

Resonant acoustic scattering from solid targets

Andrew N. Norris

Department of Mechanical and Materials Science, Rutgers University, P. O. Box 909,
Piscataway, New Jersey 08855-0909

(Received 20 October 1989; accepted for publication 2 March 1990)

An asymptotic method of solution is presented for scattering of acoustic waves from solid elastic targets. The asymptotic parameter is the ratio of the fluid density to that of the solid, and the solution is developed using the method of matched asymptotic expansions in this small quantity. The perturbations to the background rigid scattered field are regular for frequencies away from the frequencies of free vibration of the target *in vacuo*, but, near these frequencies, the perturbation is singular in that an asymptotically small value of the density ratio produces a change in the scattered field of order unity. By combining the regularly and singularly perturbed expansions, a solution is obtained that is uniformly correct at all frequencies. The elements in the uniform solution depend only upon the *in vacuo* modes and frequencies, and the Green's function for the equivalent rigid target. At no stage is it necessary to solve the fully coupled system. An analysis of the asymptotic approximation for a spherical target shows that it is equivalent in the high-frequency limit to the approximation predicted by resonance scattering theory.

PACS numbers: 43.20.Fn

INTRODUCTION

The scattering of acoustic waves from an almost rigid elastic target displays all the features of a singular perturbation problem. In order to see this, let us assume for simplicity that the elastic target has a much greater mass density than the fluid and define the small parameter ϵ as the ratio of fluid to solid density. The scattering process reduces to that of scattering from a perfectly rigid, immovable object when ϵ is zero identically. When ϵ is small but nonzero, then, for most values of frequency, the target will still scatter as a rigid object. The scattered field possesses an infinite number of damped resonance frequencies, each located very close to a frequency of free vibration of the solid body, and near every resonance frequency there is a neighborhood in frequency space, of order ϵ in width, as will be deduced later, within which the scattered field is quite different in form compared to the rigid scattered field. Thus, when viewed as a function of ϵ , the scattered field for the rigid target is a singular case in the sense that the solution for small but nonzero ϵ is not everywhere close to the unperturbed rigid response. In practice, the effect of these singular perturbations, or resonances, can alter the scattered field to such an extent that it hardly resembles the rigid response, as illustrated in Fig. 1.

This paper describes the application of the method of matched asymptotic expansions to the scattering of acoustic waves from elastic targets for which ϵ is a small parameter. Similar methods have previously been demonstrated for acoustic scattering from membranes and plates in an infinite rigid baffle¹⁻³ and from penetrable acoustic targets.⁴ Here, the scattered field is obtained in the form of an asymptotic expansion in ϵ which is composed of a rigid background combined with an "inner" solution that is valid for frequencies near the resonant frequencies of the target *in vacuo*. The term "inner" is used by analogy with boundary layer theory,

for which the inner solution usually exists in a region of space. The corresponding region of validity in the present problem is an interior layer in frequency space. The rigid background field, which assumes the form of a regular perturbation in ϵ , will be called the outer solution, again by analogy with boundary layer theory.

It is well known that an incident acoustic wave can ex-

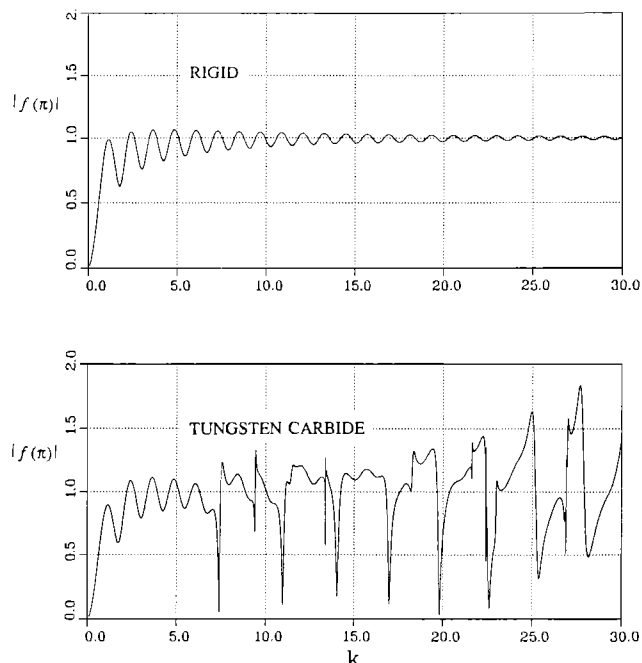


FIG. 1. The far-field backscattering amplitude [see Eq. (79)] for the rigid sphere and for a tungsten carbide sphere in water, $\epsilon = 1/13.8$. The fluid wave number k is nondimensionalized with respect to the radius of the sphere.

cite the *in vacuo* resonances of elastic spheres, cylinders, and shells, and these resonances exhibit distinctive peaks and troughs in the scattered far-field amplitude. Reference 5 provides a current and extensive bibliography on this subject. The effect of the fluid loading on elastic bodies shifts the resonant frequencies from the real-valued *in vacuo* frequencies to slightly perturbed complex values, and these may be calculated numerically for simple targets like the sphere and cylinder, for which the acoustic and elastic equations of motion are separable,⁵⁻⁸ and also for spheroids and related shapes using a *T*-matrix approach.⁹ The present method provides a means to find the perturbed complex frequencies for arbitrarily shaped targets assuming that the *in vacuo* resonances are known and that the corresponding acoustic scattering from a rigid target can be solved. In general, it is probable that both of these separate tasks can be reduced to numerical problems, each of which is far simpler than the coupled acoustic-elastic scattering problem.

The existence of identifiable resonances in the acoustic far field provides a possible means of inferring something about the target geometry and material properties. It is therefore of some importance to have a fast and stable method for modeling the scattering problem in the neighborhood of resonance frequencies for targets of complex geometry. The present method is by its very nature stable near these frequencies, and, provided one has an efficient code for generating the *in vacuo* modes and modal frequencies of the elastic body, it should be significantly faster than the numerical solution of the coupled problem. In addition to its computational possibilities, the method offers a simple means to understand the character of the scattered field near resonance. It will be shown in Sec. III that there is a complete correspondence between the inner solution near resonance and the generalized method of acoustic impedance matching.

The scattering problem is formulated in the next section, and the asymptotic solution is derived in Sec. II. Section III contains a discussion on some of the main features of the asymptotic result. The application of the general method to the specific case of a spherical target is presented in Sec. IV, and the asymptotic approximation is analytically compared with the approximate solution of resonance scattering theory.⁶

I. FORMULATION OF THE PROBLEM

The scatterer occupies the volume V with boundary S and exterior $\bar{V}$. All field variables are time harmonic, and the dependence $\exp(-i\omega t)$ will be omitted. The exterior region $\bar{V}$ is occupied by an ideal inviscid fluid of density ρ_F , sound speed c_F , and the total acoustic field there is defined by the velocity potential

$$\phi^{\text{tot}} = \phi^{\text{inc}} + \phi, \quad \mathbf{v} = \nabla \phi^{\text{tot}}, \quad (1)$$

where ϕ^{inc} is the incident wave and ϕ is the scattered field, which satisfies

$$\nabla^2 \phi + k^2 \phi = 0, \quad \mathbf{x} \text{ in } \bar{V}, \quad (2)$$

with the radiation condition that ϕ be outgoing at infinity.

The wave number in (2) is $k = \omega/c_F$, and the acoustic pressure associated with ϕ is $p = i\omega\rho_F\phi$.

