

Radiation from a point source and scattering theory in a fluid-saturated porous solid

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The time harmonic Green function for a point load in an unbounded fluid-saturated porous solid is derived in the context of Biot's theory. The solution contains the two compressional waves and one transverse wave that are predicted by the theory and have been observed in experiments. At low frequency, the slow compressional wave is diffusive and only the fast compressional and transverse waves radiate energy. At high frequency, the slow wave radiates, but with a decay radius which is on the order of cm in rocks. The general problem of scattering by an obstacle is considered. The point load solution may be used to obtain scattered fields in terms of the fields on the obstacle. Explicit expressions are presented for the scattering amplitudes of the three waves. Simple reciprocity relations between the scattering amplitudes for plane-wave incidence are also given. These hold under the interchange of incident and observation directions and are completely general results. Finally, the point source solution is Fourier transformed to get the solution for a load which is a delta function in time as well as space. We obtain a closed form expression when there is no damping. The three waves radiate from the source as distinct delta function pulses. With damping present, asymptotic approximations show the slow wave to be purely diffusive. The fast and transverse waves propagate as pulses. The pulses are Gaussian-shaped, which broaden with increasing time or radial distance.

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INTRODUCTION

We consider the problem of wave propagation in fluid-saturated porous media in the context of the Biot¹⁻³ theory. This theory has recently been firmly established by the experimental observation of the Biot slow wave.⁴ The slow wave is a consequence of the fluid-solid coupling in the Biot equations. The theory also predicts a fast compressional wave and a shear wave. These are the analogs of the longitudinal and transverse waves of elasticity. A theoretical justification for Biot's equations has been provided by Burridge and Keller.⁵

In this paper, we consider the general topics of radiation and scattering in a porous medium. The fundamental radiation problem involves an applied time harmonic point load. The solution, which is the Green function for the dynamic Biot equations, is obtained in Sec. II in terms of three potentials, one for each kind of wave. This Green function reduces to the well-known elastic point load solution in the limit of zero porosity. Our solution has the form of three radiating waves. However, the attenuation of each wave depends very much on the frequency of the applied load. The attenuation enters through the inclusion of a damping term in the original equations proportional to the difference of solid and fluid velocities. Biot^{1,2} and Geertsma and Smit,⁶ among others, have discussed the frequency dependence of the wave speeds and attenuation. In Sec. III, we examine the behavior of the point load solution as a function of ω , in particular for low and high frequencies relative to a critical frequency ω_c . Typi-

cally, $\omega_c = 0$ (10^6 cps) in rocks, so that the low-frequency results are applicable to geophysical problems. The slow wave is then diffusive. The high-frequency results are applicable to ultrasonic experiments,⁴ in which case the slow wave is propagatory.

The point load solution is then (in Sec. IV) used to derive representation integrals for the radiation from an arbitrary distribution of body forces and surface tractions. The general problem of scattering in a porous solid is considered in Sec. V. Our main result is a reciprocity relationship for scattering amplitudes from an arbitrary obstacle or inhomogeneity. Finally, in Sec. VI, we Fourier transform our Green function to obtain the solution for an applied load which is a delta function in time as well as space. This solution has been derived by Burridge and Vargas⁷ using Laplace transforms. However, our method is less cumbersome than that in Ref. 7. We also present a new result for the case of zero damping. The solution can then be expressed in closed form.

We begin by introducing the Biot equations and the three wave solutions.

I. BASIC EQUATIONS

We will use the original notation of Biot^{1,2} as much as possible. We have the stress-strain relations

$$\sigma = (P - 2N)\nabla \cdot \mathbf{u}\mathbf{I} + N[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + Q\nabla \cdot \mathbf{U}\mathbf{I}, \quad (1)$$

$$s = Q\nabla \cdot \mathbf{u} + R\nabla \cdot \mathbf{U}, \quad (2)$$

and the equations of motion

$$\rho_{11} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \rho_{12} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla \cdot \sigma + b \frac{\partial}{\partial t} (\mathbf{U} - \mathbf{u}), \quad (3)$$

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$$\rho_{12} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \rho_{22} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla s - b \frac{\partial}{\partial t} (\mathbf{U} - \mathbf{u}). \quad (4)$$

Here the vectors $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{U}(\mathbf{x}, t)$ are the solid and fluid displacements, respectively; σ is the solid stress tensor; s is a scalar proportional to the fluid pressure; and $\mathbf{I}$ is the second-order identity tensor. The shear modulus N is the same as the shear modulus of the dry frame. In addition, there are relations between the Biot elastic constants P , Q , R , N ; the bulk moduli κ_s and κ_f of the solid and fluid; and a new constant κ_b , the "jacketed" bulk modulus.⁸⁻¹⁰ We have

$$1/\kappa_s = [(1 - \phi)R - \phi Q]/(\kappa R - Q^2), \quad (5)$$

$$1/\kappa_f = [\phi\kappa - (1 - \phi)Q]/(\kappa R - Q^2), \quad (6)$$

$$1/\kappa_b = R/(\kappa R - Q^2), \quad (7)$$

where

$$\kappa = P - \frac{4}{3}N \quad (8)$$

and ϕ is the porosity or volume fraction of the fluid phase. These equations are easily inverted to give P , Q , and R in terms of κ_s , κ_f , κ_b , N , and ϕ .⁸

The densities ρ_{ij} in Eqs. (3) and (4) represent the inertia of the two phases. They are related to the fluid and solid densities ρ_f and ρ_s by¹

$$\rho_{11} + \rho_{12} = (1 - \phi)\rho_s \equiv \rho_1, \quad (9)$$

$$\rho_{12} + \rho_{22} = \phi\rho_f \equiv \rho_2. \quad (10)$$

The coupling mass $\rho_{12} < 0$ is sometimes written as

$$\rho_{12} = -(T - 1)\phi\rho_f, \quad (11)$$

where the tortuosity $T > 1$ depends upon the pore geometry.

The parameter b in Eqs. (3) and (4) represents the resistive damping due to relative motion between the fluid and solid. It incorporates viscous dissipation into the equations of motion and therefore is a major source of attenuation in waves. It is common in the literature to use $bF(\omega)$ instead of b in (3) and (4). The frequency-dependent function $F(\omega)$ was introduced by Biot² to account for the dissipation when the pore size is of the order of the viscous skin depth or smaller. However, the form of $F(\omega)$ is open to question, since in general it is a complicated viscodynamic operator. At low frequencies, such that

$$\omega < \nu/a^2, \quad (12)$$

where ν is the kinematic viscosity and a is a typical pore dimension, we have² $F(\omega) \approx 1$. With $a = 100 \mu\text{m}$, which is typical of rocks, the right-hand side of (12) is 10^6 cps for water-filled rocks. We will therefore take $F(\omega) \equiv 1$ in this paper. We note that our b is the same as in Ref. 1, but is ϕ^2 times the b in Ref. 7.

We now consider time harmonic motion with real radial frequency ω . Omitting the term $e^{-i\omega t}$ from all quantities, the equations of motion become

$$\omega^2(\tilde{\rho}_{11}\mathbf{u} + \tilde{\rho}_{12}\mathbf{U}) + \nabla \cdot \sigma = 0, \quad (13)$$

$$\omega^2(\tilde{\rho}_{12}\mathbf{u} + \tilde{\rho}_{22}\mathbf{U}) + \nabla s = 0, \quad (14)$$

where

$$\tilde{\rho}_{mn} = \rho_{mn} + (-1)^{m+n}i(b/\omega). \quad (15)$$

Thus we note that the dissipation only appears in the equations through the inertial terms.

We conclude Sec. I by describing the three kinds of traveling wave solutions to the coupled systems (13), (14), (1), and (2). The following is basically a summary of the results of Biot in his original paper.¹ The reader is referred to that paper for a full description. We shall call the two dilatational waves the fast (F) and the slow (S) waves, and the equivoluminal wave we call the transverse (T) wave. A plane wave of type α , $\alpha = F, S$, or T can be expressed as

$$(\mathbf{u}, \mathbf{U}) = (\mathbf{m}, A_\alpha \mathbf{m}) \exp(ik_\alpha \mathbf{n} \cdot \mathbf{x}), \quad |\mathbf{n}| = 1, \quad (16)$$

where

$$\mathbf{m} \parallel \mathbf{n} \text{ if } \alpha = F \text{ or } S, \quad \mathbf{m} \perp \mathbf{n} \text{ if } \alpha = T,$$

the quantity A_α is the ratio of fluid to solid displacement for each wave type, and k_α is the corresponding wavenumber. The phase speed is

$$v_\alpha = \omega/k_\alpha, \quad \alpha = F, S, T. \quad (17)$$

Expressions for A_α and v_α are given in Appendix A.

In general, when there is viscous dissipation ($b \neq 0$), the quantities A_α and v_α are complex valued functions of frequency. The slow wave is strongly attenuated, particularly at low frequency, where it behaves as a diffusion process. We will discuss the frequency dependence of these quantities in greater detail in Sec. III.

The Biot equations are valid insofar as the wavelengths of the propagating waves are much larger than the pore size. Otherwise scattering effects become important and a different theory is necessary. The long wavelength criterion is

$$\omega \ll \nu/a, \quad (18)$$

where ν is the wave speed and a the pore size. Typically, $\nu = 10^5$ cm/s and $a = 100 \mu\text{m}$, for which the right-hand side of (18) is 10^7 cps. Thus the inequality (18) is always satisfied.

