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SCATTERING OF ELASTIC WAVES BY SPHERICAL INCLUSIONS WITH APPLICATIONS TO LOW FREQUENCY WAVE PROPAGATION IN COMPOSITES

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Abstract—Scattering of plane elastic waves by a spherical inclusion is considered. A unified method of solution is presented which treats compressional and shear incidence on a similar basis. Explicit results are given for Rayleigh scattering. We apply the results of the single scattering problem to the propagation of low frequency waves in a composite containing a dilute concentration of spherical inclusions. Explicit formulae are given for the effective wave speeds and attenuations when the inclusions are voids. Both the compressional and shear wave speeds decrease initially as a function of frequency.

1. INTRODUCTION

THE PROBLEM of elastic wave scattering from a spherical inclusion has been previously considered by many people. Foremost among the treatments have been those of Ying and Truell [1] for compressional plane wave incidence and Einspruch *et al.* [2] for shear plane wave incidence. Numerical calculations of the scattering cross-sections are given in Ref. [3] for compressional incidence and Ref. [4] for shear incidence. Corrections to these papers may be found in Ref. [5]. The general procedure for solving this problem and generalizations of it are given in reference books, e.g. [6] and [7]. However, apart from Ref. [2], most treatments [6, 7, 8] just consider the case of compressional incidence. This may be because the problem of shear incidence is considered to be much more difficult. Such an attitude may be justified if one compares the analysis of Ref. [2] with that of Ref. [1]. Also, the analyses in Refs. [1] and [2] are quite different. It is desirable to have a method of solution that considers both types of incidence simultaneously.

In this paper we present a unified treatment for both compressional and shear incidence. Our method of solution uses the set of vector spherical harmonic functions defined, for example, in Ref. [9]. These functions were also adopted in Ref. [2], but the authors did not use the orthogonality of the functions to full advantage. We show that the solution for shear incidence is basically no harder to compute than that for compressional incidence. We use the procedure to compute the scattered fields in the Rayleigh or low frequency limit. We note that elegant techniques exist for treating Rayleigh scattering from arbitrary ellipsoidal inclusions [10, 11]. However, these procedures give only the first terms in the low frequency asymptotic expansions of the scattered fields. The exact method discussed here allows one to calculate higher order terms. In particular, we calculate the forward scattering amplitudes from spherical cavities correct to the fifth power in frequency.

The results for the single scattering problem are then used to consider the propagation of elastic waves in composites containing dilute concentrations of spherical inclusions. The effective complex wave numbers follow from the coherent wave equations [12, 13] which depend only upon the forward scattering amplitude of the single scattering problem. This approximation implicitly neglects multiple scattering effects, and is therefore best suited to dilute concentrations of inclusions. We derive relations for the dispersive wave speeds and attenuations at low frequency when the inclusions are empty, or voids. These relations include previous ones found by Sayers [14] for the compressional wave. The results for the shear wave are new.

2. FORMULATION AND SOLUTION OF THE SINGLE SCATTERING PROBLEM

The matrix has Lamé constants λ_1 and μ_1, and density ρ_1. We will also use the bulk modulus $K_1 = \lambda_1 + \frac{2}{3}\mu_1$. The corresponding constants for the spherical inclusion of radius a are λ_2, μ_2, ρ_2 and $K_2 = \lambda_2 + \frac{2}{3}\mu_2$.

We consider a compressional (C) or shear (S) plane wave of radial frequency ω. Thus,

$$\mathbf{u}^{\mathrm{inc}}(\mathbf{r}, t) = \begin{cases} \mathbf{a}_z e^{i(k_1 z - \omega t)}, & \alpha = C \\ \mathbf{a}_x e^{i(\kappa_1 z - \omega t)}, & \alpha = S \end{cases} \tag{2.1}$$

where α denotes the type of incident wave, $\alpha = C, S$ and $\mathbf{a}_x$, $\mathbf{a}_z$ are unit orthogonal vectors. We define the wave numbers k_j and κ_j, $j = 1, 2$

$$k_j = \omega\left(\frac{\rho_j}{\lambda_j + 2\mu_j}\right)^{1/2} \tag{2.2a}$$

$$\kappa_j = \omega\left(\frac{\rho_j}{\mu_j}\right)^{1/2}. \tag{2.2b}$$

We will omit the term $\exp(-i\omega t)$ in subsequent eqns.

We express the total field, incident plus scattered, as

$$\mathbf{u}^{\mathrm{tot}} = \begin{cases} \mathbf{u}^{\mathrm{inc}} + \mathbf{u}^{\mathrm{sc}}, & r > a \\ \mathbf{u}^{\mathrm{int}}, & r < a. \end{cases} \tag{2.3}$$

Solution

In the following, we adopt the notation of Morse and Feshbach [9]. From Ref. [9], eqn (13.3.7) (see also Ref. [2]), we have

$$\mathbf{u}^{\mathrm{inc}} = \begin{cases} \sum\limits_{n=0}^{\infty} (2n+1) i^{n-1} \mathbf{L}^1_{eon}(k_1), & \alpha = C \\ \sum\limits_{n=1}^{\infty} \dfrac{(n+1) i^n}{n(n+1)} [\mathbf{M}^1_{oin}(\kappa_1) - i\mathbf{N}^1_{ein}(\kappa_1)], & \alpha = S \end{cases} \tag{2.4}$$

where $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{N}$ are vector spherical harmonics, defined in Ref. [9], eqns (13.3.67)–(13.3.69), and also in Ref. [2], eqns (8), (9) and (10). We have taken the argument of $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{N}$ as the wave numbers k_j and κ_j, $j = 1, 2$, rather than the position vector $\mathbf{r}$, as in Ref. 9, in order to distinguish between the two types of waves.

