

ACOUSTIC RADIATION FROM A CIRCULAR PIPE WITH AN INFINITE FLANGE

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The reflection of sound from the end of a flanged pipe is solved in a manner simpler than the procedure of Nomura *et al.* (1960 *Journal of the Physical Society of Japan* **15**, 510-517). Numerical calculations give the low-frequency end correction length as $0.82159 \dots$ radii, which is very close to Rayleigh's conjectured value of 0.82 radii. The power gain in the transmitted region is illustrated as a function of angle for frequencies below the first cut-off. The related problem of the acoustic response in a flanged pipe when a plane wave is incident upon the pipe in the axial direction is also discussed. Some approximate formulae are presented for the reflection coefficients of flanged and unflanged pipes.

1. INTRODUCTION

The problem addressed here is a classic one, having been treated at length by Rayleigh [1, section 307]. We consider the radiation of sound from a flanged circular pipe, where the dimension of the flange is so much greater than the wavelength that it may be considered infinite in extent. Nomura *et al.* [2] analyzed the problem using Weber-Schafheitlin integrals and Jacobi polynomials. They derived a coupled system of two infinite sets of linear equations for the unknowns, and solved these equations numerically. Our method of solution also requires numerical computation, but only for a single system of equations for the modal amplitudes in the pipe. The present method is related to that adopted by King [3] for finding the static end correction, and his equations are obtained from ours in the limit of zero frequency.

A closed-form solution was obtained by Levine and Schwinger [4] for the associated problem of radiation from an unflanged pipe, using the Wiener-Hopf technique. The same technique was employed successfully by Ando [5] in solving the intermediate case of a pipe with a finite wall thickness. His analysis led to an infinite system of equations which may be solved in truncated form when the ratio of the inner to outer radius is not too small. However, the required truncation size grows as the ratio tends to zero, and the method appears to be ill-suited to treating the limiting case of an infinite flange. One could adapt the Wiener-Hopf technique to the present problem, and so obtain a modified Wiener-Hopf equation: however, unlike the equation of Levine and Schwinger, the resulting Wiener-Hopf equation cannot be solved in closed form. An analysis of this type was performed by Mittra and Lee [6, section 5.6] in considering the radiation from a flanged parallel-sided waveguide. They derived a modified Wiener-Hopf equation and showed how it can be reduced to a Fredholm integral equation of the second kind with a smooth kernel, which in turn may be solved by standard numerical techniques. We will not discuss the Wiener-Hopf method further, apart from noting that it does not appear to have been used in solving this particular problem.

Our solution is developed in the following section. In section 3 we derive the static end correction from a general formula for arbitrary frequency and compare our numerically computed value with previous computations [1-3, 7]. The radiated power gain function is discussed and compared with that for an unflanged pipe [4] in section 4. The solution for the related problem of a plane wave incident upon the flanged pipe from the free side is discussed in section 5. Finally, some useful approximate formulae are given in section 6 for the reflection coefficients of both flanged and unflanged pipes.

2. ANALYTICAL SOLUTION

The configuration is shown in Figure 1: a denotes the interior radius of the pipe, and all physical quantities are presumed to be symmetrical about the axis of the pipe. Both the pipe and the upper half-space are occupied by inviscid fluid of sound speed c , in which the velocity potential ϕ satisfies

$$\nabla^2 \phi + k^2 \phi = 0, \quad (1)$$

where $k = \omega/c$ and harmonic time dependence $\exp(-i\omega t)$ is everywhere omitted. The incident wave in the pipe, $z < 0$, is a wave propagating in the $+z$ direction,

$$\phi^{inc} = e^{ikz}. \quad (2)$$

In addition, the normal derivative of ϕ is everywhere zero on the solid-fluid interface and ϕ satisfies a radiation condition in $z > 0$.