The homogeneous linearly elastic solid in V is of density ρ_S and $\mathbf{C}$ is the fourth-order tensor of moduli, which possesses the usual symmetries for linearly elastic, anisotropic hyperelastic solids; i.e., $\mathbf{C}$ has at most 21 independent elements. In rectangular components the moduli are C_{ijkl} , where i, j, k , and l assume the values 1, 2, and 3, and the symmetries imply that $C_{ijkl} = C_{jikl} = C_{klij}$. The displacement field is $\mathbf{u}(\mathbf{x})$, the stress is $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C}\mathbf{e}$, where $\mathbf{e}$ is the strain tensor, $\mathbf{e} = \frac{1}{2}[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T]$, and the equations of motion are

$$\text{div } \boldsymbol{\sigma}(\mathbf{u}) + \omega^2 \rho_S \mathbf{u} = 0, \quad \mathbf{x} \text{ in } V. \quad (3)$$

The conditions relating the interior and exterior fields along the surface S are that the displacement and tractions be continuous:¹⁰

$$\left. \begin{aligned} \frac{\partial}{\partial n} \phi^{\text{inc}} + \frac{\partial}{\partial n} \phi &= -i\omega \mathbf{u} \cdot \mathbf{n}, \\ (\rho_F \phi^{\text{inc}} + \rho_F \phi) \mathbf{n} &= \frac{1}{-i\omega} \boldsymbol{\sigma} \mathbf{n}, \end{aligned} \right\} \mathbf{x} \text{ on } S, \quad (4)$$

where $\mathbf{n}$ is the unit normal to S .

Let L be a typical length scale of the scatterer, e.g., its average radius, and define the time unit $T = L/c_F$. Introduce nondimensional parameters $\mathbf{x}' = \mathbf{x}/L$, $k' = kL$, $\phi' = \phi T/L^2$, $\mathbf{u}' = \mathbf{u}/L$, and $\mathbf{C}' = \mathbf{C}/\rho_S c_S^2$, where c_S is a typical sound speed in the solid. Substituting these into Eqs. (1)–(4), and dropping the primes, Eq. (2) is recovered for ϕ , and the equation for $\mathbf{u}$ becomes

$$\text{div } \boldsymbol{\sigma}(\mathbf{u}) + k^2 c^2 \mathbf{u} = 0, \quad \mathbf{x} \text{ in } V, \quad (5)$$

where

$$c = c_F/c_S,$$

and the interface conditions are

$$\left. \begin{aligned} \frac{\partial \phi^{\text{inc}}}{\partial n} + \frac{\partial \phi}{\partial n} &= -ik \mathbf{u} \cdot \mathbf{n}, \\ \epsilon(\phi^{\text{inc}} + \phi) \mathbf{n} &= \frac{1}{-ikc^2} \boldsymbol{\sigma} \mathbf{n}, \end{aligned} \right\} \mathbf{x} \text{ on } S, \quad (6)$$

with ϵ defined as

$$\epsilon = \rho_F/\rho_S. \quad (7)$$

Let G be the Green's operator that solves the exterior Neumann radiation problem:

$$\begin{aligned} \nabla^2 f + k^2 f &= 0, \quad \mathbf{x} \text{ in } \bar{V}, \\ \frac{\partial f}{\partial n} &= g, \quad \mathbf{x} \text{ on } S, \end{aligned} \quad (8)$$

in the form $f = Gg$, and f satisfies the radiation condition at infinity. A specific example of G is presented in Sec. IV, and generally Gg may be expressed as an integral over S involving a Green's function.⁴ The field scattered from a rigid target of the same shape as V is then ϕ_0 , where

$$\phi_0 = -G \frac{\partial \phi^{\text{inc}}}{\partial n}. \quad (9)$$

The interface condition (6)₁ thus implies

$$\phi = \phi_0 - ikG\mathbf{u} \cdot \mathbf{n}, \quad (10)$$

and, consequently, the traction continuity condition (6)₂ becomes

$$\sigma(\mathbf{u})\mathbf{n} = -ikc^2\epsilon(\phi^{\text{inc}} + \phi_0 - ikG\mathbf{u}\cdot\mathbf{n}), \quad \mathbf{x} \text{ on } S. \quad (11)$$

It will be assumed for the remainder of the paper that the rigid Green's operator G is known, so that the scattering problem may be reduced to a problem within and on the surface of V , viz. Eqs. (5) and (11).

II. ASYMPTOTIC ANALYSIS

We develop a solution to (5) and (11) that is uniformly valid in frequency k under the assumption that ϵ of (7) is small; i.e., $\epsilon \ll 1$. The solution is composed of matched inner and outer asymptotic expansions^{11,12} in frequency k , the inner regions in frequency space being defined by the *in vacuo* resonance frequencies of V . The theory outlined here is closely related to an earlier theory for acoustic scattering from acoustic targets,⁴ although the form of the final uniformly asymptotic solution is actually quite a bit simpler than that of Ref. 4.

A. Outer solution

Assume a regular asymptotic expansion

$$\mathbf{u} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots \quad (12)$$

Substituting (12) into the equations of motion (5) and identifying the different order terms in ϵ implies that (5) is satisfied separately by each $\mathbf{u}_j$ for $j = 1, 2, \dots$. The boundary condition (11) becomes

$$\sigma(\mathbf{u}_j)\mathbf{n} = -ikc^2[(\phi^{\text{inc}} + \phi_0)\delta_{j1} - ikG\mathbf{n}\cdot\mathbf{u}_{j-1}]\mathbf{n}, \quad \mathbf{x} \text{ on } S, \quad (13)$$

where $j = 1, 2, \dots$, $\mathbf{u}_0 = 0$, and δ_{ij} is the Kronecker delta. These interior boundary value problems may be solved sequentially in j , and thus, in principle, the outer expansion is well defined. The associated outer expansion for the scattered field is

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \quad (14)$$

where

$$\phi_j = -ikG\mathbf{n}\cdot\mathbf{u}_j, \quad j > 0. \quad (15)$$

In practice, the rigid scattered field ϕ_0 will be all that is required of the outer solution, since the additional terms are $O(\epsilon)$ in comparison. However, problems arise at those frequencies for which (5) has nonzero solutions for zero boundary tractions. To see this, let k_m be a frequency for which there is a nonvanishing solution $\mathbf{W}_m$ to

$$\begin{aligned} \text{div } \sigma(\mathbf{W}_m) + k_m^2 c^2 \mathbf{W}_m &= 0, \quad \mathbf{x} \text{ in } V, \\ \sigma(\mathbf{W}_m)\mathbf{n} &= 0, \quad \mathbf{x} \text{ on } S. \end{aligned} \quad (16)$$

Then, k_m is a resonance frequency corresponding to free vibrations of V and $\mathbf{W}_m$ is the modal displacement, which we normalize by

$$\int_V \mathbf{W}_m \cdot \mathbf{W}_m dV = 1. \quad (17)$$

In particular, it follows from (5) and (16) that

$$\begin{aligned} \int_V \text{div} [\sigma(\mathbf{u}_j)\mathbf{W}_m - \sigma(\mathbf{W}_m)\mathbf{u}_j] dV \\ = \int_V c^2(k_m^2 - k^2)\mathbf{W}_m \cdot \mathbf{u}_j dV \\ = \int_S \mathbf{W}_m \cdot \sigma(\mathbf{u}_j)\mathbf{n} dS. \end{aligned} \quad (18)$$

For the remainder of the paper, it will be assumed that the resonant frequencies are simple, and thus the body does not exhibit degenerate modes. It can then be shown that the modes are orthonormal; i.e., $\int_V \mathbf{W}_m \cdot \mathbf{W}_n dV = \delta_{mn}$, and so a formal eigenfunction expansion for $\mathbf{u}_j$, $j = 1, 2, \dots$, may be obtained from (18) and (13):

$$\mathbf{u}_j = ik \sum_{m=1}^{\infty} \frac{((\phi_0 + \phi^{\text{inc}})\delta_{j1} - ikG\mathbf{n}\cdot\mathbf{u}_{j-1})\mathbf{n} \cdot \mathbf{W}_m}{k^2 - k_m^2} \mathbf{W}_m, \quad (19)$$

where $\mathbf{u}_0 = 0$ and

$$(f, g) := \int_S fg dS. \quad (20)$$

It is clear from (19) that $\mathbf{u}_j$, $j > 0$, become unbounded as $k \rightarrow k_m$. The outer expansions (12) and (14) thus break down near the frequencies of free vibration of V , and another type of asymptotic expansion is required there.