II. THE POINT LOAD SOLUTION

Let a point force of unit magnitude act at the vector position $\mathbf{x}'$ in the direction $\mathbf{d}$ in the solid phase. The resulting displacement fields $\mathbf{u}$ and $\mathbf{U}$ may be represented by two second-order tensors $\mathbf{u}^G(\mathbf{x}; \mathbf{x}')$ and $\mathbf{U}^G(\mathbf{x}; \mathbf{x}')$ as

$$u_i(\mathbf{x}) = u_{ij}^G(\mathbf{x}; \mathbf{x}')d_j, \quad (19)$$

$$U_i(\mathbf{x}) = U_{ij}^G(\mathbf{x}; \mathbf{x}')d_j, \quad (20)$$

or in vector notation

$$\mathbf{u} = \mathbf{u}^G \cdot \mathbf{d}, \quad (21)$$

$$\mathbf{U} = \mathbf{U}^G \cdot \mathbf{d}. \quad (22)$$

[Note: Do not confuse the semicolon subscripts in (19) and (20) with covariant differentiation.] The equations for $\mathbf{u}^G$ and $\mathbf{U}^G$ are, in vector notation,

$$\omega^2(\tilde{\rho}_{11}\mathbf{u}^G + \tilde{\rho}_{12}\mathbf{U}^G) + P\nabla\nabla \cdot \mathbf{u}^G + Q\nabla\nabla \cdot \mathbf{U}^G + N\nabla \wedge \nabla \wedge \mathbf{u}^G = -\mathbf{I}\delta(\mathbf{x} - \mathbf{x}'), \quad (23)$$

$$\omega^2(\tilde{\rho}_{12}\mathbf{u}^G + \tilde{\rho}_{22}\mathbf{U}^G) + Q\nabla\nabla \cdot \mathbf{u}^G + R\nabla\nabla \cdot \mathbf{U}^G = 0, \quad (24)$$

where δ is the Dirac delta function. The solution to Eqs. (23) and (24) is uniquely specified once we impose a radiation condition at infinity and a regularity condition at the origin.

The former requires that the fields u^G and U^G be composed of outgoing waves far from the origin. The regularity condition requires that the behavior of u^G and U^G near the origin should be less singular than that of the applied load there. The formulation of the problem is now complete.

The related problem of determining the time-dependent Green function has been considered by Burrige and Vargas.⁷ We note that Eqs. (23) and (24) are exactly equivalent to the Fourier transform of Eq. (1.1.1) in Ref. 7, with the additional substitution $\mathbf{F} \rightarrow -\mathbf{I}$ in (24). Thus the time-dependent Green function may be obtained from u^G and U^G by taking the inverse Fourier transform. In Ref. 7, the authors also obtain a solution in terms of an inverse transform, in their case the Laplace transform. However, the kernel of their inverse transform is not given explicitly in terms of simple functions [see Eq. (2.3.7) of Ref. 7]. We will show that it is possible to express u^G and U^G , and thus the kernel of the inverse Laplace transform in Ref. 7 in relatively simple forms. The time-dependent problem is discussed in Sec. VI.

We start by representing the solution in terms of three scalar potential functions ψ_α , $\alpha = F, S$, and T as follows:

$$u^G = \nabla \nabla \psi_F + \nabla \nabla \psi_S + (\nabla^2 \psi_T \mathbf{I} - \nabla \nabla \psi_T), \quad (25)$$

$$U^G = A_F \nabla \nabla \psi_F + A_S \nabla \nabla \psi_S + A_T (\nabla^2 \psi_T \mathbf{I} - \nabla \nabla \psi_T). \quad (26)$$

We also note the identity

$$-\delta(\mathbf{x})\mathbf{I} = \nabla \nabla (1/4\pi r) + (\nabla^2 \mathbf{I} - \nabla \nabla)(1/4\pi r), \quad (27)$$

where $r = |\mathbf{x}|$, and without loss of generality, we have taken $\mathbf{x}' = 0$. Substituting from Eqs. (25)–(27) into Eqs. (23) and (24), we obtain

$$\begin{aligned} \nabla \nabla \{ [\omega^2(\bar{\rho}_{11} + \bar{\rho}_{12}A_F)\psi_F + (P + QA_F)\nabla^2 \psi_F] \\ \nabla \nabla \{ [\omega^2(\bar{\rho}_{11} + \bar{\rho}_{12}A_F)\psi_F + (P + QA_F)\nabla^2 \psi_F] \\ + (\nabla^2 \mathbf{I} - \nabla \nabla) \{ \omega^2(\bar{\rho}_{11} + \bar{\rho}_{12}A_T)\psi_T \\ + N\nabla^2 \psi_T - (1/4\pi r) \} \} = 0, \end{aligned} \quad (28)$$

$$\begin{aligned} \nabla \nabla \{ [\omega^2(\bar{\rho}_{12} + \bar{\rho}_{22}A_F)\psi_F + (Q + RA_F)\nabla^2 \psi_F] \\ + [\omega^2(\bar{\rho}_{12} + \bar{\rho}_{22}A_S)\psi_S + (Q + RA_S)\nabla^2 \psi_S] \} = 0. \end{aligned} \quad (29)$$

Three equations for ψ_F , ψ_S , and ψ_T are obtained if we put the terms in curly braces to zero separately. We also note that the solutions must be spherically symmetric. Thus for $\alpha = F, S$, or T , we have

$$\nabla^2 \psi_\alpha = (r\psi_\alpha)''/r, \quad (30)$$

where the primes denote differentiation with respect to r . Our three equations are

$$\begin{aligned} [\omega^2(\bar{\rho}_{11} + \bar{\rho}_{12}A_F)(r\psi_F) + (P + QA_F)(r\psi_F)'] \\ [\omega^2(\bar{\rho}_{11} + \bar{\rho}_{12}A_F)(r\psi_F) + (P + QA_F)(r\psi_F)'] \\ [\omega^2(\bar{\rho}_{11} + \bar{\rho}_{12}A_F)(r\psi_F) + (P + QA_F)(r\psi_F)'] \end{aligned} \quad (31)$$

$$\begin{aligned} [\omega^2(\bar{\rho}_{12} + \bar{\rho}_{22}A_F)(r\psi_F) + (Q + RA_F)(r\psi_F)'] \\ + [\omega^2(\bar{\rho}_{12} + \bar{\rho}_{22}A_S)(r\psi_S) \\ + (Q + RA_S)(r\psi_S)'] = 0, \end{aligned} \quad (32)$$

$$\omega^2(\bar{\rho}_{12} + \bar{\rho}_{22}A_T)(r\psi_T) + N(r\psi_T)' = 1/4\pi. \quad (33)$$

Consider Eq. (33) first. It is an inhomogeneous equation in a single unknown. The solution can be written as the sum of

the particular solution of the inhomogeneous equation, plus the general solution to the homogeneous equation. The latter must satisfy the radiation condition at infinity. Thus we have, from Eqs. (33), (17), (A1), and (A2)

$$\psi_T = B_T G_T(r) + 1/Nk_T^2 4\pi r, \quad (34)$$

where

$$G_\alpha(r) = e^{ik_\alpha r}/4\pi r, \quad \alpha = F, S, T. \quad (35)$$

The first term on the right-hand side of Eq. (34) is the general solution with B_T a constant still to be determined, and the second term is the particular solution. The constant B_T is found from the regularity condition at the origin. Referring to Eqs. (25), (26), (34), and (35), we observe that the term $(\nabla^2 \mathbf{I} - \nabla \nabla)\psi_T$ is less singular than a delta function, if and only if,

$$B_T = -1/Nk_T^2. \quad (36)$$

Therefore, ψ_T is bounded at the origin and is given by

$$\psi_T = (1/Nk_T^2)[1/4\pi r - G_T(r)]. \quad (37)$$

We now consider the two coupled inhomogeneous equations (31) and (32). Again, the solution ψ_F and ψ_S can be written as the sums of particular plus general solutions. We find, after a little algebra and with reference to Appendix A, that

$$\psi_F = B_F G_F(r) + (1/Nk_T^2 4\pi r)[(A_S - A_T)/(A_S - A_F)], \quad (38)$$

$$\psi_S = B_S G_S(r) + (1/Nk_T^2 4\pi r)[(A_F - A_T)/(A_F - A_S)]. \quad (39)$$

The second terms on the right-hand sides of (38) and (39) are the particular solutions and the first terms the general solutions. The constants B_F and B_S are determined by the regularity condition. This amounts to requiring that ψ_F and ψ_S be bounded at the origin, and hence

$$B_F = (-1/Nk_T^2)[(A_S - A_T)/(A_S - A_F)], \quad (40)$$

$$B_S = (-1/Nk_T^2)[(A_F - A_T)/(A_F - A_S)]. \quad (41)$$

The Green tensor u^G now follows from Eqs. (25), (34), and (38)–(41) and the result

$$\nabla^2 \psi_T = (1/N)G_T(r) \quad (42)$$

as

$$\begin{aligned} u^G = (1/Nk_T^2) \{ k_T^2 G_T(r) \mathbf{I} + \nabla \nabla [G_T(r) \\ - D_F G_F(r) - D_S G_S(r)] \}, \end{aligned} \quad (43)$$

where we have defined D_F and D_S as

$$D_F \equiv (A_S - A_T)/(A_S - A_F), \quad (44)$$

$$D_S \equiv (A_F - A_T)/(A_F - A_S),$$

and for future reference we define D_T as

$$D_T \equiv D_F + D_S = 1. \quad (45)$$

The Green tensor U^G now follows from Eqs. (26) and (43) as

$$\begin{aligned} U^G = (1/Nk_T^2) \{ k_T^2 A_T G_T(r) \mathbf{I} + \nabla \nabla [A_T G_T(r) \\ - A_F D_F G_F(r) - A_S D_S G_S(r)] \}. \end{aligned} \quad (46)$$

When the point of application of the load is not at the origin, then $\mathbf{x}' \neq 0$ and the Green tensors are as above, but with $|\mathbf{x} - \mathbf{x}'|$ substituted for r . We define the Green stress tensors $\sigma_{ijk}^G(\mathbf{x}; \mathbf{x}')$ and $s_{jk}^G(\mathbf{x}, \mathbf{x}')$ through Eqs. (1) and (2). It is

evident from Eqs. (43) and (46) that u^G and U^G possess the symmetries

$$u_{i;k}^G(\mathbf{x};\mathbf{x}') = u_{k;i}^G(\mathbf{x};\mathbf{x}') = u_{i;k}^G(\mathbf{x}';\mathbf{x}), \quad (47)$$

$$U_{i;k}^G(\mathbf{x};\mathbf{x}') = U_{k;i}^G(\mathbf{x};\mathbf{x}') = U_{i;k}^G(\mathbf{x}';\mathbf{x}). \quad (48)$$

These are reciprocity relations satisfied by the Green displacement tensors. The stress tensors satisfy

$$\sigma_{ij;k}^G(\mathbf{x};\mathbf{x}') = -\sigma_{ij;k}^G(\mathbf{x}';\mathbf{x}), \quad (49)$$

$$s_{i;k}^G(\mathbf{x};\mathbf{x}') = -s_{i;k}^G(\mathbf{x}';\mathbf{x}). \quad (50)$$

The displacement and stress tensors discussed thus far have been for a point load applied in the solid part of the porous medium. The solution for a load which is applied in the fluid follows from Eqs. (23) and (24) by removing the delta function in (23) and putting it in (24). Omitting the analysis, the result is

$$u_k(\mathbf{x}) = U_{j;k}^G(\mathbf{x};\mathbf{x}'); \quad (51)$$

the fluid displacement is given by the right-hand side of (46) with A_α replaced by A_α^2 , $\alpha = F, S$, and T . These results may also be derived quite simply using the representation integral (87) derived below, with $F_i = \delta_{ij}\delta(\mathbf{x} - \mathbf{x}')$, $f_i = 0$, and letting V be all of space. In general, any distribution of loads can be handled by Eq. (87) using the fundamental solutions u^G and U^G .

