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ON THE VISCODYNAMIC OPERATOR IN BIOT'S EQUATIONS OF POROELASTICITY

A.N. NORRIS
Department of Mechanics and Materials Science
Rutgers University
P.O. Box 909
Piscataway, New Jersey 08855-0909

ABSTRACT

The equations of motion for a fluid saturated permeable solid contain a frequency dependent viscodynamic operator which follows from the solution of an oscillatory viscous flow problem in the pore space. At low freqencies, the first two terms in the asymptotic expansion of the operator depend upon the solution of a single Stokes flow problem. The second term, the inertial factor, is related to the mean square velocity in the pores. Explicit expressions are given for the low freqency permeability and inertial factor in pore geometries for which the full problem cannot be solved. Little variation is found in the inertial factor for different cylindrical pore models.

1. INTRODUCTION

The form of the equations of motion for a biconnected, fluid-saturated poroelastic medium were first proposed by Biot (1956). The same equations have been derived more recently from first principles using homogenization theory (Burridge and Keller, 1981; Levy, 1979; Auriault, 1980). Of the various macroscopic constitutive parameters in the final equation, the viscodynamic operator is certainly the most complex and least understood. This freqency dependent operator contains the inertial drag and viscous dissipative effects due to the pore fluid motion. Biot (1956) suggested characterizing the operator by considering the frictional drag at the pore surface due to the viscous pore fluid flow. In addition, he included an inertial term, to be determined independently. However, homogenization theory prescribes an explicit procedure for determining the operator, a procedure that differs from Biot's. We will point out the differences in the two schemes, and also note that Zwikker and Kosten (1949), in earlier work on the acoustics of porous media derived expressions for a complex effective inertia that is more in line with homogenization theory than Biot's scheme. This discrepancy between the two theories of Zwikker and Kosten (1949) and Biot (1956) has led to some confusion. It has generally been accepted that Biot's general set of equations of motion contains various limits including that of a nearly rigid frame, for which Zwikker and Kosten's effective inertia was proposed. However, Biot's 1956 theory, though more general than that of his predecessors', does not account correctly for the frequency dependence of the pore fluid inertia and dissipation. We wish to emphasize this aspect, which has been previously noted and discussed by Bedford et al. (1984), Auriault et al. (1985), and Attenborough (1986). An excellent review of the acoustics of porous media can be found in the review by Attenborough (1982).

The main results of this paper are in the identification of simplifications to the form of the operator at frequencies low compared with the characteristic frequency at which the viscous skin depth is of the same order as a typical pore width. In the low frequency, Stokes flow regime, the operator is characterized by the static permeability and an effective mass which can be defined by a dimensionless inertial factor. We deduce a simple but general form for the latter that depends upon the solution of the same steady flow problem which determines the static permeability. Exact solutions are obtained for a variety of cylindrical pore models. The present results may be of practical use in calculating the viscodynamic operator for realistic microgeometries. Alternatively, there are potential applications to the inverse problem of inferring the microstructure from measured data, e.g., the attenuation of the Stoneley wave in acoustic logging signals from boreholes.

Researchers in the recent past have emphasized the importance of understanding the viscodynamic operator. In particular, we note the work of Bedford (1986), who compared the fast and slow wave speeds and attenuations for a layered medium obtained by two different procedures: (1) the exact dispersion equation, and

(2) Biot's equations with the correct viscodynamic term. Excellent agreement was found. Other authors, e.g. Auriault et al. (1985), Johnson et al. (1986), have looked at the asymptotic forms of the operator at high frequencies, where the two leading terms in the expansion follow from the solution of a potential flow problem in the pores.

2. DYNAMIC POROELASTICITY AND THE VISCODYNAMIC OPERATOR

Let **u** and **w** be the macroscopic solid displacement and the relative fluid displacement, respectively. The absolute fluid displacement **U** satisfies $\mathbf{w} = \phi(\mathbf{U}-\mathbf{u})$, where ϕ is the porosity. Biot's equations of dynamic poroelasticity are (Biot, 1962)

$$\rho\partial^2 u_i/\partial t^2 + \rho_f\partial^2 w_i/\partial t^2 = \tau_{ij,j} \quad , \tag{1a}$$

$$\rho_f\partial^2 u_i/\partial t^2 + \angle w_i = -p,_i \tag{1b}$$

where $\rho=\phi\,\rho_f + (1-\phi)\rho_s$ and ρ_s, ρ_f are the solid and fluid densities, respectively. The average stress τ_{ij} and pore fluid pressure p can be related to the strains by linear equations, see for example Biot (1956). The viscodynamic operator $\angle$ is the subject of the present paper. The simplest form of $\angle$, proposed originally by Biot (1956) is

$$\angle\, w = (\mu/k_0)\,\partial w/\partial t + m\partial^2 w/\partial t^2 \equiv \angle_S w \quad , \tag{2}$$

where μ is the dynamic fluid viscosity and k_0 the permeability for steady state flow. The density m is an effective density for the pore space. It is sometimes written as

$$m = T_0\rho_f/\phi \quad , \tag{3}$$

where $T_0 > 1$, is the inertial factor or structure constant of the pore space. The first term in $\angle$ in Eq. (2) is attributable to Darcy's law for flow in a porous medium, $\nabla p = -(\mu/k_0)\,\partial w/\partial t$, while the second term represents the inertial drag offered by the frame to the pore fluid. In Biot's original notation of 1956, we have $T_0 = 1 - \rho_{12}/\phi\,\rho_f$, where $\rho_{12} < 0$.

The simple form of $\angle_S$ given in (2) is implicitly a low-frequency approximation because it uses the steady state Darcy term plus a first order inertial correction term. However, even to this approximation there are two parameters, k_0 and T_0, that depend upon the pore geometry. In the frequency domain, with time dependence exp $(-i\omega t)$,

$$\angle_S(\omega) = -\,i\,(\mu\omega/k_0) - m\,\omega^2 \ . \tag{4}$$

There is a wide diversity in terminology for the acoustics of porous media. Thus, flow resistivity $\sigma = \mu/k_0$ is often used instead of permeability (Zwikker and Kosten, 1949). The inertial factor T_0 is the same as the structure constant k of Zwikker and Kosten (1949). Carman uses the term tortuosity with respect to both permeability (Carman, 1956, p. 12) and effective conductivity (Carman, 1956, p. 46). It is in the latter sense that tortuosity is also understood by Johnson et al. (1982). Carman's definition of tortuosity is as a measure of the deviation of the streamlines from straight lines in the direction of the applied pressure gradient as a consequence of the tortuous passage of the pore fluid. As we will see from the examples below, T_0 depends upon the pore cross-section even when the streamlines are all straight and the pores cylindrical. Therefore, to call T_0 the tortuosity, flies in the face of our common understanding of the word. We prefer to call it the inertial factor, effective mass factor or even the real part of a complex tortuosity.

