

Acoustic pulse scattering by baffled membranes

Gregory A. Kriegsmann

Department of Engineering Sciences and Applied Mathematics, The Technological Institute, Northwestern University, Evanston, Illinois 60201

Andrew N. Norris

Exxon Research and Engineering Corporation, Clinton, New Jersey 08801

Edward L. Reiss

Department of Engineering Sciences and Applied Mathematics, The Technological Institute, Northwestern University, Evanston, Illinois 60201

(Received 17 February 1985; accepted for publication 6 August 1985)

Asymptotic expansions as $\epsilon \rightarrow 0$ that are uniformly valid in t are obtained for the membrane's motion and the scattered acoustic pressure field. The small parameter ϵ is the density ratio of the acoustic fluid and the membrane. For simplicity of presentation, only plane, compact incident pulses are considered. The scattered field depends on the pulse's structure. If it is a sufficiently narrow bandwidth pulse which contains none of the *in vacuo* natural frequencies of the membrane, then it is essentially reflected as though the baffled plane is completely rigid. However, if the pulse spectrum is sufficiently broad so that it contains one or more of the *in vacuo* natural frequencies of the membrane, an additional scattered field is produced. This scattered field insonifies distant observation points after the rigidly reflected pulse has arrived. It is the sum of slightly damped and oscillating outgoing spherical waves that represents the "decayed ringing" of the membrane. Application is given to the baffled circular membrane which is insonified by a normally incident pulse. Graphs of the membrane's motion and the farfield acoustic pressure are given. They demonstrate the importance of the incident pulse width on the qualitative features of the response.

PACS numbers: 43.20.Fn, 43.40.Dx

INTRODUCTION

The plane $z = 0$ is a rigid baffle that separates an acoustic fluid in the upper half-space $z > 0$ from a vacuum in the lower half-space. A thin, tightly stretched membrane occupies the region M of this plane. A pulse $P'(\mathbf{x}, t)$, which satisfies the acoustic wave equation in the upper half-space, is incident on this baffled membrane.

The scattered field depends on the pulse's structure and the membrane's geometrical and physical properties. If it is a narrow bandwidth pulse such that its spectrum is free of any of the *in vacuo* natural frequencies of the membrane, the pulse is reflected as though the entire plane $z = 0$ is essentially rigid. However, if the spectrum of the pulse contains one or more of these natural frequencies, the membrane is then in "near resonance" with the pulse, producing a scattered field in addition to the rigidly reflected field. This scattered field also depends on the physical width of the pulse. Thus, for example, when the pulse is free of the membrane, the scattered field results from the "ringing" of the membrane as it is "damped" by the back pressure of the fluid.

Since membrane (and plate) theories are valid essentially only for the "lower" modes of vibration, we assume that the pulse's bandwidth is restricted so that it can only excite the first few modes of the membrane. It may be necessary to consider the flexible region M of the plane as a three-dimensional elastic body if broader bandwidth pulses are considered.

The pulse scattering problem is formulated in Sec. I as an initial-boundary value problem for an integrodifferential equation for the lateral motion of the membrane. The scattered acoustic field is then given by an integral over the membrane's surface. This problem is then solved in Secs. II and III by first expanding the membrane's motion in its *in vacuo* normal modes. We assume that these modes are known either analytically or by numerical evaluation. The time-dependent coefficients in this expansion satisfy a coupled system of ordinary, integrodifferential equations. They are reduced to a coupled system of algebraic equations by applying the Laplace transform. An asymptotic expansion of the solution of this algebraic system is obtained in the small parameter ϵ , which is defined as the density ratio of the acoustic fluid to the membrane. Then by inverting the asymptotic expansion of the Laplace transform, we obtain an integral representation for the membrane's motion and for the scattered acoustic field. These asymptotic approximations are uniformly valid in t as $\epsilon \rightarrow 0$.

In Sec. III we obtain qualitative features of these asymptotic approximations. To simplify the presentation, we consider only plane compact pulses that excite modes corresponding to simple eigenvalues of the membrane. However, we indicate how the analysis can be extended to pulses that excite modes of multiple eigenvalues. Finally, our analysis is applied in Sec. IV to the circular membrane that is insonified by a specific, normally incident, plane compact pulse. A discussion of the response and graphs of the membrane's mo-

tion and farfield acoustic pressures are given.

The pulse scattering problem can be solved numerically by obtaining the response of the membrane to time periodic incident fields, either, for example, by a method of matched asymptotic expansions¹ or the method of normal modes,² and then numerically evaluating the inverse Fourier transform integral. Alternatively, the time-dependent scattering problem can be solved by methods which rely on the approximate decoupling of the fluid-structure equations and on the pulse's spectral content, see, e.g., Refs. 3 and 4. The finite element method is then used to determine the structure's approximate motion. A new numerical method for solving the time-dependent scattering problem is given in Ref. 5. It is a finite difference technique which uses "artificial" boundary conditions on a finite region to simulate the infinite region, $z > 0$. The use of such artificial boundary conditions for numerically solving wave propagation problems has been previously employed (see, e.g., Ref. 6). In this method the fully coupled fluid-membrane equations are solved.

I. FORMULATION

In dimensionless variables $\mathbf{x} = (x, y, z)$ and t , the pressure in the acoustic fluid $P(\mathbf{x}, t)$ satisfies the wave equation

$$\Delta_0 P = P_{tt}, \quad \text{for } z > 0, \quad (1)$$

where Δ_0 is the three-dimensional Laplacian. The dimensionless space variables are obtained by scaling with respect to a characteristic length L of the membrane. The dimensionless time t is obtained by scaling with respect to L/c_a , where c_a is the sound speed of the acoustic fluid.

The equation of motion for the dimensionless lateral deflection $w(x, y, t)$ of the membrane, which lies in the region M of the plane $z = 0$, is given by

$$\Delta w - c^2 w_{tt} = \epsilon c^2 P(x, y, 0, t), \quad \text{for } (x, y) \in M, \quad (2)$$

where Δ is the Laplacian in x and y . In addition, we have used the notation

$$c = c_a/c_m, \quad \epsilon = (\rho_a/\rho_m)L, \quad c_m = (T/\rho_m)^{1/2}, \quad (3)$$

where T and ρ_m are the membrane's tension and density per unit area, respectively, and ρ_a is the density of the acoustic fluid. The scale factors for the membrane displacement and the acoustic pressure are related by the ratio $L/(\rho_a c_a^2)$. The acoustic pressure $P(x, y, 0, t)$ acts as a driving force on the membrane.

