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Short Note

The tube wave as a Biot slow wave

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INTRODUCTION

The tube wave speed in a simple fluid-filled circular bore reduces to $v_T = v_f/(1 + K_f/\mu)^{1/2}$ as the frequency goes to zero, where $v_f = (K_f/\rho_f)^{1/2}$ is the acoustic sound speed in the fluid, ρ_f is the fluid density, K_f is the fluid bulk modulus, and μ is the formation shear modulus. Biot (1952) deduced this simple relation by considering the low-frequency asymptotic expansion of the exact dispersion relation. In 1956, Biot proposed a theory (Biot, 1956a, b) that predicts a new type of compressional bulk wave in fluid-saturated porous media. This "slow wave" is associated mainly with the motion of pore fluids. It appears that Biot never related this theory to his previous work on the bore problem, although the connection is apparent if the bore is considered as a pore. Typically, the bore radius is about 10 cm, while the relevant acoustic logging frequency is on the order of 1 kHz. With water as the bore fluid, the viscous skin depth is on the order of 100 μm. Therefore, if the bore is to be considered as a pore, the relevant form of Biot's theory is the limit in which the pore radius is large relative to the viscous skin depth of the fluid. This form is the high-frequency limit, in which the effects of the fluid viscosity are negligible and the slow-wave dissipation is relatively low.

DYNAMIC POROELASTICITY

The poroelastic equations in the absence of viscous dissipation are (Biot, 1962)

$$\rho \frac{\partial^2}{\partial t^2} \mathbf{u} + \rho_f \frac{\partial^2}{\partial t^2} \mathbf{w} = \left(K_c + \frac{\mu}{3}\right) \nabla\nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \alpha M \nabla\nabla \cdot \mathbf{w} \tag{1}$$

and

$$\rho_f \frac{\partial^2}{\partial t^2} \mathbf{u} + m \frac{\partial^2}{\partial t^2} \mathbf{w} = \alpha M \nabla\nabla \cdot \mathbf{u} + M \nabla\nabla \cdot \mathbf{w}, \tag{2}$$

where $\mathbf{u}$ and $\mathbf{w}$ are the solid and relative fluid displacements, $\rho = \phi\rho_f + (1 - \phi)\rho_s$ is the average density, ρ_s is the solid density, ϕ is the porosity, and m is the effective fluid density. m is sometimes written $m = T\rho_f/\phi$, where T (≥ 1) is the pore-space tortuosity. The shear modulus of the porous medium is μ and K_s is the bulk modulus of the solid grain. The other parameters are

$$\alpha = 1 - K/K_s, \tag{3}$$

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{\alpha - \phi}{K_s}, \tag{4}$$

and

$$K_c = K + \alpha^2 M. \tag{5}$$

Equation (5) is the Biot-Gassmann relation between the bulk moduli K_c of the saturated medium and K of the dry medium.

The two compressional-wave speeds are found by considering solutions in the form $(\mathbf{u}, \mathbf{w}) = (A, B)\hat{\mathbf{n}} \cos[\omega(t - \hat{\mathbf{n}} \cdot \mathbf{x}/c)]$, where $\hat{\mathbf{n}}$ is an arbitrary unit vector. The constants A and B, and the speed c, follow from a simple eigenvalue problem. The speeds are given by

$$c^2 = \frac{c_0^2 + \left(1 - 2\alpha \frac{\rho_f}{\rho}\right) c_1^2 \pm \left\{ \left[c_0^2 + \left(1 - 2\alpha \frac{\rho_f}{\rho}\right) c_1^2 \right]^2 + 4\left(1 - \frac{\phi\rho_f}{T\rho}\right)\left(\rho_f \frac{\alpha^2 T}{\rho\phi} c_1^2 - c_0^2\right) c_1^2 \right\}^{1/2}}{2\left(1 - \frac{\phi\rho_f}{T\rho}\right)}, \tag{6}$$

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where

$$c_0^2 = (K_c + \tfrac{4}{3}\mu)/\rho \tag{7}$$

and

$$c_1^2 = \frac{\phi M}{T\rho_f}. \tag{8}$$

The roots defined in equation (6) simplify if the elastic moduli and densities satisfy Biot's compatibility condition (Biot, 1956a), $c_0^2 = \alpha M/\rho_f$. Then, the two roots are exactly c_0 and c_2, where $c_2 = c_1[(1 - \alpha\rho_f/\rho)/(1 - \phi\rho_f/T\rho)]^{1/2}$, and it can be shown that $c_2 < c_1 < c_0$.

THE TUBE WAVE

The tube-wave limit follows from taking the limit of an isolated cylindrical pore by (1) taking the limit of vanishing porosity; (2) using the appropriate tortuosity T for a single cylindrical pore parallel to the direction of wave propagation; and (3) calculating the correct form of the dry frame modulus corresponding to the isolated pore.