We will be concerned with generalizing $\angle(\omega)$ to higher frequencies. Alternatively, one can define a frequency dependent complex density $\tilde{\rho}(\omega)$ and a complex permeability $\tilde{k}(\omega)$, by analogy with (4), through the relations

$$\angle(\omega) = -\tilde{\rho}(\omega)\omega^2 = -i\omega\mu/\tilde{k}(\omega) \quad . \tag{5}$$

We note some general analytic properties of these functions, considered as functions of complex ω. Johnson et al. (1986) have shown that $\angle(-\omega^*) = \angle(\omega)$, where the asterisk denotes complex conjugate. They have also shown that the only singularities of $\angle(\omega)$ occur on the negative imaginary ω-axis. Similar results hold for the functions $\tilde{\rho}(\omega)$ and $\tilde{k}(\omega)$, although we note that $\tilde{\rho}(\omega)$ has a simple pole at $\omega = 0$, due to the specific way in which it was defined. For real ω, we can write

$$\tilde{\rho}(\omega) = \mathrm{Re}(\tilde{\rho}(\omega)) + i\,(\mu/\omega)\,\mathrm{Re}\,(1/\tilde{k}(\omega)) \quad . \tag{6}$$

Brown (1981) has shown that for $\omega > 0$,

$$\mathrm{Re}\,(\tilde{\rho}(\omega)) \le m \text{ and } \mathrm{Re}\,(1/\tilde{k}(\omega)) \ge 1/k_0 \quad .$$

Thus, if we define a frequency dependent inertial factor $T(\omega)$ by

$$T(\omega) = (\phi/\rho_f)\,\mathrm{Re}(\tilde{\rho}(\omega)) \tag{7}$$

then $T(0) = T_0$ and Brown's results imply that $T(\omega) \le T_0$. In particular let T_∞ be the high frequency limit of $T(\omega)$, then $T_\infty \le T_0$. This infinite frequency limit of $T(\omega)$ is identical to the quantity α_∞ of Johnson et al. (1982, 1986).

Finally, we define a dimensionless frequency-dependent factor F through

$$\angle(\omega) = -i\omega(\mu/k_0)F(\omega) \tag{8}$$

or,

$$F(\omega) = k_0/\tilde{k}(\omega) \quad . \tag{9}$$

3. THE CONSISTENT THEORY FOR $\angle(\omega)$

We now describe the prescription for obtaining $\angle(\omega)$ for a given pore geometry. The procedure follows from the work of Burridge and Keller (1981), Levy (1979) and Auriault (1980), each of whom derived Biot's original equation from first principles using homogenization techniques. The idea is to introduce two length scales, one microscopic and the other macroscopic. For example, the micro length might be a typical pore diameter; and the macro length, the wavelength of elastic waves. The final equations are obtained in terms of macroscopic coordinates, but the various constants depend upon the solution of canonical "cell" problems in the microscopic coordinate system. The relevant problem for $\angle(\omega)$ considers an incompressible fluid in a pore "cell" subject to a uniform oscillatory pressure gradient. The same problem for a compressible fluid was proposed by Bedford et al. (1984). Their justification was that this procedure is entirely consistent with the form of Biot's equations of poroelasticity. Thus, we refer to this approach as the consistent theory for $\angle(\omega)$. However, we note that Bedford et al. (1984), in considering a compressible pore fluid, were over-complicating the problem (see note 17 of their paper). Homogenization theory and elementary considerations indicate that the pore fluid can be viewed as incompressible for the purposes of estimating $\angle(\omega)$.

The procedure described in this section is different from Biot's original ideas for extending $\angle_S(\omega)$ to higher frequencies (1956). The latter approach is discussed in Appendix A. However, the earlier procedure of Zwikker and Kosten (1949) for finding the complex inertia of a fluid in a rigid frame is essentially the same as the consistent theory.

Consider Eq. (1b) for the pore fluid motion with zero macroscopic pressure gradient and the solid phase oscillating uniformly as

$$\mathbf{u}(x,t) = \mathbf{u}_0 \exp(-i\omega t) \quad ,$$

where $\mathbf{u}_0$ is a constant vector. Then $\angle(\omega)$ satisfies

$$\angle(\omega)\mathbf{w}_0 = \rho_f\,\omega^2\mathbf{u}_0 \quad , \tag{10}$$

where $\mathbf{w}_0 \exp(-i\omega t)$ is the average relative displacement in the pore fluid. It is found by solving the following "cell" problem in a unit cell, denoted by the pore volume V_p and pore surface S_p:

$$-\mu\Delta_y\dot{\overline{U}} + \rho_f\ddot{\overline{U}} = -\nabla_y\overline{p} \quad , \quad \text{in } V_p ; \tag{11a}$$

$$\nabla_y \bullet \overline{U} = 0 \quad , \quad \text{in } V_p \quad ; \tag{11b}$$

$$\overline{U} = u_0 e^{-i\omega t} \quad , \quad \text{on } S_p \quad ; \tag{11c}$$

where $\overline{U}(y,t)$ is the <u>microscopic</u> fluid displacement and $\mathbf{y}$ the microscopic spatial coordinate, with operators ∇_y and $\Delta_y = \nabla_y \bullet \nabla_y$. The average displacement <w> is defined by

$$<w> = (\phi/V_p) \int_{V_p} [\overline{U}_0(y) - u_0]\, dy \tag{12}$$

where $\overline{U}(y,t) = \overline{U}_0(y)\exp(-i\omega t)$ solves the problem (11).