III. ASYMPTOTIC APPROXIMATIONS

In Sec. III, we consider our results of Sec. II in certain physical limits. We shall look at the limits of large observation distance, low and high frequency, low porosity, and stiff frame. We are specifically interested in the behavior of the wave speeds, attenuations, and the quantities A_α , $\alpha = F, S$, and T . Some of our results have appeared in the literature previously. For example, Biot^{1,2} and Geertsma and Smit⁶ discuss the frequency dependence in detail. The stiff frame limit is discussed by Johnson and Plona.¹¹

A. Farfield approximation

The farfield of the point source is defined by

$$|k_\alpha r| \gg 1, \quad \alpha = F, S, T, \quad (52)$$

where, without loss of generality, we have taken the source at the origin. Thus in the farfield we have for $\alpha = F, S$, or T ,

$$\nabla G_\alpha(r) = ik_\alpha \hat{\mathbf{r}} G_\alpha(r) [1 + O(1/|k_\alpha r|)], \quad (53)$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of the observer at $\mathbf{x}$. Using (53) in Eqs. (43) and (46), we obtain

$$u_{i;k}^G \approx (1/N) \{ (\delta_{ik} - \hat{r}_i \hat{r}_k) G_T(r) + \hat{r}_i \hat{r}_k [(k_F/k_T)^2 \times D_F G_F(r) + (k_S/k_T)^2 D_S G_S(r)] \}, \quad (54)$$

$$U_{i;k}^G \approx (1/N) \{ (\delta_{ik} - \hat{r}_i \hat{r}_k) A_T G_T(r) + \hat{r}_i \hat{r}_k [(k_F/k_T)^2 \times A_F D_F G_F(r) + (k_S/k_T)^2 A_S D_S G_S(r)] \}. \quad (55)$$

The farfield stress tensors then follow from Eqs. (1) and (2) as

$$\sigma_{ij;k}^G \approx \sum_{\alpha=F,S,T} i \left(\frac{k_\alpha^3}{k_T^2} \right) D_\alpha B_{ij;k}^\alpha(\hat{\mathbf{r}}) G_\alpha(r), \quad (56)$$

$$s_{i;k}^G \approx \sum_{\alpha=F,S} i \left(\frac{k_\alpha^3}{k_T^2} \right) D_\alpha \left(\frac{Q + RA_\alpha}{N} \right) \hat{r}_k G_\alpha(r), \quad (57)$$

where

$$B_{ij;k}^T = (\delta_{ik} \hat{r}_j + \delta_{jk} \hat{r}_i - 2\hat{r}_i \hat{r}_j \hat{r}_k) \quad (58)$$

and

$$B_{ij;k}^\alpha = \{ [(P + QA_\alpha)/N] - 2 \} \delta_{ij} + 2\hat{r}_i \hat{r}_j \hat{r}_k, \quad \alpha = F, S. \quad (59)$$

The asymptotic form of u^G and U^G clearly shows that three distinct waves radiate from the point source: two dilatational (F, S) and one equivoluminal (T). The various parameters k_α , A_α , D_α , and $B_{ij;k}^\alpha$ are all frequency-dependent complex numbers. The attenuation of the wave of type α , $\alpha = F, S$, or T depends upon the magnitude of $\text{Im}(k_\alpha)$, which in turn is a function of frequency.^{1,2} Therefore, the relative magnitude of each wave depends critically on the frequency of the source.

We note that when the source is not at the origin ($\mathbf{x}' \neq 0$) but $|\mathbf{x}'| \ll r$, then Eqs. (54)–(57) are correct if we make the substitution

$$G_\alpha(r) \rightarrow G_\alpha(r) \exp(-ik_\alpha \hat{\mathbf{r}} \cdot \mathbf{x}'). \quad (60)$$

B. Low-frequency approximation

Following Biot,¹ we define a characteristic frequency ω_c as

$$\omega_c = b/\rho_2 = b/\phi\rho_f. \quad (61)$$

We note that our ω_c is $2\pi f_c$, where f_c is the characteristic frequency in Ref. 1. Sometimes b is written as¹

$$b = \phi^2(\mu/k), \quad (62)$$

where μ is the fluid viscosity and k the intrinsic permeability of the rock. The values of b encountered in rocks are of the order of 10^5 g/cm³ s and often much greater.⁷ Therefore, ω_c is of the order of 10^6 cps in rocks and sand. The ratio ω/ω_c is very small; we can use this fact to approximate our results.

A straightforward expansion of the expressions in Appendix A yields the following:

For the transverse wave:

$$k_T = \frac{\omega}{v_{T0}} \left[1 + i \left(\frac{\omega}{\omega_c} \right) \frac{\rho_2}{2\rho} + O(\omega^2) \right], \quad v_{T0} = (N/\rho)^{1/2}, \quad (63)$$

$$A_T = 1 + i(\omega/\omega_c) + O(\omega^2).$$

For the fast compressional wave:

$$k_F = \frac{\omega}{v_{F0}} \left[1 + i \left(\frac{\omega}{\omega_c} \right) \frac{\rho_2}{2\rho} Z^2 + O(\omega^2) \right], \quad v_{F0} = (H/\rho)^{1/2}, \quad A_F = 1 + i(\omega/\omega_c)Z + O(\omega^2), \quad (64)$$

$$Z = 1 - (R + Q)/\rho_2 v_{F0}^2,$$

$$D_F = 1 + i(\omega/\omega_c)(\rho_2/\rho)(1 - Z)^2 + O(\omega^2).$$

For the slow compressional wave:

$$k_s = e^{i\pi/4} \left| \omega \omega_c \left(\frac{\rho_2 H}{PR - Q^2} \right) \right|^{1/2} + O(\omega^{3/2}),$$

$$A_s = -(P + Q)/(R + Q) + O(\omega), \quad (65)$$

$$D_s = -i(\omega/\omega_c)(\rho_2/\rho)(1 - Z)^2 + O(\omega^2).$$

The asymptotic behavior of the wavenumbers k_α , $\alpha = F, S$, and T shows that the F and T waves have attenuation proportional to ω^2 . The S wave has much greater attenuation, which goes as $(\omega/\omega_c)^{1/2}$. The form of k_s in Eq. (65) shows that the S wave is essentially a diffusion process at low frequency.¹

Looking now at the displacement ratios A_α , $\alpha = F, S$, and T , we observe that the solid and fluid displacements are in phase and of almost equal magnitude for F and T waves. The slow wave, in contrast, propagates with solid and fluid displacements out of phase and unequal in magnitude. This relative motion generates much viscous dissipation and accounts for the high attenuation of the slow wave.

The Green tensors $\mathbf{u}^G$ and $\mathbf{U}^G$ at low frequency thus split up into two propagating but attenuated waves, and a diffusive wave. The propagating F and T waves are like longitudinal and shear waves in a homogeneous viscoelastic solid, with density ρ and frequency-dependent complex Lamé constants λ^E and μ^E given by

$$\mu^E = N [1 - i(\omega/\omega_c)(\rho_2/\rho)], \quad (66)$$

$$\lambda^E + 2\mu^E = H [1 - i(\omega/\omega_c)(\rho_2/\rho)Z^2].$$

C. High-frequency approximation

In certain circumstances the frequency ratio (ω/ω_c) can be large. Referring to Eqs. (61) and (62) we see that this could occur if, for example, the fluid viscosity is small or the permeability large. The former is the case for superfluid ⁴He in the pores, the so called "superleak."¹² Large permeability is obtained in experimental situations in which the fluid can percolate easily, for example, glass beads immersed in fluid. Also, experiments are usually performed at ultrasonic frequencies, which are much higher than those encountered in geophysical applications.

Biot² has shown that at high frequencies it is necessary to include a viscous boundary layer effect. This is reflected in the equations through a frequency-dependent function $F(\omega)$ that multiplies b (see Sec. I). The relevant parameter in $F(\omega)$ is

$$\kappa_0 = a\sqrt{\omega/\nu}, \quad (67)$$

where a is a typical pore dimension and ν is the kinematic viscosity, $\nu = \mu/\rho_f$. For large κ_0 , $F(\omega)$ goes as $\kappa_0 e^{i\pi/4}$. From Eqs. (61) and (62), we have $\omega/\omega_c = \phi\omega\kappa/\nu$. Normally one has κ proportional to a^2 , so that $\omega/\omega_c \gg 1$ implies $\kappa_0 \gg 1$, and vice versa. Therefore, one would normally need to consider $F(\omega)$ explicitly in the high-frequency range. However, in the following we will assume $F = 1$. This may be reconciled with the previous statements by assuming the dominant pores to be slitlike. For such pores the relevant pore dimension a is the thickness, but the permeability will be proportional to al , where the length l is much greater than a . In this way it is possible to envisage cases where $\omega/\omega_c \gg 1$ but κ_0 can be small. We also avoid the necessity of introducing an explicit $F(\omega)$,

which requires making specific assumptions about the pore microgeometry. Finally, we note that even when $F(\omega)\alpha(i\omega)^{1/2}$, the only results below that are affected are the wave attenuations, which then grow as $\omega^{1/2}$. All the other quantities become frequency independent and thus, independent of $F(\omega)$.

