33

ON THE ACOUSTIC DETERMINATION OF THE ELASTIC MODULI OF ANISOTROPIC SOLIDS AND ACOUSTIC CONDITIONS FOR THE EXISTENCE OF SYMMETRY PLANES

By A. N. NORRIS

(*Department of Mechanics and Materials Science, Rutgers University, PO Box* 909, *Piscataway, New Jersey* 08855-0909, *USA*)

[Received 11 July 1988. Revise 29 September 1988]

SUMMARY

The twenty-one elastic moduli of a homogeneous anisotropic solid can be determined from the second-order acoustical tensors associated with wave motion in six phase directions. The directions may be quite arbitrary as long as they cannot be contained by less than three distinct planes through the origin and do not all lie on the curves formed by the intersection of the unit sphere with an elliptical cone. Two equivalent sets of conditions necessary and sufficient for the existence of a plane of material symmetry in an elastic solid are presented. The conditions are phrased in terms of acoustic waves, the first set involving polarization vectors, the second energy-flux vectors. Some consequences of the acoustic conditions are noted.

1. Introduction

THE subject of this paper concerns some general acoustic properties of homogeneous anisotropic elastic solids. The first issue addressed is the problem of determining the elastic constants of a given material. In a recent paper Van Buskirk, Cowin and Carter (**1**) proposed an acoustic-wave approach in which by a series of experiments one can obtain a sufficiency of data for determining all 21 independent moduli of an arbitrarily anisotropic specimen. Their procedure offers an alternative to the purely statical series of measurements proposed by Hayes (**2**) and may be preferable if only a single small specimen of the material is at hand. The method outlined in sections 2 and 3 generalizes that of Van Buskirk *et al.* (**1**). The nature of the acoustic data required is such that one must be able to measure the three phase speeds and associated polarizations for six different phase directions. The question of what geometrical limitations are involved in the choice of the six directions is answered by Theorem 1 in section 4.

The remainder of the paper discusses conditions necessary and sufficient for the existence of a plane of material symmetry. These conditions, derived by Cowin and Mehrabadi (**3**) and simplified by Cowin (**4, 5**), consist of algebraic relations satisfied by both the elastic stiffness and compliance tensors. The symmetry conditions are presented in section 5 in the concise form due to Cowin (**5**) although the present derivation differs slightly from his (this is based upon information provided by a referee; I have not read the forthcoming paper (**5**)). Two interpretations of the algebraic conditions are given in section 6 in terms of acoustic waves. The first interpretation, Theorem 3, has previously been noted and discussed by Cowin (**4, 5**). Theorem 4 phrases the symmetry conditions as constraints upon energy-flux vectors and appears to be new. Some consequences of these observations are explored in section 7, where it is demonstrated for a material possessing a plane of symmetry that at least three distinct coordinate bases exist in which the stiffness tensor has only 12 non-zero components.

2. Definitions and preliminary results

The following notation will be adhered to as far as possible: fourth-order tensors and 6×6 matrices are denoted by capital letters, for example, $\mathbf{C}$, $\mathbf{B}$; second-order tensors and six-dimensional vectors by lower case letters, for example, $\mathbf{d}$, $\mathbf{v}$; and unit vectors in $\mathbb{R}^3$ by Greek letters, for example, $\boldsymbol{\nu}$. The summation convention on repeated subscripts is assumed with exceptions to the rule noted.

Let C_{ijkl} be the Cartesian components of the tensor of moduli for a homogeneous elastic solid, where i, j, k and l run from 1 to 3, and $\mathbf{C}$ possesses the symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}. \tag{2.1}$$

Define the tensor $\mathbf{B}$ by its components

$$B_{ikjl} = \tfrac{1}{2}(C_{ijkl} + C_{ilkj}); \tag{2.2}$$

then $\mathbf{B}$ also has the symmetries

$$B_{ikjl} = B_{kijl} = B_{jlik}. \tag{2.3}$$

Each of $\mathbf{B}$ and $\mathbf{C}$ has at most 21 independent components. An important property of $\mathbf{B}$, and the one vital to the present purpose, is that the components of $\mathbf{B}$ uniquely define those of $\mathbf{C}$. It is clear from (2.1) and (2.2) that

$$C_{ijil} = B_{iijl} \qquad \text{(no sum)}, \tag{2.4}$$

which imply 15 components of $\mathbf{C}$ and

$$C_{iikl} = 2B_{ikil} - B_{iikl} \qquad \text{(no sum)}, \tag{2.5}$$

which yield the remaining six.