Alternatively, we can define the microscopic relative fluid displacement $\overline{w}(y,t) = \phi(\overline{U}-\mathbf{u}_o)\exp(-i\omega t)$ which satisfies the system,

$$-\mu\Delta_y\dot{\overline{w}} + \rho_f\ddot{\overline{w}} = -\phi\nabla_y\overline{p} - g \quad , \quad \text{in } V_p \quad , \tag{13a}$$

$$\nabla_y \bullet \overline{w} = 0 \quad , \quad \text{in } V_p \quad , \tag{13b}$$

$$\overline{w} = 0 \quad , \quad \text{on } S_p \quad . \tag{13c}$$

The macroscopic pressure gradient is $\mathbf{g} = -\phi\,\rho_f\,\omega^2\,\mathbf{u}_o\exp(-i\omega t)$. The average <w> is

$$<w> = (1/V_p) \int_{V_p} \overline{w}_0(y)\, dy \tag{14}$$

where $\overline{w} = \overline{w}_0(y)\exp(-i\omega t)$ solves (13). Equations (11) and (12), or (13) and (14) are two equivalent procedures for determining $\angle(\omega)$ through (10). The first, (11) and (12), is similar to the method proposed by Bedford et al. (1984) and by Auriault et al. (1985). The pressure gradient form, (13) and (14), is more akin to the original formulation of Biot (1956), but the definition of $\angle(\omega)$ in (10) is different than his, see Appendix A.

Once the frequency dependent <w> is found, the viscodynamic operator follows as

$$\angle(\omega) = \rho_f\omega^2 u_o^2[<w> \bullet \mathbf{u}_o]^{-1} \quad . \tag{15}$$

<u>Example 1: Circular Cylindrical Pores</u>

Introduce the dimensionless frequency $\Omega_c = \omega a_c^{\,2}\rho_f/\mu$, where a_c is the pore radius. Let $\mathbf{u}_o$ make an angle θ with the axis; then it is straightforward to show that

$$(<w>\bullet\mathbf{u}_o)/u_o^2 = \phi\{[J_2(\sqrt{i\Omega_c})] \,/\, [J_0(\sqrt{i\Omega_c})]\}\cos^2\theta \quad , \tag{16}$$

where J_n is the Bessel function of order n. The Darcy permeability follows from the low frequency expansion of $k(\omega)$ as

$$k_o = [(\phi\, a_c^2)/8] \cos^2\theta \quad , \tag{17}$$

and the complex frequency dependent density is

$$\tilde{\rho}(\omega) = (\rho_f/\phi)\ \{[-J_0(\sqrt{i\Omega_c})]/[J_2(\sqrt{i\Omega_c})]\} \sec^2\theta \quad . \tag{18}$$

We note that $\tilde{\rho}(\omega)$ is exactly the same as Zwikker and Kosten's (1949) effective inertia.

Example 2. Slit-Like Pores

Let $2a_s$ be the width of the slit. Define the dimensionless frequency $\Omega_s = \omega a_s{}^2\rho_f/\mu$, then if $\mathbf{u}_o$ makes an angle θ with the walls of the slit we have

$$(\langle w \rangle \cdot \mathbf{u}_o)/u_o^2 = \phi\ \{[\tan(\sqrt{i\Omega_s})/(\sqrt{i\Omega_s})] - 1\} \cos^2\theta \quad , \tag{19}$$

from which we deduce,

$$k_o = \phi(a_s^2/3) \cos^2\theta \quad , \tag{20}$$

$$\tilde{\rho}(\omega) = -(\rho_f/\phi)\ \{[\tan(\sqrt{i\Omega_s})/(\sqrt{i\Omega_s})] - 1\}^{-1} \sec^2\theta \quad . \tag{21}$$

4. NONDIMENSIONAL EQUATIONS

Consider the micro problem defined by Eq. (11). We nondimensionalize the equations by introducing a characteristic length b of the pore. For example, b is the radius of a circular cylindrical pore. Define the dimensionless frequency Ω as

$$\Omega = (\rho_f b^2/\mu)\omega \quad . \tag{22}$$

Introduce dimensionless variables x, U and p, defined by $\bar{\mathbf{U}} = \mathbf{U}\, u_o\, e^{-i\omega t}$, $\bar{p} = p(i\omega\mu u_o/b)e^{-i\omega t}$, $y = bx$. The nondimensional equations are

$$\Delta \mathbf{U} + i\Omega \mathbf{U} = -\nabla p \quad , \quad \text{in } V_p \quad ; \tag{23a}$$

$$\nabla \cdot \mathbf{U} = 0 \quad , \quad \text{in } V_p \quad ; \tag{23b}$$

$$\mathbf{U} = \hat{\mathbf{u}}_0 \quad , \quad \text{on } S_p \quad ; \tag{23c}$$

where the operators Δ and ∇ are in the dimensionless variable x and $\hat{\mathbf{u}}_0$ is the unit vector $\mathbf{u}_0/u_0$.

The complex permeability $k(\omega)$ of (5) is now

$$\tilde{k}(\omega) = (\phi b^2/i\Omega)\,(\langle \mathbf{U} \rangle \cdot \hat{\mathbf{u}}_0 - 1) \tag{24}$$

where $\langle \mathbf{U} \rangle$ is the average of U over the pore space, defined as in (14). The complex density is now, by (5),

$$\tilde{\rho}(\omega) = (\rho_f/\phi)\ /\ (1 - \langle \mathbf{U} \rangle \cdot \hat{\mathbf{u}}_0) \tag{25}$$

and the function F is

$$F(\omega) = (k_o/\phi b^2)\ [i\Omega/(\langle \mathbf{U} \rangle \cdot \hat{\mathbf{u}}_o - 1)] \quad . \tag{26}$$

5. LOW FREQUENCY ASYMPTOTIC APPROXIMATION

At low frequencies, such that $|\Omega| << 1$, we can seek a solution to (23) in the form of an asymptotic expansion,

$$\mathbf{U} = \mathbf{U}^{(0)} + (i\Omega)\mathbf{U}^{(1)} + (i\Omega)^2\mathbf{U}^{(2)} + \ldots \tag{27a}$$

$$p = p^{(0)} + (i\Omega)p^{(1)} + (i\Omega)^2 p^{(2)} + \ldots \quad . \tag{27b}$$