Since the plane $z = 0$ is acoustically rigid outside of M , we have the condition

$$P_z(x, y, 0, t) = 0, \quad (x, y) \in M. \quad (4)$$

The requirement that the acoustic and membrane velocities are continuous on the membrane's surface gives the condition

$$P_z(x, y, 0, t) = -w_{tt}(x, y, t), \quad (x, y) \in M. \quad (5)$$

The acoustic pulse which is incident on the plane $z = 0$ is denoted by $P^I(\mathbf{x}, t)$. It is a solution of the wave equation (1). If the entire plane was rigid, then the incident pulse would be reflected as the pulse $P^R(\mathbf{x}, t) \equiv P^I(x, y, -z, t)$, which is also a solution of (1). Thus, we express the total acoustic pressure in $z > 0$ as

$$P(\mathbf{x}, t) = P^I(\mathbf{x}, t) + P^R(\mathbf{x}, t) + p(\mathbf{x}, t), \quad (6)$$

where $p(\mathbf{x}, t)$ is the scattered pressure field that is caused by the membrane's presence. By inserting (6) into (1), (2), (4), and (5), we find that the scattered field satisfies the following problem:

$$\Delta_0 p = p_{tt}, \quad z > 0; \quad (7a)$$

$$p_z(x, y, 0, t) = \begin{cases} 0, & (x, y) \notin M, \\ -w_{tt}, & (x, y) \in M, \end{cases} \quad (7b)$$

$$\Delta w - c^2 w_{tt} = \epsilon c^2 [2P^I(x, y, 0, t) + p(x, y, 0, t)], \quad (x, y) \in M, \quad (7c)$$

$$w(x, y, t) = 0 \quad \text{on } B, \quad (7d)$$

where B is the boundary of M . To complete the formulation of the scattering problem, we impose the quiescent initial conditions

$$p(\mathbf{x}, 0) = p_t(\mathbf{x}, 0) = 0, \quad z > 0, \quad (7e)$$

$$w(x, y, 0) = w_t(x, y, 0) = 0, \quad (x, y) \in M, \quad (7f)$$

and the outgoing wave condition

$$p_r + p_t \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad (7g)$$

where $r \equiv |\mathbf{x}|$. Equations (7e) and (7f) imply that the incident pulse reaches the membrane at $t = 0$.

To simplify the analysis of the scattering problem (7), we now reformulate it as a problem for w . Thus we first employ the adjoint Green's function $G(\mathbf{x}, t | \mathbf{x}', t')$ given by

$$G(\mathbf{x}, t | \mathbf{x}', t') = \frac{\delta(t' - t - R)}{4\pi R} + \frac{\delta(t' - t - R_1)}{4\pi R_1}, \quad (8a)$$

where δ is the Dirac delta function and R and R_1 are defined by

$$R \equiv |\mathbf{x} - \mathbf{x}'|, \quad R_1 = [(x - x')^2 + (y - y')^2 + (z + z')^2]^{1/2}. \quad (8b)$$

The function G satisfies

$$\Delta_0 G - G_{tt} = -\delta(t - t')\delta(\mathbf{x} - \mathbf{x}'), \quad z > 0; \\ G_z = 0, \quad \text{for } z = 0; \quad G = G_t = 0, \quad \text{for } t' < t. \quad (9)$$

By combining Eqs. (7a) and (9), integrating the result over the four-dimensional region, $0 < t < t'$ and $z > 0$, applying the appropriate divergence theorem, and making use of (7b) and (8a), we obtain

$$p(\mathbf{x}, t) = -\frac{1}{2\pi} \iint_M \frac{w_{tt}(x', y', t - q)}{q} H(t - q) dx' dy', \quad (10a)$$

where q is defined by

$$q(\mathbf{x}; \mathbf{x}') \equiv [(x - x')^2 + (y - y')^2 + z^2]^{1/2}. \quad (10b)$$

Inserting (10) into (7c), (7d), and (7f) gives the required integrodifferential equation problem for w as

$$\Delta w - c^2 w_n$$

$$= \epsilon c^2 \left(2P'(x, y, 0, t) - \frac{1}{2\pi} \iint_M \frac{w_n(x', y', t - q_0)}{q_0} H(t - q_0) dx' dy' \right),$$

$$\text{for } (x, y) \in M, \quad (11a)$$

$$w(x, y, t) = 0, \quad \text{for } (x, y) \in B, \quad (11b)$$

$$w(x, y, 0) = w_t(x, y, 0) = 0, \quad (11c)$$

where H is the Heaviside function and the "cylindrical" radius q_0 is defined by

$$q_0(x, y; x', y') \equiv [(x - x')^2 + (y - y')^2]^{1/2}. \quad (11d)$$

The integral operator in (11a) is proportional to the fluid back pressure on the membrane due to the membrane's motion. Once (11) is solved for $w(x, y, t)$, the scattered acoustic pressure in the fluid is given by (10).

II. THE SOLUTION OF THE INTEGRODIFFERENTIAL PROBLEM

We solve (11) by the eigenfunction expansion method; that is, we seek solutions of (11) in the form

$$w(x, y, t) = \sum_{n=1}^{\infty} w_n(t) \psi_n(x, y), \quad (12)$$

where the $\psi_n(x, y)$ are the orthonormal, *in vacuo* eigenfunctions of the membrane. They satisfy

$$\Delta \psi_n + c^2 k_n^2 \psi_n = 0, \quad (x, y) \in M, \quad (13a)$$

$$\psi_n = 0, \quad (x, y) \in B, \quad (13b)$$

where $\mu_n^2 \equiv k_n^2 c^2$ is the eigenvalue associated with ψ_n . By virtue of the orthonormality of the eigenfunctions, the modal amplitudes $w_n(t)$, which are given by

$$w_n(t) = \langle \psi_n, w \rangle \equiv \iint_M \psi_n(x, y) w(x, y, t) dx dy, \quad (14)$$

satisfy the following infinite, coupled, system of integrodifferential equations:

$$Lw_n = 2\epsilon g_n(t) + \frac{\epsilon}{2\pi} \sum_{m=1}^{\infty} \iint_M \iint_M \psi_n(x', y') \psi_m(x, y) \times \frac{H(t - q_0)}{q_0} \frac{d^2 w_m}{dt^2}(t - q_0) dx dy dx' dy',$$

$$n = 1, 2, 3, \dots \quad (15a)$$

The operator L and the coefficients g_n are defined by

$$Lw_n \equiv \frac{d^2 w_n}{dt^2} + k_n^2 w_n, \quad (15b)$$

$$g_n(t) \equiv -\langle \psi_n, P'(x, y, 0, t) \rangle. \quad (15c)$$

From (7f) and (14) it follows that the $w_n(t)$ satisfy the initial conditions

$$w_n(0) = \frac{dw_n(0)}{dt} = 0, \quad n = 1, 2, 3, \dots \quad (16)$$

The functions $g_n(t)$ are the coefficients of the eigenfunction expansion of the incident pulse evaluated on $z = 0$.

To solve the system (15) and (16), we first take its Laplace transform. This gives the following infinite, coupled system of algebraic equations for the Laplace transform $\hat{w}_n$ of w_n :

$$D_n(s, \epsilon) \hat{w}_n(s; \epsilon) = [s^2 + k_n^2 - \epsilon a_{nn}(s)] \hat{w}_n(s; \epsilon) = 2\epsilon \hat{g}_n(s) + \epsilon \sum_{m=1}^{\infty} a_{mn}(s) \hat{w}_m(s; \epsilon), \quad n = 1, 2, 3, \dots \quad (17)$$

Here, s is the transform variable, $\hat{g}_n$ is the transform of $g_n(t)$, and the $a_{mn}(s)$ are defined by the fourfold integrals

$$a_{mn}(s) \equiv \frac{s^2}{2\pi} \iint_M \iint_M \psi_n(x, y) \psi_m(x', y') \times \frac{e^{-sq_0}}{q_0} dx dy dx' dy'. \quad (18)$$

The prime on the sum in (17) signifies the omission of the $m = n$ term.

To obtain an asymptotic expansion of the solution of (17) as $\epsilon \rightarrow 0$ that leads to an asymptotic expansion of the solution of (15) and (16) that is uniformly valid in t as $t \rightarrow \infty$, we first observe that $\hat{w}_n(s; \epsilon)$ is $O(\epsilon)$ since the right side of (17) is $O(\epsilon)$. Then we seek a solution in the form

$$\hat{w}_n(s; \epsilon) = \sum_{j=1}^{\infty} \hat{w}_n^{(j)}(s; \epsilon), \quad (19)$$

where the $w_n^{(j)}$ form an asymptotic sequence⁷ in j for each n . In particular, $w_n^{(j)} = O(\epsilon^j)$ as $\epsilon \rightarrow 0$. Thus, we obtain from (17) the asymptotic approximation

$$\hat{w}_n = 2\hat{g}_n(s)/D_n(s; \epsilon) + O(\epsilon^2). \quad (20)$$

It is *not* possible to solve (17) by a regular perturbation expansion in ϵ since this leads to an expansion for $w_n(t)$ that is unbounded as $t \rightarrow \infty$, as we can demonstrate. The zeros of $D_n(s; \epsilon)$ are approximations to the complex eigenfrequencies of the fluid-membrane system.¹

Furthermore, we wish to emphasize that the asymptotic approximation (20) is valid only when μ_n^2 is a simple, *in vacuo* eigenvalue of the membrane. This is because the $O(\epsilon^2)$ becomes $O(\epsilon)$ when μ_n^2 is not simple. To illustrate the modifications required to handle the case of a multiple eigenvalue, we suppose that the eigenfunctions ψ_{l_1} and ψ_{l_2} both share μ_l^2 as their common eigenvalue. We then rewrite (17) for $n = l_1$ and l_2 , respectively, as

$$D_{l_1} \hat{w}_{l_1}(s; \epsilon) + \epsilon a_{l_1 l_2}(s; \epsilon) \hat{w}_{l_2}(s; \epsilon) = \epsilon \hat{g}_{l_1}(s) + \epsilon \sum_{m=1}^{\infty} a_{ml_1} \hat{w}_m, \quad (21a)$$

$$D_{l_2} \hat{w}_{l_2}(s; \epsilon) + \epsilon a_{l_2 l_1}(s; \epsilon) \hat{w}_{l_1}(s; \epsilon) = \epsilon \hat{g}_{l_2}(s) + \epsilon \sum_{m=1}^{\infty} a_{ml_2} \hat{w}_m. \quad (21b)$$

The double prime indicates that both the $m = l_1$ and $m = l_2$ terms are missing in the infinite sums. This two-by-two system of equations can be solved for $\hat{w}_{l_1}$ and $\hat{w}_{l_2}$. The vanishing

of the determinant of this system yields the complex eigenfrequencies of the membrane-fluid system. This was shown in Ref. 1 for a time periodic incident wave. The general case for an eigenvalue with more degeneracy follows by similar reasoning.

We now assume that the *in vacuo* eigenvalues of the membrane are simple or that the corresponding $\hat{g}_n$ for multiple eigenvalues are zero or negligibly small. Then (20) and the convolution theorem imply that

$$\omega_n(t; \epsilon) = 2\epsilon \int_0^t g_n(\xi) d_n(t - \xi) d\xi + O(\epsilon^2), \quad (22)$$

where $d_n(t)$ is the inverse Laplace transform of $D_n^{-1}(s; \epsilon)$. To determine this function we first obtain the zeros of the nonlinear equation $D_n(s; \epsilon) = 0$. It can be shown that D_n has two zeros, which we denote by $S_1(\epsilon)$ and $S_2(\epsilon)$. Asymptotic expansions as $\epsilon \rightarrow 0$ for these roots are readily obtained as

$$S_1 = -\epsilon I_n / 2k_n + i(k_n - \epsilon R_n / 2k_n) + O(\epsilon^2), \quad (23a)$$

$$S_2 = S_1^*, \quad (23b)$$

where the * denotes complex conjugation. The quantities R_n and I_n are the real and imaginary parts of $a_{nn}(-ik_n)$, respectively. As in the analysis in Appendix A of Ref. 1, we can show that