The total potential in $z < 0$ may be expressed as a modal series,

$$\phi = \phi^{inc} + A_0 e^{-ikz} + \sum_{n=1}^{\infty} A_n \psi_n(\rho) e^{-i\xi_n z}, \quad (3)$$

where

$$\xi_n = \begin{cases} (k^2 - k_n^2)^{1/2}, & k > k_n \\ i(k_n^2 - k^2)^{1/2}, & k < k_n \end{cases}, \quad \psi_n(\rho) = \frac{J_0(k_n \rho)}{J_0(k_n a)}, \quad (4, 5)$$

and $k_n a = j_{1n}$, $n = 0, 1, 2, \dots$, are the sequential positive roots of $J_1(x) = 0$, with $j_{10} = 0$. The transmitted potential in $z > 0$ can be represented by Green's formula as

$$\phi^T(\mathbf{x}) = \iint_{\rho' < a} g(\mathbf{x}; \mathbf{x}', y', 0) V_z(\mathbf{x}', y', 0) dx' dy', \quad (6)$$

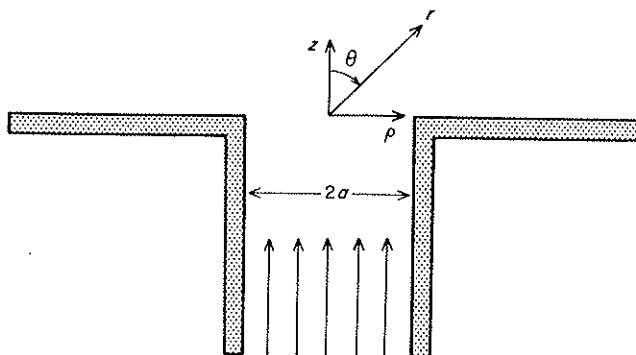


Figure 1. Schematic picture of the pipe geometry and the incident wave.

where V_z is the z -component of the velocity $\mathbf{V} = \nabla \phi$, and g is the Green function for the upper half-space with a rigid boundary condition on $z=0$. At the mouth of the pipe, $z=0, \rho < a$, both pressure and velocity must be continuous; therefore, ϕ and V_z are continuous across the mouth. In particular, ϕ^T and V_z in equation (6) can be represented by the modal sum by using equation (3), to give, for $\rho < a$,

$$\begin{aligned} 1 + A_0 + \sum_{n=1}^{\infty} A_n \psi_n(\rho) \\ = \frac{-i}{2\pi} \int_0^{2\pi} d\theta' \int_0^a \rho' d\rho' \frac{\exp[ik(\rho^2 + \rho'^2 - 2\rho\rho' \cos \theta')^{1/2}]}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos \theta')^{1/2}} \\ \times \left(k(1 - A_0) - \sum_{n=1}^{\infty} A_n \xi_n \psi_n(\rho') \right). \end{aligned} \quad (7)$$

One can now take the inner product of equation (7) with $\psi_m(\rho)$, $m=0, 1, 2, \dots$, where $\psi_0=1$, and use the orthogonality of the ψ_m to obtain a linear system of equations for the modal amplitudes,

$$\sum_{m=0}^{\infty} M_{nm}(ka) \xi_m A_m(ka) = ka \alpha_{n0}(ka) - i \delta_{n0}, \quad n=0, 1, 2, 3, \dots, \quad (8)$$

where $\delta_{nm} = 1$ if $n=m$, and 0 otherwise, and

$$M_{nm}(ka) = \alpha_{nm}(ka) + \frac{i \delta_{nm}}{\xi_m a}, \quad \alpha_{nm}(ka) = \frac{1}{2\pi^2 a^3} \iint_{\rho, \rho' < a} \frac{e^{ik\rho_0}}{\rho_0} \psi_n(\rho) \psi_m(\rho') dA dA', \quad (9, 10)$$

where ρ_0 is defined by comparison with equation (7). Equation (8) can also be derived from a similar system of modal amplitude equations for a rigid half-space with periodically positioned, infinitely deep holes [8]. Note that for $ka \ll 1$, $M_{00} = O(1/ka)$, while $M_{nm} = O(1)$ for $(n, m) \neq (0, 0)$. This leads to an unstable matrix as $ka \rightarrow 0$. Also, it is useful to distinguish the coefficient A_0 from the others since it represents the reflected amplitude of the fundamental mode. Therefore eliminate A_0 from equation (8), to give the new system,