We also write the scattered fields of eqn (2.3) as (Ref. [9], p. 1866):

$$\mathbf{u}^{\mathrm{sc}} = \sum_{mn\sigma} \{A^\sigma_{mn}\mathbf{L}^3_{\sigma mn}(k_1) + B^\sigma_{mn}\mathbf{M}^3_{\sigma mn}(\kappa_1) + C^\sigma_{mn}\mathbf{N}^3_{\sigma mn}(\kappa_1)\} \tag{2.5a}$$

$$\mathbf{u}^{\mathrm{int}} = \sum_{mn\sigma} \{R^\sigma_{mn}\mathbf{L}^1_{\sigma mn}(k_2) + S^\sigma_{mn}\mathbf{M}^1_{\sigma mn}(\kappa_2) + T^\sigma_{mn}\mathbf{N}^1_{\sigma mn}(\kappa_2)\}. \tag{2.5b}$$

The summation in eqn (2.5) is over $\sigma = e$ (even) and o (odd), $n = 0, 1, 2 \cdots$ and $m = 0, 1, 2 \cdots n$. For each $mn\sigma$, there are six unknown scalars, A, B, C, R, S and T. These follow from the six boundary conditions at the interface $r = a$: continuity of displacements (three) and continuity of the normal traction components (three).

The total displacements at $r = a$ are expressed via eqns (13.3.67)–(13.3.69) of Ref. [9] in terms of the vector harmonics $\mathbf{B}^\sigma_{mn}$, $\mathbf{P}^\sigma_{mn}$ and $\mathbf{C}^\sigma_{mn}$, defined on pp. 1898–9 of Ref. [9]. This representation is desirable because of the orthonormality properties of $\mathbf{P}^\sigma_{mn}$, $\mathbf{B}^\sigma_{mn}$ and $\mathbf{C}^\sigma_{mn}$. The normal traction at $r = a$ can be expressed in terms of these vectors using eqns (13.3.78) of Ref. [9]. However, we note a typographical error in the first of the three equations in Ref. [9], eqn (13.3.78). The error is in the square bracket following $\mathbf{B}^\sigma_{mn}$. It should read

$$\left[\frac{2\mu}{a}\frac{d}{da} j_\ell(k_c a) - \frac{2\mu}{a} j_\ell(k_c a)\right]. \tag{2.6}$$

Using the orthonormality of $\mathbf{B}^{\sigma}_{mn}$, $\mathbf{P}^{\sigma}_{mn}$ and $\mathbf{C}^{\sigma}_{mn}$, the six boundary conditions give six simultaneous equations for the six unknowns in eqn (2.5) for each $mn\sigma$.

The six equations decouple into a system of four for A^{σ}_{mn}, C^{σ}_{mn}, R^{σ}_{mn} and T^{σ}_{mn} and a system of two for B^{σ}_{mn} and S^{σ}_{mn}. The latter eqns give B^{σ}_{mn} and S^{σ}_{mn} both zero *unless* $\alpha = S$, $\sigma = o$ and $m = 1$, in which case

$$B^{o}_{1n} = \frac{(2n+1)i^n}{n(n+1)}\left[\frac{c^1_n(\kappa_1 a)\gamma^1_{n2}(\kappa_2 a) - c^1_n(\kappa_2 a)\gamma^1_{n1}(\kappa_1 a)}{c^1_n(\kappa_2 a)\gamma^3_{n1}(\kappa_1 a) - c^3_n(\kappa_1 a)\gamma^1_{n2}(\kappa_2 a)}\right] \tag{2.7a}$$

$$S^{o}_{1n} = \frac{(2n+1)i^n}{n(n+1)}\left[\frac{c^1_n(\kappa_1 a)\gamma^3_{n1}(\kappa_1 a) - c^3_n(\kappa_1 a)\gamma^1_{n1}(\kappa_1 a)}{c^1_n(\kappa_2 a)\gamma^3_{n1}(\kappa_1 a) - c^3_n(\kappa_1 a)\gamma^1_{n2}(\kappa_2 a)}\right]. \tag{2.7b}$$

The functions c^m_n and γ^m_{nj} are defined in Appendix A.

The remaining four eqns for A^{σ}_{mn}, C^{σ}_{mn}, R^{σ}_{mn}, and T^{σ}_{mn} simplify in the case of $n = 0$. A non-zero solution exists only for $\alpha = C$. Then all the constants are zero except A^{e}_{00} and R^{e}_{00}, which are

$$A^{e}_{00} = i\left[\frac{a^1_0(k_2 a)\alpha^1_{01}(k_1 a) - a^1_0(k_1 a)\alpha^1_{02}(k_2 a)}{a^1_0(k_2 a)\alpha^3_{01}(k_1 a) - a^3_0(k_1 a)\alpha^1_{02}(k_2 a)}\right] \tag{2.8a}$$

$$R^{e}_{00} = i\left[\frac{a^3_0(k_1 a)\alpha^1_{01}(k_1 a) - a^1_0(k_1 a)\alpha^3_{01}(k_1 a)}{a^1_0(k_2 a)\alpha^3_{01}(k_1 a) - a^3_0(k_1 a)\alpha^1_{02}(k_2 a)}\right]. \tag{2.8b}$$

The various functions in eqn (2.8) are defined in Appendix A.

For $n \geq 1$, the four eqns are

$$\mathbf{QX} = \mathbf{Y}_{\alpha} \qquad \alpha = C, S \tag{2.9}$$

where

$$\mathbf{X} = [A^{\sigma}_{mn}, C^{\sigma}_{mn}, R^{\sigma}_{mn}, T^{\sigma}_{mn}]^T \tag{2.10}$$

$$\mathbf{Y}_{\alpha} = \begin{cases} (2n+1)i^{(n+1)}[a^1_n(k_1 a), b^1_n(k_1 a), \alpha^1_{n1}(k_1 a), \beta^1_{n1}(k_1 a)]^T \delta_{\sigma e}\delta_{m0}, & \alpha = C \\ \dfrac{(2n+1)}{n(n+1)} i^{(n+1)}[d^1_n(\kappa_1 a), e^1_n(\kappa_1 a), \delta^1_{n1}(\kappa_1 a), \epsilon^1_{n1}(\kappa_1 a)]^T \delta_{\sigma e}\delta_{m1}, & \alpha = S \end{cases} \tag{2.11}$$

and

$$\mathbf{Q} = \begin{bmatrix} a^3_n(k_1 a) & d^3_n(\kappa_1 a) & -a^1_n(k_2 a) & -d^1_n(\kappa_2 a) \\ b^3_n(k_1 a) & e^3_n(\kappa_1 a) & -b^1_n(k_2 a) & -e^1_n(\kappa_2 a) \\ \alpha^3_{n1}(k_1 a) & \delta^3_{n1}(\kappa_1 a) & -\alpha^1_{n2}(k_2 a) & -\delta^1_{n2}(\kappa_2 a) \\ \beta^3_{n1}(k_1 a) & \epsilon^3_{n1}(\kappa_1 a) & -\beta^1_{n2}(k_2 a) & -\epsilon^1_{n2}(\kappa_2 a) \end{bmatrix}. \tag{2.12}$$

The various functions in eqns (2.11) and (2.12) are defined in Appendix A.