$$R_n = \iint_{z>0} (|\nabla \varphi_n|^2 - k_n^2 |\varphi_n|^2) dx dy dz, \quad (24a)$$

$$I_n = k_n \int_0^\pi \int_0^{\pi/2} |F_n(k_n, \hat{r})|^2 \sin \varphi' d\varphi' d\theta > 0, \quad (24b)$$

where the directivity factors $F_n(k_n, \hat{r})$ and the scattered acoustic potentials $\varphi_n(\mathbf{x})$ are produced by the membrane vibrating with frequency μ_n and mode $\psi_n(x, y)$. They are given, respectively, by

$$F_n(k_n, \hat{r}) = -\frac{1}{2\pi} \int_M \int e^{-ik_n \hat{r} \cdot \mathbf{r}} \psi_n(\xi, \eta) d\xi d\eta, \quad (25a)$$

$$\varphi_n(\mathbf{x}) = -\frac{1}{2\pi} \int_M \int \frac{e^{ik_n q(\mathbf{x}; \mathbf{x}')}}{q(\mathbf{x}; \mathbf{x}')} \psi_n(x', y') dx' dy', \quad (25b)$$

where $\hat{r} \equiv \mathbf{x}/|\mathbf{x}|$ is the unit vector in the observation direction and γ is the vector with components $(\xi, \eta, 0)$. The F_n are the Fourier transforms of the modes ψ_n with respect to the observation direction. In addition, I_n/k_n is the total cross section of φ_n and R_n is twice the corresponding dimensionless Lagrangian.

It follows from (23) by using standard residue calculus that the $d_n(t)$ are given by

$$d_n(t) = H(t) e^{-\gamma_n t} \sin(\Gamma_n t) / \Gamma_n, \quad (26a)$$

where

$$\gamma_n \equiv +I_n / 2k_n + O(\epsilon), \quad (26b)$$

$$\Gamma_n \equiv k_n - \epsilon R_n / 2k_n + O(\epsilon^2). \quad (26c)$$

Thus d_n decays on the "slow" time scale ϵt and oscillates on the "fast" scale t . It is analogous to the response of a simple damped oscillator to a Dirac delta forcing function. Combining (26) with (22), (15c), and (12) and interchanging the

order of summation and integration, we find that the membrane's displacement is given by

$$\begin{aligned} w(x, y, t, \epsilon) = & -2\epsilon \int_M \int_0^t \int_0^t P^I(\alpha, \beta, 0, \xi) \\ & \times K(x, y; \alpha, \beta; t - \xi) d\xi d\alpha d\beta + O(\epsilon^2), \end{aligned} \quad (27a)$$

where the kernel K is defined by

$$K(x, y; \alpha, \beta; t) = \sum_{n=1}^{\infty} \psi_n(x, y) \psi_n(\alpha, \beta) d_n(t). \quad (27b)$$

The scattered pressure is then obtained by inserting (27) into (10). This calculation will be performed in the next section for special incident pulses.

III. EVALUATION OF THE TRANSIENT RESPONSE

In this section we evaluate (27) and the corresponding farfield pressure, and physically interpret the results for three simple incident pulses.

A. The normally incident, plane, "spiked" pulse

This pulse is given by

$$P^I(x, y, z, t) = \delta(z + t), \quad t > 0, \quad (28)$$

so that it "touches" the membrane at $t = 0$. Inserting (28) into (27) and performing the t integration, we find that

$$w(x, y, t; \epsilon) = -2\epsilon \int_M \int K(x, y; \alpha, \beta; t) d\alpha d\beta + O(\epsilon^2). \quad (29)$$

Thus, the integral of the kernel (27b) over the membrane M is the response of the membrane to the pulse (28). Combining (27b) with (29), we obtain

$$w(x, y, t; \epsilon) = -2\epsilon \sum_{n=1}^{\infty} \langle \psi_n, 1 \rangle d_n(t) \psi_n(x, y) + O(\epsilon^2), \quad (30)$$

where the inner product $\langle \psi_n, 1 \rangle$ is defined in (14). From (30) and (26) it is evident that w is the superposition of modes whose amplitudes decay slowly in time and oscillate at a frequency slightly different from k_n . By combining (24b) and (26b), and observing from (25a) that F_n decreases as n and k_n increase, by the Riemann-Lebesgue theorem,⁸ we find that the higher-order modes decay faster than the lower-order ones.

Inserting (30) into (10) gives the scattered pressure as

$$\begin{aligned} p(\mathbf{x}, t; \epsilon) = & -\frac{\epsilon}{\pi} \sum_{n=1}^{\infty} k_n \langle \psi_n, 1 \rangle \\ & \times \text{Im} \left(\int_M \int \frac{H(t - q)}{q} e^{s_n(t - q)} \psi_n dx' dy' \right) \\ & + \frac{\epsilon}{\pi} \sum_{n=1}^{\infty} \langle \psi_n, 1 \rangle \\ & \times \int_M \int \frac{H(t - q)}{q} \delta(t - q) \psi_n(x', y') dx' dy' + O(\epsilon^2). \end{aligned} \quad (31)$$

In the farfield, $|x| = r \rightarrow \infty$ and (31) is simplified to

$$p(x, t) = 2\epsilon H(t-r) \sum_{n=1}^{\infty} k_n \langle \psi_n, 1 \rangle \frac{p_n(t-r, \hat{r}; \epsilon)}{r} + \frac{\epsilon}{\pi} \frac{H(t-r)}{r} \delta(t-r) + O(\epsilon^2), \quad (32a)$$

where the p_n are defined by

$$p_n(t-r, \hat{r}, \epsilon) \equiv e^{-\epsilon \gamma_n(t-r)} |F_n(k_n, \hat{r})| \sin[\Gamma_n(t-r) - \theta_n]. \quad (32b)$$

The phase θ_n is the argument of the complex number F_n defined in (25a).