B. Inner solution

The inner solution specifically considers frequencies near a resonance frequency k_m , $k - k_m = O(\epsilon)$. It is clear from the form of the breakdown of the outer solution that the critical dependence of the solution is through k in Eq. (5). The appearance of k in the boundary condition (11) does not seriously effect the solution near a resonance frequency. For these reasons, k in Eq. (5) will be replaced by κ , where

$$\kappa^2 = (1 + \alpha\epsilon)k_m^2, \quad (21)$$

and α is a tuning parameter that measures the distance from the *in vacuo* resonance frequency. The parameter k is kept as is in the boundary condition (11). We now assume the ansatz

$$\mathbf{u} = \mathbf{U}_0 + \epsilon\mathbf{U}_1 + \cdots \quad (22)$$

Substituting (21) and (22) into (5) and invoking (11) implies a sequence of interior problems for $\mathbf{U}_j$, $j = 0, 1, 2, \dots$:

$$\begin{aligned} \text{div } \sigma(\mathbf{U}_j) + \kappa_m^2 c^2 \mathbf{U}_j &= -\alpha k_m^2 c^2 \mathbf{U}_{j-1}, \quad \mathbf{x} \text{ in } V, \\ \sigma(\mathbf{U}_j)\mathbf{n} &= -ikc^2[(\phi^{\text{inc}} + \phi_0)\delta_{j1} - ikG\mathbf{n}\cdot\mathbf{U}_{j-1}]\mathbf{n}, \end{aligned} \quad (23)$$

$\mathbf{x} \text{ on } S,$

where $\mathbf{U}_{-1} = 0$. The first solution $\mathbf{U}_0$ satisfies a homogeneous equation in V and homogeneous boundary conditions on S , and therefore

$$\mathbf{U}_0 = A_0 \mathbf{W}_m, \quad (24)$$

where the amplitude $A_0 = A_0(\alpha)$ has yet to be determined.

Proceeding to the next order in the sequence of asymptotic equations (23) implies that $\mathbf{U}_1$ satisfies an inhomogeneous equation in V and inhomogeneous boundary conditions on S . The unique solution to this boundary-value problem must in some sense be orthogonal to the nontrivial solution

$\mathbf{W}_m$ of the homogeneous problem. The precise form of the desired solvability condition for $\mathbf{U}_1$ may be obtained by considering the identity (18) with $\mathbf{u}_j$ replaced by $\mathbf{U}_1$. Expanding the integrand in the left member of (18) and using (16), (17), (23), and (24) yields

$$-\alpha k_m^2 c^2 A_0 = k^2 c^2 a_m A_0 - i k c^2 \beta_m, \quad (25)$$

where

$$\begin{aligned} a_m(k) &= (-\mathbf{Gn} \cdot \mathbf{W}_m, \mathbf{n} \cdot \mathbf{W}_m), \\ \beta_m(k) &= (\phi^{\text{inc}} + \phi_0, \mathbf{n} \cdot \mathbf{W}_m). \end{aligned} \quad (26)$$

An explicit equation is then obtained for A_0 from (25):

$$A_0 = i k \beta_m / (\alpha k_m^2 + k^2 a_m). \quad (27)$$

Finally, the inner expansion for the radiated field is

$$\phi = \Phi_0 + \epsilon \Phi_1 + \dots, \quad (28)$$

where

$$\Phi_0 = \phi_0 - i k A_0 \mathbf{Gn} \cdot \mathbf{W}_m. \quad (29)$$

C. The uniform expansion

A solution that is uniformly valid in k may be constructed by assuming the existence of an overlap region near k_m in which both the inner and outer expansions are asymptotically valid. There is some degree of arbitrariness in exactly how one arrives at a uniform expansion, and so to allay any ambiguity, we define it as the sum of the inner and outer expansions minus the inner expansion in the overlap region.^{11,12} The inner expansion follows from (22), (24), and (27) as

$$\mathbf{u} \sim i k \beta_m / (\alpha k_m^2 + k^2 a_m) \mathbf{W}_m. \quad (30)$$

The overlap region is defined by the simultaneous limits of $\alpha \rightarrow \infty$, $\epsilon \alpha \rightarrow 0^+$, and so the inner expansion in the overlap region is

$$\mathbf{u} \sim (i k \beta_m / \alpha k_m^2) \mathbf{W}_m. \quad (31)$$

The first term in the outer expansion is $\mathbf{u} \sim \epsilon \mathbf{u}_1$, where $\mathbf{u}_1$ is defined by (19), and the uniform expansion is formed by combining (30) with the outer expansion and subtracting (31) from the result, to give

$$\begin{aligned} \mathbf{u} &\sim \epsilon \mathbf{u}_1 - [i k^3 a_m \beta_m / \alpha k_m^2 (\alpha k_m^2 + k^2 a_m)] \mathbf{W}_m \\ &\sim \epsilon \mathbf{u}_1 - [i k^3 a_m \beta_m \epsilon^2 / (k^2 - k_m^2)] \\ &\quad \times (k^2 - k_m^2 + \epsilon k^2 a_m) \mathbf{W}_m. \end{aligned} \quad (32)$$

The last expression is found after removing the inner variable α in favor of the frequency k through $\alpha = (k^2 - k_m^2) / \epsilon k_m^2$ [see Eq. (21)].

The uniform correction to the outer expansion for each mode follows from (32). Combining the effects of all the modes and eliminating $\mathbf{u}_1$ from the uniform expansion using (19) and (26)₂ yield

$$\mathbf{u} \sim \epsilon i k \sum_{m=1}^{\infty} \frac{\beta_m}{k^2 - k_m^2 + k^2 a_m \epsilon} \mathbf{W}_m. \quad (33)$$

This is the first-order asymptotic expansion for $\mathbf{u}$ that is uniformly valid for all k . The associated uniform expansion for the scattered field follows from (10) and (33) as

$$\phi \sim \phi_0 + \epsilon k^2 \sum_{m=1}^{\infty} \frac{\beta_m}{k^2 - k_m^2 + k^2 a_m \epsilon} \mathbf{Gn} \cdot \mathbf{W}_m. \quad (34)$$

III. INTERPRETATION

A. The complex resonance frequency

The first term of the inner solution defined by (22), (24), and (27) possesses a complex resonance frequency near the original real resonance frequency k_m at the complex value of k for which the uniform expression (32) becomes unbounded; i.e., the k which solves $k = k_m / [1 + \epsilon a_m(k)]^{1/2}$. Expanding the root in ϵ yields, to first order in ϵ ,

$$k = k_m - \frac{1}{2} \epsilon k_m a_m(k_m). \quad (35)$$

The imaginary part of the complex resonance frequency is negative definite, which corresponds to radiation damping caused by the fluid exterior. To see this, define Φ as the radiated field corresponding to mode m at $k = k_m$:

$$\Phi = \mathbf{Gn} \cdot \mathbf{W}_m, \quad (36)$$

and let F be the associated far-field pattern; i.e.,

$$\Phi = F(e^{ikr}/r) + O(r^{-2}), \quad r \rightarrow \infty. \quad (37)$$

It may be shown by simple means³ that

$$a_m = R + iI, \quad (38)$$

where

$$R = \int_V (|\nabla \Phi|^2 - k^2 |\Phi|^2) dV, \quad I = k \int_{4\pi} |F|^2 d\Omega > 0, \quad (39)$$

and $d\Omega$ is the element of solid angle. Equations (35), (38), and (39) imply that the imaginary part of the complex resonance frequency is always negative, as stated, and is simply related to the radiation cross section of mode m .⁴

The perturbation of the resonance frequency depends upon a_m of (26)₁. If we recall that the mode $\mathbf{W}_m$ is normalized so that the integral of $\mathbf{W}_m \cdot \mathbf{W}_m$ over V is unity, then it is clear that a_m depends strongly upon the spatial distribution of the mode. Hence, if the mode is such that the displacement is fairly evenly distributed about V , then the value of a_m is expected to be significantly smaller in magnitude than if the mode is confined to a neighborhood of the surface. It will be shown below in Sec. IV that a good example of the latter type of mode is given by the $(n,1)$ modes of a sphere,^{6,7} which are very much like Rayleigh waves, whereas the other modes, (n,l) for $l > 1$, are examples of the former. The net effect of this disparity in modal distribution is that the resonances of the $(n,1)$ modes are much broader than the (n,l) , $l > 1$ resonances.

