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AN "OPTICAL" THEOREM FOR ACOUSTIC SCATTERING BY BAFFLED FLEXIBLE SURFACES†

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The classical optical theorem for scattering by compact obstacles is a forward scattering theorem. That is, the total cross section of the obstacle is proportional to the imaginary part of the far field directivity factor evaluated in the forward scattering direction. An analogous theorem is derived in this paper for the scattering of acoustic waves by baffled membranes and plates. In this "optical" theorem the directivity factor is evaluated in the direction of the specularly reflected wave, so that it is a reflected scattering theorem.

1. INTRODUCTION

A compact flexible surface, such as a membrane or plate, is set into an infinite, rigid baffle that coincides with the plane $z=0$. The plane separates an acoustic fluid ($z>0$) from a vacuum ($z<0$). A time-harmonic plane wave in the acoustic field is incident on the plane $z=0$. The resulting acoustic pressure $P(\mathbf{x})$ can be expressed as the sum of an incident wave, the specularly reflected plane wave (corresponding to a completely rigid plane), and a scattered pressure $q(\mathbf{x})$ due to the flexible surface. Here $\mathbf{x}$ is the co-ordinate vector with components (x, y, z).

In the far field the scattered wave is spherical and is given by

$$q(\mathbf{x}) = A(\mathbf{e}_I, \hat{\mathbf{r}})(\mathrm{e}^{ikr}/r) + O(1/r^2). \tag{1.1}$$

Here the directivity factor A depends on the unit vectors $\mathbf{e}_I$ and $\hat{\mathbf{r}} \equiv \mathbf{x}/r$, where $r \equiv |\mathbf{x}|$, which are in the propagation direction of the incident plane wave and in the observation direction, respectively. In addition, the dimensionless wave number k is defined by

$$k \equiv \omega l / c_a, \tag{1.2}$$

where ω is the radian frequency of the incident wave, l is a characteristic length of the flexible surface, such as its maximum diameter, and c_a is the acoustic sound speed. The differential scattering cross section σ_d and the total scattering cross section σ_T for the baffled flexible surface are then defined by

$$\sigma_d \equiv |A(\mathbf{e}_I, \hat{\mathbf{r}})|^2, \qquad \sigma_T \equiv \int_0^{2\pi} \int_0^{\pi/2} \sigma_d \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta. \tag{1.3, 1.4}$$

In equation (1.4) the components of $\hat{\mathbf{r}}$ in spherical co-ordinates are employed.

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In this note an "optical" theorem for baffled membranes and plates is derived. For lossless membranes and plates the result is essentially an energy balance statement,

$$\sigma_T = (4\pi/k)\,\mathrm{Im}\,[A(\mathbf{e}_I, \mathbf{e}_R)]. \tag{1.5}$$

Here $\mathbf{e}_R$ is a unit vector in the propagation direction of the specularly reflected wave. The result is analogous to the classical optical theorem for acoustic or optical scattering from rigid, soft or transparent scatterers [1, 2]. However, in the classical optical theorem the vector $\mathbf{e}_R$ is replaced by $\mathbf{e}_I$ so that it is then a forward scattering theorem. In addition, a more general scattering theorem is derived which is valid for membranes and plates with dissipation. The significance of the result (1.5) is that σ_T can be determined experimentally for each $\mathbf{e}_I$ from one measurement (or evaluation) of A in the specularly reflected direction. A similar optical theorem can be derived for more general flexible surfaces, but these results are not presented here.

2. FORMULATION

The acoustic pressure and the flexible surface's motion are assumed to be proportional to $\exp(-i\omega t)$. This time factor is omitted in the subsequent analysis. Dimensionless space variables $\mathbf{x} = (x, y, z)$ are defined by dividing the dimensional variables by a characteristic length l of the flexible surface. Then the acoustic pressure $P(\mathbf{x})$ satisfies the Helmholtz equation

$$\Delta_0 P + k^2 P = 0, \tag{2.1}$$

in the upper half space $z > 0$. Here, k is defined in equation (1.2) and Δ_0 is the Laplacian in $\mathbf{x}$.