The 21 components of $\mathbf{C}$ or $\mathbf{B}$ may be represented by the usual 6×6

symmetric matrix notation, for example,

$$\mathbf{C}=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}, \tag{2.6}$$

where $C_{IJ}=C_{ijkl}$ and $I=1, 2, 3, 4, 5, 6$ correspond to $ij=11, 22, 33, 23, 13$ and 12, respectively. Equation (2.2) becomes

$$\mathbf{B}=\begin{bmatrix} C_{11} & C_{66} & C_{55} & C_{56} & C_{15} & C_{16} \\ & C_{22} & C_{44} & C_{24} & C_{46} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{45} \\ & & & \frac{1}{2}(C_{44}+C_{23}) & \frac{1}{2}(C_{45}+C_{36}) & \frac{1}{2}(C_{46}+C_{25}) \\ & & & & \frac{1}{2}(C_{55}+C_{13}) & \frac{1}{2}(C_{56}+C_{14}) \\ & & & & & \frac{1}{2}(C_{66}+C_{12}) \end{bmatrix} \tag{2.7}$$

and (2.4) and (2.5) are

$$\mathbf{C}=\begin{bmatrix} B_{11} & 2B_{66}-B_{12} & 2B_{55}-B_{13} & 2B_{56}-B_{14} & B_{15} & B_{16} \\ & B_{22} & 2B_{44}-B_{23} & B_{24} & 2B_{46}-B_{25} & B_{26} \\ & & B_{33} & B_{34} & B_{35} & 2B_{45}-B_{36} \\ & & & B_{23} & B_{36} & B_{25} \\ & & & & B_{13} & B_{14} \\ & & & & & B_{12} \end{bmatrix}. \tag{2.8}$$

3. The acoustical tensor and determining the moduli

Define the associated acoustical tensor $\mathbf{a}(\mathbf{v})$ for the direction $\mathbf{v}$ as

$$a_{ik}(\mathbf{v})=C_{ijkl}v_jv_l=B_{ikjl}v_jv_l. \tag{3.1}$$

Alternatively, let the vectors $\mathbf{v}$ and $\mathbf{a}$ in $\mathbb{R}^6$ be

$$\mathbf{v}(\mathbf{v})=[v_1^2,\ v_2^2,\ v_3^2,\ 2v_2v_3,\ 2v_1v_3,\ 2v_1v_2]^T, \tag{3.2}$$

$$\mathbf{a}(\mathbf{v})=[a_{11},\ a_{22},\ a_{33},\ a_{23},\ a_{13},\ a_{12}]^T. \tag{3.3}$$

Then by (3.1) to (3.3),

$$\mathbf{a}(\mathbf{v}) = \mathbf{B}\mathbf{v}(\mathbf{v}). \tag{3.4}$$

The acoustical tensor $\mathbf{a}(\mathbf{v})$ can be found in principle through the identity

$$a_{ik}(\mathbf{v}) = \rho \sum_{\alpha=1,3} c^{(\alpha)^2} \xi_i^{(\alpha)} \xi_k^{(\alpha)}. \tag{3.5}$$

Here ρ is the material density, $c^{(\alpha)}$, $\alpha = 1, 2, 3$, are the phase speeds for wave motion with plane phase fronts perpendicular to $\mathbf{v}$, and $\boldsymbol{\xi}^{(\alpha)}$ are the associated polarization directions. If these quantities can be measured experimentally by a series of ultrasonic inspections of a sample then $\mathbf{a}(\mathbf{v})$ may be determined. The practical issues involved in the measurements are discussed by Van Buskirk *et al.* (**1**) and also by Cowin (**4**). It is assumed for the purposes of this paper that $\mathbf{a}(\mathbf{v})$ can be found.

It is clear from (3.4) that six relations of this type are necessary to determine $\mathbf{B}$. Let $\mathbf{v}^{(I)}$, $I = 1, 2, \ldots, 6$, be a set of six unit vectors in $\mathbb{R}^3$ and let $\mathbf{v}^{(I)}$, $\mathbf{a}^{(I)}$ be the corresponding vectors in $\mathbb{R}^6$ defined through (3.2) and (3.3). Define the 6×6 matrices

$$\mathbf{A} = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}, \mathbf{a}^{(4)}, \mathbf{a}^{(5)}, \mathbf{a}^{(6)}], \tag{3.6}$$

and

$$\mathbf{V} = [\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \mathbf{v}^{(4)}, \mathbf{v}^{(5)}, \mathbf{v}^{(6)}]. \tag{3.7}$$

Then

$$\mathbf{A} = \mathbf{B}\mathbf{V}. \tag{3.8}$$

If the inverse to $\mathbf{V}$ exists such $\mathbf{V}\mathbf{V}^{-1} = \mathbf{V}^{-1}\mathbf{V} = \operatorname{diag}(1, 1, 1, 1, 1, 1)$, then

$$\mathbf{B} = \mathbf{A}\mathbf{V}^{-1}, \tag{3.9}$$

and the moduli follow from (2.8).