The total pressure at a point in the farfield consists first of the reflected spike $P^R = \delta(t-z)$ that passes the point at the instant $t = z$. It is then followed by the smaller-amplitude $[O(\epsilon)]$ scattered pressure (32) that arrives at $t = r > z$. This wave is composed of two parts. The first is an outgoing spherical spike that corresponds to the second term in (32a). It only acts at the instant $t = r$. The second component is the superposition of outgoing spherical pulses that decay slowly as functions of $t-r$ and whose angular behavior is described by the directivity factors $F_n(\mu_n, \hat{r})$. It acts, for all $t > r$, and decays to zero as $t \rightarrow \infty$. It corresponds to the "ringing" of the membrane due to the impact of the pulse. The ringing decays because of the back pressure of the fluid on the membrane. If structural damping was included in the membrane mode, then it would also contribute to the decay.

B. The normally incident, plane, compact pulse

This pulse is given by

$$P^I(x, y, z, t) = g(z+t), \quad (33)$$

where the function $g(\xi)$ is smooth for $0 < \xi < \tau$ and is identically zero for $\xi > \tau$ and $\xi < 0$. By inserting (33) into (27) and using (26) we find that the membrane's displacement is

$$w(x, y, t; \epsilon) = -2\epsilon \sum_{n=1}^{\infty} \frac{\langle \psi_n, 1 \rangle}{\Gamma_n} \psi_n(x, y) \times \left(\int_0^{\tau} g(\xi) e^{-\epsilon \gamma_n(t-\xi)} \sin[\Gamma_n(t-\xi)] d\xi \right) + O(\epsilon^2). \quad (34)$$

The interaction of the pulse with the membrane occurs in the time interval $[0, \tau]$ where it is described by the convolution integral in (34). The details of the transient response of the membrane during this interval depends on the specific form of the pulse and its convolution integral. This is demonstrated in Sec. IV for a normally incident, plane, compact pulse on a circular membrane. However, when $t > \tau$ the upper limit of the integral in (34) can be replaced by τ . Then w is given by

$$w(x, y, t; \epsilon) = -2\epsilon \sum_{n=1}^{\infty} \frac{\langle \psi_n, 1 \rangle}{\Gamma_n} \psi_n(x, y) e^{-\epsilon \gamma_n t} \times [A_n \sin(\Gamma_n t) - B_n \cos(\Gamma_n t)] + O(\epsilon^2), \quad t > \tau, \quad (35a)$$

where

$$A_n = \int_0^{\tau} g(\xi) \cos k_n \xi d\xi, \quad (35b)$$

$$B_n = \int_0^{\tau} g(\xi) \sin k_n \xi d\xi. \quad (35c)$$

Thus, after the pulse interacts with the membrane, w is again the superposition of modes which slowly decay in time and oscillate at roughly the *in vacuo* eigenfrequency k_n .

The scattering acoustic pressure is obtained by inserting (34) into (10). This gives

$$p(x, t, \epsilon) = + \frac{\epsilon}{\pi} \sum_{n=1}^{\infty} \langle \psi_n, 1 \rangle \times \int_M \int_M \frac{H(t-q)}{q} \psi_n(x', y') g(t-q) dx' dy' - \frac{\epsilon}{\pi} \sum_{n=1}^{\infty} k_n \langle \psi_n, 1 \rangle \times \int_M \int_M \frac{H(t-q)}{q} \psi_n(x', y') J(t-q) dx' dy' + O(\epsilon^2), \quad (36)$$

where the function $J(t)$ is defined by

$$J(t) = \text{Im} \left(\int_0^t g(\xi) e^{i\gamma_n(t-\xi)} d\xi \right). \quad (37)$$

In the farfield, (36) is simplified to

$$p = \frac{\epsilon}{\pi} H(t-r) \frac{g(t-r)}{r} - \frac{\epsilon}{\pi} \frac{H(t-r)}{r} \sum_{n=1}^{\infty} k_n \langle \psi_n, 1 \rangle \times \int_M \int_M \psi_n(x', y') J(t-r + \hat{r} \cdot \xi) dx' dy' + O(\epsilon^2), \quad (38)$$

where $\xi = (x', y', 0)$.

Thus, a fixed observation point in the farfield is first insonified by the reflected pulse $P^R(z, t) = g(t-z)$ for t in the interval $z < t < z + \tau$. This reflected pulse is $O(1)$, and the scattered pressure (38) is $O(\epsilon)$. The first term of the scattered pressure is a spherical pulse that insonifies the observation point for t in the interval $r < t < r + \tau$. Thus, it arrives at and leaves the observation point after the reflected pulse, and during the overlap time it is small compared to the reflected pulse. However, after the tail of the reflected pulse passes the observation point, the total pressure is given by (38) and it is $O(\epsilon)$. Moreover, when $t > r + \tau$, so that the tail of the spherical wave, which is given by the first term in (38), passes the observation point, the scattered, and hence the total pressure, is given by the second term in (38).

When $t > r + \tau$, i.e., for sufficiently large values of t , the second term is simplified to

$$p = 2\epsilon \frac{H(t-r)}{r} \sum_{n=1}^{\infty} k_n \langle \psi_n, 1 \rangle q_n(t-r, \hat{r}, \epsilon) + O(\epsilon^2), \quad t-r > \tau, \quad (39a)$$

where

$$q_n \equiv e^{-\epsilon \gamma_n(t-r)} |\hat{g}_n(S_1)| |F_n(k_n, \hat{r})| \times \sin[\Gamma_n(t-r) - \theta_n + X_n] \quad (39b)$$

and X_n is the phase of $\hat{g}_n(S_1)$. Once again, the farfield scattered pressure is the superposition of outgoing spherical pulses that decay slowly as functions of $t-r$. It represents the pressure due to the "decayed ringing" of the membrane.

C. The obliquely incident, plane, compact pulse

This pulse is given by

$$P^I(x, y, z, t) = g(a_1(x - x^*) + a_3 z + t), \quad (40)$$

where $g(\xi)$ is the same smooth, compact function as in Sec. III B. The constants a_1 and a_3 satisfy $a_1^2 + a_3^2 = 1$, $a_1 < 0$, and $a_3 > 0$. The last inequality insures that the pulse is incident on the membrane from above while the first indicates that the pulse travels from left to right in the x - y plane. Without loss of generality we have chosen the orientation of the x - y plane for this to occur. We assume that at $t = 0$ the incident pulse has just struck the membrane at the point $(x^*, y^*, 0)$ on B where the line $x = x^*$ is tangent to B (see Fig. 1). As t increases, the leading edge of the pulse $a_1(x^* - x) = t$ propagates further across M until $t = \bar{t}$, when this line is tangent to the "last" point $(\bar{x}, \bar{y}, 0)$ in M (see Fig. 1). A similar sequence of events occurs for the trailing edge of the pulse. In particular, when $t = \tau$ the trailing edge strikes $(x^*, y^*, 0)$, and

when $t = \bar{t} + \tau$ it passes through $(\bar{x}, \bar{y}, 0)$. Consequently, the membrane is insonified only for $0 < t < \bar{t} + \tau$.