The flexible surface is assumed to lie in the compact region M of the plane $z = 0$. The boundary of M is denoted by B. When the flexible surface is a membrane, its lateral deflection $w(x, y)$ satisfies

$$L_m w \equiv \Delta w + k^2 c^2 w = (l^2/T) P(x, y, 0), \quad \text{for } (x, y) \in M, \qquad w = 0, \quad \text{for } (x, y) \text{ on } B. \tag{2.2a}$$

Here, Δ is the Laplacian in x and y, $c \equiv c_a/c_m$, $c_m \equiv (T/\rho_m)^{1/2}$, ρ_m is the density per unit area of the membrane and T is the tension applied to the membrane. The acoustic pressure $P(x, y, 0)$ acts as a driving force on the membrane. When the flexible surface is a clamped plate, then its lateral deflection satisfies

$$L_p w \equiv \Delta^2 w - k^2 b^2 w = -l^4 P(x, y, 0)/D,$$
$$\text{for } (x, y) \in M, \qquad w \equiv \mathbf{n} \cdot \nabla w = 0, \quad \text{for } (x, y) \text{ on } B. \tag{2.2b}$$

In equations (2.2b) $\mathbf{n}$ is the unit outward normal to B in the plane $z = 0$, $b^2 \equiv c_a^2/c_p^2$, $c_p^2 \equiv D/\rho_p h l^2$, $D \equiv Eh^3/12(1-\nu^2)$, h is the thickness of the plate, ν is Poisson's ratio, E is Young's modulus, and ρ_p is the density of the plate. Optical theorems for more general plate equations, such as the Timoshenko-Mindlin theory, can also be obtained by the techniques used in this paper.

Since the plane $z = 0$ is acoustically rigid outside of M,

$$P_z(x, y, 0) = 0, \qquad (x, y) \notin M. \tag{2.3a}$$

Here, the subscript denotes partial differentiation. In addition, the acoustic and flexible surface motions are coupled by the requirement that their vertical velocities are continuous on M. This gives

$$P_z(x, y, 0) = l\omega^2 \rho_a w(x, y), \qquad (x, y) \in M. \tag{2.3b}$$

The incident acoustic field is given by the plane wave

$$P^I(x, y, z) = \exp\,[ik(y \sin \phi_I - z \cos \phi_I)]. \tag{2.4}$$

The unit vector in the propagation direction of the plane wave (2.4) is defined by $\mathbf{e}_I \equiv (0, \sin \phi_I, -\cos \phi_I)$. Without loss of generality, the x, y plane is oriented so that $\mathbf{e}_I$ is orthogonal to the x axis. The wave given by (2.4) is a solution of equation (2.1). The total acoustic pressure in $z > 0$ is expressed as

$$P(\mathbf{x}) = P_0(\mathbf{x}) + q(\mathbf{x}), \tag{2.5}$$

where P_0 is defined by

$$P_0(\mathbf{x}) = P^I(\mathbf{x}) + P^I(x, y, -z), \tag{2.6}$$

$P^I(x, y, -z)$ is the specularly reflected wave from the rigid plane $z = 0$, and q is the field scattered by the flexible surface. Thus, q satisfies equations (2.1) and (2.3) and it must satisfy the radiation condition as $r \to \infty$. In addition, P_0 satisfies equation (2.1) and

$$(\partial P_0/\partial z)(x, y, 0) = 0, \quad \text{for all } (x, y). \tag{2.7}$$