From the results given in Appendix A, we find that the velocities v_α and amplitude ratios A_α become real quantities to first order in (ω_c/ω) for $\alpha = F, S$, and T . We have the inequalities¹

$$A_F > A_T > 0 > A_S, \quad A_T = 1 - T^{-1} < 1, \quad (68)$$

where the tortuosity T is defined in Eq. (11). Thus the quantities D_F and $D_S = 1 - D_F$ have values between zero and one. The leading order terms in the wavenumbers are, for the T wave:

$$k_T = \frac{\omega}{v_{T\infty}} \left[1 + \frac{i\omega_c}{2\omega T} \left(\frac{v_{T\infty}^2}{v_{T0}^2} - 1 \right) + O(\omega^{-2}) \right], \quad (69)$$

where

$$v_{T\infty}^2/v_{T0}^2 = (1 - \phi\rho_f/T\rho)^{-1} > 1, \quad (70)$$

and for the F and S waves, $\alpha = F, S$:

$$k_\alpha = (\omega/v_{F0} C_{\alpha\infty}) \times \left\{ 1 + \frac{i\omega_c}{2\omega} \left[\frac{\rho_2 \rho_{22}}{\rho^2} \left(\frac{v_{T0}}{v_{T\infty}} \right)^2 \frac{|C_{\alpha\infty}^2 - 1|}{(C_{F\infty}^2 - C_{S\infty}^2)} \right] + O(\omega^{-2}) \right\}, \quad (71)$$

where $C_{\alpha\infty}$ is the limiting value as $\omega \rightarrow \infty$ of C_α , which is defined in Appendix A. It is shown in Appendix B that

$$C_{F\infty} \geq 1 > C_{S\infty}. \quad (72)$$

Hence the quantity in square brackets in Eq. (71) is less than unity for both $\alpha = F$ and S .

The attenuation of each wave tends to a constant value a_α , $\alpha = F, S$, and T in the limit $\omega \rightarrow \infty$, where

$$a_\alpha = \text{Im } k_\alpha(\omega = \infty). \quad (73)$$

The T -wave attenuation, from (69) and (73), is

$$a_T = bv_{T\infty}/2\rho T^2 v_{T0}^2. \quad (74)$$

From the definition of b in (62) we see that a_T scales roughly with $(\mu\phi^2/kT^2)$. Taking $\mu = 10^{-2}$ g/cm s for water, $\rho = 2.5$ g/cm³, $v_{T\infty} \approx V_{T0} = 2$ km/s, and putting $k = \bar{k} 10^{-8}$ cm², we have $a_T \approx \phi^2/(T^2 \bar{k})$ cm⁻¹. The critical quantity here is the product $T^2 \bar{k}$. Typically, in rocks, $\bar{k}$ is much less than unity. The tortuosity T may be much greater than unity, however. It is known for a stiff frame that¹³ $T \approx \phi(\sigma_F/\sigma)$, where σ_F and σ are the electrical conductivities of the fluid and the porous solid, respectively, assuming the grains are insulators.

The high-frequency attenuation of the α wave, $\alpha = F, S$, or T , disappears, if and only if, $v_{F\infty} = v_{F0}$, where $v_{F\infty} = C_{F\infty} v_{F0}$ is the asymptotic value of v_F as $\omega \rightarrow \infty$. This can occur for $\alpha = F$ only, and corresponds to Biot's compatibility condition.¹ In fact, when the compatibility condition holds, $v_F = v_{F0}$ and $A_F = 1$ for all frequencies.

We note that the high-frequency limit is not the same as the limit of vanishing dissipation, $b \rightarrow 0$. In the latter case, the attenuation vanishes at all frequencies. But when $b > 0$ and

$\omega \rightarrow \infty$, the attenuation of each wave tends to a constant value which is proportional to b .

D. Low porosity, high-frequency approximation

When $\phi = 0$, the bulk modulus κ_b of the dry frame defined in (7) is equal to κ_s . For arbitrary porosity, we write

$$\kappa_b = \kappa_s(1 - \phi/M), \quad (75)$$

where M is an $O(1)$ parameter defined by the microgeometry. Effective medium theory estimates of M are given in Ref. 14, for example. Substituting Eq. (74) into Eqs. (5)–(7) determines P , Q , and R . The high-frequency, low porosity ($\phi \ll 1$) approximations follow from Appendix A as

$$\begin{aligned} v_F^2 &= (\kappa_b + \frac{4}{3}N)/\rho + O(\phi), \\ v_T^2 &= N/\rho + O(\phi), \\ v_S^2 &= \frac{\kappa_f}{T\rho_f} \left[1 + \frac{\kappa_f}{\kappa_s} \left(\frac{1}{M} - 1 \right) \right]^{-1} + O(\phi), \\ A_F &= 1 + (\delta - \epsilon)/(T\epsilon - M\delta) + O(\phi), \\ A_S &= -\frac{1}{\phi} \frac{\rho_s}{\rho_f \delta} \left(\frac{T\epsilon - M\delta}{T - M} \right) + O(1), \end{aligned} \quad (76)$$

where

$$\delta = [1 + M(\kappa_s/\kappa_f - 1)]^{-1}, \quad (77)$$

$$\epsilon = (\rho_f/\rho_s)(1 + \frac{4}{3}N/\kappa_s), \quad (78)$$

and the tortuosity T is defined in (11).

The most important result in (76) is that the ratio A_S is $O(\phi^{-1})$ as $\phi \rightarrow 0$. Thus all the motion associated with the slow wave is concentrated in the vanishingly small fluid component. Referring to Eq. (44), we note that $D_F = 1 + O(\phi)$ and $D_S = O(\phi)$, so that $\mathbf{U}^G$ reduces in the limit to the purely elastic Green displacement tensor.¹⁵ The slow wave term in $\mathbf{U}^G$ of (46) remains, however, since $A_S D_S = O(1)$. This shows explicitly that the slow wave motion generated by a point force is significant only in the fluid.

We have assumed so far that the tortuosity T is independent of ϕ . This is certainly not true, but the precise dependence of T on ϕ is uncertain. Berryman¹⁶ has estimated $T = O(\phi^{-1})$ and Johnson *et al.*¹³ find $T = O(\phi^{-1/2})$ as $\phi \rightarrow 0$. Whatever the specific behavior may be, we note that if $T \rightarrow \infty$ as $\phi \rightarrow 0$, then we still have $A_S \rightarrow -\infty$ as $\phi \rightarrow 0$. Also, from (76) and (68) we see that $A_F, A_T \rightarrow 1$, and $v_S \rightarrow 0$, indicating that the fluid moves in phase with the solid and the slow wave disappears.

E. Stiff frame, high-frequency approximation

When the pore fluid is much more compressible than the frame, we have $\kappa_f \ll \kappa_b, \kappa_s, N$. The following high-frequency results then follow from Appendix A:

$$\begin{aligned} v_F &\sim \left(\frac{\kappa_b + \frac{4}{3}N}{\rho - \phi\rho_f/T} \right)^{1/2}, \quad v_S \sim \frac{v_f}{\sqrt{T}}, \quad v_T \sim \left(\frac{N}{\rho - \phi\rho_f/T} \right)^{1/2}, \\ A_F &\sim 1 - (v_F^2/T - v_S^2/M)/(v_F^2 - v_S^2), \\ A_S &\sim -[(\kappa_b + \frac{4}{3}N)/\phi\kappa_f] [(1 - v_f^2/v_F^2)/(M^{-1} - T^{-1})], \end{aligned} \quad (79)$$

where $v_f = (\kappa_f/\rho_f)^{1/2}$ is the speed of sound in the fluid and M is defined in (75). The fast and transverse waves have the

speeds of longitudinal and transverse waves in an elastic solid with moduli κ_b and N and density $\rho - \phi\rho_f/T$. The slow wave speed depends very simply on the fluid sound speed and the tortuosity T . The full ramifications of this relationship have been explored by Johnson.^{12,13}

We note from (79) that $A_S = O(\kappa_s/\kappa_f)$. Thus, as in the case of low porosity, the slow wave motion is almost completely restricted to the fluid in the pores. The energy density of a wave is roughly proportional to $\kappa_s u^2$ in the solid and $\kappa_f U^2$ in the fluid. For a slow wave, the ratio of fluid to solid energy densities is therefore $\gg 1$. Since the slow wave energy resides mainly in the fluid, it is imperative that the pore spaces be well connected if the slow wave is to propagate. Also, from (45) and (79) we have $D_F \sim 1, D_S = O(\kappa_f/\kappa_s)$. The results of Sec. II then predict that a load applied in the solid will radiate mostly as F and T waves, with relatively little energy going into slow waves. The converse of the stiff frame is that of a weak frame. We shall not consider this limit here, but note that Johnson^{11,17} has used it to explore the acoustic properties of gels.

F. High-frequency example

Ultrasonic experiments on consolidated water-saturated glass beads (Ridgefield sandstone) by Plona⁴ have shown the existence of the slow wave. In Sec. III F, we will evaluate the various high-frequency limits of the wave parameters for the data of Ref. 13. The solid (grain) and fluid parameters are $\rho_s = 2.49$, $\rho_f = 1$ g/cm³, $\kappa_s = 49.9$, and $\kappa_f = 2.25 \times 10^{10}$ dyn/cm². The bulk parameters follow from measurement of the dry frame longitudinal and transverse wave speeds.

$$v_{L,\text{dry}} = [(\kappa_b + \frac{4}{3}N)/\rho_1]^{1/2}, \quad (80)$$

$$v_{T,\text{dry}} = (N/\rho_1)^{1/2}.$$

The permeability of the sample is not given in Ref. 13. However, reported values of the permeability in similar samples^{11,18,19} give $k \approx 10^{-8}$ cm² (1 Darcy). We assume here that $k = 10^{-8}$ cm². With the viscosity of water as $\mu = 10^{-2}$ g/cm s, the characteristic frequency $f_c = \omega_c/2\pi$ follows from Eqs. (61) and (62) as $(0.14)\phi \times 10^6$ cps, where ϕ is the porosity. The center frequency of the experiments was 0.5×10^6 cps. The porosity varied from 0.33 to 0.1; therefore, the ratio ω/ω_c varied from about 10 to 40, indicating the high-frequency regime. However, the typical pore size is known to be on the order of 10^{-2} cm,¹³ which implies that κ_0 of Eq. (67) is very large. Therefore, we cannot expect to estimate the wave attenuations without making further assumptions (see Ref. 16 for a discussion of the amplitudes in these experiments), but we can use the other asymptotic results of Sec. III C. Indeed, a comparison with the wave speeds computed using the exact equations shows the high-frequency approximations to be accurate to within 1%.¹¹

The results are shown in Table I. The stiff frame approximations of Sec. III E can be shown to give reasonable approximations (accuracy $\sim 10\%$) to the results. It is interesting to note that as ϕ decreases, the amplitude ratio A_S increases in magnitude, as predicted in Sec. III D. The slow wave energy becomes concentrated in the pore space, which

is itself disappearing. Also, from Table I we note that $D_S \ll 1$, indicating that forces applied to the solid phase will not produce slow waves efficiently [see Eqs. (43) and (46)]. Direct coupling of the applied force to the pore fluid is necessary to generate slow waves. In the experiments, this was achieved using a water immersion technique with open pore boundary conditions at the porous/fluid interface. Ultrasonic waves in the water bath thus coupled to the slow wave.

