-1} - \mathbf{M}^{-1}), \quad (\text{A10})$$

where the bar denotes the complex conjugate. The derivative of the right-hand side of (A10) with respect to τ is zero, since $\mathbf{P}_1$ in (19) is real. Therefore,

$$\text{Im } \mathbf{M}(\tau) = \bar{\mathbf{M}}(\tau)\bar{\mathbf{M}}^{-1}(0)[\text{Im } \mathbf{M}(0)]\mathbf{M}^{-1}(0)\mathbf{M}(\tau). \quad (\text{A11})$$

This may be simplified using the identity, which follows from Eqs. (20) and (A4),

$$\mathbf{M}^{-1}(0)\mathbf{M}(\tau) = \mathbf{A}^{-1}(\tau). \quad (\text{A12})$$

In particular, (A11) and (A12) imply that (A2) follows from (A1) if and only if the matrix $\mathbf{P}_1 \mathbf{A}(\tau) \mathbf{P}_1$ is of rank 2.

But this is tantamount to the result (A3), and hence we have proved (A2).

We note that the initial constraint (A1) is obviously true for Gaussian wave packets, for which the stronger condition $\text{Im } M(0)$ p.d. holds. Therefore, the results (A2) and (A3) imply that an initial disturbance in the form of a Gaussian beam or a Gaussian wave packet, i.e., which satisfies (A1), remains a Gaussian for all time, $-\infty < \tau < \infty$, and as it evolves its amplitude cannot develop geometrical singularities.

- ¹ G. D. and M. A. Breazeale, "Nonlinear description of a focused Gaussian ultrasonic beam in a nonlinear medium," *J. Acoust. Soc. Am.* **81**, 51–57 (1987).
- ² S. I. Aanonsen, T. Barkve, J. N. Tjøtta, and S. Tjøtta, "Distortion and harmonic generation in the nearfield of a finite amplitude sound beam," *J. Acoust. Soc. Am.* **75**, 749–768 (1984).
- ³ T. S. Hart and M. F. Hamilton, "Nonlinear effects in focused sound beam," *J. Acoust. Soc. Am.* **84**, 1488–1496 (1988).
- ⁴ N. S. Bakhvalov, Ya. M. Zhileikin, and E. A. Zabolotskaya, *Nonlinear Theory of Sound Beams* (American Institute of Physics, New York, 1987).
- ⁵ A. T. Cates and D. G. Crighton, "Nonlinear diffraction and caustic information," *Proc. R. Soc. London A* **430**, 69–88 (1990).
- ⁶ V. Choquet-Bruhat, "Ondes asymptotiques et approches pour systems d'equations aux derivees partielles nonlineaires," *J. Math. Pure Appl.* **48**, 117–158 (1969).
- ⁷ J. K. Hunter and J. B. Keller, "Weakly nonlinear high frequency waves," *Comm. Pure Appl. Math.* **36**, 547–569 (1983).
- ⁸ D. F. Parker, "An asymptotic theory for oscillatory non-linear signals," *J. Inst. Math. Appl.* **7**, 92–110 (1971).
- ⁹ T. W. Wright, "Acceleration waves in simple elastic materials," *Arch. Rat. Mech. Anal.* **50**, 237–277 (1973).
- ¹⁰ B. R. Seymour and M. P. Mortell, "Nonlinear geometrical acoustics," in *Mechanics Today*, edited by S. Nemat-Nasser (Pergamon, New York, 1975), Vol. 2.
- ¹¹ W. D. Hayes, "Kinematic wave theory," *Proc. R. Soc. London Ser. A* **320**, 209–226 (1970).
- ¹² A. N. Anile, "Propagation of weak shock waves," *Wave Motion* **6**, 571–578 (1984).
- ¹³ J. K. Hunter and J. B. Keller, "Caustics of nonlinear waves," *Wave Motion* **9**, 429–443 (1987).
- ¹⁴ V. Cerveny, M. M. Popov, and I. Psencik, "Computation of wave fields in inhomogeneous media-Gaussian beam approach," *Geophys. J. R. Astron. Soc.* **70**, 109–129 (1982).
- ¹⁵ B. S. White, A. N. Norris, A. Bayliss, and R. Burridge, "Some remarks on the Gaussian beam summation method," *Geophys. J. R. Astron. Soc.* **89**, 579–636 (1987).
- ¹⁶ A. N. Norris, B. S. White, and J. R. Schrieffer, "Gaussian wave packets in inhomogeneous media with curved interfaces," *Proc. R. Soc. London A* **412**, 93–123 (1987).
- ¹⁷ A. N. Norris, "A theory of pulse propagation in anisotropic elastic solids," *Wave Motion* **9**, 509–532 (1987).
- ¹⁸ G. A. Deschamps, "Gaussian beam as a bundle of complex rays," *Electron. Lett.* **7**, 684–685 (1971).
- ¹⁹ A. D. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications* (McGraw-Hill, New York, 1981).