By inserting (40) into (27), interchanging the order of summation and integration, and introducing the change of variables, $\sigma = \xi - a_1(x^* - \alpha)$, we find that w is given by $w(x, y, t; \epsilon)$

$$= -2\epsilon \sum_{n=1}^{\infty} \psi_n(x, y) \iint_M \psi_n(\alpha, \beta) \times \left(\int_0^{t-x} d_n(t - \chi - \sigma) g(\sigma) d\sigma \right) d\alpha d\beta + O(\epsilon^2), \quad (41a)$$

where

$$\chi \equiv a_1(x^* - \alpha). \quad (41b)$$

In deriving this result we have replaced the lower limit by zero because g vanishes for negative arguments. The integration in (41a) is over the three-dimensional region $D(t)$, which is defined as the set of points (α, β, σ) that lie below the plane $\sigma = t + a_1(\alpha - x^*)$ and within the cylinder $D_c \equiv \{(\alpha, \beta, \sigma) | 0 < \sigma < \tau, (\alpha, \beta) \in M\}$. As t increases from zero the region $D(t)$ expands to incorporate more of the cylinder until $t = \bar{t} + \tau$. After this instant the plane is above the cylinder D_c and thus $D(t) \equiv D_c$.

It is difficult to derive the detailed qualitative features of the membrane's motion from (41), during its insonification period and for an arbitrary $g(\xi)$, other than by a numerical

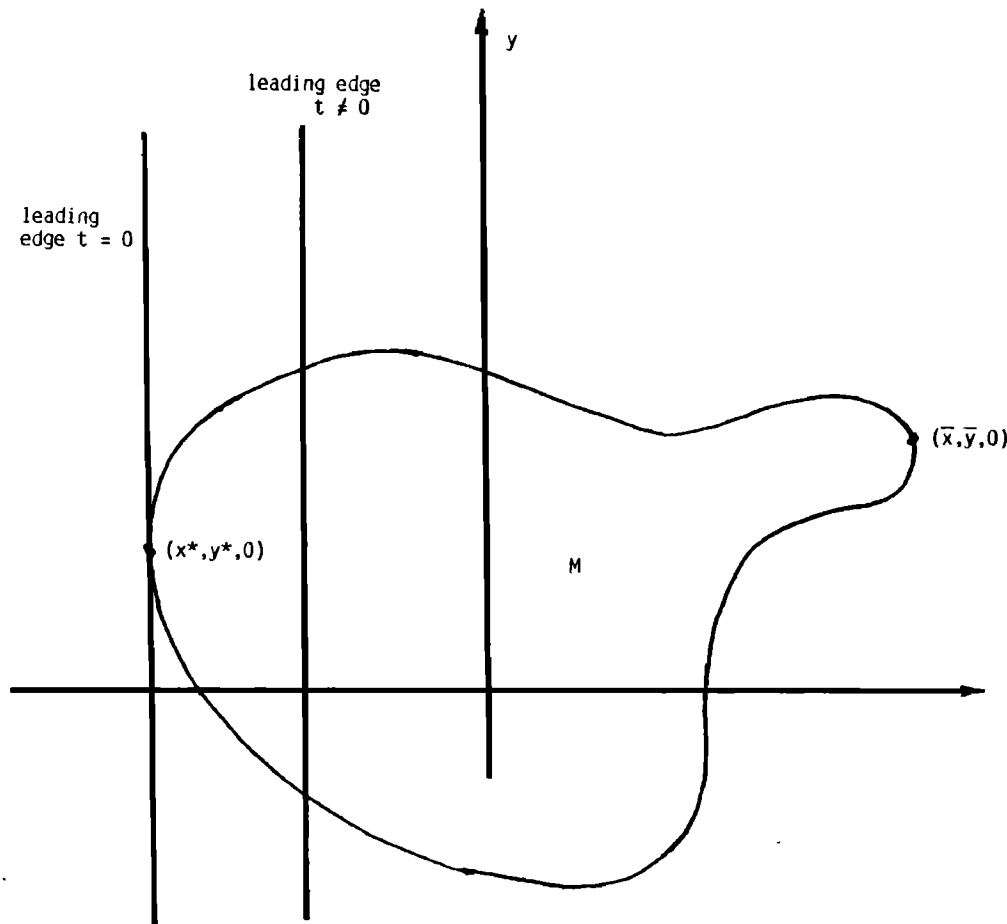


FIG. 1. An arbitrary membrane and the intersection of the pulse's leading edge with the plane $z = 0$ for two times.

evaluation of the integrals and sums in (41a). However, when $t > \bar{t} + \tau$, i.e., when the incident pulse "lifts" from the membrane, the upper σ limit in (41) can be replaced by τ . Then by interchanging the order of integration in (41) and using (26a) we obtain

$$w(x, y, t; \epsilon) = -2\epsilon \sum_{n=1}^{\infty} e^{-\epsilon \gamma_n t} \frac{\psi_n(x, y)}{k_n} \times [(C_n A_n - E_n B_n) \sin \Gamma_n t - (E_n A_n + C_n B_n) \times \cos \Gamma_n t] + O(\epsilon^2), \quad \bar{t} > t + \tau, \quad (42a)$$

$$C_n \equiv \langle \psi_n, \cos k_n \chi \rangle, \quad (42b)$$

$$E_n \equiv \langle \psi_n, \sin k_n \chi \rangle, \quad (42c)$$

and A_n and B_n are defined by (35b) and (35c). Thus, as in the previous two examples, after the pulse interacts with the membrane, w is the superposition of modes that decay slowly in time and oscillate roughly at the eigenfrequencies k_n .