IV. DYNAMIC RECIPROCITY AND A REPRESENTATION INTEGRAL

We first derive a dynamic reciprocity relationship for the Biot equations. This result is the Biot analog of the Betti-Rayleigh theorem in elasticity.¹⁵ A similar result has been derived by Vargas.²⁰ Let the time harmonic fields $\mathbf{u}^A$ and $\mathbf{U}^A$ satisfy the inhomogeneous Biot equations:

$$\omega^2(\bar{\rho}_{11}\mathbf{u}^A + \bar{\rho}_{12}\mathbf{U}^A) + \nabla \cdot \boldsymbol{\sigma}^A = -\mathbf{f}^A, \quad (81)$$

$$\omega^2(\bar{\rho}_{12}\mathbf{u}^A + \bar{\rho}_{22}\mathbf{U}^A) + \nabla \cdot \mathbf{s}^A = -\mathbf{F}^A. \quad (82)$$

Here $\mathbf{f}^A$ and $\mathbf{F}^A$ are body forces per unit volume in the solid and fluid phases, respectively. The stress tensor $\boldsymbol{\sigma}^A$ and scalar $\mathbf{s}^A$ are defined by Eqs. (1) and (2) for the fields $\mathbf{u}^A$ and $\mathbf{U}^A$. Similarly, let $\mathbf{u}^B$, $\mathbf{U}^B$, $\boldsymbol{\sigma}^B$, and $\mathbf{s}^B$ be the displacement and stress fields produced by body forces $\mathbf{f}^B$ and $\mathbf{F}^B$. Consider the integral

$$\begin{aligned} & \int_V [(\mathbf{f}_j^B \mathbf{u}_j^A + \mathbf{F}_j^B \mathbf{U}_j^A) - (\mathbf{f}_j^A \mathbf{u}_j^B + \mathbf{F}_j^A \mathbf{U}_j^B)] dV \\ &= \int_V [(\sigma_{ij}^A \mathbf{u}_i^B + s_i^A \mathbf{U}_i^B) - (\sigma_{ij}^B \mathbf{u}_i^A + s_i^B \mathbf{U}_i^A)] dV \end{aligned} \quad (83)$$

over an arbitrary volume V and the subscript comma denotes differentiation. The right-hand side of Eq. (83) follows from Eqs. (81) and (82) and the corresponding equations for system B. The integral on the right-hand side can be expressed as the divergence of some quantity by using Eqs. (1) and (2). We obtain

$$\begin{aligned} & (\sigma_{ij}^A \mathbf{u}_i^B + s_i^A \mathbf{U}_i^B) - (\sigma_{ij}^B \mathbf{u}_i^A + s_i^B \mathbf{U}_i^A) \\ &= [(\sigma_{ij}^A \mathbf{u}_i^B - \sigma_{ij}^B \mathbf{u}_i^A) + (s_i^A \mathbf{U}_i^B - s_i^B \mathbf{U}_i^A)]_{,j}. \end{aligned} \quad (84)$$

Let S be the surface of volume V . The volume integral on the right-hand side can be recast as a surface integral over S by a simple application of Green's theorem. Equation (83) then becomes

$$\begin{aligned} & \int_V [(\mathbf{f}_i^B \mathbf{u}_i^A - \mathbf{f}_i^A \mathbf{u}_i^B) + (\mathbf{F}_i^B \mathbf{U}_i^A - \mathbf{F}_i^A \mathbf{U}_i^B)] dV \\ &= \int_S [(\sigma_{ij}^A \mathbf{u}_j^B - \sigma_{ij}^B \mathbf{u}_j^A) + (s_i^A \mathbf{U}_i^B - s_i^B \mathbf{U}_i^A)] n_i dS, \end{aligned} \quad (85)$$

TABLE I. Results for the high-frequency example. The wave speeds are in units of km/s; $v_{L,\text{dry}}$ and $v_{T,\text{dry}}$ are given and $v_{F,\infty}$, $v_{S,\infty}$, and $v_{T,\infty}$ follow from Appendix A and Sec. III C. ϕ is the porosity; T is the tortuosity [Eq. (11)]; the amplitude ratios A_F , A_S , and A_T follow from (16) and Appendix A; and D_S is defined in (44). The numbers in parentheses denote powers of ten.

ϕ (%)	T	$v_{L,\text{dry}}$	$v_{T,\text{dry}}$	$v_{F,\infty}$	$v_{S,\infty}$	$v_{T,\infty}$	A_F	A_S	A_T	D_S
33.5	1.75	3.10	1.82	3.23	1.01	1.74	0.68	-12	0.43	2.0(-2)
26.6	2.00	3.83	2.28	3.89	0.97	2.20	0.66	-22	0.5	7.1(-3)
21.9	2.40	4.32	2.65	4.35	0.89	2.57	0.70	-33	0.58	3.6(-3)
16.2	3.02	4.84	2.81	4.82	0.81	2.74	0.74	-58	0.67	1.3(-3)
10.5	3.84	5.16	3.09	5.17	0.71	3.04	0.81	-85	0.74	7.8(-4)

where $\mathbf{n}$ is the unit normal out of V . This is the Betti-Rayleigh reciprocal identity for a porous medium.

Now let us consider the special case when $\mathbf{u}^B$ and $\mathbf{U}^B$ are the Green tensors for a point load. Then

$$\mathbf{u}_i^B(\mathbf{x}) = \mathbf{u}_{ik}^G(\mathbf{x}; \mathbf{x}'), \quad \mathbf{U}_i^B(\mathbf{x}) = \mathbf{U}_{ik}^G(\mathbf{x}; \mathbf{x}'), \quad (86)$$

$$\mathbf{f}_i^B(\mathbf{x}) = \delta_{ik} \delta(\mathbf{x} - \mathbf{x}'), \quad \mathbf{F}_i^B(\mathbf{x}) = 0,$$

where k denotes the direction of application of the point load. Dropping the superscripts on $\mathbf{u}^A$, $\mathbf{U}^A$, $\boldsymbol{\sigma}^A$, and $\mathbf{s}^A$, and interchanging $\mathbf{x}$ and $\mathbf{x}'$ we obtain

$$\begin{aligned} & \int_V [\mathbf{f}_i(\mathbf{x}') \mathbf{u}_{ik}^G(\mathbf{x}; \mathbf{x}') + \mathbf{F}_i \mathbf{U}_{ik}^G] dV(\mathbf{x}') \\ &+ \int_S [(\sigma_{ij} \mathbf{u}_{j,k}^G - \sigma_{ij,k}^G \mathbf{u}_j) + (s \mathbf{U}_{ik}^G - s_{,k}^G \mathbf{U}_i)] \\ &\times n_i dS(\mathbf{x}') = \begin{cases} \mathbf{u}_k(\mathbf{x}), & \text{if } \mathbf{x} \text{ is in } V, \\ 0, & \text{if } \mathbf{x} \text{ is outside } V. \end{cases} \end{aligned} \quad (87)$$

This is our representation integral. It expresses the field $\mathbf{u}(\mathbf{x})$ at any point in V in terms of the applied body forces in V and the tractions on S . A similar representation exists for the fluid displacement $\mathbf{U}$. It can be obtained from Eq. (87) by replacing the terms $G_\alpha(|\mathbf{x} - \mathbf{x}'|)$, $\alpha = F, S$, and T by $A_\alpha G_\alpha(|\mathbf{x} - \mathbf{x}'|)$ wherever they occur in the left-hand side. In other words, if Eq. (87) is rewritten as

$$\mathbf{u}_k(\mathbf{x}) = \sum_{\alpha=F,S,T} J_k^{(\alpha)}(\mathbf{x}') G_\alpha(|\mathbf{x} - \mathbf{x}'|), \quad (88)$$

where $J_k^{(\alpha)}$ are integro-differential operators and $\mathbf{x}$ is in V , then

$$\mathbf{U}_k(\mathbf{x}) = \sum_{\alpha=F,S,T} A_\alpha J_k^{(\alpha)}(\mathbf{x}') G_\alpha(|\mathbf{x} - \mathbf{x}'|). \quad (89)$$

V. SCATTERING THEOREMS

In Sec. V, we derive reciprocity relations for the scattered F , S , and T waves that are produced when a wave is incident upon an obstacle of finite extent. These relations include as a special case the known reciprocity relations for purely elastic waves.^{21,22}

We begin by writing the total fields as the sum of incident plus scattered fields:

$$(\mathbf{u}, \mathbf{U}) = (\mathbf{u}^{\text{in}}, \mathbf{U}^{\text{in}}) + (\mathbf{u}^{\text{sc}}, \mathbf{U}^{\text{sc}}). \quad (90)$$

The scattered fields must satisfy the radiation conditions that their farfields are of the form

$$\mathbf{u}^{\text{sc}} = \sum_{\alpha=F,S,T} \chi^\alpha(\hat{\mathbf{r}}) G_\alpha(r) \left[1 + O\left(\frac{1}{k_\alpha r}\right) \right], \quad r \rightarrow \infty, \quad (91)$$

$$\mathbf{U}^{\text{sc}} = \sum_{\alpha=F,S,T} A_{\alpha} \chi^{\alpha}(\mathbf{r}) G_{\alpha}(\mathbf{r}) \left[1 + O\left(\frac{1}{k_{\alpha} r}\right) \right], \quad r \rightarrow \infty. \quad (92)$$

The vector scattering amplitudes χ^{α} are functions of the direction of observation $\hat{\mathbf{r}}$, but not of the distance r . Thus they depend upon the choice of origin, but the scattered fields in (91) and (92) do not. The scattering amplitudes are polarized such that

$$\chi^{\alpha} \wedge \hat{\mathbf{r}} = 0, \quad \alpha = F, S, \quad \chi^{\alpha} \cdot \hat{\mathbf{r}} = 0, \quad \alpha = T. \quad (93)$$

We will obtain explicit representations of the χ^{α} shortly.

