

CONDITIONS UNDER WHICH THE SLOWNESS SURFACE OF AN ANISOTROPIC ELASTIC MATERIAL IS THE UNION OF ALIGNED ELLIPSOIDS

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SUMMARY

The slowness surface of an anisotropic elastic material is described as consisting of aligned ellipsoids when it is the union of three coaxial ellipsoids each common principal axis of which is a specific direction for a longitudinal plane wave. It is shown that the slowness surface has this property only when the material has orthorhombic symmetry. Five sets of conditions are obtained, one of them necessary and each sufficient, for the slowness surface of an orthorhombic elastic material to be formed from aligned ellipsoids. A complete characterization of the conditions referred to in the title of the paper is thus provided.

1. Introduction and preliminaries

IN THE theory of elastic waves in transversely isotropic media considerable simplifications spring from the adoption of one or other of the relations

$$(c_{11} - c_{44})(c_{33} - c_{44}) - (c_{13} + c_{44})^2 = 0 \quad (1.1)$$

and

$$c_{13} + c_{44} = 0 \quad (1.2)$$

between the elastic moduli (1, p. 96; 2, section 5(e)). These are the only conditions under which the slowness surface $\mathcal{S}$ of a transversely isotropic elastic material reduces to the union of three spheroids, and the associated factorization of the displacement equations of motion into wave equations is responsible for the ensuing simplifications. For media with cubic symmetry there is just one constraint on the elastic moduli,

$$c_{12} + c_{44} = 0, \quad (1.3)$$

under which $\mathcal{S}$ is composed of three spheroids (3, section 8.3).

None of equations (1.1) to (1.3) represents at all accurately the behaviour of known materials: their value is in allowing explicit calculations to be made while preserving essential qualitative features of wave propagation in transversely isotropic and cubic media. A comparable easing of the theory of small-amplitude elastic waves will result for any anisotropic medium for which $\mathcal{S}$ consists of three ellipsoids and we address in this paper the question of what conditions must be placed on the elastic moduli for this structure to occur.

The complexity of the general problem is such that some primary limitation of scope seems unavoidable. We suppose that the ellipsoids are *aligned* in the sense that, first, they have common principal axes and, second, each principal axis is a specific direction for a longitudinal plane wave. The second assumption is less restrictive than might at first appear by virtue of the existence, in every anisotropic elastic material, of at least three distinct specific directions for longitudinal plane waves (4).

Our findings are summed up in two theorems which together state that an anisotropic elastic material for which $\mathcal{S}$ consists of aligned ellipsoids is necessarily orthorhombic and that $\mathcal{S}$ assumes one of five specific forms, each requiring the elastic moduli to satisfy additional conditions. The sets of conditions are numbered (C1), ..., (C5) and we note in advance that (C3) implies a higher form of symmetry, namely transverse isotropy. In order not to have to qualify the statements of our main results on account of this one case we take the view in the present work that *transverse isotropy is a special case of orthorhombic symmetry*. In respect of solution (S3) and the accompanying conditions (C3), orthorhombic symmetry is therefore understood to subsume transverse isotropy.

The proof that $\mathcal{S}$ can be the union of aligned ellipsoids only when orthorhombic symmetry prevails is taken to its final stage in section 2 and completed in section 5. Section 3 is devoted to the investigation of conditions under which the slowness surface of an orthorhombic material consists of aligned ellipsoids and the methodology employed there is used again in section 5. We confirm in section 4 that the results for transversely isotropic and cubic materials mentioned at the outset are contained in the solutions found in section 3.

We consider initially an elastic material which is anisotropic in relation to a natural reference configuration N . The components C_{ijkl} relative to an arbitrary orthonormal basis b of the linear elasticity tensor C in N are assumed to possess the symmetries $C_{jikt} = C_{kltj} = C_{tjki}$ and to be positive definite.

The acoustical tensor $Q(\mathbf{n})$ has components

$$Q_{ij}(\mathbf{n}) = C_{piqj}n_p n_q \quad (1.4)$$

relative to b , n_i being the components of an arbitrary unit vector $\mathbf{n}$. Here, and henceforth, we reserve the letters p, q for repeated suffixes to which the summation convention applies: there is no summation on repetitions of i . Because of the properties of C , $Q(\mathbf{n})$ is symmetric and positive definite.

Crucially, $Q(\mathbf{n})$ links the speed c and the polarization $\mathbf{p}$ of a plane wave advancing in the direction of $\mathbf{n}$ through the propagation condition

$$Q(\mathbf{n})\mathbf{p} = \rho c^2 \mathbf{p}, \quad (1.5)$$

ρ being the density in N .

The slowness surface $\mathcal{S}$ of the material is the three-sheeted surface in $\mathbb{R}^3$ given by

$$s(\mathbf{n}) = \{\rho^{-1}\lambda_i(\mathbf{n})\}^{-\frac{1}{2}}\mathbf{n}, \quad i = 1, 2, 3, \text{ all } \mathbf{n} \in \mathcal{U}, \quad (1.6)$$

where $\lambda_i(\mathbf{n})$ are the eigenvalues of $Q(\mathbf{n})$, necessarily positive, and $\mathcal{U}$ denotes the set of all unit vectors. By (1.5), the speeds $c_i(\mathbf{n})$ of the three plane waves which can propagate in the direction of $\mathbf{n}$ are related to the eigenvalues by $\lambda_i(\mathbf{n}) = \rho c_i^2(\mathbf{n})$, and associated orthonormal eigenvectors $\mathbf{p}_i(\mathbf{n})$ are the polarizations of these waves.