The scattered pressure is obtained from (41a) and (10a). After the pulse lifts from the membrane, i.e., $t > \bar{t} + \tau$, the scattering pressure in the farfield is again the sum of decaying, outgoing, spherical pulses given by

$$p(x, t; \epsilon) = -2\epsilon \frac{H(t-r)}{r} \sum_{n=1}^{\infty} k_n u_n(t-r, \hat{r}, \epsilon) + O(\epsilon^2), \quad \text{for } t-r > \bar{t} + \tau, \quad (43a)$$

where the u_n are given by

$$u_n(t-r, \hat{r}, \epsilon) \equiv |F_n| \{ (C_n A_n - E_n B_n) \sin [\Gamma_n(t-r) - \theta_n] - (E_n A_n + C_n B_n) \cos [\Gamma_n(t-r) - \theta_n] \} \times e^{-\epsilon \gamma_n(t-r)}, \quad (43b)$$

and θ_n is the argument of $F_n(k_n, \hat{r})$.

IV. SCATTERING OF A NORMALLY INCIDENT, PLANE, COMPACT PULSE FROM A BAFFLED CIRCULAR MEMBRANE

The specific pulse is given by

$$g(\xi) = \begin{cases} \frac{1}{12}(10 - 15 \cos \omega_1 \xi + 6 \cos \omega_2 \xi - \cos \omega_3 \xi), & 0 \leq \xi \leq \tau, \\ 0, & \xi > \tau, \\ 0, & \xi < 0, \end{cases} \quad (44)$$

and $\omega_m \equiv 2\pi m/\tau$, $m = 1, 2, 3$. This pulse has a maximum of one at $\xi = \tau/2$. In addition, $g(\xi)$ and its first five derivatives vanish at $\xi = 0$ and $\xi = \tau$. Thus, it is a good approximation to a smooth compact pulse. The graph of $g(\xi)$ is shown in Fig. 2. The unit circular membrane is centered at $x = y = 0$. Since the incident pulse is independent of x and y , only the radially symmetric modes are excited. They are given by

$$\psi_n = \sqrt{\pi} J_0(\mu_n \rho) / J'_0(\mu_n), \quad n = 1, 2, \dots, \quad (45)$$

where μ_n is the n th zero of $J_0(\mu) = 0$. The eigenvalues μ_n^2 are simple, so that formulas (34)–(39) are valid.

The inner product $\langle \psi_n, 1 \rangle$ is evaluated directly from the properties of the Bessel function and is given by

$$\langle \psi_n, 1 \rangle = -2\sqrt{\pi}/\mu_n. \quad (46)$$

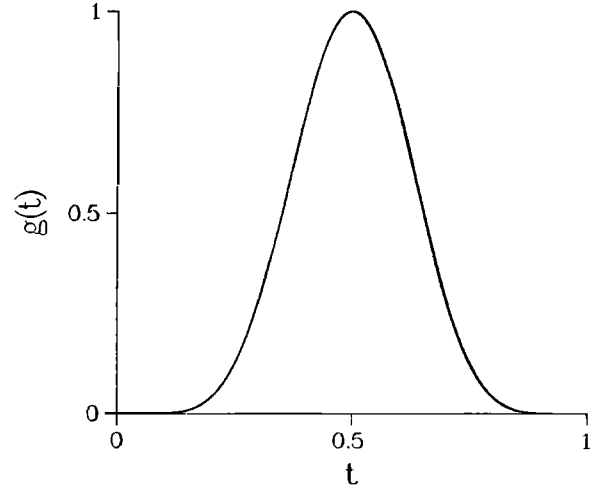


FIG. 2. The incident pulse profile for $\tau = 1.0$.

The convolution integral in (34) and the function J , which is defined in (36b), are computed numerically for the function g given in (44). The results of a numerical evaluation of the membrane's motion from (34) for $\epsilon = 0.1$, $c = 0.5$, $\tau = 1$, and $x = y = 0$ are shown in Fig. 3(a). For $0 < t < 1$, the motion is nearly sinusoidal in response to the incident pulse. For

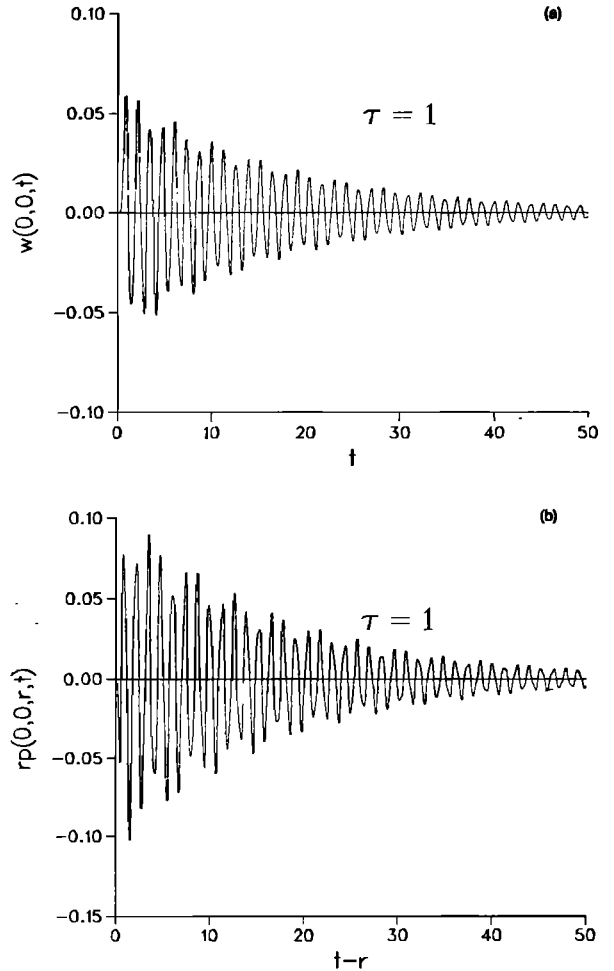


FIG. 3. (a) Membrane displacement at $x = y = 0$ for $\epsilon = 0.1$, $c = 0.5$, and $\tau = 1.0$. (b) Backscattering amplitude rp as a function of $t-r$. Same parameters as (a).