The general result expressed by (3.9) requires that both the wave speeds and polarizations be known. Noting from (3.5) that

$$\operatorname{Tr} \mathbf{a}(\mathbf{v}) = \rho \sum_{\alpha=1,3} c^{(\alpha)^2}, \tag{3.10}$$

then (3.9) in combination with (2.7) and (3.6) yields

$$\begin{bmatrix} C_{11} + C_{55} + C_{66} \\ C_{22} + C_{44} + C_{66} \\ C_{33} + C_{44} + C_{55} \\ C_{24} + C_{34} + C_{56} \\ C_{15} + C_{35} + C_{46} \\ C_{16} + C_{26} + C_{45} \end{bmatrix} = [\operatorname{Tr} \mathbf{a}^{(1)}, \operatorname{Tr} \mathbf{a}^{(2)}, \operatorname{Tr} \mathbf{a}^{(3)}, \operatorname{Tr} \mathbf{a}^{(4)}, \operatorname{Tr} \mathbf{a}^{(5)}, \operatorname{Tr} \mathbf{a}^{(6)}]\mathbf{V}^{-1}. \tag{3.11}$$

In this way six elastic constants may be obtained from the 18 speeds without recourse to the polarizations. If the latter are unknown it is not clear how the remaining 12 wave-speed data can best be used to find the other constants. At least three more speeds will probably be required since there are 21 independent moduli.

The procedure of Van Buskirk, Cowin and Carter

Van Buskirk, Cowin and Carter (**1**) proposed a particularly convenient set of six unit vectors $\mathbf{v}^{(I)}$, $I = 1, 2, \ldots, 6$, as three orthogonal vectors and their three bisectors:

$$\left.\begin{aligned} &\mathbf{v}^{(1)} = (1, 0, 0)^T, \qquad \mathbf{v}^{(2)} = (0, 1, 0)^T, \qquad \mathbf{v}^{(3)} = (0, 0, 1)^T, \\ &\mathbf{v}^{(4)} = \frac{1}{\sqrt{2}}(\mathbf{v}^{(2)} + \mathbf{v}^{(3)}), \qquad \mathbf{v}^{(5)} = \frac{1}{\sqrt{2}}(\mathbf{v}^{(1)} + \mathbf{v}^{(3)}), \qquad \mathbf{v}^{(6)} = \frac{1}{\sqrt{2}}(\mathbf{v}^{(1)} + \mathbf{v}^{(2)}). \end{aligned}\right\} \tag{3.12}$$

The matrix $\mathbf{V}$ and its inverse are

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.13}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.14}$$

The relatively simple expressions for the moduli C_{IJ} given by Van Buskirk *et al.* (**1**) then follow from (2.8), (3.3), (3.6), (3.9) and (3.14).

It is interesting that for this particular set of $\mathbf{v}^{(I)}$ an explicit tensorial relation for $\mathbf{B}$ can be obtained from the identity

$$\begin{aligned} B_{ikjl} &= B_{ikpq}\delta_{pj}\,\delta_{ql} \\ &= B_{ikpq}\Big(\sum_{\alpha=1,3} v_p^{(\alpha)} v_j^{(\alpha)}\Big)\Big(\sum_{\beta=1,3} v_q^{(\beta)} v_l^{(\beta)}\Big). \end{aligned} \tag{3.15}$$

Expanding the last expression and using (3.1) with identities of the form

$$v_p^{(\alpha)}v_q^{(\beta)} + v_q^{(\alpha)}v_p^{(\beta)} = 2v_p^{(9-\alpha-\beta)}v_q^{(9-\alpha-\beta)} - v_p^{(\alpha)}v_q^{(\alpha)} - v_p^{(\beta)}v_q^{(\beta)}, \quad (3.16)$$

where $1 \leqslant \alpha < \beta \leqslant 3$, one arrives at

$$B_{ikjl} = \sum_{\alpha=1,3} a_{ik}^{(\alpha)}\left\{v_j^{(\alpha)}v_l^{(\alpha)} - \frac{1}{\sqrt{2}}[v_j^{(\alpha)}v_l^{(\alpha+3)} + v_l^{(\alpha)}v_j^{(\alpha+3)}]\right\} + \\ + \sum_{\substack{\alpha,\beta=1,3\\ \beta>\alpha}} a_{ik}^{(9-\alpha-\beta)}[v_j^{(\alpha)}v_l^{(\beta)} + v_l^{(\alpha)}v_j^{(\beta)}]. \quad (3.17)$$

4. Geometrical limitations on the six vectors

The six unit vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(6)} \in \mathbb{R}^3$ may be quite arbitrary in general as long as $\mathbf{V}$ is invertible, which enables $\mathbf{B}$ to be found from (3.9). The geometrical constraint imposed by this requirement can be found by examining the circumstances under which it is violated.