3. FUNDAMENTAL IDENTITIES

Any solution $\psi(x, y, z)$ of equation (2.1) satisfies

$$\nabla \cdot (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) = 0, \tag{3.1}$$

where the overbar denotes complex conjugation. Integrating this identity over the region S, which is bounded by the hemisphere $r = R$, $z > 0$ and the circle $x^2 + y^2 = R^2$, $z = 0$, and applying the divergence theorem to this integral yields the "energy flux" balance,

$$R^2 \operatorname{Im}\left\{ \underset{r=R}{\int_0^{2\pi} \int_0^{\pi/2}} \psi \frac{\partial \bar{\psi}}{\partial r} \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta \right\} = \operatorname{Im}\left\{ \underset{\substack{x^2+y^2 \leqslant R^2 \\ z=0}}{\iint} \psi \frac{\partial \bar{\psi}}{\partial z} \, \mathrm{d}x \, \mathrm{d}y \right\}. \tag{3.2}$$

Setting $\psi = P$ in (3.2) and using equations (2.3a), (2.5) and (2.6) yields

$$R^2 \operatorname{Im}\left\{ \underset{r=R}{\int_0^{2\pi} \int_0^{\pi/2}} [P_0 \bar{P}_{0,r} + q\bar{q}_r + P_0 \bar{q}_r + q \bar{P}_{0,r}] \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta \right\}$$

$$= \operatorname{Im}\left\{ \iint_M P(x, y, 0) \bar{q}_z(x, y, 0) \, \mathrm{d}x \, \mathrm{d}y \right\}. \tag{3.3}$$

In deriving equation (3.3) it has been assumed that R is sufficiently large so that M is contained in the circle $x^2 + y^2 \leqslant R^2$. Then, applying equations (2.7) and (3.2) with $\psi \equiv P_0$, and using the result on the left side of equation (3.3), it can be shown that the integral involving $P_0 \bar{P}_{0,r}$ is identically zero. This gives

$$R^2 \operatorname{Im}\left\{ \int_0^{2\pi} \int_0^{\pi/2} [q\bar{q}_r + P_0 \bar{q}_r + q \bar{P}_{0,r}] \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta \right\} = \operatorname{Im} \iint_M P(x, y, 0) \bar{q}_z(x, y, 0) \, \mathrm{d}x \, \mathrm{d}y. \tag{3.4}$$

Inserting the far field approximation (1.1) in equation (3.4), and then, recalling that q satisfies equation (2.3b), gives

$$-k\sigma_T+\mathrm{Im}\left\{R^2\int_0^{2\pi}\int_0^{\pi/2}[P_0\bar{q}_r+qP_{0,r}]\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\right\}+O\left(\frac{1}{R}\right)$$

$$=\omega^2 l\rho_a\,\mathrm{Im}\left\{\iint_M P(x,y,0)\bar{w}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right\}, \tag{3.5}$$

where σ_T is defined in equation (1.4).

The right side of expression (3.5) vanishes if w satisfies either equation (2.2a) or equation (2.2b). To show this first multiply the differential equations in equations (2.2) by $\bar{w}$, and then integrate these expressions over the region M. Applying the divergence theorem the necessary number of times, and using the boundary conditions in equations (2.2), yields for the membrane and plate, respectively,

$$\iint_M P(x,y,0)\bar{w}(x,y)\,\mathrm{d}x\,\mathrm{d}y=\frac{T}{l^2}\iint_M[k^2c^2|w|^2-|\nabla w|^2]\,\mathrm{d}x\,\mathrm{d}y, \tag{3.6a}$$

$$\iint_M P(x,y,0)\bar{w}(x,y)\,\mathrm{d}x\,\mathrm{d}y=\frac{D}{l^4}\iint_M[k^2b^2|w|^2-|\Delta w|^2]\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6b}$$

Since the right sides of equations (3.6) are real, the right side of equation (3.5) vanishes and it follows from equation (3.5) that

$$k\sigma_T=\mathrm{Im}\,J+O(1/R), \tag{3.7}$$

where J is defined by

$$J=R^2\int_0^{2\pi}\int_0^{\pi/2}[P_0\bar{q}_r+q\bar{P}_{0,r}]\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\Big|_{r=R}. \tag{3.8}$$