A plane wave is said to be longitudinal when its polarization is $\pm \mathbf{n}$ and transverse when its polarization is orthogonal to $\mathbf{n}$. A unit vector $\mathbf{n}$ which is an eigenvector of $Q(\mathbf{n})$ defines a *specific direction* for a longitudinal wave and it then follows from (1.5) that

$$Q(\mathbf{n})\mathbf{n} = [\mathbf{n} \cdot \{Q(\mathbf{n})\mathbf{n}\}]\mathbf{n}. \quad (1.7)$$

In view of the orthonormality of the polarizations $\mathbf{p}_i(\mathbf{n})$, a specific direction for a longitudinal wave is also a specific direction for two transverse waves.

2. Restrictions on the elastic moduli of a material for which $\mathcal{S}$ is the union of aligned ellipsoids

The main result of this section is the following.

THEOREM 1. *The slowness surface $\mathcal{S}$ of an elastic material consists of aligned ellipsoids only if the material has orthorhombic symmetry.*

It is convenient from now on to refer all vector and tensor components to the orthonormal basis $\mathbf{e} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the members of which are directed along the common principal axes of the aligned ellipsoids constituting $\mathcal{S}$.

Since $\mathbf{e}_i$ defines a specific direction for a longitudinal plane wave, equation (1.7) yields

$$Q(\mathbf{e}_i)\mathbf{e}_i = [\mathbf{e}_i \cdot \{Q(\mathbf{e}_i)\mathbf{e}_i\}]\mathbf{e}_i. \quad (2.1)$$

The component form of (2.1), derived from (1.4), is

$$C_{ijji} = C_{iiii}\delta_{ij},$$

and on giving i and j all possible unequal values and adopting the contracted suffix notation (according to which $C_{ijkl} = c_{IJ}$ where $(i, j) = (1, 1), (2, 2), (3, 3), (2, 3), (3, 1), (1, 2)$ correspond to $I = 1, 2, 3, 4, 5, 6$ respectively), we deduce that

$$c_{15} = c_{16} = c_{24} = c_{26} = c_{34} = c_{35} = 0. \quad (2.2)$$

Next, let $\mathbf{e}$ be any member of $\mathcal{e}$. Then unit eigenvectors $\mathbf{f}$ and $\mathbf{g}$, forming with $\mathbf{e}$ an orthonormal set, are the polarizations of the transverse waves which can propagate in the direction of $\mathbf{e}$ and the ray velocities of the longitudinal and transverse waves are scalar multiples of $\mathbf{Q}(\mathbf{e})\mathbf{e}$, $\mathbf{Q}(\mathbf{f})\mathbf{e}$, $\mathbf{Q}(\mathbf{g})\mathbf{e}$ (see, for example, (5, section 21)). The ray velocity of a plane wave is directed normally to $\mathcal{S}$ at the point representing the wave (6, section 4), so the normality of a common principal axis to each ellipsoidal sheet entails all three ray velocities being codirectional with $\mathbf{e}$. There thus exists a positive scalar α such that

$$\{\mathbf{Q}(\mathbf{e}) + \mathbf{Q}(\mathbf{f}) + \mathbf{Q}(\mathbf{g})\}\mathbf{e} = \alpha\mathbf{e}. \quad (2.3)$$

Invoking (1.4) and the identity

$$e_i e_j + f_i f_j + g_i g_j = \delta_{ij},$$

and setting $\mathbf{e} = \mathbf{e}_i$, we can express the component form of (2.3) as

$$C_{pjpi} = \alpha_i \delta_{ij}.$$

Again allowing i and j to take all possible unequal values and using (2.2), we find that

$$c_{45} = c_{46} = c_{56} = 0. \quad (2.4)$$

It is easy to verify, from (1.5), that equations (2.4), together with (2.2), ensure that the transverse plane waves which can propagate along a common principal axis are polarized in the directions of the other principal axes.

It follows from (1.6) that when $\mathcal{S}$ consists of coaxial ellipsoids, each of the eigenvalues $\lambda_i(\mathbf{n})$ of $\mathbf{Q}(\mathbf{n})$ is a homogeneous linear form in n_1^2 , n_2^2 , n_3^2 , an eigenvalue

$$\lambda = q_1 n_1^2 + q_2 n_2^2 + q_3 n_3^2 \quad (2.5)$$

giving rise to an ellipsoidal sheet of $\mathcal{S}$ defined by

$$q_1 s_1^2 + q_2 s_2^2 + q_3 s_3^2 = \rho, \quad (2.6)$$

where $s_i = \mathbf{s} \cdot \mathbf{e}_i$ are the slowness coordinates. The characteristic equation,

$$\det\{\mathbf{Q}(\mathbf{n}) - \lambda \mathbf{I}\} = 0, \quad (2.7)$$

of the propagation condition (1.5) consequently contains only even powers of n_1 , n_2 , n_3 .