Substituting these expansions into (23) and comparing terms of equal order in Ω gives to order (1):

$$\Delta\mathbf{U}^{(0)} = -\nabla p^{(0)} \quad , \text{ in } V_p \ ; \tag{28a}$$

$$\nabla \cdot \mathbf{U}^{(0)} = 0 \quad , \text{ in } V_p \ ; \tag{28b}$$

$$\mathbf{U}^{(0)} = \hat{\mathbf{u}}_0 \quad , \text{ on } S_p \ . \tag{28c}$$

The solution to this is simply $\mathbf{U}^{(0)} = \hat{\mathbf{u}}_0$ and $p^{(0)}$ = constant. The $O(\Omega)$ equations then become,

$$\Delta\mathbf{U}^{(1)} + \hat{\mathbf{u}}_0 = -\nabla p^{(1)} \quad , \text{ in } V_p \ ; \tag{29a}$$

$$\nabla \cdot \mathbf{U}^{(1)} = 0 \quad , \text{ in } V_p \ ; \tag{29b}$$

$$\mathbf{U}^{(1)} = 0 \quad , \text{ on } S_p \ . \tag{29c}$$

For $j \geq 2$, the $O(\Omega^j)$ equations are

$$\Delta\mathbf{U}^{(j)} + \mathbf{U}^{(j-1)} = -\nabla p^{(j)} \quad , \text{ in } V_p \ ; \tag{30a}$$

$$\nabla \cdot \mathbf{U}^{(j)} = 0 \quad , \text{ in } V_p \ ; \tag{30b}$$

$$\mathbf{U}^{(j)} = 0 \quad , \text{ on } S_p \ . \tag{30c}$$

Referring to (24) and (27) we see that the asymptotic expansion of the effective permeability is

$$\tilde{k}(\omega) = \phi b^2\, \hat{\mathbf{u}}_0 \cdot \left[<\mathbf{U}^{(1)}> + i\Omega <\mathbf{U}^{(2)}> + \ldots \right] \ . \tag{31}$$

The Darcy's law permeability k_o, defined as $\tilde{k}(0)$ is

$$k_o = \phi\, b^2 <\mathbf{U}^{(1)} \cdot \hat{\mathbf{u}}_0> \ . \tag{32}$$

The determination of k_o therefore requires solving the system (29). The frequency dependent factor F then follows from (9), (31) and (32) as

$$F = 1 - i\Omega\, [<\mathbf{U}^{(2)} \cdot \hat{\mathbf{u}}_0> / <\mathbf{U}^{(1)} \cdot \hat{\mathbf{u}}_0>] + O(\Omega^2) \ . \tag{33}$$

<u>The Inertial Factor</u>

We now define a low frequency inertial factor T_o by analogy with $\angle_S(\omega)$ of Eq. (4), where $m = T\rho_f/\phi$. In general, we define

$$T_o = -(\phi/2\rho_f)\, [d^2\angle(\omega)/d\omega^2]\,\Big|_{\omega=0} \ . \tag{34}$$

If we write $F = F(\Omega)$, then this definition is equivalent to

$$T_o = i(\phi b^2/k_o)\,[dF(\Omega)/d\Omega]\,\Big|_{\Omega=0} \quad , \tag{35}$$

or, from (33) and (35),

$$T_o = [\ <U^{(2)}\cdot\hat{u}_0> / <U^{(1)}\cdot\hat{u}_0>^2] \quad . \tag{36}$$

Thus, it appears that the determination of T_0 requires solving for both $U^{(1)}$ and $U^{(2)}$. However, it is not necessary to solve for $U^{(2)}$ since, as shown in Appendix B,

$$<U^{(2)}\cdot\hat{u}_0> = <U^{(1)}\cdot U^{(1)}> \tag{37}$$

and thus,

$$T_o = [<U^{(1)}\cdot U^{(1)}> / <U^{(1)}\cdot\hat{u}_o>^2] \quad . \tag{38}$$

The result (37) means that both k_0 and T_0 follow from the solution to problem (29). The permeability depends upon the mean of $U^{(1)}$ in the direction of $\mathbf{u}_0$ while T_0 depends upon the mean of $|U^{(1)}|^2$. We note that

$$T_o - 1 = [<|\hat{u}_o \times U^{(1)}|^2> / <U^{(1)}\cdot\hat{u}_o>^2] + [<(\hat{u}_o\cdot U^{(1)} - <\hat{u}_o\cdot U^{(1)}>)^2> / <U^{(1)}\cdot\hat{u}_o>^2] \geq 0 \tag{39}$$

with equality if and only if $U^{(1)}$ is constant and parallel to $\hat{\mathbf{u}}_0$ everywhere in the pore. However, this is impossible, since $U^{(1)} = 0$ on S_p. Thus,

$$T_0 > 1 \quad . \tag{40}$$

The inequality $T_0 > 1$ is O(1) in the sense that $T_0 - 1 = O(1)$, not $O(\Omega)$ for $\Omega << 1$.
The definition of T_0 implies that

$$F = 1 - (i\Omega k_o/\phi b^2)\,T_o + O(\Omega^2) \tag{41}$$

is the low frequency correction factor. Alternatively, the low frequency permeability is

$$\tilde{k}(\omega) = k_o + (i\Omega k_0^2/\phi b^2)\,T_o + O(\Omega^2) \tag{42}$$

or the density is

$$\tilde{\rho}(\omega) = (i\rho_f b^2/\Omega k_o) + \rho_f T_o/\phi + O(\Omega) \quad . \tag{43}$$

6. LOW FREQUENCY EXAMPLES: CYLINDRICAL PORES

The problem of solving for $U^{(1)}$ is greatly simplified if the pore unit "cell" is assumed to be an infinite cylindrical tube of constant cross section. Let $\mathbf{e}_z$ be the unit vector parallel to the tube axis. The general solution to (29) is $\nabla p^{(1)} = \mathbf{e}_z \times (\mathbf{e}_z \times \hat{u}_0)$ and $\mathbf{U}^{(1)} = U^{(1)}\,\mathbf{e}_z \cos\theta$, where θ is the angle that $\mathbf{u}_0$ makes with the tube axis. The function $U^{(1)} = U^{(1)}(x,y)$ satisfies

$$[\partial^2 U^{(1)}/\partial x^2] + [\partial^2 U^{(1)}/\partial y^2] + 1 = 0 \quad , \quad \text{in} \quad \Omega_p \quad , \tag{44a}$$

$$U^{(1)} = 0 \quad , \quad \text{on} \quad \Gamma_p \quad , \tag{44b}$$

where Ω_p is the area in the (x,y) plane of the cross section and Γ_p is its perimeter.

This system of equations describes Poiseuille flow and is completely analogous to those obtained in the torsion problem for an elastic rod of cross-section Ω_p. Therefore, we can draw upon known solutions to the Poisson equation for several simple geometries. A good reference to the torsion problem and its solution can be found in the book of Timoshenko and Goodier (1970). We present results below for cross-sections of elliptical, rectangular, triangular and more complicated shapes. For each of these shapes, the function $U^{(1)}$ can be obtained explicitly, whereas the full problem (11) or (23), can only be solved for the elliptical or rectangular shapes. And even then, the "exact" solutions are much more complicated than the relatively simple ones below.