The limiting cases of rigid and empty spheres are easily obtained from eqns (2.9)–(2.12) as follows. Let

$$\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]^T \tag{2.13}$$

$$\mathbf{Y}_{\alpha} = [\mathbf{Y}_{1\alpha}, \mathbf{Y}_{2\alpha}]^T, \qquad \alpha = C, S \tag{2.14}$$

where X_1, X_2, etc., are vectors with two elements. Also,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \tag{2.15}$$

where $\mathbf{Q}_{ij}$ are 2×2 matrices. The rigid limit follows by letting μ_2, λ_2, $\rho_2 \to \infty$ in such a way that k_2 and κ_2 remain finite. Then, $\det(\mathbf{Q}) \sim \det(\mathbf{Q}_{11})\det(\mathbf{Q}_{22})$. From Cramer's Rule, it is clear that the system (2.9) thus reduces to

$$\mathbf{Q}_{11}\mathbf{X}_1 = \mathbf{Y}_{1\alpha}, \qquad \alpha = C, S \tag{2.16}$$

for a rigid sphere.

The limit of a spherical cavity follows by letting λ_2, μ_2, $\rho_2 \to 0$, but again keeping k_2 and κ_2 finite. Then $\det(\mathbf{Q}) \sim \det(\mathbf{Q}_{12})\det(\mathbf{Q}_{21})$, and the system of eqns reduces to

$$\mathbf{Q}_{21}\mathbf{X}_1 = \mathbf{Y}_{2\alpha}, \qquad \alpha = C, S. \tag{2.17}$$

3. LOW FREQUENCY RESULTS

The low frequency or Rayleigh regime is defined by $\kappa_1 a \ll 1$. Thus, the incident wavelength is much longer than a. In this regime the inclusion is essentially subjected to quasi-static loading. The system of eqns (2.9) may then be solved by performing regular perturbation expansions in the various terms. Similar perturbation expansions have been considered in Refs. [1] and [2]. We omit the relevant details, and refer to these papers for further discussion of the procedure.

Compressional incidence

We now present the scattering amplitudes for the scattered field defined in eqn (2.5). For compressional incidence, the only non-zero terms are, to highest order,

$$A^e_{00} = \frac{1}{3}\left(\frac{K_1 - K_2}{K_2 + \frac{4}{3}\mu_1}\right)(k_1 a)^3 + O(k_1 a)^5 \tag{3.1a}$$

$$A^e_{01} = \frac{i}{3}\left(\frac{\rho_2}{\rho_1} - 1\right)(k_1 a)^3 + O(k_1 a)^5 \tag{3.1b}$$

$$A^e_{on} = i^n\left[\frac{2^n n!}{(2n)!}\right]^2\left[\frac{-2n(n-1)(4n^2-1)(\mu_2-\mu_1)}{2n(n-1)(\mu_2-\mu_1) + [2(n^2-1)\mu_2 + (2n^2+1)\mu_1]\kappa_1^2/k_1^2}\right] \times (k_1 a)^{2n-1} + O(k_1 a)^{2n+1}, \qquad n \geq 2 \tag{3.1c}$$

$$C^e_{on} = \frac{1}{n}\left(\frac{\kappa_1}{k_1}\right)^{n+2} A^e_{on}[1 + O(k_1 a)^2], \qquad n \geq 1. \tag{3.1d}$$

We note that A^e_{00}, A^e_{01}, A^e_{02}, C^e_{01} and C^e_{02} are all $O((k_1 a)^3)$ and the rest are $O((k_1 a)^5)$ and smaller. Therefore, the scattered field, to highest order, depends only on these five terms. The scattered farfield for compressional incidence follows by expanding $\mathbf{L}^3_{\sigma mn}(k_1)$ and $\mathbf{N}^3_{\sigma mn}(\kappa_1)$ in powers of r^{-1}, and retaining only highest order terms. We obtain,

$$\mathbf{u}^{SC} \sim A^C_C(\theta)\frac{e^{ik_1 r}}{k_1 r}\,\mathbf{e}_r + A^C_S(\theta)\frac{e^{i\kappa_1 r}}{\kappa_1 r}\,\mathbf{e}_\theta \tag{3.2}$$

where

$$A^C_C = A^e_{00} - iA^e_{01}\cos\theta - A^e_{02}(1 - \tfrac{3}{2}\sin^2\theta) \tag{3.3a}$$

$$A^C_S = iC^e_{01}\sin\theta + \tfrac{3}{2}C^e_{02}\sin^2\theta. \tag{3.3b}$$

We note that this expansion of the farfield in terms of monopole ($n = 0$), dipole ($n = 1$) and quadrupole ($n = 2$) terms is only valid in the Rayleigh regime. At finite frequencies, all the spherical harmonics, $n = 0, 1, 2, \cdots$ are relevant in the farfield.

Shear incidence

For shear wave incidence, the only non-zero terms are, for $n \geq 1$

$$A^e_{in} = \frac{1}{n}\left(\frac{\kappa_1}{k_1}\right)^{n-1} A^e_{on}[1 + O(k_1 a)^2] \tag{3.4a}$$

$$C^e_{1n} = \frac{1}{n^2}\left(\frac{\kappa_1}{k_1}\right)^{2n+1} A^e_{on}[1 + O(k_1a)^2] \tag{3.4b}$$

where A^e_{on}, $n \geq 1$ are given in eqns (3.1). Also, we have from eqn (2.7)

$$B^o_{11} = \frac{1}{30}\left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{\kappa_2}{\kappa_1}\right)^2 (\kappa_1 a)^5 + O(\kappa_1 a)^7 \tag{3.5a}$$

$$B^o_{1n} = \frac{i^{n-1}(2n+1)}{n(n+1)(2n+1)}\left[\frac{2^n n!}{(2n)!}\right]^2 \left[\frac{\dfrac{\mu_2}{\mu_1} - 1}{\dfrac{\mu_2}{\mu_1} + \dfrac{n+2}{n-1}}\right] (\kappa_1 a)^{2n+1} + O(\kappa_1 a)^{2n+3}, \qquad n \geq 2. \tag{3.5b}$$