Taking account of the simplifications (2.2) and (2.4), we can write equation (2.7) as

$$\begin{aligned} \lambda^3 - (A_1 + A_2 + A_3)\lambda^2 + (A_2 A_3 + A_3 A_1 + A_1 A_2 - B_1^2 - B_2^2 - B_3^2)\lambda \\ - (A_1 A_2 A_3 - A_1 B_1^2 - A_2 B_2^2 - A_3 B_3^2 + 2B_1 B_2 B_3) = 0, \end{aligned} \quad (2.8)$$

with

$$\left. \begin{aligned} A_1 &= c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2, \\ A_2 &= c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, \\ A_3 &= c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2, \end{aligned} \right\} \quad (2.9)$$

$$\left. \begin{aligned} B_1 &= b_1n_2n_3 + x_3n_3n_1 + x_2n_1n_2, \\ B_2 &= x_3n_2n_3 + b_2n_3n_1 + x_1n_1n_2, \\ B_3 &= x_2n_2n_3 + x_1n_3n_1 + b_3n_1n_2, \end{aligned} \right\} \quad (2.10)$$

and

$$b_1 = c_{23} + c_{44}, \quad b_2 = c_{13} + c_{55}, \quad b_3 = c_{12} + c_{66}, \quad (2.11)$$

$$x_1 = c_{14}, \quad x_2 = c_{25}, \quad x_3 = c_{36} \quad (2.12)$$

(cf. (7, p. 84)). The inadmissibility in (2.8) of odd powers of n_1, n_2, n_3 implies the following set of 12 relations between the elastic moduli:

$$b_1x_2x_3 = 0, \quad b_2x_3x_1 = 0, \quad b_3x_1x_2 = 0, \quad (2.13)$$

$$\left. \begin{aligned} (c_{66}b_2 + c_{55}b_3)x_1 + c_{11}x_2x_3 &= 0, \\ (c_{44}b_3 + c_{66}b_1)x_2 + c_{22}x_3x_1 &= 0, \\ (c_{55}b_1 + c_{44}b_2)x_3 + c_{33}x_1x_2 &= 0, \end{aligned} \right\} \quad (2.14)$$

$$\left. \begin{aligned} (c_{12}b_1 - c_{22}b_2)x_3 + c_{23}x_1x_2 + x_2^2x_3 &= 0, \\ -(c_{33}b_3 - c_{13}b_1)x_2 + c_{23}x_3x_1 + x_2^2x_3 &= 0, \\ (c_{23}b_2 - c_{33}b_3)x_1 + c_{13}x_2x_3 + x_3^2x_1 &= 0, \\ -(c_{11}b_1 - c_{12}b_2)x_3 + c_{13}x_1x_2 + x_3^2x_1 &= 0, \\ (c_{13}b_3 - c_{11}b_1)x_2 + c_{12}x_3x_1 + x_1^2x_2 &= 0, \\ -(c_{22}b_2 - c_{23}b_3)x_1 + c_{12}x_2x_3 + x_1^2x_2 &= 0. \end{aligned} \right\} \quad (2.15)$$

We prove that the only solutions of (2.13) to (2.15) are

$$x_1 = x_2 = x_3 = 0 \quad (2.16)$$

and

$$\text{any two } x_i \text{ and the corresponding } b_i \text{ are zero.} \quad (2.17)$$

Equations (2.16), in conjunction with (2.12), (2.2) and (2.4), state that the elastic material under consideration has orthorhombic symmetry. We refer to the restriction of anisotropy imposed by (2.2), (2.4) and (2.17) as *quasi-orthorhombic symmetry*.

Equations (2.13) are satisfied only if one of these possibilities holds:

(i) two x_i are zero;

- (ii) one b_i and the corresponding x_i are zero;
- (iii) three b_i are zero.

We show that each of (i) to (iii) leads to either (2.16) or (2.17).

(i) Suppose that $x_1 = x_2 = 0$. Equations $(2.15)_{4,1}$ then reduce to

$$(c_{11}b_1 - c_{12}b_2)x_3 = 0, \quad (c_{12}b_1 - c_{22}b_2)x_3 = 0,$$

and we conclude that either $x_3 = 0$, in which case equations (2.16) apply, or

$$c_{11}b_1 - c_{12}b_2 = 0, \quad c_{12}b_1 - c_{22}b_2 = 0. \quad (2.18)$$

Due to the positive definiteness of $\mathbf{C}$,

$$c_{II} > 0, \quad c_{II}c_{JJ} - c_{IJ}^2 > 0, \quad I, J = 1, \dots, 6. \quad (2.19)$$

Equations (2.18) hence require that $b_1 = b_2 = 0$ and we arrive at (2.17).

(ii) Suppose that $x_1 = 0$, $b_1 = 0$. In view of $(2.19)_1$, equations (2.14) then become

$$x_2x_3 = 0, \quad b_3x_2 = 0, \quad b_2x_3 = 0,$$

and we infer that either $x_2 = 0$, $x_3 = 0$, or $x_2 = 0$, $b_2 = 0$, or $x_3 = 0$, $b_3 = 0$. The first alternative completes equations (2.16) and the other two satisfy (2.17).

(iii) Because of $(2.19)_1$, equations (2.14) simplify to

$$x_2x_3 = 0, \quad x_3x_1 = 0, \quad x_1x_2 = 0.$$

Two x_i are therefore zero and (2.17) holds.

It has now been established that $\mathcal{S}$ is formed from aligned ellipsoids only if the material has orthorhombic or quasi-orthorhombic symmetry, and to complete the proof of Theorem 1 it has to be shown that a quasi-orthorhombic material for which $\mathcal{S}$ consists of aligned ellipsoids is necessarily orthorhombic. This final stage of the proof is deferred to section 5.

3. Classification of conditions under which the slowness surface of an orthorhombic material is the union of aligned ellipsoids

For an orthorhombic elastic material equations (2.16) hold and (2.10) condense to

$$B_1 = b_1n_2n_3, \quad B_2 = b_2n_3n_1, \quad B_3 = b_3n_1n_2. \quad (3.1)$$

It is seen from (3.1) and (2.9) that when $n_1 = 0$, equation (2.8) has a root $\lambda = A_1 = c_{66}n_2^2 + c_{55}n_3^2$. Similarly, $A_2 = c_{66}n_1^2 + c_{44}n_3^2$ is a root of (2.8) when $n_2 = 0$ and $A_3 = c_{55}n_1^2 + c_{44}n_2^2$ when $n_3 = 0$. These roots correspond to the ellipses, E_1 , E_2 , E_3 respectively, which form part of the sections of $\mathcal{S}$ in the planes of symmetry with normals $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (7, p. 118). If $\mathcal{S}$ consists of three aligned ellipsoids one of the following possibilities must be realized.

