Acoustic radiation and reflection from a periodically perforated rigid solid

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An exact solution is given for the reflection of a plane wave normally incident on a rigid solid with periodically spaced semi-infinite circular holes. Analytical considerations, verified by numerical calculations, show that the reflection coefficient is unity at the cutoff frequencies defined by the periodicity of the holes. This result is independent of the volume fraction of the holes. It implies that the porous solid acts like a rigid solid at these frequencies. The problem of plane-wave incidence from the holes is also solved. A reflection coefficient of unity is obtained at the same frequencies, again implying a rigid effect. Below the first cutoff frequency, the reflection coefficient can be parametrized by a simple scalar frequency dependent quantity. This simple result can be interpreted in terms of a displaced pressure continuity condition.

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INTRODUCTION

A century ago, Lord Rayleigh[1] discussed the low-frequency reflection of acoustic waves from a porous medium. Since then there have been major developments in the theory of wave propagation in porous media and acoustic reflection from them.[2,3] A thorough review of the various theories is given by Attenborough.[4] Most previous treatments have implicitly considered randomly distributed pores, and wavelengths much larger than the pore diameter. In this article, we assume a doubly periodic arrangement of the pores, each of which is a circular tube with its axis perpendicular to the surface of the solid. The fluid is assumed to be inviscid but the wavelength of the incident wave may take on any value. In the limit of very long wavelength, the quasistatic results of Rayleigh[1] are obtained.

The leading term in the long wavelength limit can also be recovered from Biot's theory of dynamic poroelasticity.[4,5] The relevant parameters used are those of a rigid frame, with an inviscid pore fluid in straight, cylindrical channels. The latter implies an effective pore fluid mass equal to the actual mass. The reflection transmission problem is then solved by assuming the open pore boundary condition[6] at the interface. This limit of Biot's theory is one of the simplest possible. However, the Biot theory does not include effects for which the wavelengths are on the order of the pore size. It is the purpose of this article to examine the finite frequency behavior of the response. Effective boundary conditions are derived which extend the quasistatic approximation to the first cutoff frequency.

The limit of a single hole in a rigid half space can be viewed as the limit of the present problem where the porosity or volume fraction of the holes vanishes. The single hole as a radiator of sound is discussed extensively in Morse[7] for wavelengths long compared with the radius. The exact solution for finite frequencies has recently been solved,[8] using the methods of the present article. The related problem of a pipe with a finite flange has also been considered,[9] using a quite different solution method—the Weiner–Hopf technique.

I. FORMULATION

The doubly periodic porous medium is depicted in Fig. 1. The pore fluid and fluid in the upper half-space are assumed to be inviscid, with density ρ and acoustic sound speed c. Time harmonic dependence of frequency ω is assumed. The term $\exp(-i\omega t)$ will be understood but omitted throughout.

The inviscid assumption is valid in the context of viscous fluids if the pore radius R is much greater than the viscous skin depth $d = \sqrt{\eta/\rho\omega}$ where η is the viscosity. This condition is equivalent to $\omega \gg \eta/\rho R^2$. In this limit, the viscous effects are confined to a viscous boundary layer within a neighborhood d of the rigid surface. Explicit corrections could be made to take account of the dissipation in the viscous layer, but we will not attempt this here, since it is only a boundary layer correction, and does not affect the solution to first order. At higher viscosities, such that $d/R = O(1)$, the present method is invalid. The correct approach is to explicitly take into account the shear mode in the fluid. The

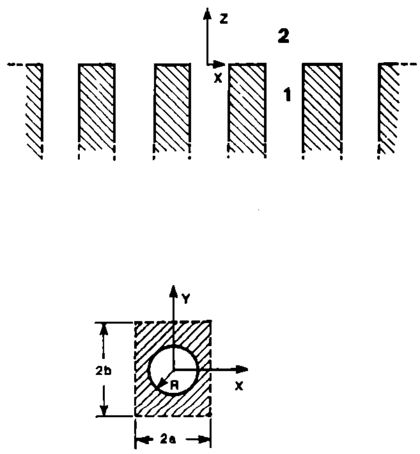

FIG. 1. Schematic diagram of the porous solid, medium 1, and the fluid half-space, medium 2.

problem is then completely analogous to the present configuration with the fluid replaced by an elastic solid. The solution is made much more difficult by the coupling between the compressional and shear modes. The related problem of a viscous fluid in an infinite tube is discussed at length by Rayleigh.[1] However, if $d/R \gg 1$, the viscosity dominates the fluid dynamics, and the modal frequencies can be approximated. This limit is considered in Rayleigh,[1] Sec. 3.5.1. Pierce[10] also discusses the effects of viscosity in Sec. 10.6 of his book.

Let $u(\mathbf{x})$ be the velocity potential in the fluid. Consider the incident wave

$$u_1^{\text{inc}} = Ae^{ikz}, \tag{1}$$

where $k = \omega/c$. This represents a plane wave incident from the porous medium (region 1 in Fig. 1) on the fluid half-space. The analysis for a plane wave normally incident from the fluid is very similar and is discussed in Appendix A. We assume

$$u = \begin{cases} u_1 = u_1^{\text{inc}} + u_1^{\text{sc}}, & z \leqslant 0^-, \\ u_2, & z \geqslant 0^+. \end{cases} \tag{2}$$

The scattered field in the pore in the unit cell of Fig. 1 satisfies

$$\begin{aligned} &\nabla^2 u_1^{\text{sc}} + k^2 u_1^{\text{sc}} = 0, \quad z<0, \quad r<R, \\ &\frac{\partial u_1^{\text{sc}}}{\partial r} = 0, \quad z<0, \quad r=R. \end{aligned} \tag{3}$$

Express u_1^{sc} as a series of eigenfunctions,

$$u_1^{\text{sc}} = \sum_{n=0}^{\infty} A_n \Psi_n(r) e^{-i\xi_n z}, \tag{4}$$

where, for $n = 0,1,2,\ldots$,

$$\Psi_n(r) = [\, J_0(k_n r)/J_0(k_n R)\,], \tag{5}$$

$$k_n = j_{1n}/R, \tag{6}$$

$$\xi_n = (k^2 - k_n^2)^{1/2}, \quad \text{Im}\,\xi_n \geqslant 0, \tag{7}$$

where J_m is the Bessel function of the first kind of order m. The numbers j_{1n} are the ordered positive roots of $J_1(x) = 0$, with $j_{10} = 0$ and, hence, $k_0 = 0$, $\xi_0 = k$, and $\Psi_0 = 1$. The normalization of Ψ_n is chosen so that

$$\int_{r<R}\!\!\int \Psi_n(r)\Psi_m(r)\,dA = \pi R^2 \delta_{mn}. \tag{8}$$

The constants A_n, $n \geqslant 0$, are the amplitudes of the reflected modes. In particular we define

$$R_{11} = A_0/A, \tag{9}$$

which is the reflection coefficient of the zeroth mode. Note that the incident wave u_1^{inc} is just the zeroth mode propagating in the positive z direction. The total field in $z<0$ can be written, from Eqs. (1), (2), and (4),

$$u_1 = Ae^{ikz} + A_0 e^{-ikz} + \sum_{n=1}^{\infty} A_n \Psi_n(r) e^{-i\xi_n z}. \tag{10}$$

Next, consider the field in the upper fluid half-space. The field is doubly periodic, defined by the unit cell problem:

$$\begin{aligned} &\nabla^2 u_2 + k^2 u_2 = 0, \quad z>0, \quad |x|<a, \quad |y|<b, \\ &\frac{\partial u_2}{\partial x} = 0, \quad z>0, \quad |x|=a, \quad |y|<b, \\ &\frac{\partial u_2}{\partial y} = 0, \quad z>0, \quad |x|<a, \quad |y|=b. \end{aligned} \tag{11}$$