Having found $U^{(1)}(x,y)$ the permeability is $k_o \cos^2\theta$, where

$$k_o = \phi b^2 \langle U^{(1)} \rangle \tag{45}$$

with

$$\langle U^{(1)} \rangle = \frac{1}{A_p} \int_{\Omega_p} U^{(1)}\, dxdy \tag{46}$$

and A_p is the cross-sectional area. The inertial factor is $T_o \sec^2\theta$, where

$$T_o = \langle {U^{(1)}}^2 \rangle / \langle U^{(1)} \rangle^2 \quad . \tag{47}$$

These results, (45)-(47), are based on the assumption that the cylindrical pores are all aligned. However, if the pores are randomly oriented, the effective permeability is $(1/3)k_o$, where the 1/3 factor comes from the average of $\cos^2\theta$. Similarly, the inertial factor for the randomly oriented distribution of pores is $3\,T_o$. Thus, the permeability decreases, and the inertial factor increases for the random pore geometry.

Example 1: Elliptical Cross-Section

Let the characteristic length b be one of the radii and the other one b/δ, where $0<\delta<\infty$. The pore surface is then $x^2 + y^2\delta^2 = 1$ in the dimensionless coordinate system $\mathbf{x} = (x,y,z)$. The solution to (44) is

$$U^{(1)} = -\,(x^2+y^2\delta^2-1)\,/[\,2(1+\delta^2)] \tag{48}$$

and thus,

$$(k_o/\phi b^2) = 1/[4(1+\delta^2)] \tag{49}$$

or

$$k_o = (\phi A_p/8\pi)\, 2\delta/(1+\delta^2) \quad . \tag{50}$$

We note that $2\delta/(1+\delta^2) \leq 1$, with equality for a circular pore. The inertial factor turns out to be

$$T_o = 4/3 \tag{51}$$

which is independent of δ.

Example 2: Rectangular Cross-Section

Let the rectangle have sides of length 2b and $2b\delta$, where $0 < \delta < \infty$. The region Ω_p in the dimensionless coordinates x and y is defined by $|x| < 1$, $|y| < \delta$. From p. 311 of Timoshenko and Goodier (1970),

$$U^{(1)} = (16/\pi^3) \sum_{n=1,3,5\ldots} (1/n^3)\,(-1)^{(n-1)/2} \left[1 - [\cosh(n\pi y/2)/\cosh(n\pi\delta/2)\right] \cos(n\pi x/2). \tag{52}$$

Using Eqs. (46), (52) and the relation

$$\sum_{n=1,3,5} (1/n^4) = (\pi^4/96) \tag{53}$$

we deduce

$$k_o/\phi b^2 = (1/3)\left[1 - (192/\delta\pi^5)\sum_{n=1,3,5\ldots}(1/n^5)\tanh(n\pi\delta/2)\right] \quad . \tag{54}$$

For large δ, we can expand to get

$$k_o/\phi b^2 = (1/3)\,[1 - (0.630/\delta)] + O(1/\delta^2) \quad . \tag{55}$$

An approximate form of (54), which incorporates the asymptotic result for $\delta >> 1$ and the exact result for a square ($\delta = 1$) that $k_o/\phi b^2 = (0.420)/3$ is

$$k_o/\phi b^2 \approx (1/3)\,[1 - (0.630/\delta) + (0.052/\delta^2)] \quad . \tag{56}$$

This turns out to be a very good approximation for the entire range $1 \le \delta < \infty$, see Fig. 1. The limit $\delta \to \infty$ corresponds to a slit-like pore of width 2b, for which $k_o/\phi b^2 = 1/3$.

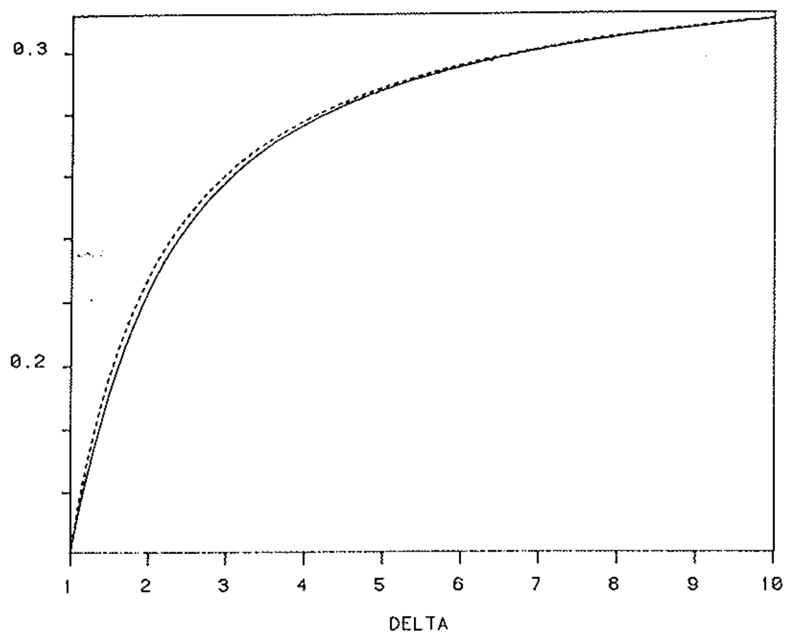

Figure 1 Comparison of the exact (Eq. (54), solid curve) and approximate (Eq. (56), dashed curve) formulae for the permeability of rectangular pores as a function of the aspect ratio. The maximum relative error is less than 3%.

The inertial factor T_o follows from (47), (52) and the relation

$$\sum_{n=1,3,5\ldots}(1/n^6) = (\pi^6/960) \tag{57}$$

as

$$T_o = \frac{(6/5)\left\{1 - 480\pi^{-6}\sum_{n=1,3,5\ldots} n^{-6}[(6/n\pi\delta)\tanh(n\pi\delta/2) - \mathrm{sech}^2(n\pi\delta/2)]\right\}}{\left\{1 - 192\delta^{-1}\pi^{-5}\sum_{n=1,3,5\ldots}[n^{-5}\tanh(n\pi\delta/2)]\right\}^2} \quad . \tag{58}$$

For large δ, we find

$$T_o = 6/5\,[1 + (0.306/\delta) + O(1/\delta^2)] \quad . \tag{59}$$

The square (δ=1) has T_o = 1.378, therefore a uniform approximation to T_o for the entire range $1 \le \delta < \infty$, that is valid in the two limits is

$$T_o = 6/5\,[1 + (0.306/\delta) - (0.158/\delta^2)] \quad . \tag{60}$$

We note from Fig. 2 that (60) is a good approximation.