1. E_1, E_2 and E_3 lie on the same ellipsoid.
2. Two of E_1, E_2, E_3 are on one ellipsoid and the other is on a different ellipsoid.
3. Each of E_1, E_2, E_3 is situated on a different ellipsoid.

The alternatives are labelled cases 1, 2, 3 and considered successively.

Case 1. Since

$$c_{66}n_2^2 + c_{55}n_3^2, \quad c_{66}n_1^2 + c_{44}n_3^2, \quad c_{55}n_1^2 + c_{44}n_2^2 \quad (3.2)$$

are obtained from a root of (2.8) of the form (2.5) by putting $n_1 = 0, n_2 = 0, n_3 = 0$ in turn, we must have

$$c_{44} = c_{55} = c_{66}. \quad (3.3)$$

This means that c_{44} is a root, say λ_3 , of (2.8).

Let

$$\left. \begin{aligned} a_1 &= c_{11} - c_{44}, & a_2 &= c_{22} - c_{44}, & a_3 &= c_{33} - c_{44}, \\ d_1 &= a_2a_3 - b_1^2, & d_2 &= a_3a_1 - b_2^2, & d_3 &= a_1a_2 - b_3^2, \\ c &= c_{44}. \end{aligned} \right\} \quad (3.4)$$

Then, bearing in mind, as always, the identity $n_1^2 + n_2^2 + n_3^2 = 1$, $A_i = a_in_i^2 + c$ and (2.8) is converted to

$$\begin{aligned} &\lambda^3 - (a_1n_1^2 + a_2n_2^2 + a_3n_3^2 + 3c)\lambda^2 \\ &\quad + \{d_1n_2^2n_3^2 + d_2n_3^2n_1^2 + d_3n_1^2n_2^2 + 2(a_1n_1^2 + a_2n_2^2 + a_3n_3^2)c + 3c^2\}\lambda \\ &\quad - \{(a_1a_2a_3 - a_1b_1^2 - a_2b_2^2 - a_3b_3^2 + 2b_1b_2b_3)n_1^2n_2^2n_3^2 \\ &\quad + (d_1n_2^2n_3^2 + d_2n_3^2n_1^2 + d_3n_1^2n_2^2)c + (a_1n_1^2 + a_2n_2^2 + a_3n_3^2)c^2 + c^3\} = 0. \end{aligned} \quad (3.5)$$

As c is a root of (3.5),

$$a_1a_2a_3 - a_1b_1^2 - a_2b_2^2 - a_3b_3^2 + 2b_1b_2b_3 = 0, \quad (3.6)$$

whereupon (3.5) factorizes as

$$\begin{aligned} &(\lambda - c)\{\lambda^2 - (a_1n_1^2 + a_2n_2^2 + a_3n_3^2 + 2c)\lambda \\ &\quad + d_1n_2^2n_3^2 + d_2n_3^2n_1^2 + d_3n_1^2n_2^2 + (a_1n_1^2 + a_2n_2^2 + a_3n_3^2)c + c^2\} = 0. \end{aligned} \quad (3.7)$$

Let

$$\lambda_1 = q_1n_1^2 + q_2n_2^2 + q_3n_3^2, \quad \lambda_2 = r_1n_1^2 + r_2n_2^2 + r_3n_3^2 \quad (3.8)$$

be the zeros of the quadratic factor in (3.7). Then, equating the coefficients of n_2^2 and n_3^2 in $\lambda_1 + \lambda_2$ and the expression for the sum of zeros, and the coefficients

of n_2^4 , n_3^4 and $n_2^2 n_3^2$ in $\lambda_1 \lambda_2$ and the expression for the product of zeros,

$$\left. \begin{aligned} q_2 + r_2 &= a_2 + 2c, & q_3 + r_3 &= a_3 + 2c, \\ q_2 r_2 &= (a_2 + c)c, & q_3 r_3 &= (a_3 + c)c, \end{aligned} \right\} \quad (3.9)$$

$$q_2 r_3 + q_3 r_2 = (a_2 + a_3 + 2c)c + d_1. \quad (3.10)$$

Similar sets of five relations connect (q_3, q_1, r_3, r_1) and (q_1, q_2, r_1, r_2) . The distinct solutions of (3.9) are

$$q_2 = a_2 + c, \quad q_3 = a_3 + c, \quad r_2 = r_3 = c, \quad (3.11)$$

and

$$q_2 = c, \quad q_3 = a_3 + c, \quad r_2 = a_2 + c, \quad r_3 = c. \quad (3.12)$$

When (3.11) (respectively (3.12)) applies, equation (3.10) imposes the condition $d_1 = 0$ (respectively $b_1 = 0$).