In addition, we have the continuity of u and the continuity of the velocity in the z direction at the interface $z = 0$,

$$u_2 = u_1, \quad z=0, \quad r<R, \tag{12}$$

$$\frac{\partial u_2}{\partial z} = \begin{cases} 0, & z=0, \quad |x|<a, \quad |y|<b, \quad r>R, \\ \dfrac{\partial u_1}{\partial z}, & z=0, \quad r<R. \end{cases} \tag{13}$$

II. GREEN FUNCTION

The fundamental unit cell problem of Eqs. (1) and (12) can be solved using $g(\mathbf{x};\mathbf{x}')$, the Green function which satisfies, for $z'>0$,

$$\nabla^2 g + k^2 g = \delta(\mathbf{x}-\mathbf{x}'), \quad z \geqslant 0, \quad |x|<a, \quad |y|<b, \tag{14}$$

$$\frac{\partial g}{\partial n} = 0, \quad \begin{cases} z>0, & |x|=a \text{ or } |y|=b, \\ z=0, & |x|\leqslant a, \quad |y|\leqslant b, \end{cases}$$

where $\partial/\partial n$ is the normal derivative. By the usual methods (e.g., Ref. 2),

$$g(\mathbf{x};\mathbf{x}') = \frac{-i}{2ab} \sum_{m,n=0}^{\infty} \frac{\cos\alpha_m x \cos\beta_n y \cos\alpha_m x' \cos\beta_n y'}{\delta_m \delta_n \rho_{mn}} \times (e^{i\rho_{mn}|z-z'|} + e^{i\rho_{mn}(z+z')}), \tag{15}$$

where

$$\begin{aligned} \alpha_m &= m\pi/a, \\ \beta_n &= n\pi/b, \\ \rho_{mn} &= (k^2 - \alpha_m^2 - \beta_n^2)^{1/2}, \quad \text{Im}(\rho_{mn}) \geqslant 0, \\ \delta_m &= \begin{cases} 2, & m=0, \\ 1, & m \neq 0. \end{cases} \end{aligned} \tag{16}$$

This Green function allows us to write the field ϕ_2 in the fluid half-space in terms of the z velocity on the circle $r<R$, $z=0$;

$$u_2(\mathbf{x}) = \int_{r<R}\!\!\int g(\mathbf{x};x',y',0)\, \frac{\partial u_2}{\partial z'}(x',y',0)\,dx'\,dy'. \tag{17}$$

The velocity in (17) can be expressed, via Eqs. (13) and (10), in terms of the unknowns A_n, $n = 0,1,2,\ldots$. Then, with $z=0$ in Eq. (17), and using the continuity condition Eq. (12), we obtain, for $r<R$,

$$A + \sum_{l=0}^{\infty} A_l \Psi_l(r) = \frac{1}{ab} \sum_{m,n=0}^{\infty} \frac{\cos\alpha_m x \cos\beta_n y}{\rho_{mn}\delta_m\delta_n} \times \int_{r<R}\!\!\int \Big(kA - \sum_{l=0}^{\infty} A_l \xi_l \Psi_l(r')\Big) \times \cos\alpha_m x' \cos\beta_n y'\,dx'\,dy'. \tag{18}$$

III. SIMPLIFICATION AND THE MATRIX EQUATION

Multiplying Eq. (18) by $\Psi_p(r)$, $p = 0,1,2,\ldots$, implies an infinite system of equations for the unknowns A_n, $n = 0,1,2,\ldots$. This system simplifies by introducing several new dimensionless variables. First, define the porosity ϕ, $0<\phi<\pi/4$,

$$\phi = \pi R^2/4ab. \tag{19}$$

Let

$$\bar{k} = kR, \tag{20a}$$

$$\bar{\xi}_l = \xi_l R, \tag{20b}$$

$$\bar{\rho}_{mn} = \rho_{mn} R, \tag{20c}$$

$$\bar{\alpha}_m = \alpha_m R, \quad \bar{\beta}_n = \beta_n R, \tag{20d}$$

$$\bar{A}_l = (A_l/A)(\xi_l/k),$$

$$K_{pmn} = \frac{1}{\pi R^2}\int_{r<R}\int \Psi_p(r)\cos\alpha_m x\cos\beta_n y\,dx\,dy$$

$$= \begin{cases} \delta_{p0}, & m=n=0, \\ \dfrac{2(\bar{\alpha}_m^2+\bar{\beta}_n^2)^{1/2}}{\bar{\alpha}_m^2+\bar{\beta}_n^2-j_{1p}^2} J_1[(\bar{\alpha}_m^2+\bar{\beta}_n^2)^{1/2}], & \\ & m^2+n^2>0, \end{cases} \tag{20e}$$

where the latter result is derived in Appendix B. Define the matrix $[M]$ and array $\{N\}$ by the elements $p, q = 0,1,2,\ldots$

$$M_{pq} = \sum_{m,n=0}^{\infty} \frac{4\phi}{\delta_m\delta_n\bar{\rho}_{mn}} K_{pmn}K_{qmn} + \frac{1}{\bar{\xi}_p}\delta_{pq}, \tag{21a}$$

$$N_p = M_{p0} - (2/\bar{k})\delta_{p0}. \tag{21b}$$

The infinite system of equations can then be written

$$[M]\{\bar{A}\} = \{N\}, \tag{22}$$

where $\{\bar{A}\}$ is the array with elements $\bar{A}_p$, $p = 0,1,2,\ldots$. In general, the matrix $[M]$ is complex and symmetric. The solution $\{\bar{A}\}$ to the system of equations (22) represents an exact solution to the scattering problem through Eqs. (10), (13), (17), and (20). Thus Eq. (10) is exact since the chosen eigenfunctions are complete. All diffraction effects, which would be important at high frequencies, are implicitly contained in this eigenfunction solution.

We now proceed to further simplify the system of equations (22) into a form more amenable to computation. First, eliminate $\bar{A}_0$ from the system using the first equation to give

$$R_{11} \equiv \bar{A}_0 = 1 - \frac{2}{M_0} + \frac{i}{2}\bar{k}\sum_{p=1}^{\infty} B_p C_p, \tag{23}$$

where

$$M_0 = \bar{k}M_{00} = 1 + \phi + 2\bar{k}\phi\left(\sum_{m=1}^{\infty}\frac{K_{0m0}^2}{\bar{\rho}_{m0}} + \sum_{n=1}^{\infty}\frac{K_{00n}^2}{\bar{\rho}_{0n}} + 2\sum_{m,n=1}^{\infty}\frac{K_{0mn}^2}{\bar{\rho}_{mn}}\right), \tag{24}$$

$$B_p = \bar{A}_p/\sqrt{j_{1p}}, \tag{25}$$

$$C_p = 2i\sqrt{j_{1p}}M_{p0}/M_0. \tag{26}$$

Let $\{B\}$ and $\{C\}$ be the arrays with elements B_p and C_p, $p = 1,2,3,\ldots$. Then $\{B\}$ satisfies the system

$$[Q]\{B\} = \{C\}, \tag{27}$$

where, for $p, q = 1,2,3,\ldots$,

$$Q_{pq} = i\sqrt{j_{1p}j_{1q}}\,[M_{pq} - (\bar{k}/M_0)M_{p0}M_{q0}]. \tag{28}$$

In general, the matrix $[Q]$ is complex and symmetric and depends upon $\bar{k}$. Similarly, $\{C\}$ is complex and depends upon $\bar{k}$. We have simplified the system in the above manner so that in the limit as $\bar{k}\to 0$, the static limit, we obtain a real, symmetric matrix problem. We will discuss this in detail below. Another limiting case of interest is the single hole limit, obtained as a/R and b/R both tend to infinity. It is shown in Appendix C how Eq. (27) reduces to the known system of equations for the single hole problem.[8] The limiting problem also provides a check on the numerical results, discussed later. A further reason for preferring Eq. (27) to Eq. (22) is that the matrix elements of $[Q]$ remain bounded at the modal frequencies $\bar{k}_{mn}$, where $\rho_{mn} = 0$. The matrix elements of $[M]$ are singular at these frequencies. However, the matrices $[M]$ and $[Q]$ both contain singular diagonal elements at the pore cutoff frequencies k_p, $p \geqslant 1$. This instability is due to the definition of M_{pq} in (21a), and can be easily corrected by using $\{A\}$ instead of $\{\bar{A}\}$, for example.