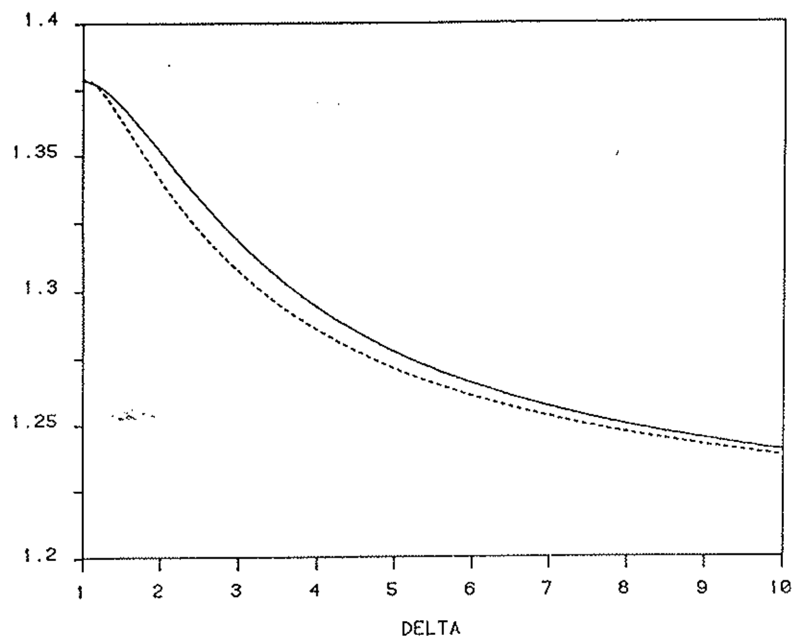

Figure 2 The inertial factor of a rectangular pore versus aspect ratio; solid curve, exact expression from Eq. (58); dashed curve, approximation in Eq. (60). The maximum relative error is less than 1%.

Example 3: The Equilateral Triangle

Let the pore cross-section be in the shape of an equilateral triangle with vertices in the (x,y) plane at (0,0), $(1,1/\sqrt{3})$ and $(1,-1/\sqrt{3})$. The area is thus $1/\sqrt{3}$ and the solution to (44) is

$$U^{(1)} = (1/4)\,(3y^2 - x^2)\,(x-1) \quad . \tag{61}$$

With b equal to the length of one side, the dimensional permeability k_o follows from (45) and (61) as $\phi b^2/80$. The inertial factor T_o follows from (47) and (61) as 10/7.

Example 4: A Tube with a Groove

The perimeter of this pore cross-section is made by considering the "tube" defined by the interior of the unit circle. A circle of radius δ, $0 \le \delta < 2$, is drawn about a point on the tube edge, and the final pore cross-section is the part of the tube exterior to the circle of radius δ, see Fig. 3.

For a given δ, define the angle α, $0 < \alpha \le \pi$, by

$$\cos(\alpha/2) = (\delta/2) \quad , \tag{62}$$

then the pore cross-sectional area is, see Fig. 4,

$$A_p = \sin\alpha - \alpha\cos\alpha \quad . \tag{63}$$

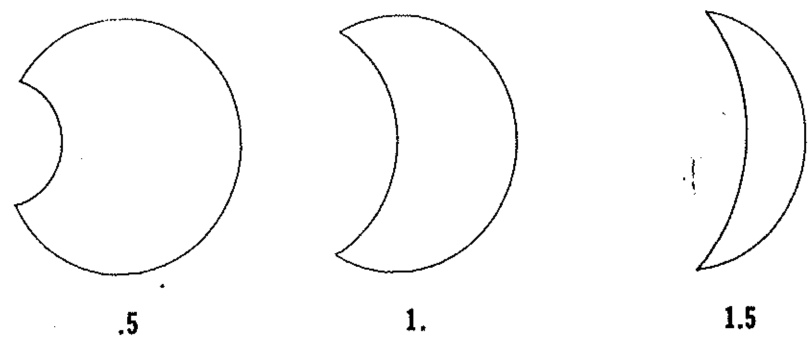

Figure 3 Examples of the cross-section of the grooved tube for $\delta = 0.5$, 1.0, and 1.5.

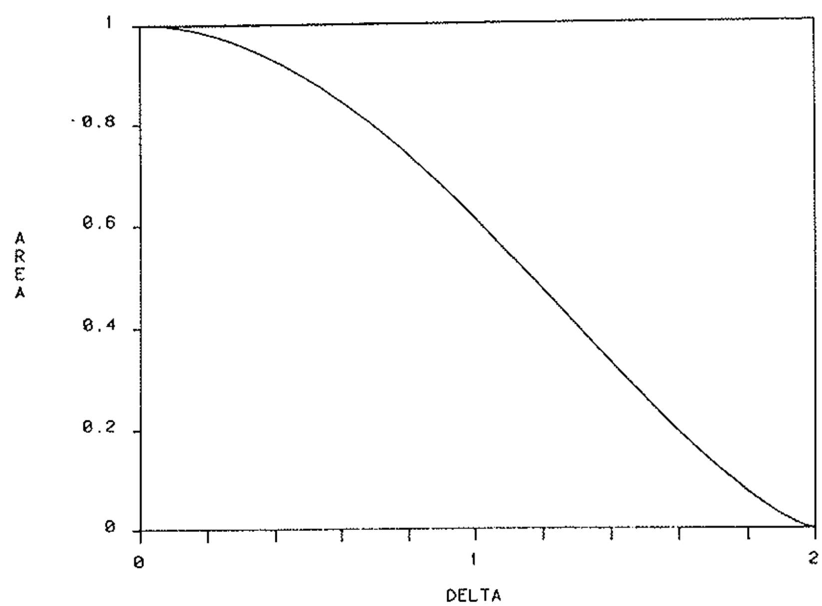

Figure 4 The area of the grooved tube as a function of δ.