In general, a target will possess modes associated with surface waves and the corresponding resonances will be relatively broad. At the other extreme are modes that have no normal surface displacement. These are called nonradiating modes and will be dealt with below in Sec. III B. However, there may also be what we will call "almost nonradiating modes." Consider an arbitrarily shaped, unsymmetric body with two points on its surface that have the same normal direction, and the line perpendicular to the surface at one passes through the other. Then, it is entirely reasonable,

especially at higher values of frequency, that there is a mode composed mainly of transverse waves bouncing back and forth between the surfaces about these points, and the resulting *normal* surface displacement will be small. The nonradiating modes of spheres and cylinders are of this type. The lack of symmetry in the general target will, however, necessitate that the mode also contain a "small" amount of compressional waves and a "small" component of normal displacement caused by the oblique incidence of the transverse waves at the surface. For reasons outlined in Sec. III B, this type of mode will always be a poor radiator of energy and will give rise to a very narrow acoustic resonance in the scattered field, and hence is not of great practical significance.

B. Nonradiating modes

It is possible that the elastic body can possess modes of free vibration for which the normal displacement on S is everywhere zero, as occurs, for instance, with torsional modes of a sphere.¹³ Such modes are nonradiating, and there has been some discussion recently on their influence upon the scattered field.¹⁴ The purpose of this subsection is to show, using rather crude but physically appealing arguments, that nonradiating modes may be safely ignored in the asymptotic theory.

Consider a nonradiating mode as the limit of a radiating mode, i.e., one for which the normal displacement $\mathbf{n} \cdot \mathbf{W}_m$ is small. In particular, let $\mathbf{n} \cdot \mathbf{W}_m$ be of the order $\delta \ll 1$ independent of ϵ ; then, it follows from (26) that a_m and β_m are of order δ^2 and δ , respectively. The influence of this mode on the scattered field is thus, by (34), of order $\delta^2/(x + i\delta^2)$, where x is roughly the relative frequency $k - k_m$, and the dependence upon ϵ has been ignored. As $\delta \rightarrow 0$, the L^2 norm of the modal contribution, i.e., the integral of $|\delta^2/(x + i\delta^2)|^2$ over x , is of order δ^2 , and so the total contribution from this mode vanishes as $\delta \rightarrow 0$. In other words, as a mode becomes nonradiating, its resonance width shrinks to zero, but the resonance height remains of fixed order. Therefore, the total energy associated with the mode vanishes. Instances of such almost nonradiating modes will be presented in the example of Sec. IV.

C. Comparison with the exact solution

An exact, formal solution for the interior problem (5) and (11) may be constructed using the modes of free vibration. Thus let

$$\mathbf{u} = \sum_{j=1}^{\infty} d_j \mathbf{W}_j. \quad (40)$$

Substituting into (11) and using (18) yields

$$d_m = \frac{\epsilon i k}{k^2 - k_m^2} \left(\beta_m + i k \sum_{j=1}^{\infty} b_{jm} d_j \right), \quad m = 1, 2, \dots, \quad (41)$$

where

$$b_{jm} = (-\mathbf{Gn} \cdot \mathbf{W}_j, \mathbf{n} \cdot \mathbf{W}_m). \quad (42)$$

Note that $b_{mm} = a_m$ and that $b_{jm} = b_{mj}$, which follows by applying Green's theorem in $\bar{V}$. The coefficients d_j can be

found, in principle, from the infinite system of linear equations

$$\sum_{j=1}^{\infty} M_{jm} d_j = \epsilon i k \beta_m, \quad m = 1, 2, \dots, \quad (43)$$

where the infinite, symmetric matrix $\mathbf{M}$ has elements

$$M_{jm} = (k^2 - k_m^2) \delta_{jm} + \epsilon k^2 b_{jm}. \quad (44)$$

The outer and inner, expansions of Sec. II may be seen to correspond to different ways of approximating the inverse $\mathbf{M}^{-1}$. Let us write

$$\mathbf{M} = \mathbf{M}_0 + \epsilon \mathbf{M}_1, \quad (45)$$

where $\mathbf{M}_0 = \text{diag}(k^2 - k_1^2, k^2 - k_2^2, \dots)$; then, the outer expansion follows from the asymptotic approximation

$$\mathbf{M}^{-1} = \mathbf{M}_0^{-1} - \epsilon \mathbf{M}_0^{-1} \mathbf{M}_1 \mathbf{M}_0^{-1} + O(\epsilon^2). \quad (46)$$

This expansion obviously breaks down when $\mathbf{M}_0$ becomes singular, which is at the real resonance frequencies k_m , $m = 1, 2, \dots$. The singularities may be removed by using the alternative expression

$$\mathbf{M} = \hat{\mathbf{M}}_0 + \epsilon \hat{\mathbf{M}}_1, \quad (47)$$

where

$$\hat{\mathbf{M}}_0 = \text{diag}(k^2 - k_1^2 + \epsilon k^2 a_1, k^2 - k_2^2 + \epsilon k^2 a_2, \dots),$$

and

$$\hat{\mathbf{M}}_{1jm} = k^2 (b_{jm} - a_m \delta_{jm}). \quad (48)$$

The first term in the uniform expansion (33) then follows from (43) with $\mathbf{M}^{-1} \sim \hat{\mathbf{M}}_0^{-1}$. An important feature of this approximation is that it only involves the diagonal elements of $\mathbf{M}$, specifically the integrals a_m . The full, exact solution must, of course, require calculation of all the coupling constants b_{jm} , $j \neq m$. If one were to calculate these terms, then the expansion of $\mathbf{M}^{-1}$ using (47) provides, with (43), a straightforward means to develop higher-order corrections to (33).

D. Relation with acoustic impedances

In the study of acoustic scattering from elastic targets for which the equations admit separable solutions, e.g., the sphere, it is well known that the complex resonance frequency occurs when the modal specific acoustic impedances of the fluid and solid are identical. It will now be shown that the same general principle applies here also, and the frequency obtained by matching impedances is precisely the frequency of Eq. (35) to the same order in ϵ .

We first define the modal specific acoustic impedances, beginning with the fluid. Let ψ be the solution to the radiation problem

$$\begin{aligned} \nabla^2 \psi + k^2 \psi &= 0, \quad \mathbf{x} \text{ in } \bar{V}, \\ \frac{\partial \psi}{\partial n} &= v_0, \quad \mathbf{x} \text{ on } S, \end{aligned} \quad (49)$$

then the associated pressure is $i k \rho_F G v_0$. The acoustic impedance is generally defined as the ratio of acoustic pressure to acoustic velocity. The specific acoustic impedance $Z_F(k, v_0)$ at frequency k for normal surface velocity v_0 is defined by a ratio of averaged surface pressure to averaged velocity:

$$Z_F = -ik\rho_F [(-Gv_0, v_0)/(v_0, v_0)]. \quad (50)$$

The modal specific acoustic impedance for mode m is defined as

$$Z_F^{(m)}(k) = Z_F(k, \mathbf{n} \cdot \mathbf{W}_m). \quad (51)$$

The specific acoustic impedance of the solid is defined similarly. Let $\mathbf{w}$ be the displacement in the solid for given normal velocity of v_0 on S and zero shear tractions on S ; i.e., $\mathbf{w}$ solves

$$\left. \begin{aligned} \operatorname{div} \boldsymbol{\sigma}(\mathbf{w}) + k^2 c^2 \mathbf{w} &= 0, \quad \mathbf{x} \text{ in } V, \\ \boldsymbol{\sigma} \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n}) \mathbf{n} &= 0, \\ -ik\mathbf{n} \cdot \mathbf{w} &= v_0, \end{aligned} \right\} \mathbf{x} \text{ on } S. \quad (52)$$

The impedance is defined again as the ratio of averaged surface pressure to averaged velocity:

$$Z_S(k, v_0) = \rho_S c^{-2} [(-\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{w}) \mathbf{n}, v_0)/(v_0, v_0)]. \quad (53)$$

Note that the definitions (50) and (53) contain ρ_F and $\rho_S c^{-2}$, respectively, which give Z_F and Z_S the correct dimensions. Finally, the modal specific acoustic impedance of the solid for mode m is

$$Z_S^{(m)}(k) = Z_S(k, \mathbf{n} \cdot \mathbf{W}_m). \quad (54)$$

The principle of matching impedances says that the complex resonance frequency associated with mode m is that value of k for which the impedances are identical:

$$Z_S^{(m)}(k) = Z_F^{(m)}(k). \quad (55)$$

Note that $Z_S(k, v_0)$ is imaginary for real k and v_0 and so $Z_S^{(m)}(k)$ is imaginary. The complex character of the resonance frequency is due to the fact that, for real k and v_0 , $Z_F(k, v_0)$ is complex valued rather than purely imaginary.