The compatible solutions for (q_2, q_3, r_2, r_3) , (q_3, q_1, r_3, r_1) and (q_1, q_2, r_1, r_2) form, with $\lambda_3 = c$, four distinct sets of roots of (2.8) corresponding to aligned ellipsoids. With reference to the definitions (2.11) and (3.4), the first two are

$$\left. \begin{aligned} \lambda_1 &= c_{11}n_1^2 + c_{22}n_2^2 + c_{33}n_3^2, \\ \lambda_2 &= \lambda_3 = c_{44}, \end{aligned} \right\} \quad (S1)$$

subject to the conditions

$$\left. \begin{aligned} c_{44} &= c_{55} = c_{66}, \\ (c_{11} - c_{44})(c_{22} - c_{44}) - (c_{12} + c_{44})^2 &= 0, \\ (c_{11} - c_{44})(c_{33} - c_{44}) - (c_{13} + c_{44})^2 &= 0, \\ (c_{11} - c_{44})(c_{22} - c_{44})(c_{33} - c_{44}) &= (c_{12} + c_{44})(c_{13} + c_{44})(c_{23} + c_{44}), \end{aligned} \right\} \quad (C1)$$

and

$$\left. \begin{aligned} \lambda_1 &= c_{44}n_1^2 + c_{22}n_2^2 + c_{33}n_3^2, \\ \lambda_2 &= c_{11}n_1^2 + c_{44}(n_2^2 + n_3^2), \\ \lambda_3 &= c_{44}, \end{aligned} \right\} \quad (S2)_1$$

subject to the conditions

$$\left. \begin{aligned} c_{44} &= c_{55} = c_{66}, \\ c_{12} &= c_{13} = -c_{44}, \\ (c_{22} - c_{44})(c_{33} - c_{44}) - (c_{23} + c_{44})^2 &= 0. \end{aligned} \right\} \quad (C2)_1$$

The other eigenvalues, $(S2)_2$ and $(S2)_3$, and the allied conditions, $(C2)_2$ and $(C2)_3$, are formed from $(S2)_1$ and $(C2)_1$ by changing the suffixes of the elastic

moduli and the components of $\mathbf{n}$ according to the following scheme:

$$\begin{aligned} (S\cdot)_1, (C\cdot)_1 &: 1 \ 2 \ 3 \ 4 \ 5 \ 6 \\ (S\cdot)_2, (C\cdot)_2 &: 2 \ 3 \ 1 \ 5 \ 6 \ 4 \\ (S\cdot)_3, (C\cdot)_3 &: 3 \ 1 \ 2 \ 6 \ 4 \ 5 \end{aligned} \quad (3.13)$$

It can easily be confirmed that each set of conditions satisfies (3.6).

The slowness surfaces represented by (S1) and (S2), via (2.5) and (2.6), have two coincident spherical sheets and one spherical and one spheroidal sheet respectively. Unnormalized polarization vectors relating to (S1) and (S2)₁ are

$$\left. \begin{aligned} \mathbf{p}_1 &= a_1^{\frac{1}{2}} n_1 \operatorname{sgn} b_1 \mathbf{e}_1 + a_2^{\frac{1}{2}} n_2 \operatorname{sgn} b_2 \mathbf{e}_2 + a_3^{\frac{1}{2}} n_3 \operatorname{sgn} b_3 \mathbf{e}_3, \\ \mathbf{p}_2 \text{ and } \mathbf{p}_3 &\text{ span the plane perpendicular to } \mathbf{p}_1, \end{aligned} \right\} \quad (P1)$$

and

$$\mathbf{p}_1 = a_2 n_2 \mathbf{e}_2 + b_1 n_3 \mathbf{e}_3, \quad \mathbf{p}_2 = \mathbf{e}_1, \quad \mathbf{p}_3 = -b_1 n_3 \mathbf{e}_2 + a_2 n_2 \mathbf{e}_3. \quad (P2)_1$$

With regard to (P1), it should be noted that the signs of a_1 , a_2 and a_3 are required by (C1)_{3,4} to be the same. If each of them is negative then (C1)₅ implies that at least one of b_1 , b_2 , b_3 is negative. But if $b_3 < 0$ we have $c_{11}c_{22} < c_{44}^2 < c_{12}^2$, in violation of (2.19)₂, so a_1 , a_2 and a_3 must be positive.

Case 2. Since two of the expressions (3.2) are now specializations of a homogeneous linear form in n_1^2 , n_2^2 , n_3^2 , two of c_{44} , c_{55} , c_{66} must be equal. Suppose that

$$c_{44} = c_{55}. \quad (3.14)$$

Then the roots of (2.8) are

$$\left. \begin{aligned} \lambda_1 &= vn_1^2 + wn_2^2 + zn_3^2, \\ \lambda_2 &= c_{66}(n_1^2 + n_2^2) + c_{44}n_3^2, \\ \lambda_3 &= c_{44}(n_1^2 + n_2^2) + zn_3^2, \end{aligned} \right\} \quad (3.15)$$

where v , w , y and z are to be determined. When (3.14) holds, E_1 and E_2 are situated on the ellipsoid associated with λ_2 , and E_3 on the ellipsoid associated with λ_3 .

Equating the coefficients of n_1^2 , n_2^2 and n_3^2 in the expressions for the sum of roots derived from (3.15) and from (2.8) and (2.9) results in

$$v = c_{11}, \quad w = c_{22}, \quad y + z = c_{33} + c_{44}. \quad (3.16)$$

Forming next from (3.15) and from (2.8), (2.9), and (3.1) the sum of products of pairs of roots, it is found that, due to (3.16)_{1,2}, the coefficients of n_1^4 and n_2^4 agree. Equality of the coefficients of n_3^4 , $n_1^2 n_3^2$, $n_2^2 n_3^2$ and $n_1^2 n_2^2$ provides the