IV. LOW-FREQUENCY ASYMPTOTIC EXPANSION

The quasistatic case of the wavelength much larger than the cell size occurs when $\bar{k} \ll 1$, for $\phi = O(1)$. With regards to Eq. (27), we assume the ansatz

$$[Q] = [Q^{(0)}] + \bar{k}[Q^{(1)}] + \bar{k}^2[Q^{(2)}] + \cdots,$$
$$\{B\} = \{B^{(0)}\} + \bar{k}\{B^{(1)}\} + \bar{k}^2\{B^{(2)}\} + \cdots, \tag{29}$$
$$\{C\} = \{C^{(0)}\} + \bar{k}\{C^{(1)}\} + \bar{k}^2\{C^{(2)}\} + \cdots,$$

where $[Q^{(0)}]$, $[Q^{(1)}]$, etc., are independent of $\bar{k}$. Explicit expansion of Eq. (28) gives, for $n = 0,1,2,\ldots$,

$$Q_{pq}^{(2n)} = \sqrt{j_{1p}j_{1q}}L_{pqn} + \binom{-1/2}{n}\frac{1}{j_{1p}^{2n}}\delta_{pq}, \tag{30a}$$

$$Q_{pq}^{(2n+1)} = 0, \tag{30b}$$

where

$$L_{pql} = 2\phi\binom{-1/2}{l}\left(\sum_{m=1}^{\infty}\frac{K_{pm0}K_{qm0}}{\bar{\alpha}_m^{(2l+1)}} + \sum_{n=1}^{\infty}\frac{K_{p0n}K_{q0n}}{\bar{\beta}_n^{(2l+1)}} + 2\sum_{m,n=1}^{\infty}\frac{K_{pmn}K_{qmn}}{(\bar{\alpha}_n^2+\bar{\beta}_n^2)^{(l+1/2)}}\right). \tag{31}$$

The first two terms in the expansion of $\{C\}$ are

$$C_p^{(0)} = \frac{2\sqrt{j_{1p}}}{1+\phi}L_{p00}, \tag{32a}$$

$$C_p^{(1)} = \frac{i}{1+\phi}L_{000}C_p^{(0)}. \tag{32b}$$

Inserting the ansatz equation (29) into Eq. (27), and equating terms of different order in $\bar{k}$, yields a sequence of problems for the arrays $\{B^{(m)}\}$, $n = 0,1,2,\ldots$. The first two are

$$[Q^{(0)}]\{B^{(0)}\} = \{C^{(0)}\}, \tag{33a}$$

$$[Q^{(0)}]\{B^{(1)}\} = \{C^{(1)}\} - [Q^{(1)}]\{C^{(0)}\}. \tag{33b}$$

The zeroth-order equation (33a) is a real, symmetric system. It is interesting to note, using Eqs. (30b) and (32b), that the solution to the first-order system equation (33b) is trivially

$$B_p^{(1)} = \frac{i}{1+\phi}L_{000}B_p^{(0)}. \tag{34}$$

We turn our attention specifically to the reflection coefficient R_{11} of Eq. (23), expanding it as

$$R_{11} = R_{11}^{(0)} + i\bar{k}R_{11}^{(1)} + (i\bar{k})^2 R_{11}^{(2)} + \cdots . \tag{35}$$

It is easy to show

$$R_{11}^{(0)} = -\left(\frac{1-\phi}{1+\phi}\right), \tag{36}$$

$$R_{11}^{(1)} = \frac{1}{1+\phi}\left(\sum_{p=1}^{\infty} \sqrt{j_{1p}}\, L_{p00} B_p^{(0)} - \frac{2}{(1+\phi)} L_{000}\right). \tag{37}$$

Define the quasistatic effective increase in length of the pores as $L(0)$, where

$$R_{11} = -\left(\frac{1-\phi}{1+\phi}\right) e^{i2kL(0)} + O(\bar{k}^2). \tag{38}$$

Therefore, by Eqs. (35)–(38),

$$\frac{L(0)}{R} = \frac{L_{000}}{1-\phi^2} - \frac{1}{2(1-\phi)} \sum_{p=1}^{\infty} \sqrt{j_{1p}} L_{p00} B_p^{(0)}. \tag{39}$$

It is also clear from Eqs. (23b), (32b), (34), and (35) that $R_{11}^{(2)}$ is real and may be determined from these equations. A simpler method is to use Eqs. (38) and (54) below, to yield

$$R_{11}^{(2)} = \frac{2(1-\phi)^2}{3-\phi}\left(\frac{L(0)}{R}\right)^2. \tag{40}$$

V. REFLECTION AND TRANSMISSION COEFFICIENTS

For arbitrary frequency, the transmitted field in the fluid half-space follows from Eqs. (10), (13), (17), and (20e) as

$$u_2(\mathbf{x}) = A \sum_{m,n=0}^{\infty} \frac{\bar{k}}{\bar{\rho}_{mn}} F_{mn}(\bar{k}) \cos \alpha_m x \cos \beta_n y\, e^{i\rho_{mn} z}, \tag{41}$$

where

$$F_{mn}(\bar{k}) = \frac{4\phi}{\delta_m \delta_n}\left(K_{0mn} - \sum_{l=0}^{\infty} K_{lmn} \bar{A}_l\right). \tag{42}$$

The numbers F_{mn} are the amplitudes of the different order transmitted plane waves. Only a finite number of these are propagating at a given frequency. The propagating modes satisfy $\rho_{mn}^2 > 0$, and the remaining evanescent modes have $\rho_{mn}^2 < 0$. The only mode that propagates at all frequencies is the (00) or plane wave in the z direction. Define the transmission coefficient

$$T_{12}(\bar{k}) = F_{00}(\bar{k}). \tag{43}$$

Then by Eqs. (42) and (9), we have the general relation

$$T_{12}(\bar{k}) = \phi[1 - R_{11}(\bar{k})]. \tag{44}$$

The plane-wave transmission coefficient for normal incidence from the fluid half-space follows from Appendix A as

$$T_{21}(\bar{k}) = \bar{A}'_0. \tag{45}$$

From Eqs. (9), (45), and (A9), we deduce the general result

$$T_{21}(\bar{k}) = 1 - R_{11}(\bar{k}). \tag{46}$$

Define the reflection coefficients R_{22} for incidence from the fluid as

$$R_{22}(\bar{k}) = 1 + F'_{00}(\bar{k}), \tag{47}$$

where the scattered field u_2^{sc} of Appendix A is

$$u_2^{sc} = A \sum_{m,n=0}^{\infty} \frac{\bar{k}}{\bar{\rho}_{mn}} F'_{mn}(\bar{k}) \cos \alpha_m x \cos \beta_n y\, e^{i\rho_{mn} z}, \tag{48}$$

with the modal amplitudes given by

$$F'_{mn} = -F_{mn}. \tag{49}$$

It is straightforward to show that

$$R_{22}(\bar{k}) = 1 - \phi + \phi R_{11}(\bar{k}). \tag{50}$$

Equations (44), (46), and (50) describe all four fundamental reflections and transmission coefficients in terms of a single one, in this case R_{11}.