The six associated vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(6)} \in \mathbb{R}^6$ are linearly dependent, or equivalently $\mathbf{V}$ is not invertible, if there exists a non-zero vector $\mathbf{b} \in \mathbb{R}^6$ such that

$$\mathbf{b}^T\mathbf{V} = \mathbf{0}. \quad (4.1)$$

Hence, from (4.1) and (3.2),

$$b_1v_1^2 + b_2v_2^2 + b_3v_3^2 + 2b_4v_2v_3 + 2b_5v_1v_3 + 2b_6v_1v_2 = 0, \quad (4.2)$$

or equivalently

$$\mathbf{v}^T\mathbf{b}\mathbf{v} = 0, \quad (4.3)$$

where

$$\mathbf{b} = \begin{bmatrix} b_1 & b_6 & b_5 \\ b_6 & b_2 & b_4 \\ b_5 & b_4 & b_3 \end{bmatrix}. \quad (4.4)$$

Let $\mathbf{p}$ be the orthogonal matrix that diagonalizes $\mathbf{b}$ as

$$\mathbf{pbp}^T = \mathrm{diag}\,(d_1, d_2, d_3), \quad (4.5)$$

where d_1, d_2, and d_3 are the eigenvalues of $\mathbf{b}$; then (4.2) becomes

$$d_1\mu_1^2 + d_2\mu_2^2 + d_3\mu_3^2 = 0, \quad (4.6)$$

and

$$\boldsymbol{\mu} = \mathbf{p}\mathbf{v}. \quad (4.7)$$

Equations (4.2) and (4.6) represent a quadratic surface in $\mathbb{R}^3$. Three possibilities arise: (i) all three eigenvalues are non-zero, (ii) one is zero, or (iii) two are zero.

Case (i). If all three of d_1, d_2 and d_3 are non-zero, one must be of opposite sign to the other two. Specifically let d_1 and d_2 be positive and $d_3 = -1$, then the surface is

$$d_1\mu_1^2 + d_2\mu_2^2 = \mu_3^2, \tag{4.8}$$

which is an elliptical cone. The intersection of the cone with the unit sphere forms two similar curves whose projections in the plane $\mu_3 = 0$ are the ellipse

$$(1+d_1)\mu_1^2 + (1+d_2)\mu_2^2 = 1. \tag{4.9}$$

Thus, if all six $\mathbf{v}^{(I)}$ lie on an elliptical cone, the image vectors $\mathbf{v}^{(I)}$ do not span $\mathbb{R}^6$. Equivalently, the $\mathbf{v}^{(I)}$ are linearly dependent if the projections of the six $\mathbf{v}^{(I)}$ on some plane lie on an ellipse.

Case (ii). Let $d_3 = 0$; then the surface is

$$d_1\mu_1^2 + d_2\mu_2^2 = 0, \tag{4.10}$$

which implies that d_1 and d_2 are of opposite sign, and the surface is two planes through the origin.

Case (iii). Let $d_2 = d_3 = 0$; then the surface is

$$d_1\mu_1^2 = 0, \tag{4.11}$$

which is a single plane.

Further degeneracy must be contained in these three cases. The following summarizes the results.

THEOREM 1. *A set of necessary and sufficient conditions that the* 6×6 *matrix* $\mathbf{V}$ *formed from* $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(6)}$ *be invertible are that the six vectors do not lie on a cone through the origin and cannot be contained in less than three distinct planes through the origin.*

The results of Theorem 1 are also relevant to the acoustical measurement of tensorial quantities other than the elastic moduli. For instance, Kohn and Rice (**6**) showed that the long-wavelength elastic scattering from an anisotropic defect in a uniform isotropic medium is characterized by 22 parameters. These are the excess mass of the defect and the 21 independent components of a tensor $\mathbf{D}$ possessing the symmetries $D_{ijkl} = D_{jikl} = D_{klij}$. This fourth-rank tensor depends upon the deviation in the modulus tensors of the medium and the defect, and upon the shape and orientation of the defect. If the excess mass is known, the scalar $D_{ijkl}\eta_i\eta_j\mu_k\mu_l$ can be related to the scattered longitudinal amplitude in the direction $\boldsymbol{\mu}$ for an incident plane wave in the direction $\boldsymbol{\eta}$. Therefore, the 36 separate measurements of the scattered longitudinal-to-longitudinal amplitude for six directions of incidence and six directions of scattering gives enough data to determine all 21 components of $\mathbf{D}$, provided the two sets of six vectors are independent in the sense of Theorem 1. For example, the directions $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(6)}$ of (3.12) could be used for both the directions of incidence and scattering.