The condition (55) may be expressed, using (52) and the definitions of the acoustic impedances, as

$$(\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{w}) \mathbf{n}, \mathbf{n} \cdot \mathbf{W}_m) = \epsilon i k c^2 a_m(k), \quad (56)$$

where $\boldsymbol{\sigma}(\mathbf{w})$ follows from (52) with $v_0 = \mathbf{n} \cdot \mathbf{W}_m$. When $k = k_m$, the left member of (56) vanishes since $\boldsymbol{\sigma}(\mathbf{W}_m) \mathbf{n} = 0$ and thus $Z_S^{(m)}(k_m) = 0$. The surface integral may be expanded near $k = k_m$ so that the root of (56) becomes

$$k - k_m = \epsilon i k_m c^2 a_m(k_m) / (\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{w}') \mathbf{n}, \mathbf{n} \cdot \mathbf{W}_m) + O(\epsilon^2). \quad (57)$$

Here, $\mathbf{w}' = d\mathbf{w}/dk$ at $k = k_m$, and the equations for $\mathbf{w}'$ follow by differentiating (52), using $\mathbf{w} = (-ik_m)^{-1} \mathbf{W}_m + (k - k_m) \mathbf{w}' + \dots$, and then setting $k = k_m$:

$$\left. \begin{aligned} \operatorname{div} \boldsymbol{\sigma}(\mathbf{w}') + k_m^2 c^2 \mathbf{w}' &= -2ic^2 \mathbf{W}_m, \quad \mathbf{x} \text{ in } V, \\ \boldsymbol{\sigma}(\mathbf{w}') \mathbf{n} - [\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{w}') \mathbf{n}] \mathbf{n} &= 0, \\ \mathbf{n} \cdot \mathbf{w}' &= 0, \end{aligned} \right\} \mathbf{x} \text{ on } S. \quad (58)$$

Application of an identity similar to (18), and use of (16) implies that the integral in (57) is equal to $-2ic^2$, and so the complex resonant frequency is again given by $k - k_m = -\epsilon i k_m c^2 a_m(k_m) + O(\epsilon^2)$, in complete accord with (35). This result shows that the shifted position of the real resonant frequency and the width of the associated resonance agree asymptotically with predictions based upon impedance matching.

The resonant amplitude may also be determined

through the use of impedances. The usual type of pointwise impedance boundary condition is

$$\frac{\partial \phi^{\text{tot}}}{\partial n} = \frac{ik\rho_F}{Z} \phi^{\text{tot}}, \quad (59)$$

where Z is the surface acoustic impedance, assumed constant. The impedance is not uniform for the general case under consideration here, but there is an averaged impedance boundary condition

$$\left(\frac{\partial \phi^{\text{tot}}}{\partial n}, \mathbf{n} \cdot \mathbf{W}_m \right) = \frac{ik\rho_F}{Z_S^{(m)}} (\phi^{\text{tot}}, \mathbf{n} \cdot \mathbf{W}_m). \quad (60)$$

Let the total field be

$$\phi^{\text{tot}} = \phi^{\text{inc}} + \phi_0 + \phi^e, \quad (61)$$

where ϕ^e is the extra field to be determined. It follows from (26) and (60) that

$$Z_S^{(m)} \left(\frac{\partial \phi^e}{\partial n}, \mathbf{n} \cdot \mathbf{W}_m \right) - ik\rho_F (\phi^e, \mathbf{n} \cdot \mathbf{W}_m) = ik\rho_F \beta_m(k). \quad (62)$$

If it is assumed that the extra field ϕ^e is generated by the resonant mode m ; i.e.,

$$\frac{\partial \phi^e}{\partial n} = -ik\mathbf{n} \cdot \mathbf{W}_m A, \quad (63)$$

then the amplitude A follows from (50) and (62) as

$$A = \rho_F \beta_m(k) / (Z_F^{(m)} - Z_S^{(m)})(\mathbf{n} \cdot \mathbf{W}_m, \mathbf{n} \cdot \mathbf{W}_m), \quad (64)$$

which can easily be shown to be asymptotically equivalent to A_0 of (27).

IV. EXAMPLE: A SPHERICAL TARGET

A. General theory

The formalism of Sec. II is easily applied to the case of an isotropic spherical target subject to plane-wave incidence. Let $\theta = 0$ be the direction of incidence, where θ , $0 \leq \theta \leq \pi$ is the spherical polar angle. Owing to the symmetry of the scattering configuration, there is no dependence upon the azimuthal angle in any of the field variables. The rigid Green operator for an arbitrary function g defined upon S , the unit sphere (i.e., we are explicitly taking the reference length L equal to the sphere radius) can be expressed by first representing g in terms of spherical harmonics of the required symmetry:

$$g = \sum_{n=0}^{\infty} g_n P_n(\cos \theta). \quad (65)$$

Then, the associated rigid exterior field is

$$Gg = \sum_{n=0}^{\infty} \frac{h_n^{(1)}(kr)}{k h_n^{(1)'}(k)} g_n P_n(\cos \theta), \quad (66)$$

where $h_n^{(1)}$ are spherical Hankel functions. In particular, an incident plane wave of amplitude B is

$$\phi^{\text{inc}} = \sum_{n=0}^{\infty} e_n j_n(kr) P_n(\cos \theta), \quad e_n = (i)^n (2n+1)B, \quad (67)$$

and so the rigid scattered field is

$$\phi_0 = - \sum_{n=0}^{\infty} e_n \frac{j'_n(k)}{h_n^{(1)'}(k)} h_n^{(1)}(kr) P_n(\cos \theta). \quad (68)$$

The modes of free vibration of an elastic sphere may be split into families such that, for each $n = 0, 1, 2, \dots$, there is an infinite number of modal frequencies $k = k_{nl}$, $l = 1, 2, 3, \dots$, and define k_L and k_T by $k_{nl} = (c_L/c_F)k_L = (c_T/c_F)k_T$, where c_L and c_T are the longitudinal and transverse wave speeds of the isotropic solid. The modal frequencies are roots of⁷

$$d_{12}^n d_{33}^n - d_{13}^n d_{32}^n = 0, \quad (69)$$

where the terms d_{ij}^n are defined in Appendix A, and the associated modal displacement is

$$\mathbf{W}_{nl} = \nabla \phi_L + \nabla \wedge \nabla \wedge (\mathbf{x}\psi), \quad (70)$$

where

$$\begin{aligned} \phi_L &= \gamma j_n(k_L r) P_n(\cos \theta), \\ \psi &= \lambda \gamma j_n(k_T r) P_n(\cos \theta). \end{aligned} \quad (71)$$

For $n > 0$,

$$\lambda = - \frac{d_{12}^n}{d_{13}^n} = - \frac{d_{32}^n}{d_{33}^n}, \quad (72)$$

and γ in (71) is the normalization constant required by (17). It is shown in Appendix B that

$$\begin{aligned} \gamma &= [(2n+1)/4\pi]^{1/2} \left\{ F_n(k_L) + k_L j_n(k_L) j'_n(k_L) \right. \\ &\quad + \lambda^2 n(n+1) \left[F_n(k_T) + j_n^2(k_T) \right. \\ &\quad \left. \left. + k_T j_n(k_T) j'_n(k_T) + \frac{2}{\lambda} j_n(k_L) j_n(k_T) \right] \right\}^{-1/2}, \end{aligned} \quad (73)$$

where $F_n(x)$ is defined in (B6). For $n = 0$, the characteristic equation (69) factors into $d_{12}^0 = 0$, or $d_{33}^0 = 0$. The latter set of roots do not correspond to actual modes, since any ψ in (70) that does not depend upon θ automatically gives zero displacement everywhere in V . The only physical modes for $n = 0$ are the roots of $d_{12}^0 = 0$, and these are pure compressional modes with $\lambda = 0$ in (71) and (73).