The limit of a circular pore is $\alpha = \pi$. Define polar coordinates (r,θ) about the center of the circle of radius δ, with $\theta = 0$ as the axis of symmetry, see Fig. 3. Then the solution to (44) is

$$U^{(1)} = (1/4)\,(\delta^2 - r^2)\,[1 - 2\cos\theta/r] \quad . \tag{64}$$

The permeability follows from (45) and (64) as

$$k_o/\phi b^2 = 1 - (12\alpha + 2\alpha\cos 2\alpha - 7\sin 2\alpha) / [16(\sin\alpha - \alpha\cos\alpha)] \tag{65}$$

where b is understood to be the radius of the original ungrooved circular tube. Thus, $k_o/\phi b^2 \to 1/8$ as the groove disappears ($\delta \to 0$). The inertial factor is from (47) and (64),

$$T_o = (\phi b^2/k_o)^2 (1/A_p) \{[3\alpha + 3\sin\alpha - 36\alpha\cos^2(\alpha/2) - 216\alpha\cos^4(\alpha/2) - 96\alpha\cos^6(\alpha/2) - 34\sin\alpha\cos^2\alpha + 376\sin\alpha\cos^4(\alpha/2)] / 144 + 4\cos^4(\alpha/2)[-(1/2)(\alpha + \sin\alpha)\, ln\,[2\cos(\alpha/2)] + 2\int_0^{\alpha/2} \cos^2\theta\, ln\,[2\cos\theta]\, d\theta]\} \quad (66)$$

The inertial factor is plotted in Fig. 5. Note that it increases from 4/3, the circular value, to about 1.42 as the groove becomes more pronounced.

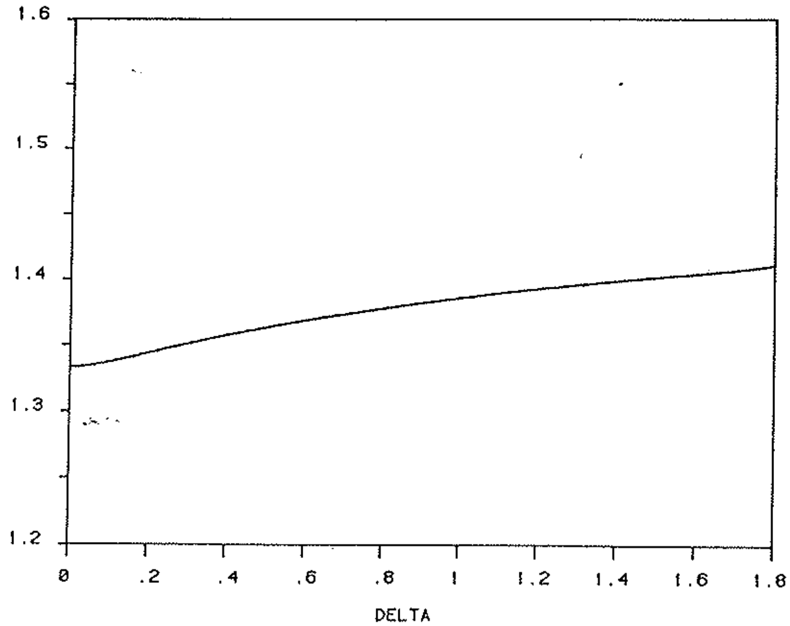

Figure 5 The inertial factor T_o of the grooved tube.

Example 5: Distributions of Pores

The above results can be used to consider more complicated pore geometries comprising networks of these simple shapes - for example, if we allow the cross-section to change with axial distance. A simple case of this is a network of circular pores in series. Let R be the variable radius, then it is well known (Johnson et al., 1986) that the permeability is

$$k_o = (\phi/8)\,(1/\langle R^{-4}\rangle\langle R^2\rangle) \quad (67)$$

where $\langle\ \rangle$ denotes the average. Thus k_o is dominated by the narrowest sections. It is easy to show that the same network has

$$T_o = (4/3)\,\langle R^{-2}\rangle\langle R^2\rangle \quad (68)$$

which will also depend strongly on narrow throats. It thus seems possible for T_o to assume a very large value, $T_o \gg 1$, in such a pore geometry.

Conversely, a network of circular pores of different radii in parallel gives

$$k_o = (\phi/8)\,[\langle R^4\rangle / \langle R^2\rangle] \quad , \quad (69)$$

$$T_o = (4/3)\,[\langle R^2\rangle\langle R^6\rangle / \langle R^4\rangle^2] \quad . \tag{70}$$

Now, k_0 is dominated by the larger radii and T_0 also depends upon the distribution. However, it is clear that $T_0 \approx 4/3$.

Next, consider a slit-like pore of constant cross-section in the flow direction, but of varying width d in the transverse direction. Thus, the flow is in the z-direction and the slit is defined by the region between $y=\pm d(x)/2$. We find that

$$k_0 = (\phi/12)\,[\langle d^3\rangle / \langle d\rangle] \quad , \tag{71}$$

$$T_o = (6/5)\,[\langle d\rangle\langle d^5\rangle / \langle d^3\rangle^2] \quad . \tag{72}$$

This case is analogous to the circular pores in parallel, where T_0 depended weakly upon the distribution. As an example, consider the periodic sawtooth profile $d = d_0(1+\varepsilon x/L)$ on the fundamental period $-L<x<L$. We find, from (72),

$$T_o = (6/5)\{1 + (4/3)\,[\varepsilon^2/(1+\varepsilon^2)^2]\} \quad , \tag{73}$$

which is maximum for $\varepsilon = 1$ giving $T_0 = 8/5$. Alternatively, a periodic sinusoidal profile $d = d_0(1 + \varepsilon \sin\pi x/L)$, $-L<x<L$ produces

$$T_o = (6/5)\,[1 + (16\varepsilon^2 - 3\varepsilon^4) / (2\{2+3\varepsilon^2\}^2)] \quad . \tag{74}$$

This has a maximum of $T_0 = 14/9$ at $\varepsilon = 8/15$. It appears that T_0 is always of order unity, but still greater than one for cylindrical pores in parallel. However, if the slit cross-section varies in the direction of flow, but is constant in the transverse direction, i.e., $d=d(z)$, then

$$k_o = (\phi/12)\,(1/\langle d^{-3}\rangle\langle d\rangle) \tag{75}$$

and

$$T_o = (6/5)\,\langle d^{-1}\rangle\langle d\rangle \quad . \tag{76}$$

This case is analogous to the circular pores in series.

These results for distributed pore diameters and widths indicate that the same pore geometry that most strongly limits permeability, i.e., narrow constrictions or throats, also produces a large inertial factor. This correspondence suggests the possibility of a universal scaling relation involving k_0, T_0 and maybe one or more other parameters. This idea is pursued by Johnson et al. (1986), who postulate a relation between k_0, α_∞ and Λ, where α_∞ is the infinite frequency limit of $T(\omega)$, defined in Eq. (7), and Λ is a length parameter. We will not develop the possible connections among these quantities, except to note that $T_0 = 4/3\ \alpha_\infty$ for tubes in series.