Let the obstacle or inhomogeneity be surrounded by the closed surface Σ . For example, if the object is an infinitesimally thin crack, then Σ may be the two faces of the crack. Or, if the object is an elastic inhomogeneity then Σ may be its outer surface. Now consider the representation integral (87) with $\mathbf{u}$, $\mathbf{U}$, σ , and s replaced by the fields $\mathbf{u}^{\text{sc}}$, $\mathbf{U}^{\text{sc}}$, σ^{sc} , and s^{sc} of the scattered waves. Let V be the volume between Σ and the surface of a sphere of radius Δ which is centered at the point of observation and encloses Σ . As $\Delta \rightarrow \infty$, the integral over the surface of the large sphere vanishes because of the radiation conditions (91) and (92) and we obtain

$$\begin{aligned} u_k^{\text{sc}}(\mathbf{x}) = & \int_{\Sigma} \{ [\sigma_{ij}^{\text{sc}}(\mathbf{x}') u_{j,k}^G(\mathbf{x}; \mathbf{x}') - \sigma_{ij,k}^G u_j^{\text{sc}}] \\ & + [s^{\text{sc}} U_{i,k}^G - s_{i,k}^G U_i^{\text{sc}}] \} n_i dS(\mathbf{x}'), \end{aligned} \quad (94)$$

where $\mathbf{n}$ is the unit normal into Σ . A similar expression for $\mathbf{U}^{\text{sc}}(\mathbf{x})$ follows from (88) and (89). Equation (94) is an exact representation of the scattered field at the position $\mathbf{x}$ outside Σ in terms of the fields on Σ . This equation could be used as the starting point in the derivation of an integral equation for the scattered fields. For example, if Σ is the surface of a crack, the equation is obtained by letting $\mathbf{x}$ approach Σ and then applying the boundary conditions on the crack faces. This procedure is outlined in Ref. 23 for the elastic case.

The farfield scattering amplitudes now follow from Eqs. (91) and (94) and the results of Sec. III A. We obtain

$$\begin{aligned} \chi_k^T(\hat{\mathbf{r}}) = & \int_{\Sigma} \{ (1/N) (\delta_{jk} - \hat{r}_j \hat{r}_k) [\sigma_{ij}^{\text{sc}}(\mathbf{x}') + s^{\text{sc}} A_T \delta_{ij}] \\ & + ik_T B_{ij,k}^T(\hat{\mathbf{r}}) u_j^{\text{sc}} \} n_i \exp(-ik_T \hat{\mathbf{r}} \cdot \mathbf{x}') dS(\mathbf{x}') \end{aligned} \quad (95)$$

and for $\alpha = F$ or S ,

$$\begin{aligned} \chi_k^{\alpha}(\hat{\mathbf{r}}) = & \hat{r}_k D_{\alpha} \left(\frac{k_{\alpha}}{k_T} \right)^2 \int_{\Sigma} \left\{ \frac{1}{N} [\sigma_{ij}^{\text{sc}}(\mathbf{x}') + s^{\text{sc}} A_{\alpha} \delta_{ij}] \hat{r}_j \right. \\ & + ik_{\alpha} \left[u_j^{\text{sc}} B_{ij,l}(\hat{\mathbf{r}}) \hat{r}_l + \left(\frac{Q + RA_{\alpha}}{N} \right) U_i^{\text{sc}} \right] \} n_i \\ & \times \exp(-ik_{\alpha} \hat{\mathbf{r}} \cdot \mathbf{x}') dS(\mathbf{x}'). \end{aligned} \quad (96)$$

We note that the scattering amplitudes of Eqs. (95) and (96) automatically satisfy the radiation conditions (93).

Consider two fields A and B , as in Eq. (85), but with no body forces present in V , which we take to be the volume interior to Σ . Then if the material of the inhomogeneity or obstacle possesses a symmetric stress tensor, we have from Eq. (85)

$$\begin{aligned} \gamma(A; B) \equiv & \int_{\Sigma} [(\sigma_{ij}^A u_j^B - \sigma_{ij}^B u_j^A) \\ & + (s^A U_i^B - s^B U_i^A)] n_i dS = 0. \end{aligned} \quad (97)$$

This relation holds, for example, when the medium inside is an anisotropic Biot material, or an anisotropic elastic material, which is a particular case of the Biot material. The result (97) is independent of the boundary conditions on Σ , which could be a closed or open pore or the intermediate permeable condition,²⁴ for example.

Now, splitting both fields A and B into sums of incident plus scattered fields as in (90), we obtain

$$\begin{aligned} \gamma(A, \text{in}; B, \text{in}) + \gamma(A, \text{sc}; B, \text{in}) \\ + \gamma(A, \text{in}; B, \text{sc}) + \gamma(A, \text{sc}; B, \text{sc}) = 0. \end{aligned} \quad (98)$$

The first integral in (98) can be transformed into an integral over the interior of Σ using (84). This integral is zero since A, in and B, in are regular in Σ . Similarly, the last integral in (98) can be transformed into a volume integral over the region external to Σ and a surface integral at infinity. Both integrals are zero, the first because the scattered fields are regular in the exterior region and the second due to the radiation conditions (91)–(93). This leaves us with

$$\begin{aligned} \int_{\Sigma} [(\sigma_{ij}^{A, \text{in}} u_j^{B, \text{sc}} - \sigma_{ij}^{B, \text{sc}} u_j^{A, \text{in}}) + (s^{A, \text{in}} U_i^{B, \text{sc}} - s^{B, \text{sc}} U_i^{A, \text{in}})] n_i dS \\ = \int_{\Sigma} [\sigma_{ij}^{B, \text{in}} u_j^{A, \text{sc}} - \sigma_{ij}^{A, \text{sc}} u_j^{B, \text{in}} \\ + (s^{B, \text{in}} U_i^{A, \text{sc}} - s^{A, \text{sc}} U_i^{B, \text{in}})] n_i dS. \end{aligned} \quad (99)$$

Now, let the incident fields be plane waves:

$$\mathbf{u}^{A, \text{in}} = \mathbf{a} \exp(ik_{\alpha} \mathbf{p} \cdot \mathbf{x}), \quad |\mathbf{p}| = 1, \quad (100)$$

$$\mathbf{u}^{B, \text{in}} = \mathbf{b} \exp(ik_{\beta} \mathbf{q} \cdot \mathbf{x}), \quad |\mathbf{q}| = 1, \quad (101)$$

where $\alpha, \beta = F, S$, or T . The vectors $\mathbf{a}$ and $\mathbf{b}$ are polarized in accordance with (16). Substituting for the incident fields in (99) and using Eqs. (1), (2), (95), and (96), we obtain the reciprocity relations

$$k_{\beta}^2 D_{\beta} \mathbf{a} \cdot \chi^{\alpha(B)}(-\mathbf{p}) = k_{\alpha}^2 D_{\alpha} \mathbf{b} \cdot \chi^{\beta(A)}(-\mathbf{q}). \quad (102)$$

Here, the vector $\chi^{\alpha(B)}$ is the scattering amplitude of type α due to the incident field B . Also, the quantities D_{α} are defined in (44) and (45). We note that (102) includes all possible scattering processes. For example, when there is no mode conversion ($\alpha = \beta$), we have

$$\mathbf{a} \cdot \chi^{\alpha(B)}(-\mathbf{p}) = \mathbf{b} \cdot \chi^{\alpha(A)}(-\mathbf{q}). \quad (103)$$

The reciprocity relations (102) include the elastodynamic reciprocity relations²¹ as a special limit. In this case, $\alpha, \beta = F$ or T and $D_F \equiv 1$.

It is possible to derive “optical” theorems in nonabsorbing media which relate total scattering cross sections to forward scattering amplitudes. Tan²² has derived these relations for elastodynamics. The key to any optical theorem is the conservation of energy in an infinite sphere: The energy of the plane wave coming in is equal to the energy going out. The whole procedure is not applicable when there is attenuation in the host medium. Since this is the case with the Biot porous solid, it is not feasible to obtain optical theorems, except in two cases: first, when the Biot medium is not

damped ($b = 0$), but this is an unrealistic assumption. The second case is when the Biot compatibility condition holds (see Sec. III C and Ref. 1). Under these special circumstances, the fast wave radiates to infinity and it is possible to derive an optical theorem. Owing to the limited applicability of such a theorem, we shall not present it here.

VI. THE TIME-DEPENDENT POINT LOAD SOLUTION

The results of Secs. I–V are for a time harmonic point load. Any other time-dependent point load may be considered by taking the appropriate Fourier transform of $\mathbf{u}^G$ and $\mathbf{U}^G$ with respect to ω . In particular, if the applied load is a delta function at time $t = 0$, the corresponding solid displacement tensor is

$$\mathbf{u} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{u}^G e^{-i\omega t} d\omega. \quad (104)$$

This is the same solution considered by Burridge and Vargas,⁷ who obtained the solution using Laplace transforms. We will show how the results in Ref. 7 can be obtained readily from our time harmonic solution.