If the sound speed ratios c and b are complex numbers so that the membrane and plate materials are dissipative, then a similar analysis applied to equations (3.6) and (3.5) yields

$$\mathrm{Im}\left\{\iint_M P(x,y,0)\bar{w}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right\}\equiv\Gamma\|w\|^2, \tag{3.9}$$

$$\|w\|^2\equiv\iint_M|w|^2\,\mathrm{d}x\,\mathrm{d}y,\qquad \Gamma\equiv(k/l)^2\begin{Bmatrix}T\,\mathrm{Im}\,c^2 & \text{for the membrane}\\ (D/l^2)\,\mathrm{Im}\,b^2 & \text{for the plate}\end{Bmatrix}. \tag{3.10}$$

Then the equation for σ_T corresponding to equation (3.7) is

$$k\sigma_T=\mathrm{Im}\,J+\omega^2 l\rho_a\Gamma\|w\|^2+O(1/R). \tag{3.11}$$

4. ASYMPTOTIC EVALUATION OF THE INTEGRAL J

The integral J is evaluated by first substituting equations (2.4), (2.6), and (1.1) into equation (3.8) to obtain

$$J=-\mathrm{i}kR\int_0^{2\pi}\int_0^{\pi/2}[(\mathbf{e}_I\cdot\hat{\mathbf{r}})A\,\mathrm{e}^{\mathrm{i}kR\psi_I}+(\mathbf{e}_R\cdot\hat{\mathbf{r}})A\,\mathrm{e}^{\mathrm{i}kR\psi_R}+\bar{A}\,\mathrm{e}^{-\mathrm{i}kR\psi_I}+\bar{A}\,\mathrm{e}^{-\mathrm{i}kR\psi_R}]$$
$$\times[1+O(1/R)]\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta, \tag{4.1}$$

where $\mathbf{e}_R\equiv(0,\sin\phi_I,\cos\phi_I)$ is the unit vector in the direction of the specularly reflected plane wave. The phase functions ψ_I and ψ_R are defined by

$$\psi_I\equiv 1-\sin\phi_I\sin\theta\sin\phi+\cos\phi_I\cos\phi,\qquad \psi_R\equiv 1-\sin\phi_I\sin\theta\sin\phi-\cos\phi_I\cos\phi. \tag{4.2a, b}$$

Since $R\to\infty$ the method of stationary phase [3-5] is used to evaluate J asymptotically. For the integrals in equation (4.1) which contain $\exp[\pm\mathrm{i}kR\psi_R]$ the major contributions come from the stationary points of ψ_R: i.e., from the roots of

$$\partial\psi_R/\partial\phi=\partial\psi_R/\partial\theta\equiv 0 \tag{4.3}$$

in the rectangle $\Omega=\{(\theta,\phi)|0\leqslant\theta\leqslant 2\pi, 0\leqslant\phi\leqslant\pi/2\}$. An analysis of equations (4.3) shows that ψ_R has three stationary points. The first is $\phi=0$, $\theta=0$, which lies on the boundary of Ω, and hence has a contribution [4] of $O(1/R^{3/2})$ as $R\to\infty$. The second point is $\phi=\pi-\phi_I$, $\theta=3\pi/2$ which is exterior to Ω. Its contribution is at most [4] $O(1/R^2)$ as $R\to\infty$. The third and most important point is $\theta=\pi/2$, $\phi=\phi_I$ which gives $\hat{\mathbf{r}}\equiv\mathbf{e}_R$. The contribution [3] of this point is $O(1/R)$ as $R\to\infty$ which dominates the effects of the other two stationary points. It follows from the two dimensional stationary phase formula that

$$J_1\equiv\iint_\Omega[(\mathbf{e}_R\cdot\hat{\mathbf{r}})A\,\mathrm{e}^{\mathrm{i}kR\Psi_R}+\bar{A}\,\mathrm{e}^{-\mathrm{i}kR\Psi_R}][1+O(1/R)]\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$$
$$=-(4\pi/kR)\,\mathrm{Im}\,\{A(\mathbf{e}_I,\mathbf{e}_R)\}+O(1/R^{3/2}). \tag{4.4}$$