relations

$$c_{44}(y+z) + yz = 2c_{33}c_{44} + c_{44}^2, \quad (3.17)$$

$$\left. \begin{aligned} (c_{44} + c_{66})y + (c_{11} + c_{66})z &= c_{33}(c_{11} + c_{66}) + c_{44}(c_{44} + c_{66}) - (c_{13} + c_{44})^2, \\ (c_{44} + c_{66})y + (c_{22} + c_{66})z &= c_{33}(c_{22} + c_{66}) + c_{44}(c_{44} + c_{66}) - (c_{23} + c_{44})^2, \end{aligned} \right\} \quad (3.18)$$

$$(c_{11} - c_{66})(c_{22} - c_{66}) - (c_{12} + c_{66})^2 = 0. \quad (3.19)$$

The solutions of (3.16)₃ and (3.17) are

$$y = c_{33}, \quad z = c_{44}, \quad (3.20)$$

and

$$y = c_{44}, \quad z = c_{33}. \quad (3.21)$$

When (3.20) applies, equations (3.18) become

$$\left. \begin{aligned} (c_{11} - c_{44})(c_{33} - c_{44}) - (c_{13} + c_{44})^2 &= 0, \\ (c_{22} - c_{44})(c_{33} - c_{44}) - (c_{23} + c_{44})^2 &= 0. \end{aligned} \right\} \quad (3.22)$$

When (3.21) holds, they reduce to

$$c_{13} + c_{44} = 0, \quad c_{23} + c_{44} = 0. \quad (3.23)$$

Lastly, we form from (3.15) and from (2.8), (2.9) and (3.1) the product of roots. The coefficients of n_1^6 and n_2^6 are identical, and when account is taken of (3.19) and the relation $yz = c_{33}c_{44}$, satisfied by (3.20) and (3.21), agreement is also secured between the coefficients of n_3^6 , $n_1^4n_2^2$ and $n_1^2n_2^4$. Equality of the coefficients of $n_1^4n_3^2$, $n_1^2n_3^4$, $n_2^4n_3^2$ and $n_2^2n_3^4$ reproduces (3.22) or (3.23) according as (3.20) or (3.21) is in force. There remain the coefficients of $n_1^2n_2^2n_3^2$ which, when equated, give

$$\begin{aligned} 2c_{44}c_{66}y + (c_{11} + c_{22})c_{66}z &= c_{11}c_{22}c_{33} + c_{33}c_{66}^2 + 2c_{44}^2c_{66} \\ &\quad - c_{11}(c_{23} + c_{44})^2 - c_{22}(c_{13} + c_{44})^2 \\ &\quad - c_{33}(c_{12} + c_{66})^2 \\ &\quad + 2(c_{12} + c_{66})(c_{13} + c_{44})(c_{23} + c_{44}). \end{aligned} \quad (3.24)$$

When (3.20) and the attendant conditions (3.19) and (3.22) hold, equation (3.24) can be recast as

$$(|c_{11} - c_{44}|^{\frac{1}{2}}|c_{22} - c_{66}|^{\frac{1}{2}} \pm |c_{11} - c_{66}|^{\frac{1}{2}}|c_{22} - c_{44}|^{\frac{1}{2}})^2(c_{33} - c_{44}) = 0,$$

and, after multiplication by

$$(|c_{11} - c_{44}|^{\frac{1}{2}}|c_{22} - c_{66}|^{\frac{1}{2}} \mp |c_{11} - c_{66}|^{\frac{1}{2}}|c_{22} - c_{44}|^{\frac{1}{2}})^2,$$

this can be expressed as

$$(c_{11} - c_{22})^2(c_{33} - c_{44})(c_{44} - c_{66})^2 = 0.$$

Clearly, $c_{11} = c_{22}$, or $c_{33} = c_{44}$, or $c_{44} = c_{66}$. When $c_{44} = c_{66}$, however, the equalities (3.3) hold and the solution (S1) is recovered, and when $c_{33} = c_{44}$, (3.20) is the same as (3.21). When (3.21) and the accompanying conditions (3.19) and (3.23) apply, equation (3.24) is satisfied identically.

The following sets of eigenvalues representing aligned ellipsoids have thus been found:

$$\left. \begin{aligned} \lambda_1 &= c_{11}(n_1^2 + n_2^2) + c_{33}n_3^2, \\ \lambda_2 &= c_{66}(n_1^2 + n_2^2) + c_{44}n_3^2, \\ \lambda_3 &= c_{44}, \end{aligned} \right\} \quad (\text{S3})_1$$

subject to the conditions

$$\left. \begin{aligned} c_{11} &= c_{22}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55}, \quad c_{66} = \frac{1}{2}(c_{11} - c_{12}), \\ (c_{11} - c_{44})(c_{33} - c_{44}) &= (c_{13} + c_{44})^2, \end{aligned} \right\} \quad (\text{C3})_1$$

and

$$\left. \begin{aligned} \lambda_1 &= c_{11}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, \\ \lambda_2 &= c_{66}(n_1^2 + n_2^2) + c_{44}n_3^2, \\ \lambda_3 &= c_{44}(n_1^2 + n_2^2) + c_{33}n_3^2, \end{aligned} \right\} \quad (\text{S4})_1$$

subject to the conditions

$$\left. \begin{aligned} -c_{13} &= -c_{23} = c_{44} = c_{55}, \\ (c_{11} - c_{66})(c_{22} - c_{66}) - (c_{12} + c_{66})^2 &= 0. \end{aligned} \right\} \quad (\text{C4})_1$$

Additional solutions, (S3)₂, (S4)₂ and (S3)₃, (S4)₃, are obtained by replacing (3.14) by $c_{55} = c_{66}$ and $c_{44} = c_{66}$ respectively. They are adapted from (S3)₁, (S4)₁ by permuting the suffixes of the elastic moduli and the components of $\mathbf{n}$ as specified in (3.13).