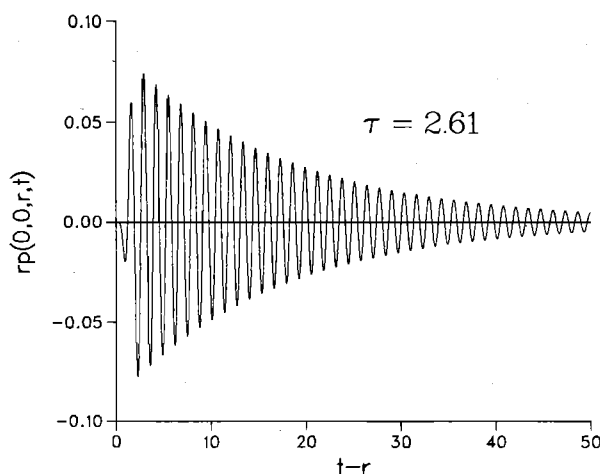


FIG. 4. Same as Fig. 3(b), except $\tau = 2.61$.

$t \gg \tau$, the membrane "rings" and its displacement slowly decays to zero, as predicted by (35a).

In Fig. 3(b) the corresponding farfield backscattering amplitude rp is graphed from (39a) as a function of $t - r$. The response in Fig. 3(a) and (b) is oscillatory with a slowly decaying amplitude. The additional quasiperiodicity is caused by the superposition of modes as described by (39). The dominant period is that of the fundamental mode, $T_1 = 2\pi c/\mu_1 \approx 1.31$. Its dominance is due to the proximity of τ to T_1 . As τ approaches T_1 or some integral multiple of T_1 , the coupling to the fundamental mode will become stronger. This is evident in Fig. 4 where $\tau = 2.61$. Figure 4 illustrates the typical "ringing" and exponential decay of an isolated mode. Also, for a given pulse shape $g(\xi)$, as τ increases the broader pulse has more of its energy located in the low-frequency portion of the spectrum. Thus, the broader pulse will not significantly excite any particular mode in the superposition. This is confirmed by the results presented in Fig. 5 where the backscattered amplitude is shown for the same values of ϵ and c as in Fig. 3(a) and (b), but for $\tau = 5.0$. The additional modulation is not present.

On the other hand, for a fixed pulse shape $g(\xi)$, decreasing τ puts more energy into the high-frequency portion of the

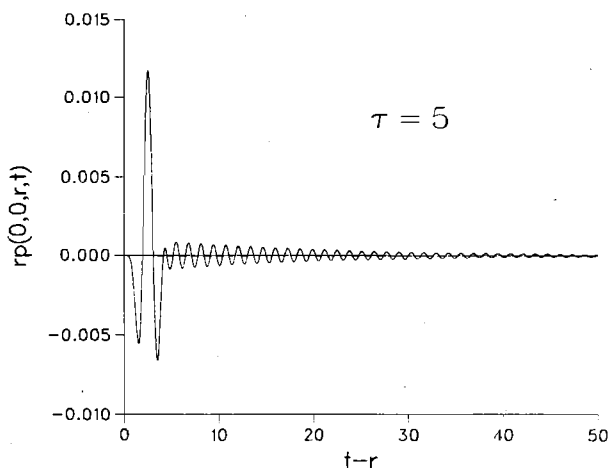


FIG. 5. Same as Fig. 4, except $\tau = 5.0$.

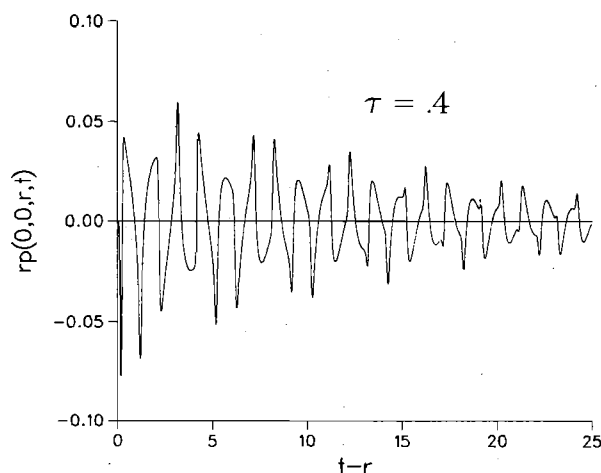


FIG. 6. Same as Fig. 4, except $\tau = 0.4$.

pulse's spectrum. Thus, the sharper pulse will excite more of the membrane's modes and make the response more quasi-periodic in character. This is illustrated by the results presented in Fig. 6 where the backscattered amplitude is shown for the same values of ϵ and c as in Fig. 4, but for $\tau = 0.4$. The jagged, almost random response indicates the presence of many frequencies in the waveform. However, we observe that as $\tau \rightarrow 0$ the pulse width vanishes but the maximum value of the pulse, which is independent of τ , equals 1. Thus, the limiting pulse as $\tau \rightarrow 0$ does not correspond to a delta function. Hence, the small τ response shown in Fig. 6 does not necessarily give the qualitative features of the response to a delta function given by (32).

ACKNOWLEDGMENTS

This research was supported by the Air Force Office of Scientific Research under Grant No. AFOSR 80-0016A, by the National Science Foundation under Grant No. MCS 83 00578, and by the Office of Naval Research under Contract No. N00014-83-C-0518.

¹G. A. Kriegsmann, A. Norris, and E. L. Reiss, *J. Acoust. Soc. Am.* **75**, 685 (1984).

²P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968).

³T. L. Geers, "Transient response analysis of submerged structures," in *Finite Element Analysis of Transient Structural Behavior*, edited by T. Belytschko (ASME, New York, 1975), AMD Vol. 14, pp. 59-84.

⁴T. L. Geers, *J. Acoust. Soc. Am.* **64**, 1500 (1978).

⁵G. A. Kriegsmann and C. L. Scandrett, "Numerical studies of acoustic pulse scattering by baffled two-dimensional membranes," *J. Acoust. Soc. Am.* **79**, 9-17 (1986).

⁶G. A. Kriegsmann and C. S. Morawetz, *SIAM J. Sci. Statist. Comput.* **1**, 371 (1980).

⁷E. T. Copson, *Asymptotic Expansions* (Cambridge U.P., Cambridge, 1967).

⁸F. W. J. Olver, *Introduction to Asymptotic and Special Functions* (Academic, New York, 1974).