Now consider the energy flux balances. For incidence from the porous side it is easily shown that the energy balance is

$$\phi = \phi |R_{11}|^2 + |T_{12}|^2 + k\left(\sum_{(1)} \frac{\phi |\bar{A}_n|^2}{\xi_n} + \sum_{(2)} \frac{|F_{mn}|^2}{\rho_{mn}}\right) \tag{51}$$

and, for incidence from the fluid side,

$$1 = |R_{22}|^2 + \phi |T_{21}|^2 + k\left(\sum_{(1)} \frac{\phi |\bar{A}_n|^2}{\xi_n} + \sum_{(2)} \frac{|F'_{mn}|^2}{\rho_{mn}}\right). \tag{52}$$

The modal sum (1) is over those $n \neq 0$, for which $j_{1n} < kR$, and sum (2) over $(m,n) \neq (0,0)$ for which $m^2/a^2 + n^2/b^2 < k^2/\pi^2$. Let $\bar{k}_{mn}$ be the dimensionless cutoff frequency of mode (m,n),

$$\bar{k}_{mn} = (\alpha_m^2 + \beta_n^2)^{1/2} R$$
$$= 2\sqrt{\pi\phi}\,[m^2(b/a) + n^2(a/b)]^{1/2}. \tag{53}$$

If $\bar{k} < \min(\bar{k}_{10}, \bar{k}_{01}) \leqslant \pi < j_{11} = 3.83...$, then both modal sums are zero and the energy balances can be expressed, using Eqs. (44), (46), and (50), as

$$|R_{11}|^2 + \phi |1 - R_{11}|^2 = 1 \tag{54}$$

for incidence from the pores, and as

$$\phi |1 - R_{11}|^2 + |1 - \phi + \phi R_{11}|^2 = 1 \tag{55}$$

for incidence from the fluid half-space. Equations (54) and (55) are easily seen to be equivalent.

VI. BEHAVIOR OF R_{11} NEAR THE CUTOFF FREQUENCIES $\bar{k}_{mn}$

For any m, n, we have from Eq. (42)

$$R_{11}(\bar{k}) = 1 - \frac{F_{mn}(\bar{k})}{K_{0mn}} \frac{\delta_m \delta_n}{4\phi} - \sum_{p=1}^{\infty} \frac{K_{pmn}}{K_{0mn}} \bar{A}_p(\bar{k}). \tag{56}$$

Also, Eq. (23) can be rewritten as

$$R_{11}(\bar{k}) = 1 - \frac{2}{\bar{k} M_{00}} - \sum_{p=1}^{\infty} \frac{M_{p0}}{M_{00}} \bar{A}_p(\bar{k}). \tag{57}$$

Define $\epsilon \equiv \bar{\rho}_{mn}$; then, as $|\epsilon| \to 0$, Eq. (21a) implies

$$M_{pq} = \left(\frac{4\phi}{\delta_m \delta_n} K_{pmn} K_{qmn}\right) \frac{1}{\epsilon} + O(1). \tag{58}$$

Thus, for $p \geqslant 1$,

$$M_{p0}/M_{00} = 2K_{pmn}/K_{0mn} + O(\epsilon). \tag{59}$$

Define

$$S_{mn} \equiv \sum_{p=1}^{\infty} \frac{K_{pmn}}{K_{0mn}} \overline{A}_p(\overline{k}_{mn}) \tag{60}$$

then, Eqs. (56) and (57) become, in the neighborhood of $\overline{k}_{mn}$,

$$R_{11}(\overline{k}) = 1 - S_{mn} - (F_{mn}/K_{0mn}) \times (\delta_m \delta_n / 4\phi) + O(\epsilon), \tag{61}$$

$$R_{11}(\overline{k}) = 1 - 2S_{mn} + O(\epsilon). \tag{62}$$

Eliminating S_{mn} from Eqs. (61) and (62) gives

$$R_{11}(\overline{k}) = 1 - \frac{F_{mn}}{K_{0mn}} \frac{\delta_m \delta_n}{2\phi} + O(\epsilon). \tag{63}$$

However, if ϵ is real and $0 < \epsilon \ll 1$, then, (51) implies that $|F_{mn}| = O(1)$. Equation (63), in combination with Eqs. (44), (46), and (50), shows that in the limit, as $\overline{k} \to \overline{k}_{mn}$,

$$R_{11}, R_{22} \to 1, \tag{64a}$$

$$T_{12}, T_{21} \to 0. \tag{64b}$$

The above analysis is for arbitrary m and n. In particular, it does not depend upon the porosity ϕ, or indeed the pore shape. Any symmetric, cylindrical pore would have the same effect. However, as $\phi \to 0$, we must be careful to scale ϵ, accordingly. In this limit the width of the "resonances" at $\overline{k}_{mn}$ will shrink. The limit of $\phi \to 0$ is actually quite a pathological one; see Fig. 2. As ϕ tends to zero, the modes $\overline{k}_{mn}$ become infinitely dense and the lowest one tends to zero. This produces a very jagged appearance in $|R_{11}(\overline{k})|$, for example. At zero frequency we know that $R_{11} = -1$ as $\phi \to 0$. But the above results say that there will be a frequency very close to zero at which $R_{11} = 1$. This type of behavior is characteristic of the limit of a continuous spectrum from a discrete spectrum; see, for example, Ref. 11, Sec. 4.13.

The limits in Eq. (64) indicate that the interface of the porous solid acts like a rigid membrane at the cutoff frequen-