Solution (S3) describes a slowness surface with one spherical and two spheroidal sheets, and (S4) also displays two spheroidal sheets. Unnormalized polarization vectors for (S3)₁ and (S4)₁ are

$$\left. \begin{aligned} \mathbf{p}_1 &= (c_{11} - c_{44})(n_1\mathbf{e}_1 + n_2\mathbf{e}_2) + (c_{13} + c_{44})n_3\mathbf{e}_3, \\ \mathbf{p}_2 &= -n_2\mathbf{e}_1 + n_1\mathbf{e}_2, \\ \mathbf{p}_3 &= -(c_{13} + c_{44})n_3(n_1\mathbf{e}_1 + n_2\mathbf{e}_2) + (c_{11} - c_{44})(n_1^2 + n_2^2)\mathbf{e}_3, \end{aligned} \right\} \quad (\text{P3})_1$$

and

$$\left. \begin{aligned} \mathbf{p}_1 &= (c_{11} - c_{66})n_1\mathbf{e}_1 + (c_{12} + c_{66})n_2\mathbf{e}_2, \\ \mathbf{p}_2 &= -(c_{12} + c_{66})n_2\mathbf{e}_1 + (c_{11} - c_{66})n_1\mathbf{e}_2, \\ \mathbf{p}_3 &= \mathbf{e}_3. \end{aligned} \right\} \quad (\text{P4})_1$$

Case 3. When the expressions (3.2) are specializations of distinct roots of (2.8) it is apparent from (2.9) that

$$\lambda_i = A_i + t_i n_i^2, \quad (3.25)$$

where t_i are as yet unknown. However, equating the sum of roots as given by (3.25) and (2.8) yields

$$t_1 n_1^2 + t_2 n_2^2 + t_3 n_3^2 = 0,$$

whence $t_i = 0$ and $\lambda_i = A_i$. We deduce from equation (2.8) that $B_i = 0$ and then from (3.1) that $b_i = 0$.

Case 3 accordingly delivers a single set of aligned ellipsoids, linked to the eigenvalues

$$\left. \begin{aligned} \lambda_1 &= c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2, \\ \lambda_2 &= c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, \\ \lambda_3 &= c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2, \end{aligned} \right\} \quad (S5)$$

and subject to the conditions

$$c_{23} + c_{44} = 0, \quad c_{13} + c_{55} = 0, \quad c_{12} + c_{66} = 0. \quad (C5)$$

Associated polarizations are

$$\mathbf{p}_1 = \mathbf{e}_1, \quad \mathbf{p}_2 = \mathbf{e}_2, \quad \mathbf{p}_3 = \mathbf{e}_3. \quad (P5)$$

The results of this section can be summarized as follows.

THEOREM 2. *If the slowness surface of an orthorhombic elastic material consists of aligned ellipsoids, the elastic moduli satisfy one of the sets of conditions (C1),..., (C5), the eigenvalues and corresponding eigenvectors of $\mathbf{Q}(\mathbf{n})$ being given by (S1),..., (S5) and (P1),..., (P5). Conversely, if the elastic moduli of an orthorhombic material conform to one of (C1),..., (C5), then $\mathcal{S}$ consists of aligned ellipsoids represented by the corresponding member of (S1),..., (S5).*

The first part of the theorem has been proved. To establish the second part we need only rearrange the derivations of (S1),..., (S5) so as to confirm that, subject to (C1),..., (C5) respectively, (S1),..., (S5) are the roots of (2.8).

4. Transversely isotropic and cubic materials

When the elastic moduli of an orthorhombic elastic material are related by

$$c_{11} = c_{22}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55}, \quad c_{66} = \frac{1}{2}(c_{11} - c_{12}), \quad (4.1)$$

the material is transversely isotropic. Adjoining (4.1) to each of the conditions (C1),..., (C5), we find that (C2) and (C5) are brought into conflict with the basic inequality (2.19)₂, while (C3) and (C4) impose, in addition to (4.1), only one restriction on the moduli, namely (1.1) in the case of (C3) and (1.2) in the case of (C4). Conditions (C1) require that, extra to (1.1), $\frac{1}{2}(c_{11} - c_{12}) = c_{44}$.

Solutions (S3) and (S4) duly reproduce the equations of the spheroidal sheets of $\mathcal{S}$ given in (2, section 5(e)).

The connections between the moduli of an orthorhombic material appropriate to cubic symmetry are

$$c_{11} = c_{22} = c_{33}, \quad c_{12} = c_{13} = c_{23}, \quad c_{44} = c_{55} = c_{66}. \quad (4.2)$$

When (4.2) are combined with the conditions obtained in section 3, (C2) and (C4) become incompatible with (2.19)₂ and (C1) and (C3) can apply only to an isotropic material. Conditions (C5) enforce the single relation (1.3) supplementary to (4.2) and (S5) then specifies the form of $\mathcal{S}$ in agreement with (3, equation (8.10)).

Theorem 2 thus includes as special cases the results stated in the opening paragraph of section 1.

5. Quasi-orthorhombic materials

Turning finally to the proof of the italicized statement at the end of section 2, we suppose, in conformity with (2.17), that

$$x_1 = x_2 = 0, \quad b_1 = b_2 = 0.$$

The definitions (2.10) then simplify to

$$B_1 = x_3 n_3 n_1, \quad B_2 = x_3 n_2 n_3, \quad B_3 = b_3 n_1 n_2, \quad (5.1)$$

and we see, with the use of (2.9), that the expressions

$$c_{22}n_2^2 + c_{44}n_3^2, \quad c_{11}n_1^2 + c_{55}n_3^2, \quad c_{55}n_1^2 + c_{44}n_2^2 \quad (5.2)$$

are roots of equation (2.8) when $n_1 = 0$, $n_2 = 0$, $n_3 = 0$ respectively. As in section 3 these roots correspond to ellipses in the planes orthogonal to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and the same three mutually exclusive possibilities are the only ones that can arise. Now labelled cases 4, 5, 6, they are considered in turn.