$$\sum_{m=1}^{\infty} \left(M_{nm} - \frac{\alpha_{n0} \alpha_{m0}}{M_{00}} \right) B_m = \frac{\alpha_{n0}}{(1 - ika \alpha_{00})}, \quad n=1, 2, 3, \dots, \quad (11)$$

where

$$B_m(ka) = (\xi_m / 2k) A_m(ka). \quad (12)$$

The complex-valued quantities $\alpha_{nm} = \alpha_{nm}$ can be simplified to [8]

$$\alpha_{nn}(x) = \beta_n^{(1)}(x) + j_{1n}^2 \beta_n^{(2)}(x) \quad \text{for } n \geq 0, \quad (13a)$$

$$\alpha_{nm}(x) = \frac{j_{1n}^2 \beta_n^{(1)}(x) - j_{1m}^2 \beta_m^{(1)}(x)}{j_{1n}^2 - j_{1m}^2} \quad \text{for } n, m \geq 0, n \neq m, \quad (13b)$$

where

$$\beta_n^{(q)}(x) = i \int_0^x \frac{2J_1^2(s) s ds}{(x^2 - s^2)^{1/2} (s^2 - j_{1n}^2)^q} + \int_x^\infty \frac{2J_1^2(s) s ds}{(s^2 - x^2)^{1/2} (s^2 - j_{1n}^2)^q}. \quad (13c)$$

In particular, $\text{Im } \alpha_{00}(x) = x^{-1} - J_1(2x)/x^2$ and $\alpha_{00}(0) = 8/3\pi$ [1, section 302]. Quantities similar to the α_{nm} are occasionally encountered in the theory of acoustics: for instance, they are related to the acoustic resistance and reactance factors of Morse and Ingard [9, equation (10.2.14)] by $\chi_{mn}^1(x) + i\theta_{mn}^1(x) = 2j_{1m}[j_{1n}J_0(j_{1m})J_0(j_{1n})]^{-1}[\alpha_{mn}(x) - \alpha_{n0}(x)]$.

The fundamental reflection coefficient $R(ka) \equiv A_0(ka)$ is

$$R(ka) = -1 + 2(\alpha_{00} - \sum_{n=1}^{\infty} \alpha_{n0} B_n) / M_{00}. \quad (14)$$

As mentioned before, the problem under consideration is equivalent to radiation from a deep, circular hole in a smooth, rigid surface, which is the limiting case of a periodically perforated half-space when the volume fraction occupied by the holes approaches zero. The reflection of acoustic waves from a rigid half-space with periodically spaced deep holes was considered in reference [8] by using a method of analysis similar to that employed here. The present result for the reflection coefficient can be seen to follow from the results of reference [8] in the appropriate limit.

3. THE END CORRECTION

The end correction $L(ka)$ is defined through

$$R(ka) = -|R(ka)| e^{i2kL}. \quad (15)$$

Nomura *et al.* [2] have plotted $L(ka)/a$ and the magnitude of R for $ka < j_{11} = 3.8317$. We have also calculated these quantities by truncating the complex system (11) at successively larger sizes until convergence was observed. Truncation sizes of less than 100 sufficed, and we saw no difference between our results and the curves of reference [2]. Approximate formulae for both $|R|$ and L/a are discussed below in section 6.

In the long-wavelength limit $ka \rightarrow 0$, $R \rightarrow -\exp[i2kL(0)]$, where the quasi-static end correction is

$$L(0)/a = \alpha_{00}(0) - \sum_{n=1}^{\infty} \alpha_{n0}(0) B_n(0), \quad (16)$$

and the real numbers $B_n(0)$ satisfy the real symmetric system,

$$\sum_{m=1}^{\infty} M_{nm}(0) B_m(0) = \alpha_{n0}(0), \quad n = 1, 2, 3, \dots \quad (17)$$