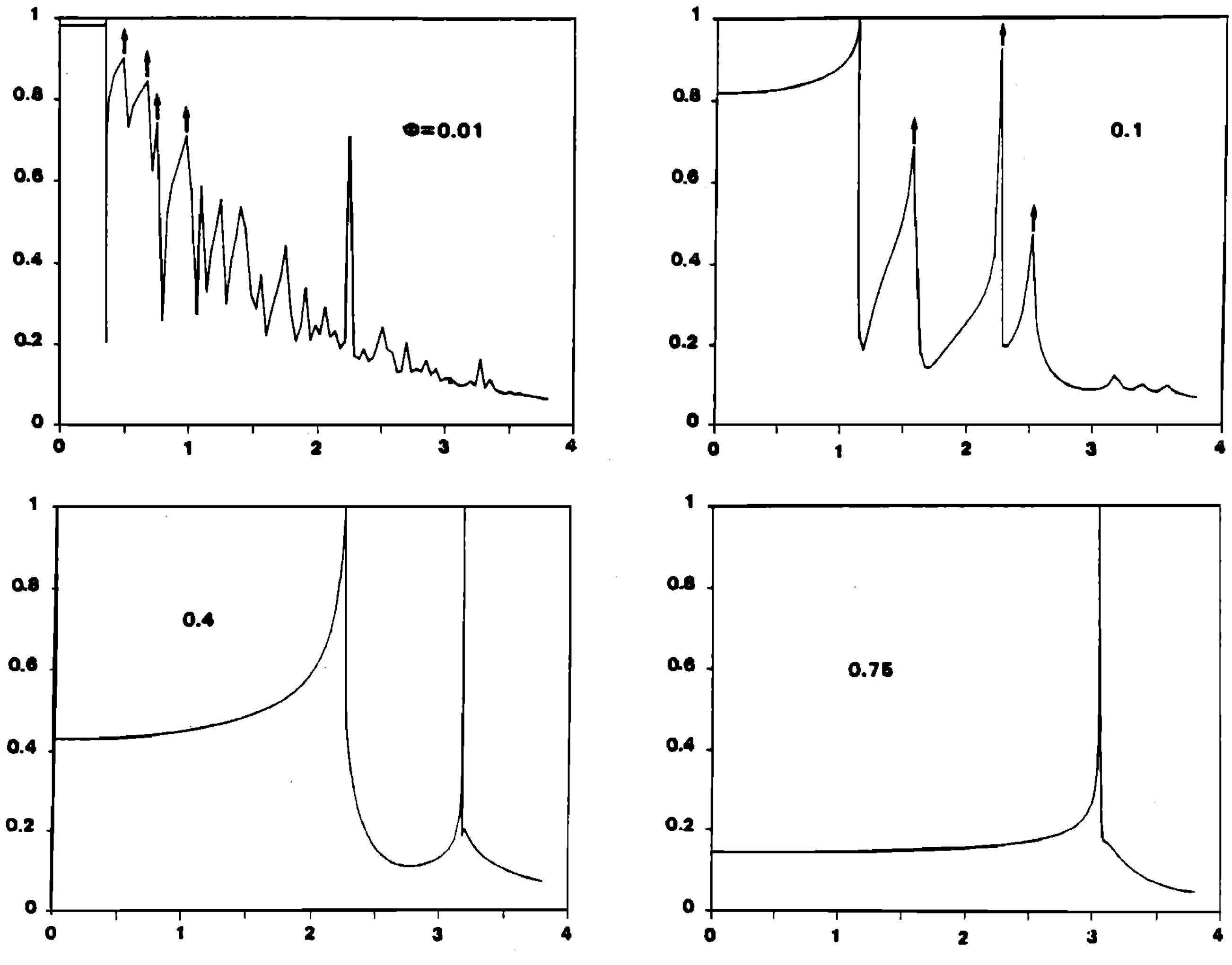

FIG. 2. The magnitude of R_{11} versus frequency $\overline{k} = kR$ for several values of the porosity ϕ. The magnitude actually goes to unity at each of the modal frequencies $\overline{k}_{mn}$. However, these peaks are not shown in the plots because of the large step size chosen for $\overline{k}$ relative to the resonance widths. This discrepancy is most evident for smaller values of ϕ, when the frequencies $\overline{k}_{mn}$ are very dense and the peaks exceedingly narrow.

cies $\bar{k}_{mn}$. No energy is transmitted across the interface when the wave is incident from either side.

VII. EFFECTIVE BOUNDARY CONDITION

For $\bar{k} < \min(\bar{k}_{10}, \bar{k}_{01})$, we know that R_{11} satisfies Eqs. (54) and (55). In addition, as $\bar{k} \to 0$, R_{11} has the simple form of Eq. (38). We also know, from Eq. (64), that $R_{11} \to 1$ as $\bar{k} \to \min(\bar{k}_{10}, \bar{k}_{01})$. We now propose to use these various pieces of information to devise a simple, physical description for $R_{11}(\bar{k})$ below the first cutoff frequency.

Consider the two coefficients R_{11} and T_{12}. Equation (44) is a general relation between them, which expresses the conservation of fluid mass. At zero frequency, we also have the static continuity of pressure condition across the interface, that

$$1 + R_{11}(0) = T_{12}(0). \tag{65}$$

Solving the simultaneous Eqs. (44) and (65) gives $R_{11}(0) = R_{11}^{(0)}$, where $R_{11}^{(0)}$ is defined in Eq. (36). Consider the following generalization of Eq. (65) to nonzero frequencies:

$$e^{i\alpha} + R_{11}(\bar{k})e^{i\beta} = T_{12}(\bar{k}), \tag{66}$$

where α and β are real numbers that depend upon $\bar{k}$, and both equal zero when $\bar{k} = 0$. Let $R_{11} = R_{11}^{(\alpha\beta)}$ be the solution to the simultaneous equations (44) and (66),

$$R_{11}^{(\alpha\beta)} = -[(e^{i\alpha} - \phi)/(e^{i\beta} + \phi)]. \tag{67}$$

Note that both Eqs. (54) and (55) are satisfied if α and β are related by

$$1 + \cos(\alpha - \beta) - \cos\alpha - \cos\beta = 0. \tag{68}$$

A trivial solution to Eq. (68) is $\alpha = \beta = 0$, for which $R_{11}^{(00)} \equiv R_{11}^{(0)}$. However, Eq. (68) is also satisfied if (i) $\alpha \neq 0$, $\beta = 0$, or (ii) $\alpha = 0, \beta \neq 0$. In general,

$$|R_{11}^{(\alpha\beta)}|^2 = \frac{1 + \phi^2 - 2\phi\cos\alpha}{1 + \phi^2 + 2\phi\cos\beta} \tag{69}$$

and, therefore,

$$|R_{11}^{(\alpha\beta)}| \geqslant |R_{11}^{(\alpha 0)}| \geqslant (1 - \phi)/(1 + \phi), \tag{70a}$$

$$|R_{11}^{(\alpha\beta)}| \geqslant |R_{11}^{(0\beta)}| \geqslant (1 - \phi)/(1 + \phi). \tag{70b}$$

In particular, $R_{11}^{(\alpha\beta)} = 1$ if $\alpha - \beta = \pm\pi$.

The effective boundary condition equation (66), in conjunction with Eq. (44), completely determines the four re-

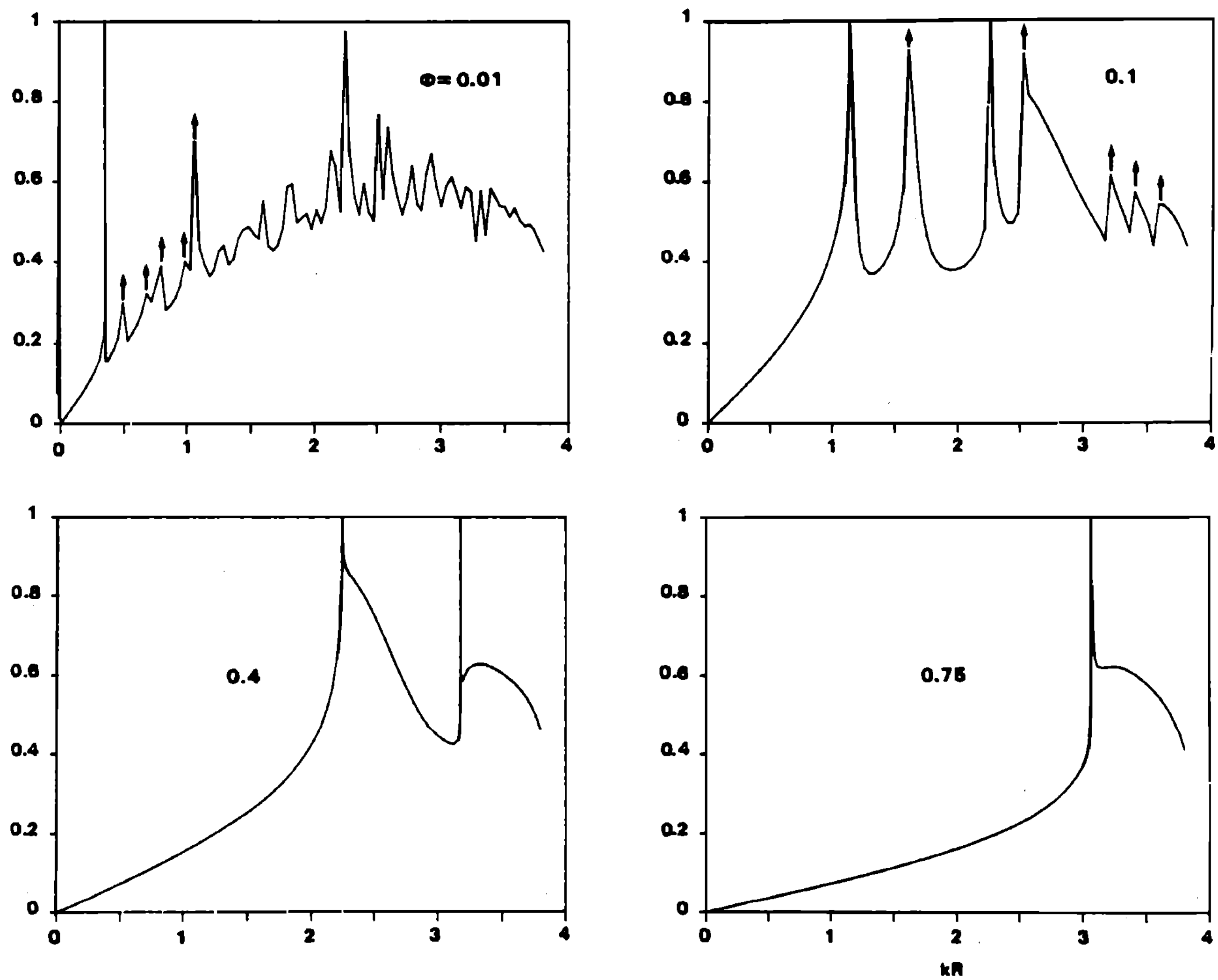

FIG. 3. The phase θ_{11}/π where $R_{11} = -|R_{11}|\exp(i\theta_{11})$ vs $\bar{k}$ for the same values of ϕ as in Fig. 2. The value of ϕ_{11} is π at each of k_{mn}. However, this is not always indicated in the plots for the same reasons as given in Fig. 2.