Case 4. Each of (5.2) derives from an eigenvalue of the form (2.5), so

$$c_{11} = c_{22} = c_{44} = c_{55} = c, \quad (5.3)$$

and $\lambda_3 = c$ is a root of (2.8). There follow from (2.8) the relation

$$(c_{66} - c + b_3)(c_{33} - c) - 2x_3^2 = 0 \quad (5.4)$$

and the formulae

$$\begin{aligned} \lambda_1 + \lambda_2 &= c_{66}(n_1^2 + n_2^2) + c_{33}n_3^2 + c, \\ \lambda_1\lambda_2 &= \{(c_{33} - c)(c_{66} - c) - x_3^2\}(n_1^2 + n_2^2)n_3^2 \\ &\quad + \{(c_{66} - c)^2 - b_3^2\}n_1^2n_2^2 + \{c_{66}(n_1^2 + n_2^2) + c_{33}n_3^2\}c, \end{aligned}$$

for the sum and product of the other roots.

Taking λ_1 and λ_2 to be given by (3.8) and equating coefficients of like monomials in n_1, n_2, n_3 leads to

$$\left. \begin{aligned} q_1 + r_1 &= q_2 + r_2 = c_{66} + c, & q_1 r_1 &= q_2 r_2 = c_{66} c, \\ q_3 + r_3 &= c_{33} + c, & q_3 r_3 &= c_{33} c, \end{aligned} \right\} \quad (5.5)$$

$$q_1 r_2 + q_2 r_1 = c_{66}^2 + c^2 - b_3^2, \quad (5.6)$$

$$q_1 r_3 + q_3 r_1 = q_2 r_3 + q_3 r_2 = c_{33} c_{66} + c^2 - x_3^2. \quad (5.7)$$

Equations (5.5) admit three distinct solutions:

$$q_1 = q_2 = q_3 = c, \quad r_1 = r_2 = c_{66}, \quad r_3 = c_{33}, \quad (5.8)$$

$$q_1 = q_2 = c, \quad q_3 = c_{33}, \quad r_1 = r_2 = c_{66}, \quad r_3 = c, \quad (5.9)$$

$$q_1 = q_3 = c, \quad q_2 = c_{66}, \quad r_1 = c_{66}, \quad r_2 = c, \quad r_3 = c_{33}. \quad (5.10)$$

When (5.8) or (5.9) applies, equation (5.6) can be written as

$$(c_{66} - c + b_3)(c_{66} - c - b_3) = 0.$$

By (5.3) and (2.11)₃, the second factor on the left is $-(c_{11}c_{22})^{\frac{1}{2}} - c_{12}$ which is forced by (2.19) to be negative. The first factor is therefore zero and we see from (5.4) that $x_3 = 0$. Equations (2.16) accordingly hold and the material is orthorhombic. When (5.10) applies, equation (5.7)₂ requires x_3 to vanish and again the prevailing symmetry is orthorhombic.

Case 5. Equation (3.14) is a sufficient condition for two of (5.2) to relate to ellipses on the same ellipsoidal sheet of $\mathcal{S}$ and the roots of (2.8) are then of the form

$$\left. \begin{aligned} \lambda_1 &= c_{11}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, \\ \lambda_2 &= c_{44}(n_1^2 + n_2^2) + yn_3^2, \\ \lambda_3 &= vn_1^2 + wn_2^2 + zn_3^2. \end{aligned} \right\} \quad (5.11)$$

We proceed, as in case 2, to equate the coefficients of corresponding terms in the expressions for $\lambda_1 + \lambda_2 + \lambda_3$, $\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2$ and $\lambda_1\lambda_2\lambda_3$ supplied by (5.11) and by (2.8), (2.9) and (5.1). The alternatives (3.20) and (3.21) once more emerge from $\lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2$, together with the equalities

$$v = w = c_{66}$$

and

$$\left. \begin{aligned} c_{66}y + c_{44}z &= c_{33}c_{66} + c_{44}^2 - x_3^2, \\ (c_{11} - c_{66})(c_{22} - c_{66}) - b_3^2 &= 0. \end{aligned} \right\} \quad (5.12)$$

When (3.20) applies, equation (5.12)₁ reduces to $x_3 = 0$.

When (3.21) holds, equations (5.12) ensure that all but one of the coefficients

in $\lambda_1\lambda_2\lambda_3$ agree. The exceptional pair provide the relation

$$c_{44}(c_{11} + c_{22})(c_{44} - c_{33} - c_{66}) + c_{33}(c_{11}c_{22} + c_{66}^2) - c_{33}b_3^2 + 2(b_3 - c_{66})x_3^2 = 0,$$

which, with the aid of (2.11)₃, (3.21) and (5.12), can be simplified to

$$(c_{11} + c_{22} + 2c_{12})x_3^2 = 0.$$

The vanishing of the first factor on the left would contravene (2.19)₂, so again $x_3 = 0$.

Case 6. The reasoning employed earlier for case 3 shows that when (5.2) come from distinct roots of (2.8), $B_i = 0$. Hence, from (5.1)_{1,2}, $x_3 = 0$ and the proof of Theorem 2 is complete.

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