The static end correction was considered by Rayleigh [1, section 307] who showed that the first term on the right side of equation (16) provides an upper bound, $L(0)/a < \alpha_{00}(0) = 8/3\pi = 0.8488$. He also gave $\pi/4 = 0.7854$ as a lower bound, valid for a very short pipe. Using trial functions [1, Appendix A], Rayleigh obtained a better upper bound as 0.8242, on the basis of which he conjectured the real value to be about 0.82. Daniell [7] used an educated approximation to the axial velocity at the mouth, involving three and four parameters, and concluded that the end correction is less than 0.82168, and "is probably extremely close to this value". Two alternative, formally exact, procedures were derived by King [3] for the end correction. Both required solving infinite systems of equations, and gave monotonically increasing and decreasing values, respectively, as the truncation size grew. It can be shown fairly readily that expressions (16) and (17) correspond to equations (46.2) and (46.1) in King's paper [3], respectively.

We find $L(0)/a = 0.82159 \dots$, with convergence, as illustrated in Figure 2.

4. ACOUSTIC RADIATION

The far field amplitude in the fluid is defined by

$$\phi^r(\mathbf{x}) \sim f(\theta) e^{ikr}/r, \quad r \rightarrow \infty, \quad z > 0, \quad (18)$$

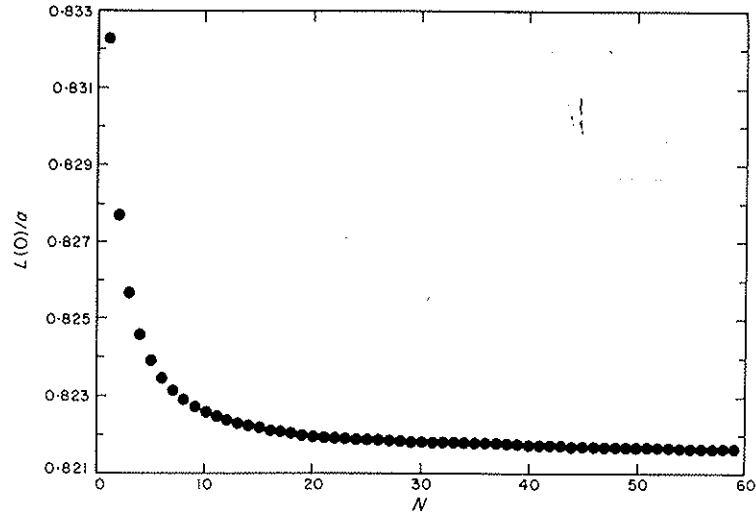


Figure 2. Convergence of the approximant $\alpha_{00} - \sum_{n=1}^N B_n(0)\alpha_{n0}(0)$ to the infinite sum in equation (16), versus N .

where

$$f(\theta) = -ia \frac{J_1(ka \sin \theta)}{\sin \theta} \left(1 - R + \sum_{m=1}^{\infty} \frac{2(ka \sin \theta)^2}{j_{1m}^2 - (ka \sin \theta)^2} B_m \right). \quad (19)$$

The forward amplitude is $f(0) = (-i/2)ka^2(1 - R)$, and at low frequency $f(\theta)$ is independent of angle and equal to $-ika^2$, to first order. Following Levine and Schwinger [4] we define the power gain function $G(\theta)$ as the average power radiated into a unit solid angle about the direction θ divided by the total power radiated,

$$G(\theta) = P(\theta)/(P_{\text{rad}}/2\pi), \quad P(\theta) = |f(\theta)|^2, \quad P_{\text{rad}} = \pi a^2(1 - |R|^2), \quad (20-22)$$

where expression (22) is valid only for $ka < j_{11}$. In particular, the forward gain in this frequency range is

$$G(0) = \frac{1}{2}(ka)^2 |1 - R|^2 / \{1 - |R|^2\}. \quad (23)$$

Figures 3 and 4 show $G(\theta)$ for the flanged and unflanged pipes, respectively. In comparing these functions one should note that, according to equation (20), $\int G d\Omega = 2\pi$, where Ω is solid angle and the integral is over the upper hemisphere. In Levine and Schwinger's paper on the unflanged pipe [4] the radiation is in all directions, and so $\int G d\Omega = 4\pi$ for their power gain function. Hence, at low frequencies both gain functions are approximately unity, and at higher frequencies our $G(\theta)$ is approximately one half that of Levine and Schwinger, as expected. Nomura *et al.* [2], in considering the flanged pipe, defined their gain function $g(\theta)$ in the same manner as Levine and Schwinger; thus, $g \approx 2$ at low frequencies, and $g = 2G$, where G is defined in equation (20). We found excellent agreement when our results were compared with the curves of $g(\theta)$ in reference [2].