Therefore, the only terms of significance in the farfield are A^e_{11}, A^e_{12}, C^e_{11} and C^e_{12} which are all $O((\kappa_1 a)^3)$. The farfield is

$$\mathbf{u}^{SC} \sim A^S_C(\theta)\, \frac{e^{ik_1 r}}{k_1 r} \sin\phi\, \mathbf{e}_r + \mathbf{A}^S_S(\theta, \phi)\, \frac{e^{i\kappa_1 r}}{\kappa_1 r} \tag{3.6}$$

where

$$A^S_C = \left[\frac{1}{3}\left(1 - \frac{\rho_2}{\rho_1}\right)\sin\theta + \frac{5(\mu_2 - \mu_1)\kappa_1/k_1 \sin 2\theta}{4(\mu_2 - \mu_1) + (6\mu_2 + 9\mu_1)\kappa_1^2/k_1^2}\right](k_1 a)^3 \tag{3.7a}$$

$$\mathbf{A}^S_S = -iC^e_{11}\sqrt{2}\,\mathbf{B}^e_{11}(\theta, \phi) - C^e_{12}\sqrt{6}\,\mathbf{B}^e_{12}(\theta, \phi) \tag{3.7b}$$

and $\mathbf{B}^e_{11}$ and $\mathbf{B}^e_{12}$ are vector spherical harmonic functions as defined in Ref. [9].

$$\sqrt{2}\,\mathbf{B}^e_{11} = \cos\theta \cos\phi\, \mathbf{e}_\theta - \sin\phi\, \mathbf{e}_\phi \tag{3.8a}$$

$$\sqrt{6}\,B^e_{12} = 3\cos 2\theta \cos\phi\, \mathbf{e}_\theta - 3\cos\theta \sin\phi\, \mathbf{e}_\phi. \tag{3.8b}$$

We note that the amplitude A^S_C equals the amplitude A^C_S, in accordance with known reciprocity relations between scattered elastodynamic waves [15].

4. SCATTERED ENERGY

Consider a sphere of radius r concentric with the scattering sphere. The flux of energy out of this sphere due to the scattered field $\mathbf{u}^{sc}$ defined in eqn (2.3) is

$$\text{Flux at } r = \text{Average over a period of} \int -(\mathrm{Re}\,\dot{\mathbf{u}}^{sc})\cdot(\mathrm{Re}\,\tau^{sc}_r)r^2\, d\Omega. \tag{4.1}$$

Here τ^{sc}_r is the stress in the radial direction due to $\mathbf{u}^{sc}$, the dot denotes the time derivative, and $d\Omega$ is the incremental solid angle. The integral in eqn (4.1) is the power (force times velocity) expended over the surface of the sphere. In the limit as $r \to \infty$, we obtain the flux at infinity as

$$\Sigma_\alpha = \sum_{mn} \{f_n|A^e_{mn}|^2\|\mathbf{P}^e_{mn}\|^2 + g_n|C^e_{mn}|^2\|\mathbf{B}^e_{mn}\|^2\} \tag{4.2}$$

where $\alpha = C$ or S, and

$$f_n = -i\frac{\omega}{2}\lim_{r\to\infty}[\overline{a^3_n(k_1 r)}\alpha^3_{n1}(k_1 r)r^2] = \frac{\mu_1}{2}\frac{\omega}{K_1}\left(\frac{\kappa_1}{k_1}\right)^3 \tag{4.3a}$$

$$g_n = -i\frac{\omega}{2}\lim_{r\to\infty}[\overline{e^3_n(\kappa_1 r)}\epsilon^3_{n1}(\kappa_1 r)r^2] = \frac{n}{2}(n+1)\mu_1\omega/\kappa_1 \tag{4.3b}$$

Also,

$$|X|^2 = \bar{X}X$$

where the bars denote complex conjugate. The norm $\|\mathbf{P}^{\sigma}_{mn}\|$ is defined by

$$\|\mathbf{P}^{\sigma}_{mn}\|^2 = \int_0^{2\pi}\int_0^{\pi} \overline{\mathbf{P}^{\sigma}_{mn}}(\theta, \phi)\cdot\mathbf{P}^{\sigma}_{mn}(\theta, \phi)\sin\theta\, d\theta\, d\phi \tag{4.4}$$

where $\mathbf{P}^{\sigma}_{mn}$ is the vector spherical harmonic defined in Ref. [9], pp. 1898–9, and also in Ref. [2]. The norm $\|\mathbf{B}^{\sigma}_{mn}\|$ is defined similarly. From Ref. [9], p. 1900

$$\|\mathbf{P}^{\sigma}_{mn}\|^2 = \|\mathbf{B}^{\sigma}_{mn}\|^2 = \frac{4\pi}{\epsilon_m(2n+1)}\frac{(n+m)!}{(n-m)!} \tag{4.5}$$

where $\epsilon_m = 1$ for $m = 0$, 2 for $m > 0$, is the Neumann factor. We note that the constants $\mathbf{B}^{\sigma}_{mn}$ of eqn (2.5) do not contribute to the flux at infinity. We also note that $m \equiv 0$ for $\alpha = C$, and $m \equiv 1$ for $\alpha = S$. Thus,

$$\Sigma_\alpha = \frac{2\pi\omega}{\epsilon_m}\frac{\mu_1}{\kappa_1}\left[\sum_{n=m}^{\infty}\left(\frac{\kappa_1}{k_1}\right)^3\frac{(n+m)!}{(n-m)!}\frac{|A^e_{mn}|^2}{(2n+1)} + \sum_{n=1}^{\infty}\frac{(n+m)!}{(n-m)!}\frac{n(n+1)}{(2n+1)}|C^e_{mn}|^2\right],$$
$$m = 0 \quad \text{for} \quad \alpha = C$$
$$m = 1 \quad \text{for} \quad \alpha = S. \tag{4.6}$$