Once the modal displacement is known, it is straightforward to compute the quantities $a_m(k)$ and $\beta_m(k)$ of Eq. (26) with $m \rightarrow nl$, since these only involve surface integrals of the type (B3) and (B4). For each mode let us define the constant

$$\begin{aligned} \delta_{nl} &= [4\pi/(2n+1)]^{1/2} \gamma \\ &\quad \times [k_L j'_n(k_L) + \lambda n(n+1) j_n(k_T)]. \end{aligned} \quad (74)$$

The quantity δ_{nl} should not be confused with the Kronecker delta; it will only be used in this section and always with the subscripts nl . Equations (26)₁, (65), (66), (70), and (71) imply that

$$a_{nl}(k) = - [h_n^{(1)}(k)/k h_n^{(1)'}(k)] \delta_{nl}^2, \quad (75)$$

while (26)₂, (65)–(67), (70), (71), and the Wronskian relation for spherical Bessel functions¹⁵ yield

$$\beta_{nl}(k) = [ie_n/k^2 h_n^{(1)'}(k)] [4\pi/(2n+1)]^{1/2} \delta_{nl}. \quad (76)$$

In the same manner, it follows that

$$\mathbf{Gn} \cdot \mathbf{W}_{nl} = \left(\frac{2n+1}{4\pi} \right)^{1/2} \delta_{nl} \frac{h_n^{(1)}(kr)}{h_n^{(1)'}(k)} P_n(\cos \theta). \quad (77)$$

Combining all of these results and using (29) yield the uniform asymptotic approximation to the scattered field:

$$\begin{aligned} \phi &\sim \sum_{n=0}^{\infty} \left(\frac{-j'_n(k)}{h_n^{(1)}(k)} - \frac{i\epsilon}{h_n^{(1)}(k) h_n^{(1)'}(k)} \right) \\ &\quad \times \sum_{l=1}^{\infty} \frac{a_{nl}}{(k^2 - k_{nl}^2 + \epsilon k^2 a_{nl})} \\ &\quad \times e_n h_n^{(1)}(kr) P_n(\cos \theta). \end{aligned} \quad (78)$$

The first term in the large parentheses represents the rigid scattered field.

B. Comparison with previous results

A far-field form function $f(k, \theta)$ and its n th spherical harmonic $f_n(k)$ may be defined by

$$\phi \sim B f(k, \theta) (e^{ikr}/r), \quad r \rightarrow \infty; \quad f = \sum_{n=0}^{\infty} f_n P_n(\cos \theta). \quad (79)$$

Define the phases $\xi_n(k)$ by

$$e^{i2\xi_n} = -h_n^{(2)'}(k)/h_n^{(1)'}(k), \quad n = 0, 1, 2, \dots \quad (80)$$

Then, it follows from (78)–(80), the Wronskian relation, and the asymptotic behavior of Hankel functions¹⁵ that

$$\begin{aligned} f_n &= \frac{(2n+1)}{k} e^{i2\xi_n} \left(e^{-i\xi_n} \sin(\xi_n) \right. \\ &\quad \left. + \frac{1}{1 - ix_n} \sum_{l=1}^{\infty} \frac{i\epsilon k^2 a_{nl}}{k^2 - k_{nl}^2 + \epsilon k^2 a_{nl}} \right), \end{aligned} \quad (81)$$

where

$$x_n(k) = \frac{-k^2}{2} (|h_n^{(1)}(k)|^2)'. \quad (82)$$

The scattering of acoustic waves from an elastic sphere is a classical problem with an exact analytical solution¹⁶ and a comprehensive literature, which is documented by Brill and Gaunard.¹⁷ Motivated by previous analyses in the quantum theory of scattering, Flax *et al.*⁶ and others⁵ have rearranged the solution into an approximate form suggestive of a rigid background field with an additional resonance contribution. In particular, Flax *et al.*⁶ obtained

$$\begin{aligned} f_n &= \frac{(2n+1)}{k} e^{i2\xi_n} \left(e^{-i\xi_n} \sin(\xi_n) \right. \\ &\quad \left. + \sum_{l=1}^{\infty} \frac{\frac{1}{2} \Gamma'_{nl}}{k_{nl}^r - k - \frac{1}{2} i \Gamma'_{nl}} \right), \end{aligned} \quad (83)$$

where k_{nl}^r is close to, but not exactly the same as, k_{nl} , and Γ'_{nl} is defined in Ref. 6. Comparing (81) and (83), the two expressions are seen to be very similar if $x_n \sim 0$ and

$$\Gamma'_{nl} = -i\epsilon k_{nl} a_{nl}. \quad (84)$$

The former condition is satisfied if $k_{nl} \gg 1$, since $x_n(k) = O(k^{-1})$ for $k \gg 1$. In the same limit, i.e., $k_{nl} \gg 1$, it is clear from (75) that

$$a_{nl}(k_{nl}) \sim (i/k_{nl})\delta_{nl}^2, \quad (85)$$

and so $a_{nl}(k_{nl})$ is almost purely imaginary, and (84) becomes very simply

$$\Gamma_{nl}' \sim \epsilon \delta_{nl}^2. \quad (86)$$

In summary, the uniform asymptotic approximation (81) is equivalent to the approximation (83) of resonance scattering theory⁶ if the resonance frequency k_{nl} is large.

It is well known that the exact complex resonance frequencies are roots of⁷

$$k_T^{-2} \left(\frac{d_{12}^n d_{33}^n - d_{13}^n d_{32}^n}{d_{22}^n d_{33}^n - d_{23}^n d_{32}^n} \right) = - \frac{\epsilon h_n^{(1)}(k)}{k h_n^{(1)'}(k)}, \quad (87)$$

where d_{ij}^n are defined in Appendix A. Equation (69) for the frequencies of free vibration follows by setting $\epsilon = 0$, while the complex roots of (87) for $\epsilon \ll 1$ can be found by a regular perturbation procedure. The resulting expression, when compared with (35) and (75), implies the identity

$$\delta_{nl}^2 = -2 \frac{k_T^2}{k} (d_{22}^n d_{33}^n - d_{23}^n d_{32}^n) \bigg/ \frac{d}{dk} (d_{12}^n d_{33}^n - d_{13}^n d_{32}^n) \bigg|_{k=k_{nl}}, \quad (88)$$

which may be shown analytically for $n = 0$, and we have verified (88) numerically for the remaining 22 roots of Table I with $n \geq 1$.

C. Numerical results

Since the exact solution for the elastic sphere can be obtained as a sum of partial waves,¹⁶ this canonical problem is used here as a check on the asymptotic theory. Any physical quantity associated with the scattered field depends upon the singularities in k space associated with the perturbed complex resonance frequencies of Eq. (35). The acid test of the asymptotic approximation is therefore whether or not it accurately predicts the locations of these poles in the complex k plane. The computations summarized in Table I are for a tungsten carbide sphere in water, an example that has been discussed extensively in the literature.^{5-7,17-19} The relevant densities are $\rho_F = 1 \text{ g/cm}^3$, $\rho_S = 13.80 \text{ g/cm}^3$, so that $\epsilon = 0.072464\dots$, and the wave speeds are $c_F = 1.476 \times 10^5 \text{ cm/s}$, $c_L = 6.860 \times 10^5 \text{ cm/s}$, and $c_T = 4.185 \times 10^5 \text{ cm/s}$.