Define $\bar{\theta}$ as the angle for which $\int_0^{\bar{\theta}} G(\theta) \sin(\theta) d\theta = 1/2$ for the flanged pipe at a given frequency. Thus, half of the energy radiated into the fluid half-space is contained in the cone of semi-angle $\bar{\theta}$. Obviously, $\bar{\theta} \approx 60^\circ$ at low frequencies, and $\bar{\theta}$ will decrease at high frequencies. The analogous cone angle for half the radiated energy from an unflanged pipe is defined by $\int_0^{\bar{\theta}} G(\theta) \sin(\theta) d\theta = 1$, where $G(\theta)$ is Levine and Schwinger's [4] gain

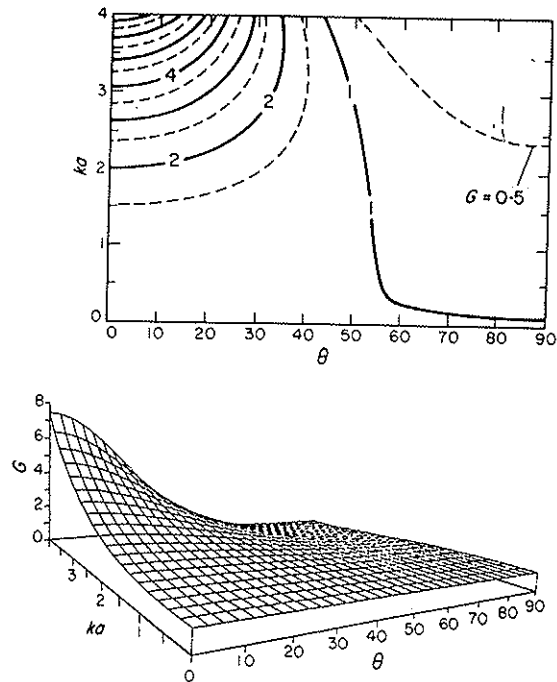


Figure 3. The power gain function $G(\theta)$ for the flanged pipe in the frequency range $0 \leq ka \leq j_{11}$, with a contour plot of discrete values of G .

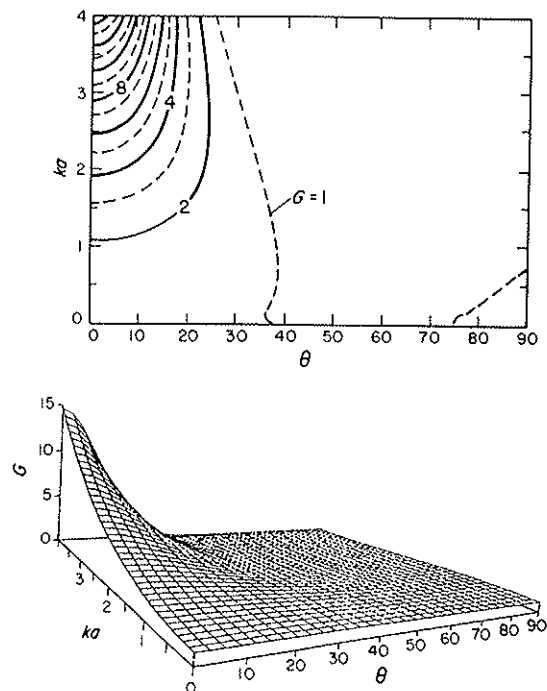


Figure 4. The power gain function for the unflanged pipe from Levine and Schwinger [4] in the same frequency range as Figure 3.