A. An exact result for no dissipation

We consider first the case of no dissipation present in the system ($b = 0$). Then, the integral in (104) can be obtained in closed form. As discussed in Sec. III C, the wave speeds are real, as are the quantities A_α and D_α , $\alpha = F, S$, and T . The inequalities (68) hold and D_F and D_S are between zero and unity. The time harmonic Green tensor in (43) depends on ω only through the wavenumbers k_α defined in (17). Substituting for $\mathbf{u}^G$ in (104), we have in the first term

$$\frac{1}{2\pi N} \int_{-\infty}^{\infty} G_T(r) e^{-i\omega t} d\omega = \frac{1}{4\pi N r} \delta\left(t - \frac{r}{v_T}\right), \quad (105)$$

which follows from the definition of $G_\alpha(r)$ in (35). We note that the wave speeds v_α , $\alpha = F, S$, and T are equal to the infinite frequency speeds when dissipation is present. Thus, referring to Sec. III C,

$$v_T = v_{T\infty}, \quad v_\alpha = C_{\alpha\infty} v_{F0}, \quad \alpha = F, S. \quad (106)$$

In particular, we have for $\alpha = F$ or S ,

$$\begin{aligned} & \frac{-D_\alpha}{2\pi N} \int_{-\infty}^{\infty} \frac{G_\alpha(r)}{k_\alpha^2} e^{-i\omega t} d\omega \\ &= \frac{-v_T^2 D_\alpha}{8\pi^2 N r} \int_{-\infty}^{\infty} \exp\left[-i\omega\left(t - \frac{r}{v_\alpha}\right)\right] \frac{d\omega}{\omega^2} \\ &= \frac{v_T^2 D_\alpha}{4\pi N r} \left(t - \frac{r}{v_\alpha}\right) H\left(t - \frac{r}{v_\alpha}\right), \end{aligned} \quad (107)$$

where H is the Heaviside step function; $H(x) = 0$, $x < 0$ and $H(x) = 1$, $x > 0$. Combining Eqs. (105) and (107), we obtain

$$\begin{aligned} 4\pi N \mathbf{u} = & \mathbf{I} \frac{1}{r} \delta\left(t - \frac{r}{v_T}\right) \\ & + v_T^2 \nabla \nabla \frac{1}{r} \left[D_F \left(t - \frac{r}{v_F}\right) H\left(t - \frac{r}{v_F}\right) \right. \\ & + D_S \left(t - \frac{r}{v_S}\right) H\left(t - \frac{r}{v_S}\right) \\ & \left. - \left(t - \frac{r}{v_T}\right) H\left(t - \frac{r}{v_T}\right) \right]. \end{aligned} \quad (108)$$

Evaluating the derivatives in Eq. (108), we can write $\mathbf{u}$ explicitly as

$$\begin{aligned} u_{ij} = & \frac{1}{4\pi N r} \left\{ (\delta_{ij} - \hat{r}_i \hat{r}_j) \delta\left(t - \frac{r}{v_T}\right) \right. \\ & + \hat{r}_i \hat{r}_j \left[\left(\frac{v_T}{v_F}\right)^2 D_F \delta\left(t - \frac{r}{v_F}\right) \right. \\ & + \left(\frac{v_T}{v_S}\right)^2 D_S \delta\left(t - \frac{r}{v_S}\right) \\ & + (\delta_{ij} - 3\hat{r}_i \hat{r}_j) v_T^2 \frac{t}{r^2} \left[H\left(t - \frac{r}{v_T}\right) \right. \\ & \left. \left. - D_F H\left(t - \frac{r}{v_F}\right) - D_S H\left(t - \frac{r}{v_S}\right) \right] \right\}. \end{aligned} \quad (109)$$

In analogy with the time harmonic case, the fluid displacement $\mathbf{U}$ follows from (109) by replacing $\delta(t - r/v_\alpha)$ and $H(t - r/v_\alpha)$ by $A_\alpha \delta(t - r/v_\alpha)$ and $A_\alpha H(t - r/v_\alpha)$, for each $\alpha = F, S$, and T .

Several comments are in order regarding Eq. (109). First we note that three δ -function singularities propagate with speeds v_F , v_T , and v_S . The magnitude of each singularity is proportional to D_α , $\alpha = F, T$, and S . As each singularity passes a given point in space, a discontinuous function proportional to (t/r^3) is “turned on.” Suppose, for example, that $v_F > v_T > v_S$ and consider the series of events at a fixed value of r . The fast δ -function pulse passes at $t = r/v_F$, leaving a displacement after it which is proportional to $-D_F t$. Then, the transverse pulse passes at $t = r/v_T$, changing the proportionality constant to $(1 - D_F)t = D_S t$. Finally, at $t = r/v_S$, the slow pulse arrives, leaving zero displacement after it. The fluid displacement $\mathbf{U}$ undergoes a similar displacement history. After the arrival of the F pulse the displacement is proportional to $(-A_F D_F t)$. This proportionality changes to $(A_S D_S t)$ when the T pulse passes and becomes zero at $t = r/v_S$. We note from Eq. (68) and the definition of D_F and D_S in (44) that the sense of $\mathbf{u}$ changes at $t = r/v_T$ while the sense of $\mathbf{U}$ remains the same. Thus all motion is confined to the interval between the arrivals of the first (fast) pulse at $t = r/v_F$ and the final (slow) pulse at $t = r/v_S$. Finally, we note that $\mathbf{u}$ reduces to the well-known result for pure elastodynamics¹⁵ when $D_S \rightarrow 0$, as discussed in Sec. III D.

B. Approximate results when dissipation is present

The presence of viscous damping ($b \neq 0$) in the system makes the inverse transform in (104) impossible to do in closed form. However, it is possible to extract the dominant terms for large values of t and r . This has been done by Bur-

ridge and Vargas⁷ using inverse Laplace transforms. We now rederive the results of Ref. 7 in compact form using (104) and our results in Sec. III.

It can be shown^{1,7} that, in general, the F - and T -wave attenuation is zero only at zero frequency. The slow wave is attenuated at all frequencies, and so its contribution will be exponentially small in comparison with the F and T waves. Therefore, we first consider the farfield approximation to u^G in (54) with only the F - and T -wave terms present. We expand the quantities k_F , k_T , and D_F about their zero frequency values using (63) and (64). Then, substituting from (54) into (104) and using the identity

$$\int_{-\infty}^{\infty} \exp \left[-i\omega \left(t - \frac{r}{v_{T0}} \right) \right] \exp \left(\frac{-\omega^2 v_2 r}{v_{T0}^3} \right) d\omega = \left(\frac{\pi v_{T0}^3}{v_2 r} \right)^{1/2} \exp \left(-\frac{(r - tv_{T0})^2}{4v_2 r / v_{T0}} \right) \quad (110)$$

and a similar one for the F wave, we obtain

$$u_{ij} \sim (\rho/2\rho_2) [b^{1/2}/(2\pi t)^{3/2}] \{ (\delta_{ik} - r_i r_j) N^{-3/2} \times \exp [-(r - tv_{T0})^2/4v_2 t] + \hat{r}_i \hat{r}_j H^{-3/2} \times \exp [-(r - tv_{F0})^2/4v_3 t] \}, \quad (111)$$

where

$$v_2 = (N/2b)(\rho_2/\rho)^2 \quad (112)$$

and

$$v_3 = (H/2b)(\rho_2 Z/\rho)^2. \quad (113)$$

The other quantities in Eqs. (110)–(113) are defined in Sec. III B. In going from (110) to (111) we have substituted $r = tv_{T0}$ for r in the pre-exponential factor. This is valid since any error is exponentially small. A similar substitution has been made for the F -wave term.

The result in (111) neglects terms $O(t^{-5/2})$ and smaller. The diffusion coefficients v_2 and v_3 are the same as those defined by Burridge and Vargas⁷; Eq. (111) agrees with the result in Ref. 7. We note that the two terms in (111) have their maxima at distinct times. We refer to Burridge and Vargas⁷ for a discussion of this result.

The slow wave diffusion process can be obtained by putting the S part of u^G in (43) into (104). The low-frequency expansions of Eqs. (63)–(65) are then used to approximate the integrand. Let u^D , with D for diffusion, be the displacement field. Then

$$u^D \approx \frac{-1}{4\pi b} \left(\frac{R+Q}{H} \right)^2 \nabla \nabla \left(\frac{J}{r} \right), \quad (114)$$

where

$$J = \int_{-\infty}^{\infty} \frac{-i}{2\pi\omega} \exp \left[r \left(\frac{\omega}{v_1} \right)^{1/2} e^{i3\pi/4} - i\omega t \right] d\omega \quad (115)$$

and

$$v_1 = (PR - Q^2)/bH. \quad (116)$$

The branch cut of $(\omega)^{1/2}$ in (115) is defined such that $(\omega)^{1/2} = i|\omega|^{1/2}$ for $\omega < 0$. The integral in (115) is to be understood as a principal value integral. It can be shown that

$$J = \frac{2}{\pi} \int_0^{\infty} e^{-s^2} \sin \left(\frac{sr}{v_1^{1/2}} \right) \frac{ds}{s} = \text{erf}[r/(4v_1 t)^{1/2}] \quad (117)$$

Combining Eqs. (114) and (117) we get

$$u^D \approx -\frac{1}{4\pi b} \left(\frac{R+Q}{H} \right)^2 \nabla \nabla \left[\frac{1}{r} \text{erf} \left(\frac{r}{(4v_1 t)^{1/2}} \right) \right] \quad (118)$$

and, from Eq. (65),

$$U^D \approx -[(P+Q)/(R+Q)]u^D. \quad (119)$$

These results agree with those of Burridge and Vargas,⁷ except for a minus sign.

The diffusion constant v_1 of (116) is equal to the diffusivity C_D of Rice and Cleary.²⁵ The equivalence has been demonstrated by Chandler and Johnson.²⁶ These authors discuss the behavior of C_D as a function of the physical parameters. Dutta and Odé²⁷ also find that the same diffusion constant governs the quasistatic slow wave. We note that in the stiff frame limit (see Sec. III E), we have

$$v_1 \approx \phi \kappa_f / b = v_f^2 / \omega_c, \quad (120)$$

where v_f is the speed of sound in the fluid. The other diffusion constants v_2 and v_3 of (112) and (113), unlike v_1 , depend upon the fluid and bulk densities $\rho_f = \rho_2/\phi$ and ρ . Therefore, they cannot be derived from a quasistatic analysis which ignores inertial effects. Rather, they are a consequence of the full Biot equations.

C. Precursors

In addition to the wave packets discussed above, there may also be discontinuous precursors. There are three in number, corresponding to F , T , and S waves. They are a consequence of the fact that the time harmonic waves can have finite attenuations at “infinite” frequency, i.e., for $\omega/\omega_c \gg 1$. In this respect, we note the caveats in Sec. III C and below. Under the assumption that $F(\omega) = 1$ as $\omega \rightarrow \infty$, the time harmonic Green tensors are given by the right-hand sides of (54) and (55) with k_α , D_α , and A_α , $\alpha = F, S$, and T , given by their asymptotic values discussed in Sec. III C. Specifically, $u^G(\omega) \rightarrow u^{G,\infty}(\omega)$, where

$$u_{ij}^{G,\infty} \equiv \frac{1}{4\pi N r} \left[(\delta_{ij} - \hat{r}_i \hat{r}_j) \exp \left(\frac{i\omega r}{v_{T\infty}} \right) e^{-a_\infty r} + \hat{r}_i \hat{r}_j \sum_{\alpha=F,S} \left(\frac{v_{T\infty}}{v_{\alpha\infty}} \right)^2 D_{\alpha\infty} \exp \left(\frac{i\omega r}{v_{\alpha\infty}} \right) e^{-a_\infty r} \right]. \quad (121)$$

The subscript ∞ indicates the infinite frequency ($\omega \rightarrow \infty$) or zero dissipation ($b \rightarrow 0$) limits of a quantity, which have been discussed in Sec. III C. The attenuation coefficients a_α , $\alpha = F, S$, and T follow from Eqs. (69), (71), and (73).