5. Necessary and sufficient conditions for a plane of symmetry

The remainder of this paper is concerned with the identification of material symmetry in elastic solids. Having determined the tensor **C** by the acoustic procedure discussed above, or by other means (**2**), one would like to determine if the material possesses a plane of symmetry. This problem was first solved by Cowin and Mehrabadi (**3**) and the following simplified version is due to Cowin (**4, 5**).

THEOREM 2. *The necessary and sufficient conditions that the direction* ξ *be normal to a plane of symmetry are*

$$C_{ijkl}\xi_j\xi_k\xi_l = (C_{pqrs}\xi_p\xi_q\xi_r\xi_s)\xi_i, \tag{5.1}$$

$$C_{ijkl}v_jv_k\xi_l = (C_{pqrs}v_pv_r\xi_q\xi_s)\xi_i, \tag{5.2}$$

for all directions $\mathbf{v}$ *perpendicular to* ξ.

The original theorem of Cowin and Mehrabadi (**3**) phrased the solution in a slightly different manner. In addition to the relations (5.1) and (5.2) they gave the conditions

$$C_{ikkj}\xi_j = (C_{pkkq}\xi_p\xi_q)\xi_i, \tag{5.3}$$

$$C_{ijkk}\xi_j = (C_{pqkk}\xi_p\xi_q)\xi_i. \tag{5.4}$$

Let $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ be two orthogonal directions each orthogonal to ξ. Then (5.3) follows from (5.1) by adding (5.2) for $\mathbf{v} = \mathbf{v}^{(1)}$ and $\mathbf{v} = \mathbf{v}^{(2)}$, and using the identity

$$\delta_{jk} = \xi_j\xi_k + v_j^{(1)}v_k^{(1)} + v_j^{(2)}v_k^{(2)}. \tag{5.5}$$

In regard to (5.4), for $\alpha = 1$ and 2

$$\begin{aligned} C_{ijkk}\xi_jv_i^{(\alpha)} &= C_{ijkl}(\xi_k\xi_l + v_k^{(1)}v_l^{(1)} + v_k^{(2)}v_l^{(2)})\xi_jv_i^{(\alpha)} \\ &= C_{ijkl}\xi_k\xi_l\xi_jv_i^{(\alpha)} + C_{ijkl}v_i^{(\alpha)}v_k^{(1)}v_l^{(1)}\xi_j + C_{ijkl}v_i^{(\alpha)}v_k^{(2)}v_l^{(2)}\xi_j. \end{aligned} \tag{5.6}$$

The first term on the right-hand side of (5.6) vanishes using (5.1) and the second and third terms are zero by virtue of (5.2). Therefore $C_{ijkk}\xi_j$ is perpendicular to $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$, so it must be parallel to ξ, and (5.4) follows.

The relations (5.3) and (5.4) are therefore consequences of (5.1) and (5.2). The former are useful (**3, 4**) if one is interested in finding the planes of symmetry of a given elastic tensor **C**. Thus (5.3) and (5.4) imply that ξ must be a common eigenvector of the second-order tensors C_{ikkj} and C_{ijkk}. Any common eigenvectors that also satisfy (5.1) and (5.2) are automatically normal to symmetry planes. Conversely, the material has no planes of symmetry if C_{ikkj} and C_{ijkk} have distinct eigenvectors. The six-vector corresponding to the second-rank tensor C_{ikkj} is equal to the left-hand side of (3.11) and thus can be found using only wave speeds. Alternatively, C_{ikkj} can be determined from the acoustical tensors associated with three orthogonal directions. Let $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$ be mutually orthogonal, then it follows from (3.1) that

$$C_{ikkj} = a_{ij}(\mathbf{v}^{(1)}) + a_{ij}(\mathbf{v}^{(2)}) + a_{ij}(\mathbf{v}^{(3)}). \tag{5.7}$$

Some information may therefore be gleaned from the acoustical data for three orthogonal directions. Suppose that the material is known to possess orthotropic symmetry, or three planes of symmetry, and the orientations of the symmetry axes are to be determined. Then the three eigenvectors of C_{ikkj} correspond to the desired axes. Fedorov (**7**) discussed further the tensor C_{ikkj} which he called the convoluted tensor for the elastic moduli. The full ramifications of (5.1) to (5.4) were investigated in detail by Cowin and Mehrabadi (**3**).