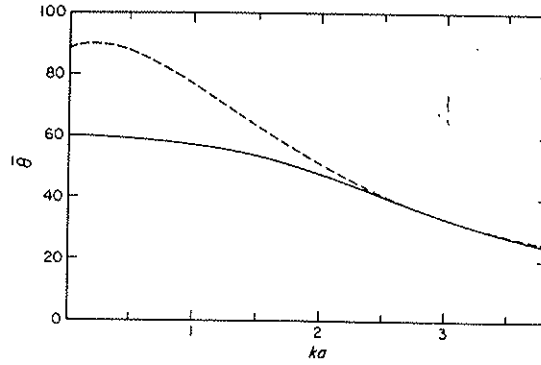


Figure 5. Plots of the angle $\bar{\theta}$ that defines the cone containing half of the radiated energy; for the flanged pipe (—) and the unflanged pipe (---).

function. In this case, $\bar{\theta} \approx 90^\circ$ for small ka . The angles $\bar{\theta}$ for both the flanged and unflanged pipes should be very close at higher frequencies, when the presence or absence of a flange is not expected to effect the radiation. These expectations are borne out by Figure 5, showing the two angles as functions of frequency. It is interesting to note that $\bar{\theta}$ for the unflanged pipe initially increases above 90° as ka increases from 0.

5. INCIDENCE OF A PLANE WAVE ON A FLANGED PIPE

The complementary scattering problem is that of a plane wave incident in the axial direction upon the flanged pipe, such that the incident field in $z > 0$ is $\phi^{inc} = e^{-ikz}$. Denote the solutions to the original problem and this problem as solutions I and II, respectively. Let solution III be $\phi = e^{-ikz} + e^{ikz}$ throughout the fluid in $z > 0$, and in the pipe, $z < 0$; see Figure 1. This solution satisfies all the boundary conditions, and equals the sum of solutions I and II. Therefore, the total potential for solution II follows from equation (3) as

$$\phi = (1 - A_0) e^{-ikz} - \sum_{n=1}^{\infty} A_n \psi_n(\rho) e^{-i\xi_n z}, \quad z < 0, \quad (24)$$

$$\phi = e^{-ikz} + e^{ikz} - \phi^T, \quad z > 0, \quad (25)$$

where ϕ^T , defined in equation (6), is the transmitted field for the original radiation problem.

6. DISCUSSION

At low frequencies, the first correction to the reflection coefficient $R = -1$ comes from the end correction to the phase of R . The magnitude, on the other hand, follows from equation (23) with $G(0) = 1$ as

$$|R| = 1 - (ka)^2 + o(ka)^2. \quad (26)$$

It is often useful for engineering applications to have a simple but accurate formula for $R(ka)$ over a reasonable range of frequency. An *ad hoc* way to extend expression (26) to higher values of ka is to attempt to approximate the numerically calculated values of $|R|$. One approach in this regard is to take a rational function approximation of the form

$$|R| = \{1 + \alpha ka + \beta(ka)^2\} / \{1 + \alpha ka + (1 + \beta)(ka)^2\}, \quad (27)$$

which automatically satisfies equation (26), and where α and β are chosen to best fit the calculated values. We find that $(\alpha, \beta) = (0.323, -0.077)$ provide the closest fit over the range $0 < ka < 3.8$, as shown in Figure 6. The same curve-fitting procedure for the magnitude of the reflection coefficient for an unflanged pipe must satisfy, for small ka [4],

$$|R| = 1 - \frac{1}{2}(ka)^2 + o(ka)^2. \quad (28)$$

This can again be achieved with a rational function approximation of the form (27), but this time with $(\frac{1}{2} + \beta)$ instead of $(1 + \beta)$ in the denominator. Using the exact expression of Levine and Schwinger [4], we found the optimal choice over the same range of ka to be $(\alpha, \beta) = (0.200, -0.084)$, with resulting accuracy as indicated in Figure 6. As for the phase, $L(ka)/a$ defined in equation (15), it was determined that