First consider the limit of $\phi \to 0$. Then $\alpha = O(\phi)$ and $\phi M = O(1)$, where O means "order of." On the basis of the Hashin-Shtrikman bounds, $\alpha > \phi$ for all $\phi > 0$. Keeping T of order unity, the two speeds of equation (6) reduce to c_0 and c_1 of equations (7) and (8). c_0 is the compressional-wave speed in a homogeneous elastic solid. c_1 is the Biot slow-wave speed.

The tortuosity T for wave motion parallel to the axis of a cylindrical pore of arbitrary cross-section reduces to unity in the limit when the viscous skin depth is vanishingly small, which is the present high-frequency limit of the Biot theory. This result follows from the simple observation that as the viscous skin depth vanishes, there is no coupling between the solid and fluid motions in the axial direction. Thus, the effective inertia ρ_{12} (Biot, 1956a, b), representing the drag of the fluid on the frame, is zero. Because $T = 1 - \rho_{12}/\phi\rho_f$ (Biot, 1962), it follows that $T = 1$. Usually, it has been assumed that $T \to \infty$ as $\phi \to 0$, so that $c_1 \to 0$. In this limit, the pore space becomes very poorly connected and highly "tortuous." However, the present limit process maintains the connectivity of the pore and actually minimizes T. With $T = 1$, c_1 becomes [from equations (4) and (8)]

$$c_1 = v_f \Big/ \left(1 + \frac{K_f}{K_0}\right)^{1/2}, \tag{9}$$

where the modulus K_0 is defined as

$$\frac{1}{K_0} = \lim_{\phi \to 0} \frac{\alpha - \phi}{\phi K_s}. \tag{10}$$

In order to obtain K, the dry frame modulus, in the limit of $\phi \to 0$, consider the porous medium as a thick annular elastic shell with inner radius a and outer radius b. The length of the shell is arbitrary, and the porosity of this medium is just the volume fraction occupied by the pore, defined by $r < a$. The solid occupies $a < r < b$, and therefore the porosity is $\phi = (a/b)^2$. The dry bulk modulus is found by subjecting the porous medium to a hydrostatic pressure. Thus, consider the static elasticity problem of the shell subject to uniform pressure on the exterior surface. This surface is the cylindrical surface $r = b$, plus the two end caps $a < r < b$. The inner pore surface $r = a$ is left free of traction. If p is the applied pressure, define the dry bulk modulus K by $\Delta V/V = -p/K$, where ΔV is the incremental volume change from the original V. The solution is

$$\frac{1}{K} = \frac{1}{1-\phi}\left(\frac{1}{K_s} + \frac{\phi}{\mu_s}\right), \tag{11}$$

where μ_s is the solid shear modulus. This result is valid within the range of linear elasticity for $0 \le \phi < 1$.

The modulus K_0 follows from equations (3), (10), and (11) as $K_0 = \mu_s$. The limiting value of the Biot slow-wave speed becomes, from equation (9), $c_1 = v_f/(1 + K_f/\mu_s)^{1/2}$ which is exactly the long-wavelength limit of the tube-wave speed for a bore in a formation with shear modulus μ_s. Thus, the tube wave is the limiting case of the Biot slow wave when the bore is considered as an isolated pore.

AN ADDITIONAL RESULT

The ratio of amplitudes B/A of the assumed plane-wave solution $(\mathbf{u}, \mathbf{w}) = (A, B)\hat{\mathbf{n}} \cos[\omega(t - \hat{\mathbf{n}} \cdot \mathbf{x}/c)]$ is known to be negative for the Biot slow wave (Biot, 1956a). In the limit under consideration, as $\phi \to 0$,

$$\frac{B}{A} = \frac{\rho_s \mu_s}{\rho_f K_s} - \left(1 + \frac{\mu_s}{K_f}\right)\left(1 + \frac{4\mu_s}{3K_s}\right). \tag{12}$$

For example, if $v_p = 3\,000$ m/s, $v_s = 2\,000$ m/s, $\rho_s = 2.5$, $v_f = 1\,500$ m/s, and $\rho_f = 1$, then $B/A = -117/11$. The relative fluid displacement $\mathbf{w}$ is equal to $\phi(\mathbf{U} - \mathbf{u})$, where $\mathbf{U}$ is the absolute average fluid displacement. Therefore, as $\phi \to 0$ in the pore-to-bore limit, the ratio of solid-to-fluid displacement in the direction of propagation, i.e., in the axial direction, is $u/U = \phi/(1 + B/A)$. This displacement goes to zero in the limit as $\phi \to 0$, as would be expected, since the solid displacement is then averaged over an infinite volume compared with the borehole volume. The result is still interesting for $\phi > 0$, because it indicates how the fluid and solid interact in the presence of more than one pore or bore.

CONCLUSION

The tube wave is a limiting case of the Biot slow wave when the bore is considered as an isolated pore in a homogeneous, porous medium. The correspondence follows naturally from Biot's theory of dynamic poroelasticity, which predicts an additional slow compressional wave in the bulk porous medium. The slow-wave speed reduces exactly to the zero-frequency, tube-wave speed when the appropriate parameters are taken in Biot's theory for a porous medium with straight pores and vanishing porosity. The tube wave is further evidence of the consistency of Biot's theory of dynamic poroelasticity.

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