The scattering cross-section γ_α, $\alpha = C, S$ is defined by dividing the scattered flux by the flux of the incident wave per unit area:

$$\text{Flux of incident wave per unit area} = \begin{cases} \frac{1}{2}\omega\mu_1\kappa_1^2/k_1, & \alpha = C \\ \frac{1}{2}\omega\mu_1\kappa_1, & \alpha = S. \end{cases} \tag{4.7}$$

$$\gamma_\alpha = \frac{4\pi}{\epsilon_m\kappa_1^2}\left(\frac{k_1}{\kappa_1}\right)^{1-m}\left[\sum_{n=m}^{\infty}\left(\frac{\kappa_1}{k_1}\right)^3\frac{(n+m)!}{(n-m)!}\frac{|A^e_{mn}|^2}{(2n+1)} + \sum_{n=1}^{\infty}\frac{(n+m)!}{(n-m)!}\frac{n(n+1)!}{(2n+1)}|C^e_{mn}|^2\right],$$
$$m = 0 \quad \text{for} \quad \alpha = C$$
$$m = 1 \quad \text{for} \quad \alpha = S. \tag{4.8}$$

Low frequency results

The flux of the scattered field in the Rayleigh regime follows from eqn (4.6) by including only the terms $n = 0$, 1 and 2. The remaining terms give contributions of higher order in $(\kappa_1 a)$. The scattering cross-section for compressional incidence then follows as

$$\gamma_C = \frac{4\pi}{9} g_C k_1^4 a^6\{1 + O(k_1 a)^2\} \tag{4.9}$$

where g_C is the same quantity that appears in eqns (27) and (28) of Ref. [1].

$$g_C = \left(\frac{K_1 - K_2}{K_2 + \frac{4}{3}\mu_1}\right) + \frac{1}{3}\left[1 + 2\left(\frac{\kappa_1}{k_1}\right)^3\right]\left(\frac{\rho_2}{\rho_1} - 1\right)^2$$
$$+ 2\left[2 + 3\left(\frac{\kappa_1}{k_1}\right)^5\right]\left[\frac{10(\mu_1 - \mu_2)}{4(\mu_2 - \mu_1) + (6\mu_2 + 9\mu_1)\kappa_1^2/k_1^2}\right]^2. \tag{4.10}$$

This agrees with eqn (28) of Ref. [1] if (κ_2/κ_1) in the second term in that equation is replaced by $(\kappa_2/\kappa_1)^2$.

The scattering cross-section for shear incidence is

$$\gamma_S = \frac{4\pi}{9} g_S k_1^4 a^6 \{1 + O(k_1 a)^2\} \tag{4.11}$$

where

$$g_S = \frac{\kappa_1}{k_1}\left\{\frac{1}{3}\left[1 + \left(\frac{\kappa_1}{k_1}\right)^3\right]\left(\frac{\rho_2}{\rho_1} - 1\right)^2 + \frac{3}{10}\frac{\kappa_1^2}{k_1^2}\left[2 + 3\left(\frac{\kappa_1}{k_1}\right)^5\right] \times \left[\frac{10(\mu_2 - \mu_1)}{4(\mu_2 - \mu_1) + (6\mu_2 + 9\mu_1)\kappa_1^2/k_1^2}\right]^2\right\}. \tag{4.12}$$

This result agrees with a similar result of Ref. [5]. The result of Ref. [2] for γ_S is known to contain errors [5]. Plots of g_C and g_S are shown in Fig. 1 for empty inclusions or cavities. The ordinate is $\kappa_1/k_1 = V_C/V_S$, where V_C and V_S are the compressional and shear wave speeds of the matrix. We note that $(V_C/V_S)^2 = 1 + (1 - 2\nu)^{-1}$, where ν is Poisson's ratio, and therefore $V_C/V_S > \sqrt{2}$.

5. LOW FREQUENCY FORWARD SCATTERING AMPLITUDES

At low frequencies, the amplitudes of the scattered fields in the forward direction ($\theta = 0$) follow from eqns (3.2) and (3.6). It is clear from these equations and also from symmetry considerations, that the forward scattered field consists only of a wave of the same type as the incident field. The forward compressional amplitude is $A_C^C(0)$, which follows from eqn (3.3a). The forward shear amplitude is polarized in the same direction as the incident wave, and has amplitude $A_S^S(0) \equiv \mathbf{A}_S^S(0, 0)\cdot \mathbf{e}_\theta$, which follows from eqns (3.7b) and (3.8).

Now, the above forward amplitudes are necessarily real quantities, by virtue of the Rayleigh approximation which makes both $A_C^C(0)$ and $A_S^S(0)$ of order $(k_1 a)^3$. The imaginary parts of the amplitudes are of higher order and could be calculated by extending the perturbation scheme. However, the latter course is not necessary, since we can invoke the optical theorem [16] to compute the imaginary parts of the forward scattering amplitudes. The optical theorem is a general result which, in the present circumstances, yields

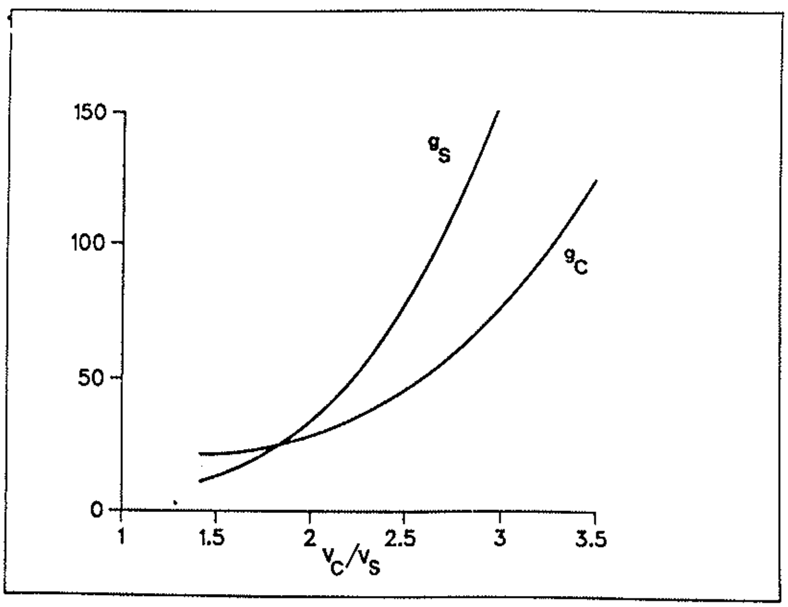

Fig. 1. The dimensionless, normalized low frequency scattering cross-sections for compressional and shear waves incident on a spherical cavity.

$$\mathrm{Im}A_C^C(0) = \frac{k_1^2}{4\pi}\gamma_C \tag{5.1a}$$

$$\mathrm{Im}A_S^S(0) = \frac{\kappa_1^2}{4\pi}\gamma_S. \tag{5.1b}$$

These relations follow from eqns (2.6) and (2.7) of Ref. [16]. However, we note that the scattering amplitudes of Ref. [16] have dimensions of length, whereas ours are dimensionless.