The real resonance frequencies of free vibration less than $k = 32$ are listed in the left column of Table I. The complex resonance frequencies are slightly shifted from these values, and both the exact and asymptotic results are shown in Table I. Note the uniformly excellent agreement, particularly for the imaginary parts of the frequencies. The real part of the perturbations are generally much less than the imaginary parts because even for the lowest modal frequency of a radiating mode, $k_{21} = 7.46$, the approximation (85) is reasonable, and it becomes more accurate as k_{nl} increases. Also shown in Table I are the numbers δ_{nl}^2 , which

TABLE I. Comparison of the exact and asymptotic theories for the first 24 modes of free vibration of a tungsten carbide sphere in water. The real, *in vacuo* resonance frequencies are k_{nl} , where n is the spherical harmonic (there are actually no $l = 1$ modes for $n = 0$ and 1, but this ordering convention is adopted in order to remain consistent with Ref. 7). The associated exact complex resonance frequencies are k_{nl}^{ex} and the predictions of the asymptotic theory are $k_{nl}^{as} = k_{nl} - (\epsilon/2)k_{nl}a_{nl}(k_{nl})$, which follow from Eqs. (35) and (73)–(75). The numbers δ_{nl} are defined in Eq. (74). Some of the roots have been previously reported: k_{nl} and k_{nl}^{ex} , $n = 0, 1, \dots, 7$, were presented by Gaunaurd and Überall (Ref. 7), although we note that errors in k_{23} , k_{15}^{ex} , and k_{31}^{ex} are here corrected; k_{81}^{ex} , k_{91}^{ex} , and $k_{10,1}^{ex}$ were given by Williams and Marston.

n	l	k_{nl}	$k_{nl}^{ex} - k_{nl}$	$k_{nl}^{as} - k_{nl}$	δ_{nl}^2
0	3	11.469 727	(−0.009 861 − i0.114 176)	(−0.009 944 − i0.114 050)	3.171 712
0	6	28.031 617	(−0.002 842 − i0.077 151)	(−0.002 752 − i0.077 143)	2.131 854
1	2	9.415 125	(−0.002 056 − i0.019 219)	(−0.002 086 − i0.019 208)	0.530 402
1	3	18.274 795	(−0.004 295 − i0.081 086)	(−0.004 461 − i0.081 033)	2.236 591
1	4	21.635 143	(−0.000 860 − i0.012 550)	(−0.000 584 − i0.012 581)	0.347 239
1	5	30.268 063	(−0.000 032 − i0.001 354)	(−0.000 045 − i0.001 353)	0.037 349
2	1	7.466 222	(−0.012 034 − i0.080 989)	(−0.012 081 − i0.080 799)	2.157 548
2	2	13.405 441	(−0.000 412 − i0.004 907)	(−0.000 379 − i0.004 913)	0.134 121
2	3	22.984 267	(−0.001 934 − i0.054 967)	(−0.002 416 − i0.054 911)	1.509 868
2	4	26.854 539	(−0.002 136 − i0.042 597)	(−0.001 601 − i0.042 632)	1.173 412
3	1	11.042 275	(−0.011 066 − i0.112 265)	(−0.011 246 − i0.112 079)	2.969 068
3	2	17.873 799	(−0.000 002 − i0.000 017)	(−0.000 001 − i0.000 018)	0.000 478
3	3	26.985 319	(−0.001 101 − i0.038 497)	(−0.001 449 − i0.038 466)	1.054 402
4	1	14.097 086	(−0.010 193 − i0.134 169)	(−0.010 566 − i0.133 974)	3.530 248
4	2	22.422 603	(−0.000 212 − i0.003 919)	(−0.000 182 − i0.003 922)	0.106 317
4	3	30.769 789	(−0.000 732 − i0.028 428)	(−0.000 943 − i0.028 413)	0.776 732
5	1	16.959 476	(−0.009 434 − i0.152 263)	(−0.010 014 − i0.152 063)	3.990 697
5	2	26.903 860	(−0.000 536 − i0.012 046)	(−0.000 467 − i0.012 052)	0.326 172
6	1	19.732 099	(−0.008 795 − i0.168 743)	(−0.009 581 − i0.168 538)	4.409 033
6	2	31.260 255	(−0.000 860 − i0.022 432)	(−0.000 750 − i0.022 438)	0.606 531
7	1	22.455 371	(−0.008 255 − i0.184 422)	(−0.009 238 − i0.184 212)	4.806 767
8	1	25.148 206	(−0.008 108 − i0.199 700)	(−0.008 960 − i0.199 425)	5.192 789
9	1	27.820 672	(−0.007 671 − i0.214 400)	(−0.008 730 − i0.214 339)	5.571 299
10	1	30.478 666	(−0.007 467 − i0.229 000)	(−0.008 537 − i0.229 041)	5.944 529

are independent of the fluid parameters. There is obviously a large range in the values of δ_{nl}^2 . Thus the extremely small value for the (3,2) mode means that it is almost nonradiating and so is expected to have little influence on the scattered field for reasons given in Sec. III. Conversely, the $(n,1)$ modes have uniformly large, i.e., of order unity, values of δ_{nl}^2 and are expected to strongly affect the scattered field. These expectations are borne out by Fig. 2, which compares the exact backscattering amplitude of Fig. 1 with the amplitude given by the uniform asymptotic expansion. The latter was computed from Eq. (78) using the 24 radiating modes of Table I and the exact rigid scattering of Fig. 1, which was calculated by summing the first 50 spherical harmonics.

V. DISCUSSION

The test case of the sphere illustrates the accuracy of the asymptotic method; however, the full potential of this technique lies in its application to nonsymmetric 3-D targets for which "exact" numerical methods would be prohibitively time consuming and costly. One possible use for which the technique is well suited would be in determining the scattering from a target of a given shape, of which the interior is a highly complex body. One might also want to slightly modify the interior and see the result of such alterations on the response. The conventional approach to tackling such a project is to solve the fully coupled fluid-elastic scattering problem each time using one of various alternative numerical schemes. With the present method, one need only compute the exterior Green's operator G once, since the target shape is fixed. In practice, this could be achieved by discretizing the surface S and computing a matrix Green's function that gives the pressure at every point for a force at any single point, subject to the condition that the normal velocity is zero everywhere on S . Then, for each realization of the interior, the modes of free vibration could be found using existing fast and efficient codes. The modes and the exterior Green's function are then combined to compute the surface integrals of (26), after which the uniform expansion of (34) can be quickly determined.

The key step in applying the asymptotic method is in calculating the terms β_m and a_m of (26). The magnitude of a_m is closely related to the magnitude of the normal surface

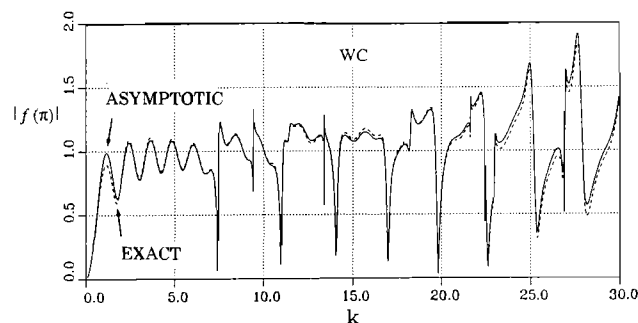


FIG. 2. The backscattering amplitude for a tungsten carbide sphere in water. The dashed curve is the exact result, as shown in Fig. 1, and the solid curve is the asymptotic approximation obtained using the first 24 modes of the sphere, i.e., the modes of resonance frequency $k_{nl} < 32$. The mode associated with each feature can be identified by referring to Table I.