A similar analysis holds for the integrals in equation (4.1) which contain $\exp[\pm\mathrm{i}kR\Psi_I]$. Again, there are three stationary points with the same qualitative properties as described above. The dominant interior point is at $\theta=3\pi/2$, $\phi=\phi_I$. This corresponds to $\hat{\mathbf{r}}=-\mathbf{e}_I$ or the backscattered direction. Thus, the stationary phase formula gives

$$J_2\equiv\iint_\Omega[(\mathbf{e}_I\cdot\hat{\mathbf{r}})A\sin\phi\,\mathrm{e}^{\mathrm{i}kR\Psi_I}+\bar{A}\sin\phi\,\mathrm{e}^{-\mathrm{i}kR\Psi_I}][1+O(1/R)]\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$$
$$=-(4\mathrm{i}\pi/kR)\,\mathrm{Re}\,\{A(\mathbf{e}_I,-\mathbf{e}_I)\,\mathrm{e}^{2\mathrm{i}kR}\}+O(1/R^{3/2}). \tag{4.5}$$

By substituting expressions (4.4) and (4.5) into equation (4.1) an asymptotic evaluation is obtained for J as $R\to\infty$. Then by inserting this result into equations (3.7) and (3.11) it follows that,

$$\sigma_T=(4\pi/k)\,\mathrm{Im}\,\{A(\mathbf{e}_I,\mathbf{e}_R)\} \tag{4.6}$$

for the lossless membrane and plate and

$$\sigma_T=(4\pi/k)\,\mathrm{Im}\,\{A(\mathbf{e}_I,\mathbf{e}_R)\}+\omega\rho_a c_a\Gamma\|w\|^2 \tag{4.7}$$

for the dissipative membrane and plate. These are the required optical theorems.

Since the acoustic fluid and flexible surface are coupled, the scattering problem cannot be solved in a closed analytic form, even for simple geometries. For example, solving the coupled problem by the method of normal modes [6, 7] yields an infinite system of linear algebraic equations which in general cannot be solved explicitly without truncation. Thus, an explicit expression for the directivity factor, A, cannot be obtained. This makes the results (4.6)-(4.7) particularly useful since σ_T can be evaluated for each $\mathbf{e}_I$ from one measurement of A in the direction of $\mathbf{e}_R$.

Numerical, variational [7], and asymptotic [8] methods can be employed to solve the scattering problem approximately. In particular, we have recently developed an asymptotic technique for "heavy" membranes of arbitrary shape using the method of matched asymptotic expansions [8]. Specifically, we have obtained asymptotic approximations for the directivity factor and the total cross section which are uniformly valid in the frequency of the incident wave. For frequencies near simple eigenfrequencies, k_n, of the membrane *in vacuo*, these approximations are reduced to

$$A(\phi, \theta) = A_n F(\phi, \theta), \qquad \sigma_T = |A_n|^2 I/k_n, \tag{4.8, 4.9}$$

where the amplitude A_n and the "shape factor" $F(\phi, \theta)$ are defined by

$$A_n = \frac{2\beta_n}{(2\alpha + R) + \mathrm{i}I}, \qquad F(\phi, \theta) = \frac{-1}{2\pi} \iint_M \mathrm{e}^{-\mathrm{i}k_n(\hat{\mathbf{r}}\cdot\mathbf{x})}\, \psi_n(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{4.10, 4.11}$$