Combining the above findings, we may write the forward scattering amplitudes, correct to first order in their real and imaginary parts. In the following formulae we have dropped the subscript 1 on parameters associated with the matrix. Also, we have found it useful to define K^* and μ^* as

$$K^* = \tfrac{4}{3}\mu \tag{5.2a}$$

$$\mu^* = \frac{\mu}{6}\left(\frac{9K + 8\mu}{K + 2\mu}\right). \tag{5.2b}$$

Then,

$$A_C^C(0) = \frac{(ka)^3}{3}\left\{\left(\frac{\rho_2}{\rho} - 1\right) - \left(\frac{K_2 - K}{K_2 + K^*}\right) - \frac{4}{3}\left(\frac{\mu_2 - \mu}{K + K^*}\right)\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right) - B(ka)^2 + i\,\frac{(ka)^3}{3}\right.$$
$$\left.\times\left[\left(\frac{K_2 - K}{K_2 + K^*}\right)^2 + \frac{1}{3}\left[1 + 2\left(\frac{\kappa}{k}\right)^3\right]\left(\frac{\rho_2}{\rho} - 1\right)^2 + \frac{8}{3}\left(\frac{k}{\kappa}\right)^4\left[2 + 3\left(\frac{\kappa}{k}\right)^5\right]\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right)^2\left(\frac{\mu_2}{\mu} - 1\right)^2\right]\right\} \tag{5.3}$$

$$A_S^S(0) = \frac{(\kappa a)^3}{3}\left\{\left(\frac{\rho_2}{\rho} - 1\right) - \left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right)\left(\frac{\mu_2}{\mu} - 1\right) - D(ka)^2 + i\,\frac{(ka)^3}{3}\left[\frac{1}{3}\left[1 + 2\left(\frac{\kappa}{k}\right)^3\right]\right.\right.$$
$$\left.\left.\times\left(\frac{\rho_2}{\rho} - 1\right)^2 + \frac{2}{15}\left(\frac{k}{\kappa}\right)^2\left[2 + 3\left(\frac{\kappa}{k}\right)^5\right]\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right)\left(\frac{\mu_2}{\mu} - 1\right)^2\right]\right\}. \tag{5.4}$$

Here, B and D are real constants that depend upon the higher order terms in the asymptotic expansions of A_{mn}^σ and C_{mn}^σ. We have not computed them here because of the excessive algebra required. We note, however, that B has been computed for the special case of a cavity by Sayers [14, eqn (6)]. We have verified his result and also calculated D for a cavity. Thus, defining $\eta = \kappa/k$,

$$B = -[\tfrac{16}{15} - \tfrac{5}{12}\eta^2 + \tfrac{3}{16}\eta^4] + \tfrac{7}{15}[1 - \tfrac{19}{12}\eta^2]^{-1} - \tfrac{5}{9}[5 - \tfrac{9}{4}\eta^2][1 - \tfrac{9}{4}\eta^2]^{-2} \tag{5.5}$$

$$D = -(\tfrac{1}{9} + \tfrac{29}{60}\eta^2) + \tfrac{3}{8}\eta^2[1 - \tfrac{5}{2}(2 - \eta^2 + \tfrac{9}{8}\eta^4)(1 - \tfrac{9}{4}\eta^2)^{-2}] + \tfrac{14}{15}\eta^4(1 - \tfrac{19}{12}\eta^2)^{-1}. \tag{5.6}$$

The three separate terms in (5.6) correspond to the $O(ka)^5$ contributions for C_{11}^e, C_{12}^e and C_{13}^e, respectively.

6. LOW FREQUENCY WAVES IN A DILUTE CONCENTRATION OF INCLUSIONS

Consider a random distribution of identical spherical inclusions of materials in a uniform matrix of material 1. We wish to calculate the effective wave speed and attenuation of the coherent waves propagating through the composite. At low concentrations of inclusion we can use the following dispersion relations [12, 13, 14, 17]

$$k^2 = k_1^2 + \frac{N}{V}\frac{4\pi}{k}A_C^C(0) \tag{6.1a}$$

$$\kappa^2 = \kappa_1^2 + \frac{N}{V}\frac{4\pi}{\kappa}A_S^S(0). \tag{6.1b}$$

Here k and κ are the effective wavenumbers for the composite. They are related to the effective wave speeds V_C and V_S by

$$V_C = \omega/k \tag{6.2a}$$

$$V_S = \omega/\kappa. \tag{6.2b}$$

The effective attenuations are determined from the imaginary parts of V_C and V_S. The number N in eqn (6.1) is the number of spheres per volume V. The forward scattering amplitudes $A_C^{\mathcal{C}}(0)$ and $A_S^{\mathcal{S}}(0)$ are defined for a sphere of material 2 in the effective medium with elastic moduli λ, μ and $K = \lambda + \frac{2}{3}\mu$, and density $\rho = \rho_1 + \phi(\rho_2 - \rho_1)$, where ϕ is the volume fraction of material 2. The effective wave speeds may also be written

$$V_C = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} \tag{6.3a}$$

$$V_S = \left(\frac{\mu}{\rho}\right)^{1/2}. \tag{6.3b}$$

The coherent wave equations from which eqn (6.1) follows, is a weak scattering theory in that all multiple scattering effects are smoothed out. It also assumes that the single scattering is "small." Therefore we will only use eqn (6.1) in the low frequency or Rayleigh regime, $|\kappa a| \ll 1$. Equation (6.1) is also valid only for dilute concentrations, $\phi \ll 1$. At higher concentrations, there remains some controversy over the correct form of the eqns analogous to (6.1) [18]. We therefore limit the application of (6.1) to dilute concentrations. We note, also, that the effective inertia of the composite is just the spatial average of the density. In this respect, we agree with McCoy [19] but not with Datta [20], who obtains a different effective inertia.