6. Equivalent acoustic conditions for a plane of symmetry

Consider a wave represented by the displacement field

$$u_i(\mathbf{x}, t) = \mu_i F(\boldsymbol{\gamma} \cdot \mathbf{x}/c - t), \tag{6.1}$$

where $\boldsymbol{\mu}$ is the polarization direction, $\boldsymbol{\gamma}$ is the phase direction, c is the phase speed and F is some C^2 function. The following relations must be satisfied (for example, (**7**)):

$$C_{ijkl}\gamma_j\gamma_k\mu_l = (C_{pqrs}\gamma_p\gamma_r\mu_q\mu_s)\mu_i, \tag{6.2}$$

$$\rho c^2 = C_{ijkl}\gamma_i\gamma_k\mu_j\mu_l. \tag{6.3}$$

The wave is called longitudinal if $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are parallel and transverse if they are orthogonal. The conditions (5.1) and (5.2) can then clearly be rephrased in the following statement which is due to Cowin (**4, 5**).

THEOREM 3. *The direction* ξ *is normal to a plane of symmetry if it is a longitudinal direction and if a transverse wave polarized in the* ξ*-direction exists for any phase direction perpendicular to* ξ.

Kolodner (**8**) proved that three or more longitudinal directions exist in every anisotropic solid. Consequently the first condition in Theorems 2 and 3 is satisfied by at least three directions. If for one of these directions the plane normal to it supports transverse waves in the ξ-direction for any phase vector in the plane then the plane is one of symmetry. Fedorov (**7**) observed that the acoustic conditions of Theorem 3 are satisfied when ξ is normal to a plane of symmetry, although he does not appear to have observed that these same conditions are sufficient for ξ to be normal to a symmetry plane. It is also possible to state a set of conditions which are equivalent to Theorems 2 and 3 but which involves waves propagating in the direction ξ only. This requires the notion of the energy-flux vector and is as follows.

THEOREM 4. *Necessary and sufficient conditions that the direction* ξ *be normal to a plane of symmetry are that it be a longitudinal direction and that the energy-flux vector be in the direction* ξ *for any* combination *of transverse waves with phase direction* ξ.

Proof. Sufficiency will be demonstrated first. Equation (5.1) follows from the fact that ξ is a longitudinal direction. Let $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ be the other eigenvectors of $C_{ijkl}\xi_j\xi_l$, so that $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \xi)$ form an orthonormal triad. The most general form of transverse wave motion with phase direction ξ is of the form

$$\mathbf{u} = \mathbf{v}^{(1)}F_1(\phi^{(1)}) + \mathbf{v}^{(2)}F_2(\phi^{(2)}), \tag{6.4}$$

where

$$\phi^{(\alpha)} = \xi \,.\, \mathbf{x}/c^{(\alpha)} - t, \qquad \alpha = 1 \text{ or } 2, \tag{6.5}$$

F_1 and F_2 are arbitrary C^2 functions, and $c^{(1)}$, $c^{(2)}$ are phase speeds. The i-component of the energy flux vector is

$$\begin{aligned} -\sigma_{ij}\frac{\partial u_j}{\partial t} &= -C_{ijkl}\frac{\partial u_j}{\partial t}\frac{\partial u_k}{\partial x_l} \\ &= \sum_{\alpha,\beta=1,2} c^{(\beta)-1}C_i^{(\alpha\beta)}F'_\alpha(\phi^{(\alpha)})F'_\beta(\phi^{(\beta)}), \end{aligned} \tag{6.6}$$

where

$$C_i^{(\alpha\beta)} = C_{ijkl}v_j^{(\alpha)}v_k^{(\beta)}\xi_l, \qquad \alpha,\, \beta = 1, 2. \tag{6.7}$$

The vector in (6.6) is parallel to ξ by hypothesis. Selecting in turn F'_1 and F'_2 to be identically zero implies that (5.2) holds for $\mathbf{v} = \mathbf{v}^{(2)}$ and $\mathbf{v} = \mathbf{v}^{(1)}$, respectively. Consequently,

$$C_i^{(12)}v_i^{(2)} = C_i^{(21)}v_i^{(1)} = 0, \tag{6.8}$$

and in addition, as a result of (5.1),

$$C_i^{(12)}\xi_i = C_i^{(21)}\xi_i = 0. \tag{6.9}$$

When F'_1 and F'_2 are both non-zero, equation (6.6) implies that

$$c^{(1)}C_i^{(12)}v_i^{(\alpha)} + c^{(2)}C_i^{(21)}v_i^{(\alpha)} = 0, \qquad \alpha = 1, 2. \tag{6.10}$$

Therefore, from (6.8) to (6.10),

$$C_i^{(12)} = C_i^{(21)} = 0, \tag{6.11}$$

and (5.2) follows since any $\mathbf{v}$ perpendicular to ξ can be written as a linear combination of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$.