Here R is a positive quantity that is defined in reference [8], and β_n and I are defined by

$$\beta_n = \iint_M P^I(x, y, 0)\psi_n(x, y)\, \mathrm{d}x\, \mathrm{d}y, \qquad I = k_n \iint_M |F(\phi, \theta)|^2 \sin\phi\, \mathrm{d}\phi\, \mathrm{d}\theta. \tag{4.12a, b}$$

In equation (4.11) the notation $\mathbf{x} = (x, y, 0)$ and $\hat{\mathbf{r}} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ is used. The parameter α used in equation (4.10) is defined by

$$\alpha = (k - k_n)/\varepsilon k_n, \tag{4.13}$$

where $\varepsilon \equiv l\rho_a/\rho_m$ and $k_n = \omega_n l/c_a$. The function $\psi_n(x, y)$ is the eigenmode of the membrane *in vacuo* which corresponds to the frequency ω_n.

The results stated in equations (4.8)-(4.13) are valid when $\varepsilon \ll 1$, which corresponds to a "heavy" membrane, and $\alpha = O(1)$, which implies that k is near k_n. The optical theorem (4.6) can be verified for this physical situation. By applying equations (4.8)-(4.13) and observing that $F(\phi_I, \pi/2) \equiv -\beta_n/2\pi$, an asymptotic expression for σ_T is obtained. It is equal to the value of σ_T given by the optical theorem evaluated for $\varepsilon \ll 1$ and k near k_n, as is expected. In addition, similar agreement is obtained by using our asymptotic results [8] that are valid for k bounded away from k_n and $\varepsilon \ll 1$. We omit all the details of these calculations.

5. EXTENSIONS

We have extended the present analysis to other scattering problems where the flexible surface is no longer backed by a vacuum. For example, if a membrane or plate is backed by a rigid cavity filled with an acoustic fluid, then the optical theorems (4.6)-(4.7) remain valid. Now, however, A is the directivity factor of the flexible surface-cavity-baffle system. If the entire half space $z < 0$ is filled with an acoustic fluid of density ρ_b and sound speed

c_b, the optical theorems (4.6) and (4.7) are again valid when σ_T is replaced by $\hat{\sigma}_T$, which is defined by

$$\hat{\sigma}_T = \int_0^{2\pi} \int_0^{\pi} |A|^2 \nu(\phi) \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta. \tag{5.1}$$

Here the function $\nu(\phi)$ is defined by

$$\nu(\phi) \equiv \left\{ \begin{matrix} 1, & 0 \leq \phi \leq \pi/2 \\ c_a\rho_b/c_b\rho_a, & \pi/2 < \phi \leq \pi \end{matrix} \right\}. \tag{5.2}$$

REFERENCES

1. A. ISHIMARU 1978 *Wave Propagation and Scattering in Random Media*, Volume 1. New York: Academic Press.
2. M. BORN and E. WOLF 1964 *Principles of Optics*. New York: Macmillan.
3. D. S. JONES and M. KLEIN 1958 *Journal of Mathematical Physics* **37**, 1–28. Asymptotic expansion of multiple integrals and the method of stationary phase.
4. L. SIROVICH 1971 *Techniques of Asymptotic Analysis*. New York: Springer-Verlag.
5. N. S. BLEISTEIN and R. A. HANDELSMAN 1975 *Asymptotic Evaluation of Integrals*. New York: Holt, Rinehart, and Winston.
6. B. E. SANDMAN and J. VIEIRA 1975 *Proceedings of the 14th Midwestern Mechanics Conference* **8**, 295–307. The experimental and theoretical dynamics of fluid-loaded plates.
7. P. M. MORSE and K. U. INGARD 1968 *Theoretical Acoustics*. New York: McGraw-Hill.
8. G. A. KRIEGSMANN, A. N. NORRIS and E. L. REISS 1984 *Journal of the Acoustical Society of America* **75**, 685–694. Acoustic scattering by baffled membranes.