Since we are restricted to $\phi \ll 1$, we will just compute the derivatives $dK/d\phi$ and $d\mu/d\phi$ at $\phi = 0$. The values of K and μ at $\phi > 0$ but $\phi \ll 1$ then follow by linear extrapolation, e.g.

$$\mu(\phi) = \mu_1 + \phi\left(\left.\frac{d\mu}{d\phi}\right|_{\phi=0}\right) + O(\phi^2) \tag{6.4}$$

We first obtain the initial slope of the shear modulus. Differentiating eqn. (6.1b) with respect to ϕ, using $\phi = \frac{4}{3}\pi a^3 N/V$, and putting $\phi = 0$, gives

$$\left.\frac{1}{\rho}\frac{d\rho}{d\phi}\right|_{\phi=0} - \left.\frac{1}{\mu}\frac{d\mu}{d\phi}\right|_{\phi=0} = \frac{3}{(\kappa_1 a)^3} A_S^{\mathcal{S}}(0). \tag{6.5}$$

Then, using eqn (5.4) for $A_S^{\mathcal{S}}(0)$ and the fact that $d\rho/d\phi = \rho_2 - \rho_1$, we get

$$\left.\frac{d\mu}{d\phi}\right|_{\phi=0} = (\mu_2 - \mu)\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right) + \mu D(ka)^2 - i\frac{(ka)^3}{9}\left\{\mu\left[1 + 2\left(\frac{\kappa}{k}\right)^3\right]\left(\frac{\rho_2}{\rho} - 1\right)^2 + \frac{2}{5}\mu\left(\frac{k}{\kappa}\right)^2\left[2 + 3\left(\frac{\kappa}{k}\right)^5\right]\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right)^2\left(\frac{\mu_2}{\mu} - 1\right)^2\right\} \tag{6.6}$$

where μ^* is defined in eqn (5.2b). A similar eqn for the bulk modulus initial slope follows from differentiating eqn (6.1a) and using eqn (6.6):

$$\left.\frac{dK}{d\phi}\right|_{\phi=0} = (K_2 - K)\left(\frac{K + K^*}{K_2 + K^*}\right) + [KB + K^*(B - D](ka)^2 - i\,\frac{(ka)^3}{9}\left\{K\left[1 + 2\left(\frac{\kappa}{k}\right)^3\right]\right.$$
$$\left.\times\left(\frac{\rho_2}{\rho} - 1\right)^2 + \mu\,\frac{112}{15}\left(\frac{k}{\kappa}\right)^2\left[2 + 3\left(\frac{\kappa}{k}\right)^5\right]\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right)^2\left(\frac{\mu_2}{\mu} - 1\right)^2 + 3(K + K^*)\left(\frac{K_2 - K}{K_2 + K^*}\right)^2\right\} \tag{6.7}$$

where K^* is defined in eqn (5.2a).

The first terms on the right-hand sides of eqns (6.6) and (6.7) are the well-known [21] results for the static moduli of a composite. Therefore, the effective real wave speeds are given by the effective static elastic moduli [19]. The density variation does not enter into these moduli, as one would expect since inertial effects are zero in the static limit. The low frequency attenuation depends upon the imaginary parts of the complex moduli $K(\phi)$ and $\mu(\phi)$. From eqns (6.6) and (6.7) it is clear that the density variation plays a role in the attenuation. Finally, we note that the initial slopes of K and μ for composites containing spherical cavities or pores follow from eqns (6.6) and (6.7) by setting K_2, μ_2 and ρ_2 to zero.

The effect of porosity on the wave speeds at low values of porosity may be written out more explicitly. At porosity ϕ, where $\phi \ll 1$, let the complex wave speeds be $V'_C - i\alpha_C V_C^2/\omega$ and $V'_S - i\alpha_S V_S^2/\omega$, where V_c and V_s are the values at $\phi = 0$ and ω is the frequency. Thus, α_C and α_S are the compressonal and shear attenuation. From eqns (6.1), we have

$$\frac{V'_C}{V_C} = 1 + \frac{\phi}{2}(A + B(ka)^2) \tag{6.8}$$

$$\alpha_C = \frac{N}{V}\frac{\gamma_C}{2} = \phi k^4 a^3\,\frac{g_C}{6} \tag{6.9}$$

$$\frac{V'_S}{V_S} = 1 + \frac{\phi}{2}(E + D(ka)^2) \tag{6.10}$$

$$\alpha_S = \frac{N}{V}\frac{\gamma_S}{2} = \phi k^4 a^3\,\frac{g_S}{6}\,. \tag{6.11}$$

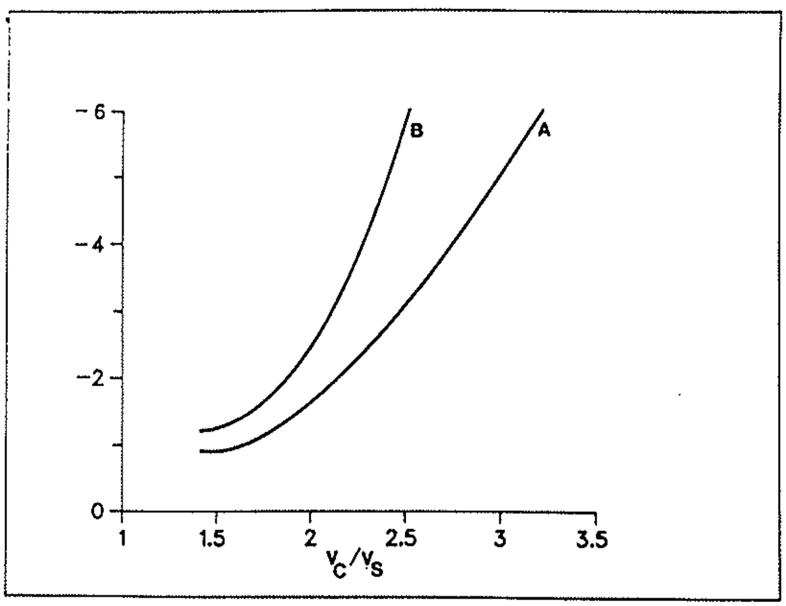

Fig. 2. The constants pertaining to low frequency dispersion of a compressional wave in a dilute suspension of spherical cavities.