The displacement field ($u^G - u^{G,\infty}$) is $O(\omega^{-1})$ as $\omega \rightarrow \infty$. Therefore, its inverse Fourier transform can have finite step discontinuities but not δ -function singularities. However, since

$$\int_{-\infty}^{\infty} u_{ij}^{G,\infty} e^{-i\omega t} \frac{d\omega}{2\pi} \\ = \frac{1}{4\pi N r} \left[(\delta_{ij} - \hat{r}_i \hat{r}_j) \delta \left(t - \frac{r}{v_{T\infty}} \right) e^{-a_\tau r} \right. \\ \left. + \sum_{\alpha=F,S} \left(\frac{v_{T\infty}}{v_{\alpha\infty}} \right)^2 D_{\alpha\infty} \delta \left(t - \frac{r}{v_{\alpha\infty}} \right) e^{-a_\alpha r} \right] \quad (122)$$

and

$$\mathbf{u}^G = \mathbf{u}^{G,\infty} + (\mathbf{u}^G - \mathbf{u}^{G,\infty}), \quad (123)$$

it follows that (122) represents the most singular part of $\mathbf{u}$ as defined in (104). At the arrival of the pulses in (122),

$$a_\alpha r = t a_\alpha v_{\alpha\infty} = O(\omega_c t), \quad (124)$$

and since ω_c is very large for rocks, $\sim 10^6$ cps, it follows that the singularity strengths become negligible very quickly. References 7 and 20 also discuss the discontinuous precursors and reach the same conclusion about their fast decay.

Finally, we note that the existence of traveling singularities is due to the finite attenuation at infinite frequencies. Frequency-dependent models of the function $F(\omega)$ usually have $F(\omega) \sim (i\omega)^{1/2}$ (e.g., Biot²) and hence produce infinite attenuation as $\omega \rightarrow \infty$. The effect of this is to kill the singularities of (122). The low-frequency diffusive pulses are unchanged. Realistically, we can expect the damping to increase with frequency. Therefore, not much emphasis should be placed on the precursors. The dominant sources of motion are the low-frequency contributions (111) and (118).

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APPENDIX A: THE BIOT WAVE SPEEDS AND RATIOS

Transverse wave:

$$v_T^2 = (N/\rho)(\tilde{\gamma}_{22}/A), \quad (A1)$$

$$A_T = -\tilde{\gamma}_{12}/\tilde{\gamma}_{22}. \quad (A2)$$

Fast and slow waves, $\alpha = F$ or S :

$$v_\alpha^2 = (H/\rho)C_\alpha^2, \quad (A3)$$

$$A_\alpha = -\frac{\sigma_{1j} - \tilde{\gamma}_{1j}C_\alpha^2}{\sigma_{2j} - \tilde{\gamma}_{2j}C_\alpha^2}, \quad j = 1 \text{ or } 2, \quad (A4)$$

where

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \quad (A5)$$

$$\tilde{\gamma}_{ij} = \tilde{\rho}_{ij}/\rho, \quad (A6)$$

$$H = P + 2Q + R, \quad (A7)$$

$$\sigma_{11} = P/H, \quad \sigma_{12} = Q/H, \quad \sigma_{22} = R/H, \quad (A8)$$

$$A = \tilde{\gamma}_{11}\tilde{\gamma}_{12} - \tilde{\gamma}_{12}^2, \quad (A9)$$

$$C = \sigma_{11}\sigma_{22} - \sigma_{12}^2, \quad (A10)$$

$$B = \tilde{\gamma}_{11}\sigma_{22} + \tilde{\gamma}_{22}\sigma_{11} - 2\tilde{\gamma}_{12}\sigma_{12}, \quad (A11)$$

$$D = (B^2 - 4AC)^{1/2}, \quad (A12)$$

and

$$C_{F,S}^2 = (B \pm D)/2A. \quad (A13)$$

APPENDIX B

We will show that $C_{F\infty}^2 > 1$. Let B_∞ , D_∞ , and A_∞ be the limiting values of B , D , and A in Appendix A as $\omega \rightarrow \infty$. Then

$$1/C_{F\infty}^2 = (B_\infty - D_\infty)/2C \quad (B1)$$

and

$$D_\infty = (B_\infty - 2C)^2 + 4C(\rho_2/\rho)^2 Z^2 > |B_\infty - 2C|, \quad (B2)$$

with equality, if and only if, $Z = 0$, where Z is defined in Eq. (64). We note that $Z = 0$ is the compatibility condition.¹ Two cases now arise: First, if

$$B_\infty - 2C > 0, \quad (B3)$$

then (B1) and (B2) imply that $C_{F\infty} > 1$ since $C > 0$.¹ If

$$B_\infty - 2C < 0, \quad (B4)$$

then (B1) and (B2) give

$$1/C_{F\infty}^2 < (B_\infty - C)/C. \quad (B5)$$

However, from (B4) we see that the right-hand side of (B5) is less than 1, and so $C_{F\infty} > 1$. When $Z = 0$, we can easily show that (B3) holds and hence $C_{F\infty} = 1$. Similarly it can be shown that $C_{S\infty} < 1$.

¹M. A. Biot, "Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range," *J. Acoust. Soc. Am.* **28**, 168-178 (1956).

²M. A. Biot, "Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range," *J. Acoust. Soc. Am.* **28**, 179-191 (1956).

³M. A. Biot, "Generalized theory of acoustic propagation in porous media," *J. Acoust. Soc. Am.* **34**, 1254-1264 (1962).

⁴T. J. Plona, "Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies," *Appl. Phys. Lett.* **36**, 259-261 (1980).

⁵R. Burridge and J. B. Keller, "Poroelectricity equations derived from microstructure," *J. Acoust. Soc. Am.* **70**, 1140-1146 (1981).

⁶J. Geertsma and D. C. Smit, "Some aspects of elastic wave propagation in fluid-saturated porous solids," *Geophysics* **26**, 169-181 (1961).

⁷R. Burridge and C. A. Vargas, "The fundamental solution in dynamic poroelasticity," *Geophys. J. R. Astron. Soc.* **58**, 61-90 (1979).

⁸M. A. Biot and D. G. Willis, "The elastic coefficients of the theory of consolidation," *J. Appl. Mech.* **24**, 594-601 (1957).

⁹R. D. Stoll, "Acoustic waves in marine sediments," in *Physics of Sound in Marine Sediments*, edited by L. Hampton (Plenum, New York, 1974).

¹⁰S. Feng and D. L. Johnson, "High-frequency acoustic properties of a fluid/porous solid interface. I. New surface mode," *J. Acoust. Soc. Am.* **74**, 906-914 (1983).

¹¹D. L. Johnson and T. J. Plona, "Acoustic slow waves and the consolidation transition," *J. Acoust. Soc. Am.* **72**, 556-565 (1982).

¹²D. L. Johnson, "Equivalence between fourth sound in liquid He II at low temperatures and the Biot slow wave in consolidated porous media," *Appl. Phys. Lett.* **37**, 1065-1067 (1980).

¹³D. L. Johnson, T. J. Plona, C. Scala, F. Pasierb, and H. Kojima, "Tortuosity and acoustic slow waves," *Phys. Rev. Lett.* **49**, 1840-1844 (1982).

¹⁴J. G. Berryman, "Elastic wave propagation in fluid-saturated porous media," *J. Acoust. Soc. Am.* **69**, 416-424 (1981).

¹⁵J. D. Achenbach, *Wave Propagation in Elastic Solids* (North-Holland, Amsterdam, 1973), p. 82.

¹⁶J. G. Berryman, "Elastic waves in fluid-saturated porous media," in *Macroscopic Properties of Disordered Media*, edited by R. Burridge, S. Childress, and G. Papanicolaou (Springer-Verlag, Berlin, 1982).

¹⁷D. L. Johnson, "Elastodynamics of gels," *J. Chem. Phys.* **77**, 1531-1539 (1982).

¹⁸N. C. Dutta, "Theoretical analysis of observed second bulk compressional wave in a fluid-saturated porous solid at ultrasonic frequencies," *Appl. Phys. Lett.* **37**, 898-900 (1981).

¹⁹J. G. Berryman, "Confirmation of Biot's Theory," *Appl. Phys. Lett.* **37**, 382-384 (1981).

- ²⁰C. A. Vargas, "Dynamic poroelasticity," Ph.D. dissertation, New York University (1975).
- ²¹T. H. Tan, "Reciprocity relations for scattering of plane, elastic waves," *J. Acoust. Soc. Am.* **61**, 928–931 (1977).
- ²²T. H. Tan, "Theorem on the scattering and absorption cross section for scattering of plane, time-harmonic, elastic waves," *J. Acoust. Soc. Am.* **59**, 1265–1267 (1976).
- ²³B. Budiansky and J. R. Rice, "An integral equation for dynamic elastic response of an isolated 3-D crack," *Wave Motion* **1**, 187–192 (1979).
- ²⁴H. Deresiewicz and R. Skalak, "On uniqueness in dynamic poroelasticity," *Bull. Seismol. Soc. Am.* **53**, 783–788 (1963).
- ²⁵J. R. Rice and M. P. Cleary, "Some basic stress diffusion solutions for fluid-saturated porous media with compressible constituents," *Rev. Geophys. Space Phys.* **14**, 227–241 (1976).
- ²⁶R. N. Chandler and D. L. Johnson, "The equivalence of quasistatic flow in fluid-saturated porous media and Biot's slow wave in the limit of zero frequency," *J. Appl. Phys.* **52**, 3391–3395 (1981).
- ²⁷N. C. Dutta and H. Odé, "Attenuation and dispersion of compressional waves in fluid-filled porous rocks with partial gas saturation," *Geophysics* **44**, 1777–1788 (1979).