flection and transmission coefficients. The basis for Eq. (66) was first, as an obvious generalization of Eq. (65), and second, the energy balances Eqs. (54) and (55). We can interpret Eq. (66) physically as a displaced pressure continuity condition. For example, if $\beta = 0$ then it says that the transmitted pressure at $z = 0$ equals the sum of the reflected pressure at $z = 0$ plus the incident pressure at $z = \alpha/k$. The low-frequency results Eq. (38), along with Eq. (67), imply that for $R_{11}^{(\alpha\beta)}$ to be equal to R_{11}, we must have

$$\alpha/(1-\phi) - \beta/(1+\phi) = 2kL(0) + O(\bar{k}^2) \quad (71)$$

as $\bar{k} \to 0$. If we let $\beta = 0$, then this determines the initial behavior of α as a function of $\bar{k}$. As $\bar{k} \uparrow \min(\bar{k}_{10}, \bar{k}_{01})$, we should have $\alpha \to \pi$, since $R_{11} \to 1$. Alternatively, we could let $\alpha = 0$ and determine β as a function of $\bar{k}$ such that $R_{11}^{(\alpha\beta)} = R_{11}$. If we define $\bar{\alpha}$ and $\bar{\beta}$ such that

$$R_{11}^{(\bar{\alpha}0)} = R_{11}^{(0\bar{\beta})} = R_{11}(\bar{k}), \quad (72)$$

then

$$e^{i\bar{\alpha}} = \phi - (1+\phi)R_{11} \quad (73)$$

and

$$e^{i\bar{\beta}} = -\phi - (1-\phi)/R_{11}. \quad (74)$$

Either of these frequency dependent parameters allows us to interpret R_{11} through the effective boundary condition equation (66). However, the exact values of $\bar{\alpha}$ and $\bar{\beta}$ have to be determined from Eqs. (73) and (74), which involves solving the infinite complex system. One would hope that a simpler, more direct method exists to determine the real quantity $\bar{\alpha}$ or $\bar{\beta}$.

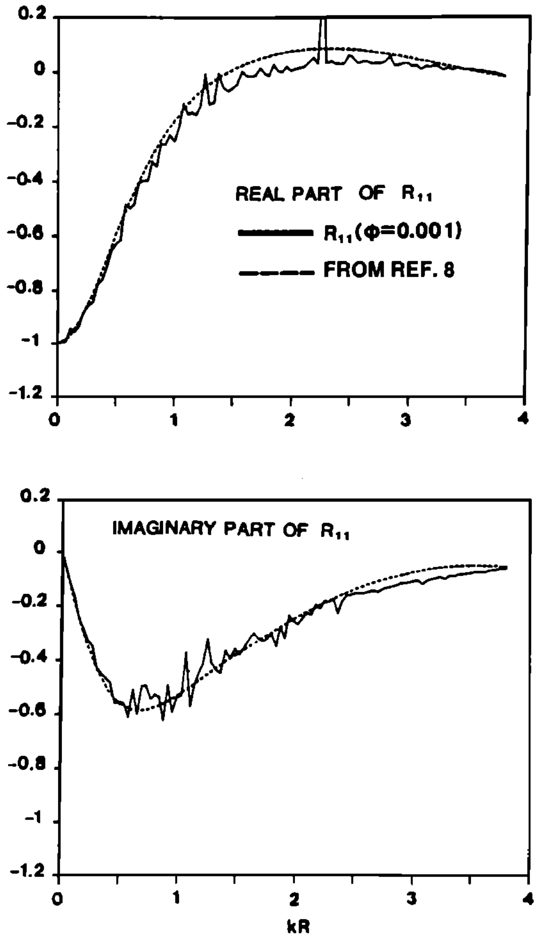

FIG. 4. The real and imaginary parts of R_{11} vs $\bar{k}$ for $\phi = 0.001$ compared with the same quantity for the single hole from Ref. 8. Again, the limitations described above apply at $\bar{k}_{mn}$.

VIII. NUMERICAL RESULTS AND DISCUSSION

All of the results shown are for a square array of pores. The determination of the quasistatic effective length increase $L(0)$ of Eq. (38) required solving the real, symmetric, infinite system equation (33a). It was found, mainly by trial and error and also on the basis of similar calculations for the single hole problem,[8] that truncation by a 75×75 system was adequate for convergence. A larger truncation size, typically 100×100, was used in solving the complex, symmetric, finite frequency system equation (27). For each value of $\bar{k}$ and $\bar{\phi}$, it was necessary to solve the truncated system equation (27). This sets a limit to the number of points that could be considered in $(\bar{k},\phi)$ space. We thus restricted our attention to $0 \leqslant \bar{k} \leqslant j_{11}$, the first cutoff frequency of the pores. The frequency step size was, by necessity, relatively large. Hence, it is not always apparent from the figures shown that, for example, $R_{11} \to 1$ as $\bar{k} \to \bar{k}_{mn}$. Independent checks on several arbitrary modal frequencies $\bar{k}_m$ did, however, show complete agreement with our analytical findings.

We note the plots of R_{11} in Figs. 2 and 3 for small values of the porosity ϕ. This seemingly erratic behavior, superimposed upon a smooth background, is typical of waveguide phenomena involving very many modal frequencies. As $\phi \to 0$, the peaks at $\bar{k}_{mn}$ still have finite magnitude, in fact $R_{11} \to 1$ at these frequencies, but the widths of the peaks disappear. This is analogous to the vanishing of the resonance peaks in the acoustic scattering cross section of a soft or hard target as the impedance mismatch between the target and host medium becomes large.[12] In the limit of $\phi \to 0$, the reflection coefficient reduces to the coefficient for a single hole in a rigid half-space; see Fig. 4. Plots of other related quantities are shown in Figs. 5–8. Note that the results shown are for an inviscid acoustic fluid. The effect of viscosity would be

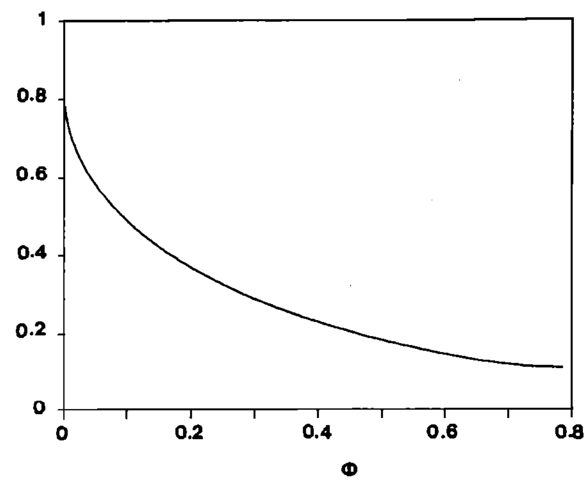

FIG. 5. Plot of the quasistatic end correction $L(0)/R$ vs ϕ.