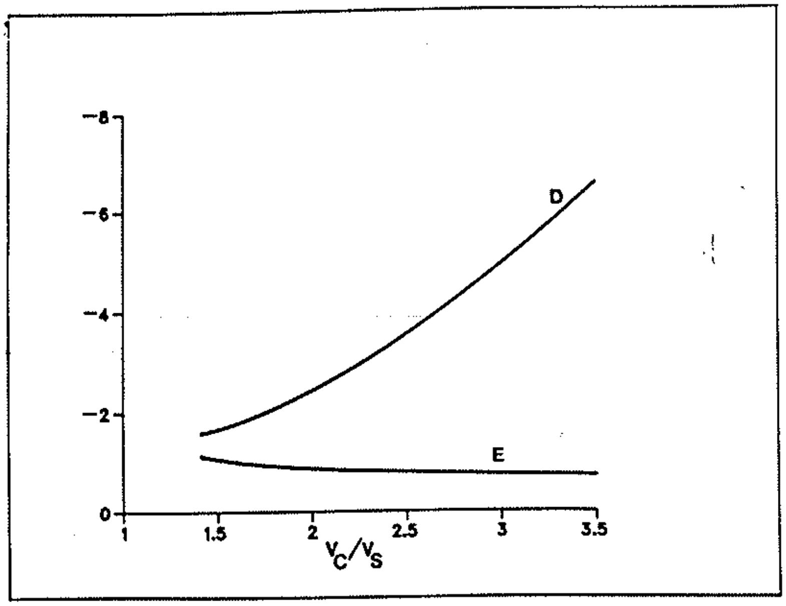

Fig. 3. Same as Fig. 2, but for the effective shear wave speed.

Here,

$$A = \left(1 - \frac{\rho_2}{\rho}\right) + \left(\frac{K_2 - K}{K_2 + K^*}\right) + \frac{4}{3}\left(\frac{\mu_2 - \mu}{K + K^*}\right)\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right) \tag{6.12}$$

$$E = \left(1 - \frac{\rho_2}{\rho}\right) + \left(\frac{\mu_2}{\mu} - 1\right)\left(\frac{\mu + \mu^*}{\mu_2 + \mu^*}\right). \tag{6.13}$$

The constants g_C and g_S are defined in eqns (4.10) and (4.12) and B and D are defined in eqns (5.5) and (5.6) for cavities.

Equations similar to eqns (6.8) and (6.9) have been given by Sayers [14] for the particular case of cavities. We have checked that our eqns agree with his in this case. Equations (6.10) and (6.11) are new. Plots of A, B, E and D are shown in Figs. 2 and 3 for cavities. The ordinate in these figs. is $\kappa/k = V_C/V_S$, which must exceed $\sqrt{2}$. We note that A, B, E and D are all negative. Thus, the speeds decrease with increasing porosity and increasing frequency at low values of each. For example, in aluminum, $V_C/V_S = 2$, which is typical of many metals, we have $A = -13/8$, $B = -883/360$, $E = -7/8$, $D = -1739/720$, so that the wave speeds are

$$\frac{V_c'}{V_c} = 1 - \phi[0.81 + 1.23(ka)^2] \tag{6.14}$$

$$\frac{V_s'}{V_s} = 1 - \phi[0.44 + 1.21(ka)^2]. \tag{6.15}$$

These eqns predict the dispersion of both wave speeds at low frequencies. From Ref. [22] we can expect them to be reasonable for ka less than approximately 0.4. At higher frequencies, alternate theories are to be preferred [22, 23, 24].

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APPENDIX A: SOME FUNCTIONS

The following functions are the coefficients of $\mathbf{P}^{\sigma}_{mn}$, $\mathbf{B}^{\sigma}_{mn}$ and $\mathbf{C}^{\sigma}_{mn}$ in eqns (13.3.67)–(13.3.69) of Ref. [9].

$$a^1_n(x) = j'_n(x) \tag{A1}$$

$$b^1_n(x) = \sqrt{n(n+1)}\,\frac{j_{n(x)}}{x} \tag{A2}$$

$$c^1_n(x) = x b^1_n(x) \tag{A3}$$

$$d^1_n(x) = \sqrt{n(n+1)}\,b^1_n(x) \tag{A4}$$

$$e^1_n(x) = \sqrt{n(n+1)}\,a^1_n(x) + b^1_n(x). \tag{A5}$$

The functions $a^3_n, \ldots, e^3_n$ are given by eqns (A1)–(A5) with $j_n(x)$ replaced by the spherical Hankel functions $h_n(x)$.

The following functions are the coefficients of $\mathbf{P}^{\sigma}_{mn}$, $\mathbf{B}^{\sigma}_{mn}$ and $\mathbf{C}^{\sigma}_{mn}$ in eqns (13.3.78) of Ref. [9]. There is an error in the first of these eqns. The correction is noted in eqn (2.6).

$$\alpha^1_{nj}(kr) = k[2\mu_j j''_n(kr) - \lambda_j j_n(kr)] \tag{A6}$$

$$\beta^1_{nj}(kr) = 2\mu_j k\sqrt{n(n+1)}(kr)^{-2}[kr\,j'_n(kr) - j_n(kr)] \tag{A7}$$

$$\gamma^1_{nj}(kr) = \frac{kr}{2}\,\beta^1_{nj}(kr) \tag{A8}$$

$$\delta^1_{nj}(kr) = \sqrt{n(n+1)}\,\beta^1_{nj}(kr) \tag{A9}$$

$$\epsilon^1_{nj}(kr) = \mu_j k\sqrt{n(n+1)}[j''_n(kr) + (n^2 + n - 2)j_n(kr)/(kr)^2]. \tag{A10}$$

The functions α^3_{nj}---ϵ^3_{nj} are given by eqns (A6)–(A10) by replacing the spherical Bessel functions $j_n(kr)$ by the spherical Hankel functions $h_n(kr)$.