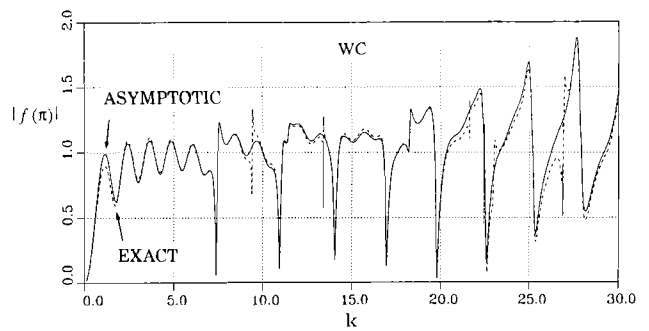


FIG. 3. The backscattering amplitude for a tungsten carbide sphere. The exact solution is the dashed line, and the solid curve is the asymptotic result using the 12 modes of frequency $k_{nl} < 32$, which are of sufficient breadth that $\delta_{nl}^2 > 2$. This removes very narrow resonances but retains the significant Rayleigh-wave resonances (see Table I).

displacement and will be small if this displacement is small. Therefore, even before the scattered field is calculated, some estimate of the radiation strength of each mode may be surmised from the surface integrals $(\mathbf{n} \cdot \mathbf{W}_m, \mathbf{n} \cdot \mathbf{W}_m)$, subject to the normalization of Eq. (17). In the example of the sphere considered in Sec. IV, the resonance width is very accurately given by $(\epsilon/2)\delta_{nl}^2$ [see Eqs. (35) and (85)], and the amplitude is proportional to δ_{nl}^2 also. Thus the broader resonances are also the stronger resonances, and a reasonable approximation might be to disregard very narrow resonances, emphasizing the effect of the strong resonances. Figure 3 shows the asymptotic backscattering amplitude computed using only the first 12 resonances below $k = 32$, which have $\delta_{nl}^2 > 2$ (see Table I). The strong $(n,1)$ Rayleigh resonances remain, but some of the other resonances of lesser significance, such as (1,2) and (2,2), have disappeared from the approximate solution. Despite this, the overall appearance of the asymptotic result in Fig. 3 is very good considering that it uses only half of the modes in that frequency range. In general, the narrow modes will not be significant since internal dissipation tends to wipe them out, leaving only the broader resonances.⁵

In conclusion, the numerical results show quite convincingly that the present asymptotic method is suitable for targets such as tungsten carbide in water, with $\epsilon = 0.07$. Future work will examine the applicability of the theory to targets for which ϵ is not quite so small.

APPENDIX A

The elements d_{12}^n and d_{13}^n , $i = 1, 2, 3$, for an elastic sphere are⁷

$$d_{12}^n = [2n(n+1) - k_T^2]j_n(k_L) - 4k_L j_n'(k_L),$$

$$d_{13}^n = 2n(n+1)[k_T j_n'(k_T) - j_n(k_T)],$$

$$d_{22}^n = k_L j_n'(k_L),$$

$$d_{23}^n = n(n+1)j_n(k_T),$$

$$d_{32}^n = 2[j_n(k_L) - k_L j_n'(k_L)],$$

$$d_{33}^n = 2k_T j_n'(k_T) + [k_T^2 - 2(n^2 + n - 1)]j_n(k_T).$$

APPENDIX B

Each mode W_m of an arbitrary isotropic elastic body V may be expressed in terms of potentials

$$W_m = \nabla \phi_L + \nabla \wedge \mathbf{f}, \quad (\text{B1})$$

where $\text{div } \mathbf{f} = 0$, and both ϕ_L and $\mathbf{f}$ satisfy Helmholtz equations. Substituting (B1) into (17) and integrating by parts yields the normalization constraint

$$\int_V (k_L^2 \phi_L^2 + k_T^2 f^2) dV + \int_S \mathbf{n} \cdot [\frac{1}{2} \nabla (\phi_L^2 + f^2) + 2\phi_L \nabla \wedge \mathbf{f} - (\mathbf{f} \cdot \nabla) \mathbf{f}] dS = 1. \quad (\text{B2})$$

For the particular case of the sphere, ϕ_L and $\mathbf{f}$ follow from (70)–(72). Using the identities¹⁵

$$\int_{-1}^1 P_n^2(\cos \theta) d \cos \theta = \frac{2}{2n+1}, \quad (\text{B3})$$

$$\int_{-1}^1 \left(\frac{d}{d\theta} P_n(\cos \theta) \right)^2 d \cos \theta = \frac{2n(n+1)}{2n+1}, \quad (\text{B4})$$

it is fairly easy to show that the volume integral in (B2) becomes

$$\begin{aligned} \int_V (k_L^2 \phi_L^2 + k_T^2 f^2) dV \\ = \frac{4\pi\gamma^2}{2n+1} [F_n(k_L) + \lambda^2 n(n+1)F_n(k_T)], \end{aligned} \quad (\text{B5})$$

where²⁰

$$\begin{aligned} F_n(x) &= \frac{1}{x} \int_0^x j_n^2(t) t^2 dt \\ &= \frac{1}{2} \{ [x^2 - n(n+1)] j_n^2(x) \\ &\quad + x^2 j_n'^2(x) + x j_n(x) j_n''(x) \}. \end{aligned} \quad (\text{B6})$$

The surface integrals in (B2) can be evaluated directly by again using (B3) and (B4), and, when combined with (B5), the normalization factor γ of (73) is obtained.

- ¹F. G. Leppington, "Scattering of sound waves by finite membranes and plates near resonance," *Q. J. Mech. Appl. Math.* **29**, 527–546 (1976).
- ²I. D. Abrahams, "Scattering of sound by finite elastic surfaces bounding ducts or cavities near resonance," *Q. J. Mech. Appl. Math.* **35**, 91–101 (1982).
- ³G. A. Kriegsmann, A. N. Norris, and E. L. Reiss, "Acoustic scattering by baffled membranes," *J. Acoust. Soc. Am.* **75**, 685–694 (1984).
- ⁴G. A. Kriegsmann, A. N. Norris, and E. L. Reiss, "Scattering by penetrable acoustic targets," *Wave Motion* **6**, 501–516 (1984).
- ⁵G. C. Gaunaurd, "Elastic and acoustic resonance wave scattering," *Appl. Mech. Rev.* **42**, 143–192 (1989).
- ⁶L. Flax, G. C. Gaunaurd, and H. Überall, "Theory of resonance scattering," in *Physical Acoustics*, edited by W. P. Mason and R. N. Thurston (Academic, New York, 1981), Vol. XV, pp. 191–294.
- ⁷G. C. Gaunaurd and H. Überall, "RST analysis of monostatic and bistatic acoustic echoes from an elastic sphere," *J. Acoust. Soc. Am.* **73**, 1–12 (1983).
- ⁸D. Brill and G. C. Gaunaurd, "Acoustic resonance scattering by a penetrable cylinder," *J. Acoust. Soc. Am.* **73**, 1448–1455 (1983).
- ⁹M. F. Werby, H. Überall, A. Nagl, S. H. Brown, and J. W. Dickey, "Bistatic scattering and identification of resonances of elastic spheroids," *J. Acoust. Soc. Am.* **84**, 1425–1436 (1988).
- ¹⁰M. C. Junger and D. Feit, *Sound, Structures, and Their Interaction* (MIT, Cambridge, MA, 1972).
- ¹¹J. D. Cole, *Perturbation Methods in Applied Mathematics* (Blaisdell, Waltham, MA, 1968).
- ¹²C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
- ¹³A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1944), pp. 284, 285, and 288.
- ¹⁴D. S. Jones, "Low frequency scattering by a body in lubricated contact," *Q. J. Mech. Appl. Math.* **36**, 111–138 (1983).
- ¹⁵M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
- ¹⁶J. J. Faran, "Sound scattered by solid cylinders and spheres," *J. Acoust. Soc. Am.* **23**, 405–418 (1951).
- ¹⁷D. Brill and G. C. Gaunaurd, "Resonance theory of elastic waves ultrasonically scattered from an elastic sphere," *J. Acoust. Soc. Am.* **81**, 1–21 (1987).
- ¹⁸K. L. Williams and P. L. Marston, "Axially focused (glory) scattering due to surface waves generated on spheres: Model and experimental confirmation using tungsten carbide spheres," *J. Acoust. Soc. Am.* **78**, 722–728 (1985).
- ¹⁹K. L. Williams and P. L. Marston, "Backscattering from an elastic sphere: Sommerfeld–Watson transformation and experimental confirmation," *J. Acoust. Soc. Am.* **78**, 1093–1102 (1985).
- ²⁰I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980), Eq. (5.54.2).