7. CONCLUSIONS

A consistent formulation for determining the viscodynamic operator in Biot's equations has been described. At low frequencies the operator depends upon the static permeability and a dimensionless inertial factor both of which are determined from the solution of a Stokes flow problem. The inertial factor is always greater than unity. We have obtained explicit solutions to the Stokes flow problem for a variety of cylindrical pore models of constant cross-section. It is found that the inertial factor varies between 1.2 and 1.4, when these types of pores are aligned parallel to the wave direction. Random orientation, without inter-pore flow, will increase this by a factor of three but pore-core connections will cause a decrease. Networks of cylindrical pores in series and parallel can also lead to an enhanced inertial factor. In particular, narrow constrictions and throats, of the type that restrict through-flow, can increase T_0 significantly.

ACKNOWLEDGMENTS

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APPENDIX A: BIOT'S ORIGINAL THEORY FOR $\angle(\omega)$

Biot (1956) suggested extending the low frequency form of $\angle_S$ defined in Eq. (4) by correcting the drag term $(-i\omega\mu/k_0)$ to account for the frequency dependence. He proposed

$$\angle(\omega) = \angle_B(\omega) = -\omega^2 m - (i\omega\mu/k_0)\, F_B(\omega) \tag{A.1}$$

where F_B is a frequency dependent correction factor. Biot considered only simple pore models for which F_B could be calculated explicitly. We now generalize his formulation to include arbitrary pore shapes. Our purpose is to compare this with the "consistent" theory for $\angle(\omega)$ described earlier.

Consider the pore fluid Eq. (13) with the oscillatory pressure gradient $\mathbf{g} = -\phi\rho_f\, \omega^2\, \mathbf{u}_0 \exp(-i\omega t)$. The stress tensor in the fluid is

$$t_{ij} = -\phi\, \overline{p}\, \delta_{ij} - i\omega\mu(\overline{w}_{i,j} + \overline{w}_{j,i}) \quad . \tag{A.2}$$

The drag of the pore boundary S_p on the fluid is the vector $<\tau> \exp(-i\omega t)$, where

$$<\tau_i> = -(1/V_p)\int_{S_p} t_{ij} n_j dS \tag{A.3}$$

and $\mathbf{n}$ is the unit normal to the fluid/solid interface, directed out of V_p. By the divergence theorem and Eq. (13a),

$$<\tau> \;=\; \omega^2\, \rho_f(\phi u_0 + <w>) \quad , \tag{A.4}$$

where $<w>$ is defined in Eq. (12). We note that $\phi\mathbf{u}_0 + <w> = \phi<U>$, where $<U>$ is the average absolute fluid displacement in the pore space. The average drag in the direction of $\mathbf{u}_0$ divided by the average relative velocity in the same direction is

$$[<\tau>\bullet u_0/(-i\omega<w>\bullet u_0)] = i\omega\, \rho_f\, \phi\, [u_0^2/(<w>\bullet u_0)] + i\omega\, \rho_f = [\phi\mu/\tilde{k}(\omega)] + i\omega\, \rho_f \tag{A.5}$$

which follows from (5) and (15). The value of this ratio at zero frequency is $\phi\mu/k_0$, which is the Darcy coefficient. The frequency dependent factor F_B is defined by normalizing the above ratio to be unity at $\omega = 0$. Thus, $F_B = 1$ at zero frequency, and at other frequencies,

$$F_B = [k_0/\tilde{k}(\omega)] + [i\omega\rho_f k_0/\phi\mu] = F(\omega) + [i\omega\rho_f k_0/\phi\mu] \tag{A.6}$$

where F is defined in (9). Substituting F_B into the definition of $\angle_B$ gives

$$\angle_B = (-i\omega\mu F/k_0) - (\omega^2\rho_f/\phi)\,(T_0 - 1) = \angle(\omega) - (\omega^2\rho_f/\phi)\,(T_0 - 1) \tag{A.7}$$

where $m = T_0\rho_f/\phi$ and $T_0 > 1$ (Brown, 1980). Thus, Biot's prescription for $\angle$ gives a larger effective inertia than that of the consistent theory.

Finally, we note that for circular cylindrical pores, Biot's factor follows from (5), (18), (A7) and the relation

$$J_2(x) = (2/x)\, J_1(x) - J_0(x) \tag{A.8}$$

as

$$F_B = (\sqrt{i\Omega_c}/4)\, [J_1(\sqrt{i\Omega_c})\, /\, J_2(\sqrt{i\Omega_c})]\, \cos^2\theta \tag{A.9}$$

in agreement with Biot (1956). The result for slit-like pores follows from (5), (21) and (A7) as

$$F_B = (i\Omega_s^2/3)\, [\, 1 - \sqrt{i\Omega_s}\, \cot(\sqrt{i\Omega_s})]^{-1} \tag{A.10}$$

also in agreement with Biot's original results.

APPENDIX B: PROOF OF EQUATION (37)

We now prove the assertion (37). Consider the identities

$$\int_{V_p} [U_k^{(1)} U_{k,j}^{(2)} - U_k^{(2)} U_{k,j}^{(1)}]_{,j}\, dV = \int_{V_p} (U^{(1)}\cdot\Delta U^{(2)} - U^{(2)}\cdot\Delta U^{(1)})\, dV$$

$$= \int_{S_p} [U_k^{(1)} U_{k,j}^{(2)} - U_k^{(2)} U_{k,j}^{(1)}] n_j\, dS = 0 \tag{B.1}$$

where the last one follows from the boundary conditions $U^{(1)} = U^{(2)} = 0$ on S_p. Substituting for $\Delta U^{(1)}$ and $\Delta U^{(2)}$ using (29) and (30), we have

$$\int_{V_p} [U^{(1)} \cdot U^{(1)} - U^{(2)} \cdot \hat{u}_o]\, dV = \int_{V_p} [U^{(2)} \cdot \nabla p^{(1)} - U^{(1)} \cdot \nabla p^{(2)}]\, dV =$$

$$\int_{S_p} [p^{(1)} U^{(2)} - p^{(2)} U^{(1)}] \cdot n\, dS + \int_{V_p} [p^{(2)} \nabla \cdot U^{(1)} - p^{(1)} \nabla \cdot U^{(2)}]\, dV \quad . \tag{B.2}$$

The surface integral is zero by virtue of the no slip boundary conditions, and the final volume integral vanishes because of the incompressibility conditions (29b) and (30b). This completes the proof of (37).