It remains to show that the conditions of Theorem 4 are necessary if ξ is normal to a plane of symmetry. To this end it will be shown that (6.11) is a consequence of (5.1) and (5.2). Equation (5.2) for

$$\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{v}^{(1)} + \mathbf{v}^{(2)}), \qquad \mathbf{v} = \mathbf{v}^{(1)}, \qquad \mathbf{v} = \mathbf{v}^{(2)},$$

in combination with (5.1), yields after some elimination

$$C_i^{(12)} + C_i^{(21)} = 0. \tag{6.12}$$

Equations (6.11) follow from (6.8), (6.9) and (6.12), thus completing the proof.

The proof of Theorem 4 relies upon the condition that *any* transverse wave motion has energy flux in the ξ-direction. It is not enough to assume that each transverse mode has energy flux parallel to ξ as this does not imply (6.11). These relations follow from the requirement that the cross term in the flux be parallel to ξ for a combination of transverse waves. The cross term turns out to be identically zero, so that the flux of the sum of the two distinct transverse modes is equal to the sum of the individual fluxes. The longitudinal wave automatically has energy flux in its direction of propagation.

7. Discussion

As mentioned above, Kolodner (**8**) showed for an arbitrarily anisotropic elastic solid that there are always at least three directions which satisfy (5.1). Let ξ be one of these directions and let $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ be the associated orthogonal eigenvectors of $C_{ijkl}\xi_j\xi_k$. With no loss in generality let $\mathbf{v}^{(1)} = (1, 0, 0)^T$, $\mathbf{v}^{(2)} = (0, 1, 0)^T$ and $\xi = (0, 0, 1)^T$. Then (5.1) implies that

$$C_{i333} = C_{3333}\delta_{i3}, \tag{7.1}$$

while the eigenvalue equations for $\mathbf{v}^{(\alpha)}$ yield

$$C_{i33\alpha} = C_{\alpha 33\alpha}\delta_{i\alpha}, \qquad \alpha = 1, 2 \quad \text{(no sum on } \alpha). \tag{7.2}$$

These relations imply that

$$C_{34} = C_{35} = C_{45} = 0, \tag{7.3}$$

and so there are only 18 independent elastic constants in this particular basis. The remaining three parameters can be thought of as the Euler angles that define the basis $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \xi)$. The fourth-order tensor still has 21 independent constants but has only 18 when referred to this particular coordinate system. It has long been known (**7, 9, 10**) that three of the 21 elastic constants can be set to zero by an appropriate choice of coordinate system. However, the existence of at least three such systems does not appear to have been previously noted.

Consider a material possessing a single plane of symmetry, that is, monoclinic, for which it follows from (5.1) and (5.2) (**3**), or by other means, for example, (**9**), that the eight moduli C_{14}, C_{15}, C_{24}, C_{25}, C_{34}, C_{35}, C_{46} and C_{56} are all zero. Equation (7.3) also holds for the particular basis chosen, and therefore only twelve of the moduli are non-zero when the coordinate axes are defined by the polarization axes for phase vector in the normal direction. The tensor $\mathbf{C}$ for a monoclinic solid thus has 15 independent parameters: 12 non-zero moduli plus three Euler angles to define the basis (**7, 9**). Alternatively, there are 13 non-zero moduli plus two Euler angles to define the normal direction.

There are two other specific bases with respect to which a monoclinic solid has only 12 non-zero moduli, although one of these bases may coincide with the one previously discussed. The two bases are defined by the longitudinal axes in the plane $\boldsymbol{\xi}\cdot\mathbf{x}=0$, and their existence can be proved as follows. Let $\mathbf{v}$ be any direction in this plane, then it follows from (5.2) that two of the eigenvectors of $C_{ijkl}v_jv_l$ lie in the plane because the third is automatically in the $\boldsymbol{\xi}$-direction. Define the map $\mathbf{v}\to f(\mathbf{v})\mathbf{v}$, where

$$f(\mathbf{v})=C_{ijkl}v_iv_jv_kv_l, \tag{7.4}$$

which maps the unit circle in the plane $\boldsymbol{\xi}\cdot\mathbf{x}=0$ onto a continuous curve about the origin in the same plane, with reflection symmetry about the origin. The curve must therefore have at least four points at which its normal is in the same direction as $\mathbf{v}$. Two of these points are images of the other two and the pair of distinct directions defines longitudinal axes in the plane of symmetry. Let $\mathbf{v}^{(1)}$ be one such direction and define the basis $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \boldsymbol{\xi})$, where $\mathbf{v}^{(2)}$ is orthogonal to $\mathbf{v}^{(1)}$ and $\boldsymbol{\xi}$, that is, it is the vector orthogonal to $\mathbf{v}^{(1)}$ in the plane of symmetry. Again let $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\boldsymbol{\xi}$ be the (x_1, x_2, x_3)-coordinate axes, respectively, so that

$$C_{14}=C_{15}=C_{24}=C_{25}=C_{34}=C_{35}=C_{46}=C_{56}=0,$$

as before. Furthermore, a longitudinal wave exists for phase direction $\mathbf{v}^{(1)}$, so

$$C_{i111}=C_{1111}\,\delta_{i1}. \tag{7.5}$$

The additional identity $C_{16}=0$ follows from this equation with $i=2$.