$$L(ka)/a = [0.82159 - 0.49(ka)^2]/[1 - 0.46(ka)^3] \quad (29)$$

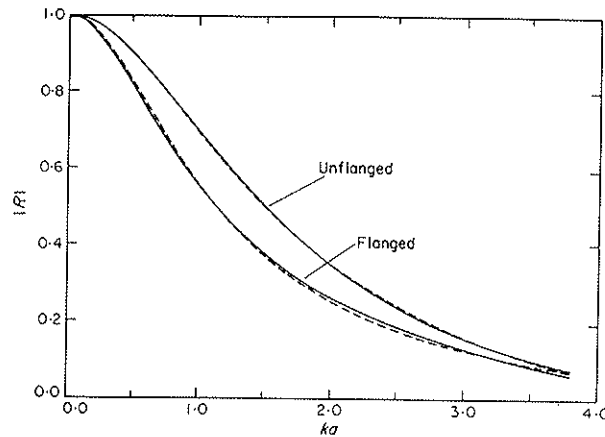


Figure 6. Magnitudes of the reflection coefficient $R(ka)$ for the flanged and unflanged pipe. The solid lines are the result of numerical calculations and the dashed curves are rational function approximations. The results for the unflanged pipe are from Levine and Schwinger [4].

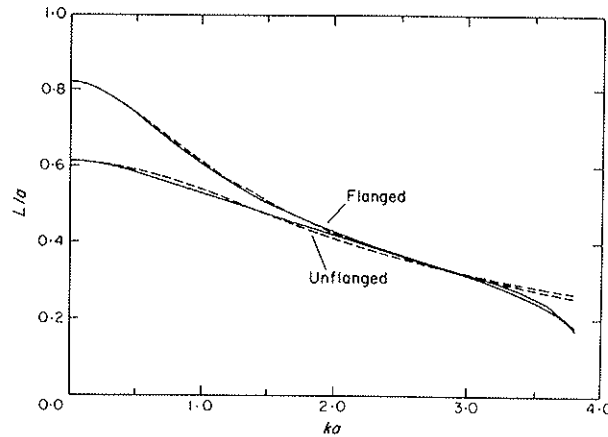


Figure 7. The phase, $L(ka)/a$ as defined in equation (15), of the reflection coefficients for the flanged and unflanged pipe. The solid and dashed lines are as in Figure 6, and the results for the unflanged pipe are from Levine and Schwinger [4].

gives a reasonable approximation for the flanged pipe over most of the range (0.0, 3.8). Comparison with the exact formula of Levine and Schwinger [4] shows that

$$L(ka)/a = [0.6133 + 0.027(ka)^2]/[1 + 0.19(ka)^2], \quad (30)$$

is adequate to the same degree of approximation for the unflanged pipe. These results are summarized in Figure 7. It should be noted that expressions (27) and (30) are (2, 2) Padé approximants, whereas expression (29) is a (2, 3) approximant. This choice was deliberate, based upon trial and error fitting with different Padé approximations of low order.

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REFERENCES

1. LORD RAYLEIGH 1945 *The Theory of Sound* (volume 2). New York: Dover; second edition.
2. Y. NOMURA, I. YAMAMURA and S. INAWASHIRO 1960 *Journal of the Physical Society of Japan* **15**, 510-517. On the acoustic radiation from a flanged circular pipe.
3. L. V. KING 1936 *Philosophical Magazine* **21**, 128-144. On the electrical and acoustic conductivities of cylindrical tubes bounded by infinite flanges.
4. H. LEVINE and J. SCHWINGER 1948 *Physical Review* **73**, 383-406. On the radiation of sound from an unflanged circular pipe.
5. Y. ANDO 1969/70 *Acustica* **22**, 219-225. On the sound radiation from semi-infinite circular pipe of certain wall thickness.
6. R. MITTRA and S. W. LEE 1971 *Analytical Techniques in the Theory of Guided Waves*. New York: Macmillan.
7. P. J. DANIELL 1915 *Philosophical Magazine* **30**, 137-146. The coefficient of end-correction—Part I.
8. A. N. NORRIS and H. A. LUO 1987 *Journal of the Acoustical Society of America* **82**, 2113-2122. Acoustic radiation and reflection from a periodically perforated rigid solid.
9. P. M. MORSE and K. U. INGARD 1968 *Theoretical Acoustics*. New York: McGraw-Hill.