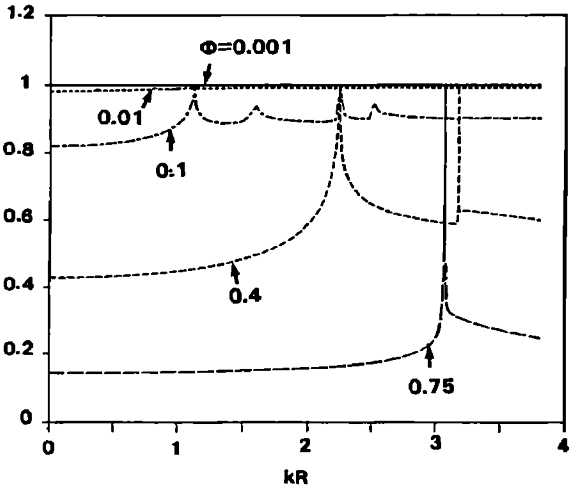

FIG. 6. The magnitude of R_{22} vs $\bar{k}$ for different values of ϕ. The magnitude should be unity at $\bar{k} = \bar{k}_{mn}$, but this is not always indicated because of the sparsity of points.

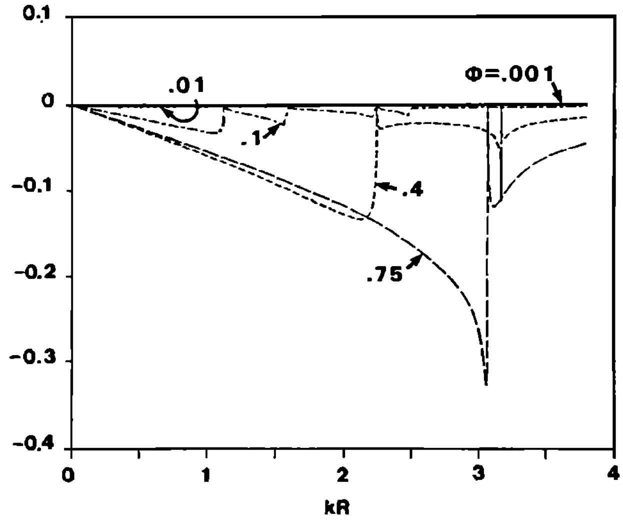

FIG. 7. The phase θ_{22} defined by $R_{22} = |R_{22}|\exp(i\theta_{22})$ versus frequency for different porosities. The value of θ_{22} is actually zero at $\bar{k} = \bar{k}_{mn}$, but this is not always apparent in the plots. The curves correspond to ϕ as in Fig. 6.

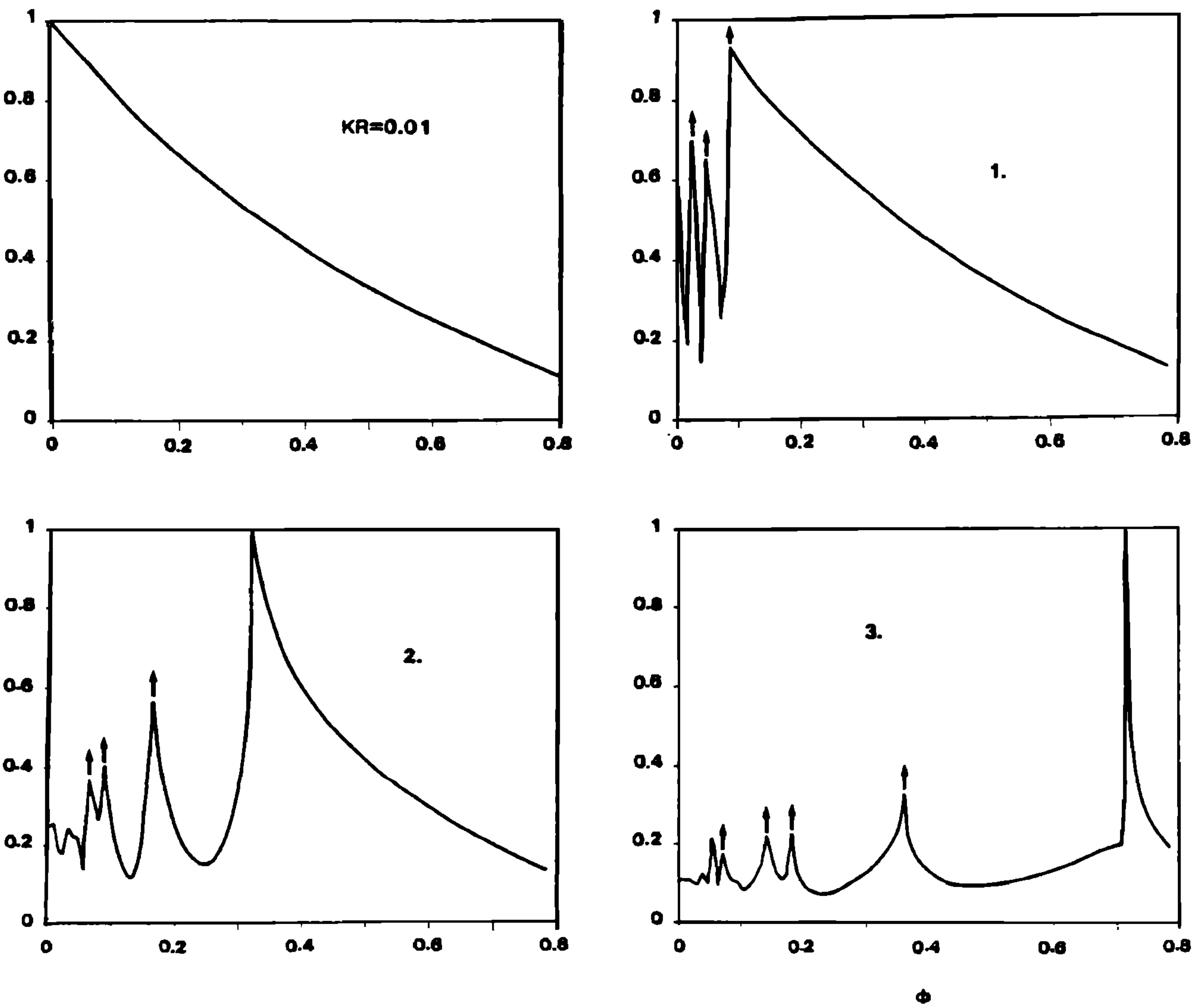

FIG. 8. The magnitude of R_{11} vs ϕ for several values of $\bar{k}$. The same caveat applies at the "resonance" porosities $\phi_{mn} = \bar{k}^2/4\pi(m^2 + n^2)$. Note the quasistatic form of $|R_{11}|$ at $\bar{k} = 0.01$.

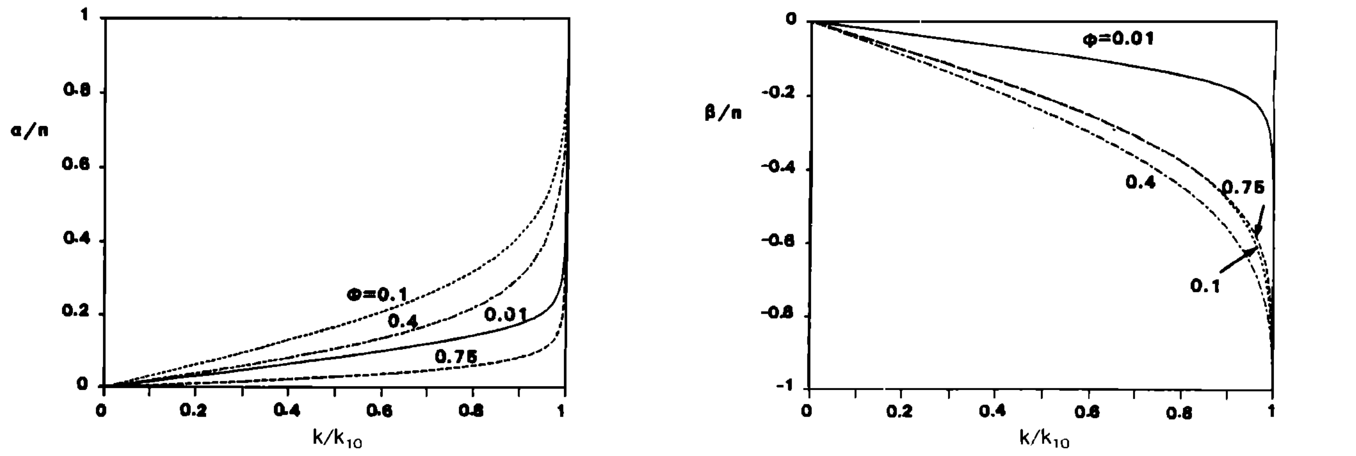

FIG. 9. The quantities $\bar{\alpha}$ and $\bar{\beta}$ of Eqs. (73) and (74). The curves show $\bar{\alpha}/\pi$ and $\bar{\beta}/\pi$ vs k/k_{10} for different values of ϕ.

to broaden the resonances and lower the magnitudes at resonance. The displaced pressure continuity condition parameters $\bar{\alpha}$ and $\bar{\beta}$ are plotted in Fig. 9.