In summary, these results may be stated as follows.

THEOREM 5. *An elastic solid has at least three coordinate systems with respect to which there are only* 18 *non-zero elastic constants. If the solid possesses a plane of symmetry, three of the coordinate systems have as common direction the normal to the plane of symmetry and the solid has* 12 *non-zero moduli when referred to these coordinate systems.*

It is well known (**7, 9, 10**) that there exists a coordinate system in which a monoclinic solid has only 12 non-zero moduli. Theorem 5 goes further in showing the existence of at least three such coordinate systems. It should be pointed out that two or possibly all three of these coordinate systems may coincide for certain values of the elastic constants. Furthermore, these coordinate systems have no relation to symmetry coordinate systems (**3**). The distinction is obvious when one considers the compliance tensor, $\mathbf{S}=\mathbf{C}^{-1}$. Both $\mathbf{S}$ and $\mathbf{C}$ have eight zero components for a monoclinic solid referred to a symmetry system. While $\mathbf{C}$ has nine zero components in the acoustic coordinate systems, $\mathbf{S}$ will generally have only eight zero components in the same systems. However, the compliance tensor is positive definite and hence strongly elliptic (**11**). It therefore possesses the property that (**8**)

$$S_{ijkl}\xi_j\xi_k\xi_l=(S_{pqrs}\xi_p\xi_q\xi_r\xi_s)\xi_i \tag{7.6}$$

TABLE 1. *A characterization of material anisotropies*

Type of material symmetry	Minimum number of elastic constants	Number of Euler angles†	Number of independent parameters‡
Triclinic	18	3	21
Monoclinic	12	3	15
Orthotropic	9	3	12
Hexagonal	6	3	9
Tetragonal	6	3	9
Transverse isotropy	5	2	7
Cubic	3	3	6
Isotropic	2	0	2

† This is the number of parameters necessary to define a coordinate system in which the stiffness or compliance tensor has the minimum number of elastic constants.

‡ This is the minimum number of parameters characterizing the elastic tensors. It is the sum of the entries in the previous two columns.

for at least three distinct $\boldsymbol{\xi}$. If there exists a plane of symmetry then at least two of these directions lie in that plane. The results of Theorem 5 therefore apply equally to **S** and **C**, although there is no acoustical interpretation for the preferred coordinate axes that minimizes the number of non-zero components in **S**. In conclusion, Table 1 summarizes the present discussion and offers a slightly different way of looking at the distinct material symmetries in elastic solids.

Acknowledgments

I am indebted to Professor J. E. Taylor of Rutgers University for help in section 4 and grateful to the referees for providing valuable suggestions and references. This work was supported by the National Science Foundation through grant MSM 85-16256.

REFERENCES

1. W. C. VAN BUSKIRK, S. C. COWIN and R. CARTER, JR., *J. Mater. Sci.* **21** (1986) 2759.
2. M. A. HAYES, *ibid.* **4** (1969) 10.
3. S. C. COWIN and M. M. MEHRABADI, *Q. Jl Mech. appl. Math.* **40** (1987) 451.
4. S. C. COWIN, In *Proceedings of the IUTAM Symposium on Elastic Wave Propagation* (eds M. F. McCarthy and M. A. Hayes; North–Holland, Amsterdam 1988).
5. ——, *Q. Jl Mech. appl. Math.* **42** (1989) 249.
6. W. KOHN and J. R. RICE, *J. appl. Phys.* **50** (1979) 3346.
7. F. I. FEDOROV, *Theory of Elastic Waves in Crystals* (Plenum, New York 1968).
8. I. I. KOLODNER, *J. acoust. Soc. Am.* **40** (1966) 730.

9. L. D. LANDAU and E. M. LIFSHITZ, *Theory of Elasticity* (Pergamon Press, New York 1959).
10. S. G. LEKHNITSKII, *Theory of Elasticity of an Anisotropic Body* (Holden–Day, San Francisco 1963).
11. M. E. GURTIN, In *Handbuch der Physik,* VIa/2 (Springer, Berlin 1972).