Finally, we note that the results for the reflection and transmission coefficients R_{11}, R_{22}, T_{12}, and T_{21} can be combined to consider reflection and transmission through a porous panel. Let l be the panel thickness and let R, T be the reflection and transmission coefficients for the unsupported panel, i.e., with free space on either side. Then, neglecting multiple reflections of all modes other than the fundamental ($n=0$), it is easy to show

$$T = T_{21}T_{12}e^{ikl}/(1 - R_{11}^{2}e^{i2kl}), \tag{75}$$

$$R = R_{22} + TR_{11}e^{ikl}. \tag{76}$$

For very thin panels with $kl \ll 1$, R and T simplify on the basis of Eq. (38) and the results of Sec. V, to give $T \sim 1$, $R \sim -i[(1-\phi)/2\phi(1+\phi)][(1+\phi)^2 2kL(0) + (1+\phi^2) \times kl] \equiv -ik\bar{l}$. In terms of the equivalent circuit theory of Morse[7] (p. 365), the panel has a characteristic impedance $Z = \rho c(1+R)/(1-R)$. For thin panels, $Z = \rho c(1 - i2k\bar{l})$. The panel acts like an impedance ρ in series with a small resistance of order ω^2. The small resistance here is analogous to the resistance due to power radiation from a duct into free-space [Morse,[7] Eq. (2.3.4)]. However, the resistance is due to conversion from the fundamental mode to the higher, nonpropagating modes. Radiation effects are unimportant. The present results could be augmented by radiation loss for a panel of finite lateral dimensions.

IX. CONCLUSIONS

The exact solution for plane-wave reflection from a periodic fluid saturated porous solid has been derived and numerical results presented. Our major finding is the finite frequency effect that the interface acts like a rigid membrane at the discrete frequencies defined by the double periodicity of the pore spacing. This effective rigidity is independent of the porosity ϕ, and is also expected to be independent of the pore shape. We have also identified a simple, displaced pressure continuity condition that depends upon a single real frequency dependent quantity. This new condition is in the form of an obvious generalization of the static condition, but is only valid for frequencies up to the first cutoff. The simplicity and utility of this effective interface condition suggests that further work is necessary in the area of deriving effective interface conditions from first principles, so that no arbitrary parameters are required.

ACKNOWLEDGMENTS

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APPENDIX A: INCIDENCE FROM THE FLUID

Consider the plane wave

$$u_2^{\text{inc}} = Ae^{-ikz} \tag{A1}$$

incident upon the porous half-space. Assume the solution

$$u = \begin{cases} u_1, & z \leqslant 0, \\ u_2 = u_2^{\text{inc}} + Ae^{ikz} + u_2^{\text{sc}}, & z \geqslant 0. \end{cases} \tag{A2}$$

Note that u_2^{sc} does not include the rigid reflection Ae^{ikz}. Therefore, u_2^{sc} should vanish when the pores disappear. Express u_1 as

$$u_1 = \sum_{n=0}^{\infty} A_n'\Psi_n(r)e^{-i\xi_n z}. \tag{A3}$$

Proceeding as before, we can derive an integral relation similar to Eq. (17). Use of the continuity condition then implies the equation analogous to Eq. (18),

$$\sum_{l=0}^{\infty} A_l'\Psi_l(r) = 2A - \frac{1}{ab}\sum_{l=0}^{\infty} A_l'\xi_l \sum_{m,n=0}^{\infty} \frac{\cos\alpha_m x \cos\beta_n y}{\rho_{mn}\delta_m\delta_n} \times \int_{r'<R}\int \Psi_l(r')\cos\alpha_m x' \cos\beta_m y'\, dx'\, dy'. \tag{A4}$$

The orthogonality of the Ψ_n leads to the system

$$[M]\{\bar{A}'\} = \{N'\}, \tag{A5}$$

where $[M]$ is the same as before [Eq. (21a)], and

$$\bar{A}'_p = (A'_p/A)(\xi_p/k), \tag{A6}$$

$$N'_p = 2\delta_{p0}. \tag{A7}$$

Noting that

$$N'_p = M_{p0} - N_p, \tag{A8}$$

where N_p is defined in Eq. (21b), it is clear that the solution to the system (A5) can be written in terms of the solution to Eq. (22) as

$$\bar{A}'_p = \delta_{p0} - \bar{A}_p. \tag{A9}$$

APPENDIX B: DERIVATION OF EQ. (20e)

It is easily seen from its definition in Eq. (20e) that, because of the symmetry involved,

$$K_{pmn} = \frac{1}{\pi R^2} \int_{r<R} \int \Psi_p(r) e^{i(\alpha_m x + \beta_n y)}\, dx\, dy. \tag{B1}$$

Let $\gamma = (\alpha_m^2 + \beta_n^2)^{1/2}$ and introduce polar coordinates, so that

$$K_{pmn} = \frac{1}{\pi R^2} \int_0^R \Psi_p(r) r\, dr \int_0^{2\pi} e^{i\gamma r \cos(\theta - \theta')}\, d\theta. \tag{B2}$$

From Eq. (5) and Ref. 13, Eq. (9.1.21),

$$K_{pmn} = \frac{2}{R^2 J_0(k_p R)} \int_0^R J_0(k_p r) J_0(\gamma r) r\, dr. \tag{B3}$$

Equation (20e) then follows from Ref. 13, Eq. (11.3.29).

APPENDIX C: THE SINGLE HOLE LIMIT

As $a/R, b/R \to \infty$, the porosity $\phi \to 0$ and the double sum in Eq. (21a) becomes in the limit a double integral. This can be reduced to a single integral using polar coordinates. After some straightforward manipulation it can be shown that the elements of $[Q]$ and $\{C\}$ in the fundamental equation (27) tend, in the present limit, to $(p,q = 1,2,...)$:

$$C_p = 2\alpha_{p0}\sqrt{j_{1p}}/(1 - i\bar{k}\alpha_{00}), \tag{C1}$$

$$Q_{pq} = (1 - \bar{k}^2/j_{1p}^2)^{1/2}\delta_{pq} + \sqrt{j_{1p}j_{1q}} \times\{\alpha_{pq} + i\bar{k}\,[\alpha_{p0}\alpha_{q0}/(1 - i\bar{k}\alpha_{00})]\}, \tag{C2}$$

where, for $m,n = 0,1,2,...$,

$$\alpha_{mn}(\bar{k}) = 2\int_0^\infty \frac{s^3 J_1^2(s)\, ds}{(s^2 - \bar{k}^2)^{1/2}(s^2 - j_{1m}^2)(s^2 - j_{1n}^2)}. \tag{C3}$$

The real and imaginary parts of $\alpha_{0n}(\bar{k})$ are proportional to the quantities χ_n and θ_n, defined in Eq. (28.8) of Morse.[7]

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